The following preprints in various stages of being prepared:


   This manuscript describes how to correctly calculate the nonlinear coefficients in the case of an electromagnetic pulse propagating in a two-component plasma. We also demonstrate that other values given in the literature are incorrect. We correct the predictions for such electromagnetic propagation and discuss the astrophysical consequences.
   The above results have been presented as short talks at two meetings, the APS plasma physics meeting in Nov. 1988 and at the Grossman general relativity meeting in Australia in Aug. 1988.

   This problem has become much more complex than first envisioned. In particular, the longitudinal electric field is found to be much larger than first estimated. In this limit, it seems that any charge separation leads to an intense longitudinal electrical field. The consequences of this are being explored numerically in order to determine how to correctly formulate the expansion.
Reduction Of 3 x 3 Polynomial Bundles

And New Types Of Integrable 3-Wave Interactions

by

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1. Our aim is to show that the group of reductions proposed by Mikhailov (A.V. Mikhailov, 1981) can be effectively used in constructing of new versions of integrable nonlinear evolution equations (NLEF) in $1 + 1$ dimensions. We illustrate this by two examples, which lead to new integrable versions of the well known 3-wave interaction (V.E. Zakharov and S.V. Manakov, 1975) and (D.J. Kaup, 1976).

Let us consider a matrix Lax pair, first order in $\frac{d}{dx}$ and $\frac{d}{dt}$ of the form:

\[
(1 \frac{d}{dx} + U(x,t,\lambda)) \psi(x,t,\lambda) = 0 \tag{1}
\]

\[
(1 \frac{d}{dt} + V(x,t,\lambda)) \psi(x,t,\lambda) = 0 \tag{2}
\]

Following (A.V. Mikhailov, 1981) we will say, that it possesses a $\mathbb{Z}_N$ group of reductions if $U$ and $V$ satisfy the relations:
$K^{-1} U(x,t,\lambda) = U(x,t,\lambda_\omega), \quad K^{-1} V(x,t,\lambda) = V(x,t,\lambda_\omega)$  

(3)

where $K$ is a constant matrix such, that $K^N = I$ and $\omega = \exp(2\pi i/N)$. In what follows we shall also impose the involution (or $Z_2$ - reduction):

$$U^+(x,t,\lambda) = B_1^{-1} U(x,\varepsilon\lambda) B_1,$$

(4)

and the same for $V(x,t,\lambda)$, where $B_1^+ B_1^{-1} = I$ and $\varepsilon = \pm 1$. We limit ourselves to the simplest possible case, when $U$ and $V$ are $3 \times 3$ matrix - valued functions, depending polynomially in $\lambda$.

2. As first example we shall consider $U$ and $V$ to be quadratic on $\lambda$:

$$U(x,t,\lambda) = \sum_{k=0}^{2} U_k(t,x) \lambda^k, \quad V(x,t,\lambda) = \sum_{k=0}^{2} V_k(t,x) \lambda^k$$

(5)

where

$$U_0(x,t) = \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ 0 & 0 & u_{33} \end{bmatrix}, \quad U_1(x,t) = \begin{bmatrix} 0 & 0 & u_{13} \\ 0 & 0 & u_{23} \\ u_{31} & u_{32} & 0 \end{bmatrix},$$

(6)

$$V_0(x,t) = \begin{bmatrix} v_{11} & v_{12} & 0 \\ v_{21} & v_{22} & 0 \\ 0 & 0 & v_{33} \end{bmatrix}, \quad V_1(x,t) = \begin{bmatrix} 0 & 0 & v_{13} \\ 0 & 0 & v_{23} \\ v_{31} & v_{32} & 0 \end{bmatrix},$$

(7)

and
are constant diagonal matrices, whose entries are pairwise different:
\(a_1 \neq a_2 \neq a_3 \neq a_1\) and \(b_1 \neq b_2 \neq b_3 \neq b_1\). Obviously, \(U\) and \(V\) defined by (5)-(8) satisfy (3) with \(N=2\) and \(K = \text{diag}(1,1,-1)\). Imposing the involution (4) with \(B_1 = \text{diag}(1,\gamma,\alpha)\), \(\alpha, \gamma\) - real constants, leads to:

\[
u_{21} = \gamma u_{12}^*, \quad u_{31} = au_{13}^*, \quad u_{32} = \frac{\alpha}{\gamma} u_{23}^* \]

\[
u_{21} = \gamma v_{12}^*, \quad v_{31} = av_{13}^*, \quad v_{32} = \frac{\alpha}{\gamma} v_{23}^* \]

\[
u_{2j} = \nu_{jj}^*, \quad v_{jj} = v_{jj}^*, \quad a_j = a_j^*, \quad b_j = b_j^* \]

After some calculations we find, that the compatibility condition for (1),(2) gives us the following expressions for \(v_{ij}\) in terms of \(u_{ij}\):

\[
u_{13} = \eta_{13} u_{13}, \quad \nu_{23} = \eta_{23} u_{23} \]

\[
u_{12} = \eta_{12} u_{12} + \frac{\alpha \kappa}{\gamma} u_{13}^* u_{23} \]

where

\[
\eta_{2j} = \frac{b_j - b_1}{a_j - a_1}, \quad \kappa = \frac{\eta_{23} - \eta_{13}}{a_1 - a_2} \]

and the following NLEE for \(u_{ij}\):

\[
-\nu_{2} = \text{diag}(a_1, a_2, a_3), \quad -\nu_{2} = \text{diag}(b_1, b_2, b_3) \]
\[
\begin{align*}
1 \left( \partial_t - \eta \partial_x \right) u &= (\eta - \eta_2) u_{12} u_{23} - \frac{\alpha \kappa}{\gamma} u_1 u_{23}^2 \\
&\quad + u_1 \left[ \left( \eta_1 \left( u_{11} - u_{22} \right) + v_{33} - v_{11} \right) \right] \\
1 \left( \partial_t - \eta_2 \partial_x \right) u &= \gamma (\eta - \eta_2) u_{12} u_{23} - \kappa \alpha |u_1|^2 u_{23} \\
&\quad + u_2 \left[ \left( \eta_2 \left( u_{22} - u_{33} \right) + v_{33} - v_{22} \right) \right] \\
1 \left( \partial_t - \eta_1 \partial_x \right) u &= \frac{\alpha \kappa}{\gamma} \left[ i \partial_x \left( u_{13} u_{23}^* \right) + u_1 u_{23}^* \left( u_{11} - u_{22} \right) \right] \\
&\quad + u_3 \left[ \left( \eta_1 \left( u_{11} - u_{22} \right) + v_{33} - v_{11} \right) \right]
\end{align*}
\]

In (10) (12) we have written down only the relations (NLEE's) for \( u_{ij} \), \( u_{ij} \) with \( i < j \); the corresponding ones with \( i > j \) are obtained from them by complex conjugation and the use of (9). As regards the diagonal elements \( u_{ij} \) and \( v_{ij} \), they satisfy:

\[
i(v_{jj},x - u_{jj},t) + F_{jj}(x,t) = 0
\]

\[
F_{11} = - F_{22} = \alpha \kappa (u_{12} u_{13} u_{23} - u_{12} u_{13} u_{23}) \quad F_{33} = 0
\]

We can fix up the diagonal terms \( u_{jj} \) and \( v_{jj} \) by choosing the gauge of the Lax pair. There are many possibilities to do this:
\[ u_{jj} = u_j |u_{13}|^2 + \gamma v_j |u_{23}|^2 \]
\[ v_{jj} = u_j n_{13} |u_{13}|^2 + \gamma v_j n_{23} |u_{23}|^2 \quad j = 1, 2, 3 \] (14)

Then equations (13) are direct consequence of (12), if the constants \( u_j, v_j \) are related by:

\[ u_j = \frac{a_2 - a_3}{a_1 - a_3} v_j + \theta_j, \quad \theta_1 = -\theta_2 = -\frac{a}{\gamma(a_1 - a_3)}, \quad \theta_3 = 0 \] (15)

Note that only differences of \( u_j \) and \( v_j \) occurs in (12), and by a phase transformation on \( u_{ij} \) and \( v_{ij} \), one can also transform some of the constants in (15) to zero. For example, one may easily phase transform all three \( u_j \)'s to be equal or all three \( v_j \)'s to be equal. But because of (13), with \( F_{jj} \) nonzero in general, one may never phase transform all three \( u_j \)'s or all three \( v_j \)'s to be zero.

Thus the first two equations in (12) contain in addition to the usual bilinear in \( u_{ij} \) nonlinearities, also cubic terms. In the third equation in (12) the usual bilinear in \( u_{ij} \) term appears under an \( x \)-derivative; here we also have cubic terms.

Note, that both reductions on \( U \) and \( V \) commute between themselves, so the total reduction group is \( Z_2 \times Z_2 \).

As an example, let us choose \( a_3 = 0, a_2 = -a_1, b_1 = b_2 = 0, b_3 = -ca_1, a_1 = c/k, u_{13} = E, u_{23} = F, u_{12} = N, \gamma = -1, v_2 = v_1 + 1/c, v_3 = v_2 - 3/(2c), \) and \( \omega = 1 \), then (12) reduces to
\[ i (\partial_t - c \partial_x) E = - cNF \]
\[ i (\partial_t + c \partial_x) F = - cN*E \]  
\[ i \partial_t N = i 2 \partial_x (F*E) + F*E (|E|^2 - |F|^2) \frac{1}{c} \]
\[ + N (|E|^2 + |F|^2) \]  

These equations remind one of two counterpropagating electromagnetic beams, \( E \) and \( F \), interacting via some density modulation, \( N \). And these are another example of integrable interactions between long waves and short waves (D.J. Benny, 1977) and (Alan C. Newell, 1978).

3. Now consider \( U \) and \( V \) to be cubic in \( \lambda \):

\[ U(x,t,\lambda) = \sum_{k=0}^{3} U_k(x,t) \lambda^k, \]
\[ V(x,t,\lambda) = \sum_{k=0}^{3} V_k(x,t) \lambda^k \]  

and such, that, \( U_0, U_3, V_0 \) and \( V_3 \) are diagonal:

\[ U_0 = \text{diag} (u_{11}, u_{22}, u_{33}), \quad U_3 = \text{diag} (a_1, a_2, a_3) \]
\[ V_0 = \text{diag} (v_{11}, v_{22}, v_{33}), \quad V_3 = \text{diag} (b_1, b_2, b_3) \]  

where \( a_j, b_j \) are the same constants as in (8) above. The matrices \( U_1, U_2, V_1, V_2 \) are given by:
With this choice $U$ and $V$ automatically satisfy the reduction condition (3) with $N=3$ and $K=\text{diag}(1, \omega, \omega^2)$, $\omega = \exp(2\pi i/3)$. We can also impose the involution (4)

$$
V_1(x,t) = \begin{bmatrix}
0 & v_{12} & 0 \\
0 & 0 & v_{23} \\
v_{31} & 0 & 0
\end{bmatrix}, \quad V_2(x,t) = \begin{bmatrix}
0 & 0 & v_{13} \\
v_{21} & 0 & 0 \\
0 & v_{32} & 0
\end{bmatrix}
$$

(20)

$$
U_1(x,t) = \begin{bmatrix}
0 & u_{12} & 0 \\
0 & 0 & u_{23} \\
u_{31} & 0 & 0
\end{bmatrix}, \quad U_2(x,t) = \begin{bmatrix}
0 & 0 & u_{13} \\
u_{21} & 0 & 0 \\
0 & u_{32} & 0
\end{bmatrix}
$$

(19)

With this choice $U$ and $V$ automatically satisfy the reduction condition (3) with $N=3$ and $K=\text{diag}(1, \omega, \omega^2)$, $\omega = \exp(2\pi i/3)$. We can also impose the involution (4)

$$
\begin{bmatrix}
0 & 0 & \varepsilon_1 \\
0 & 1 & 0 \\
\varepsilon_1 & 0 & 0
\end{bmatrix}, \quad \varepsilon_1 = \pm 1,
$$

which gives:

$$
\begin{align*}
\varepsilon_1 u_{12} &= -u_{23}, \\
\varepsilon_1 u_{31} &= -u_{31}, \\
\varepsilon_1 u_{13} &= u_{13} \\
\varepsilon_1 u_{21} &= -u_{32}, \\
\varepsilon_1 u_{11} &= -u_{33}, \\
\varepsilon_1 u_{22} &= -u_{22}
\end{align*}
$$

(21)

$$
\begin{align*}
a_3 &= -a_1, \\
a_2 &= -a_2
\end{align*}
$$

and analogous relations for $v_{ij}$ and $b_j$. The compatibility condition now gives:
\[ v_{12} = -\eta_{12} u_{12} + \kappa \varepsilon_1 u_{13} u_{21}^*; \quad v_{13} = \eta_{13} u_{13} \]
\[ v_{31} = \eta_{13} u_{31} + \kappa \varepsilon_1 |u_{21}|^2; \quad v_{21} = \eta_{12} u_{21} \]  
(22)

where \( \eta_{ij} \) and \( \kappa \) are expressed through \( a_j, b_j \) as in (11). The corresponding NLEE have the form:

\[ i(\partial_t - \eta_{13} \partial_x)u_{13} = \varepsilon_1 (\eta_{12} - \eta_{23}) |u_{12}|^2 + \kappa u_{13} (u_{12} u_{21} + u_{12}^* u_{21}^*) \]
\[ + u_{13} [\eta_{13} (u_{11} - u_{33}) + v_{33} - v_{11}] \]
\[ i(\partial_t - \eta_{12} \partial_x)u_{21} = - \varepsilon_1 (\eta_{13} - \eta_{23}) u_{12}^* u_{31} - \kappa u_{21} (u_{12}^* u_{21}^* + u_{13} u_{31}) \]
\[ + u_{21} [\eta_{12} (u_{22} - u_{11}) + v_{11} - v_{22}] \]
\[ i(\partial_t - \eta_{12} \partial_x)u_{12} = i \kappa \varepsilon_1 (u_{13} u_{21}^*) + \kappa \varepsilon_2 u_{13} u_{21} (u_{11} - u_{22}) \]
\[ + u_{12} [\eta_{12} (u_{11} - u_{22}) + v_{22} - v_{11}] \]
\[ i(\partial_t - \eta_{13} \partial_x)u_{31} = i \kappa \varepsilon_1 (|u_{21}|^2) + \kappa \varepsilon_1 |u_{21}|^2 (u_{33} - u_{11}) \]
\[ + u_{31} [\eta_{13} (u_{33} - u_{11}) + v_{11} - v_{33}] \]  
(23)

For the diagonal elements \( u_{jj}, v_{jj} \) we get

\[ u_{jj,t} - v_{jj,x} = 0, \quad j = 1, 2, 3 \]  
(24)
If we fix the gauge in the simplest possible way, choosing \( v_{jj} = u_{jj} = 0 \), then what we get is a modification of the 3-wave equations with additional cubic nonlinearities in the first two equations in (23); the nonlinearities in the two last equations of (23) just acquire additional \( x \)-derivative.

However, we can choose another gauge by requiring that the corresponding linear problems (1), (2) become equivalent to a Riemann-Hilbert problem with canonical normalization. This leads to:

\[
\begin{align*}
  u_{11} &= \frac{u_{12} u_{21}}{a_1 - a_2} + \frac{u_{13} u_{31}}{a_1 - a_3} - \frac{\epsilon_1 u_{13} |u_{21}|^2}{(a_1 - a_2)(a_1 - a_3)}, \\
  u_{22} &= -\frac{u_{12} u_{21}}{a_2 - a_3} - \frac{u_{12} u_{21}}{a_1 - a_2} + \frac{\epsilon_1 u_{13} |u_{21}|^2}{(a_1 - a_3)(a_2 - a_3)} \quad (25)
\end{align*}
\]

and

\[
\begin{align*}
  v_{11} &= \frac{n_{12} u_{12} u_{21}}{a_1 - a_2} + \frac{n_{13} u_{13} u_{31}}{a_1 - a_3} + \theta_{11} \epsilon_1 u_{13} |u_{21}|^2, \\
  v_{22} &= -\frac{n_{23} u_{12} u_{21}}{a_2 - a_3} - \frac{n_{12} u_{12} u_{21}}{a_1 - a_2} + \theta_{22} \epsilon_1 u_{13} |u_{21}|^2, \\
  \theta_{11} &= \frac{3a_1 \kappa - n_{23}}{(a_1 - a_2)(a_1 - a_3)}, \quad \theta_{22} = \frac{-3a_2 \kappa + n_{13}}{(a_1 - a_2)(a_2 - a_3)} \quad (26)
\end{align*}
\]
In deriving the last line of (26) we have used also, that \[ \sum_{j=1}^{3} s_j = 0; \] \[ u_{33} \]

3 defined by \( \sum_{j=1}^{3} u_{jj} = \sum_{j=1}^{3} v_{jj} = 0 \). From (27), (26) we see, that such choice

of the gauge leads to an additional quartic and quintic nonlinearities in the

WLE (23).

At the end we give the explicit form of the WLE (23) in the simplest

possible case, when \( u_{jj} = v_{jj} = 0 \) and moreover \( \eta_{12} = \eta_{23} = c = \text{real} \) and \( \eta_{13} = 0 \). Denoting

\[ u_{12} = E(x,t), \quad u_{13} = n(x,t), \quad u_{23} = -\varepsilon_1 E(x,t) \]

\[ u_{21} = F(x,t), \quad u_{31} = iN(x,t), \quad u_{32} = \varepsilon_1 F(x,t) \] (27)

we get the following system for the two real \( n(x,t), N(x,t) \) and the two

complex-valued functions \( E(x,t) \) and \( F(x,t) \):

\[ \partial_{t} n = g n (E*F* + EF) \]

\[ \partial_{t} N = g \varepsilon_1 \partial_{x} (F*F) \]

\[ \partial_{t} F = -g E* (F*F) - i g N n F \]

\[ \partial_{t} E = i \varepsilon_1 g \partial_{x} (n F*) \] (28)

Here \( g = -i\kappa \) is a real-valued constant.

The question for possible physical applications of these equations is

open.
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References


