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ON THE ENERGY RELEASE RATE FOR DYNAMIC TRANSIENT ANTI-PLANE SHEAR CRACK PROPAGATION IN A GENERAL LINEAR VISCOELASTIC BODY

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J.R. WALTON

OFFICE OF NAVAL RESEARCH
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On the Energy Release Rate for Dynamic Transient Anti-Plane Shear Crack Propagation in a General Linear Viscoelastic Body

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Summary.

The problem of a semi–infinite mode III crack that suddenly begins to propagate at a constant speed is considered for a general linear viscoelastic body. Simple closed form expressions are derived for the Laplace transform of the stress, displacement, and stress intensity factor. A Barenblatt type failure zone is assumed to exist at the crack tip and an expression for the total energy flux into the failure zone per unit crack advance, G(t), is derived. The short time asymptotic form of G is constructed and the rate at which it converges to its steady–state limit is studied. It is shown that the rate of convergence to steady–state is dependent upon crack speed and material properties. Moreover, it is found that whether or not a failure zone is incorporated into the model significantly influences both quantitatively and qualitatively the short and long time behavior of G. This difference is important to predictions of stable vs unstable crack speeds based upon a critical work input to the failure zone fracture criterion.

1. Introduction.

Several analytical studies of dynamically propagating cracks in linear viscoelastic material have appeared in the literature since Willis(1967) presented an analysis of the dynamic, steady–state propagation of a semi–infinite, mode III (anti–plane shear) crack in an infinite viscoelastic body. Employing transform methods and the Wiener–Hopf technique, Willis constructed the dynamic stress intensity factor (SIF) for a standard linear solid material model and general crack face loadings. Subsequently, Atkinson and List(1972) introduced transient effects into the problem by assuming that the crack, initially at rest, begins to propagate at a constant speed under the action of suddenly applied loads on the crack faces. Also utilizing the Wiener–Hopf method, they derived an expression for the Laplace transform of the time dependent SIF from which long and short time asymptotic approximations and numerical Laplace inversion calculations were obtained. However, their analysis was limited to consideration of constant applied load along the crack faces and the required Wiener–Hopf factorization was effected only for a Maxwell material model and the Achenbach–Chao(1962) three parameter approximation to the standard linear solid.

Somewhat later, Atkinson and Coleman(1977) used a matched asymptotic expansion technique to develop an approximate analysis of the steady–state propagation of a semi–infinite mode I (plane strain or plane stress) crack propagating in a clamped viscoelastic strip. Shortly thereafter, Atkinson(1979) presented an approximate analysis of the mode I counterpart to the mode III problem considered by Atkinson and List(1972). Their argument, ostensibly valid for fairly general material models, involved slightly modifying the exact elastic result of Baker(1962) in order to approximate the Laplace transform of the actual dynamic viscoelastic SIF. The dominant term for each of the short and long time asymptotic expansions of this approximate SIF was derived for each of three different applied crack face loads: a constant, a delta function point force and an exponentially decreasing form. Also in that paper, Atkinson reconsidered the mode III problem and extended the Atkinson and List analysis to handle the above three types of crack face loadings. However, consideration was limited again to the Achenbach–Chao material model. Atkinson also constructed an expression for the energy release rate (ERR) based upon a local (i.e. at the crack tip) work argument and the singular stress field.

Also in that same year, Atkinson and Popelar(1979) presented an analysis of the transient constant crack speed mode III problem for a viscoelastic strip. Constitutive relations in terms of differential operators were assumed and the external load consisted either of constant displacement of the upper and lower layer boundaries or constant tractions on them. The crack faces were assumed to be stress free. Again the Wiener–Hopf method was used to construct an exact expression for the Laplace transform of the SIF. The required Wiener–Hopf factorization was carried out modulo a term involving a Cauchy type integral. Atkinson and Popelar then restricted attention to numerically approximating the Cauchy integral for the steady–state limit case and assumed a standard linear solid material model.

A year later, Atkinson and Popelar(1980) addressed the more difficult mode I problem for a viscoelastic strip. Again by use of the Wiener–Hopf method, a formal expression for the Laplace transform of the SIF was constructed containing a complicated Cauchy integral. As with the corresponding mode III case, the integral was studied numerically in the limiting special case of steady–state crack propagation in a standard linear solid.

Somewhat later, Walton(1982) examined further the steady–state mode III problem considered by Willis(1967). Utilizing the Riemann–Hilbert rather than the Wiener–Hopf methodology, he constructed a simple closed form expression for the SIF valid for general crack face loadings and very general material models. More specifically, constitutive equations expressed in terms of convolution integrals rather than differential operators were adopted and the results were shown to be valid irrespective of any assumed time rate of decay of the viscoelastic shear modulus. In contrast, constitutive relations in terms of constant coefficient differential operators, necessarily force an exponentially decaying modulus thereby preventing consideration of the important class of power–law models which more effectively represent the mechanical response of many real viscoelastic materials, such as rubber, than do exponentially decaying functions.

Subsequently, Walton(1983) extended the above analysis to determine the angular dependence of the stress field in a neighborhood of the crack tip. In particular, it was shown that the asymptotic stress field at the crack tip has the same angular dependence as the corresponding dynamic elastic problem. Only the SIF differs between the elastic and viscoelastic fields.

Walton(1985) next considered the steady–state mode III problem for a viscoelastic strip. Again utilizing the Riemann–Hilbert method, a closed form expression for the SIF was constructed for general loadings and shear modulus. The form of the solution exhibits clearly the combined effects of material properties, crack speed and layer thickness upon the SIF.

More recently, Walton(1987a) reconsidered steady–state mode III crack propagation in an infinite viscoelastic body in order to investigate the implications of including a failure zone model of Barenblatt(1962) type into the determination of the ERR from a global energy balance calculation. A simple closed form expression for the ERR, which in this case is just the work done by the tractions in the failure zone, was derived under the same mild conditions on the shear modulus assumed in Walton(1982) and for a fairly broad class of crack face and failure zone loadings. It was then observed that whether or not a failure zone is incorporated into the model greatly influences both qualitatively and quantitatively the dependence of the ERR upon crack speed and material properties. In particular, calculations based upon a failure zone seem to reflect more closely experimental observations of cracks rapidly propagating in real viscoelastic material.[1]

The methods of Walton(1987a) can be applied to the calculation of the ERR for a wide variety of dynamic viscoelastic crack problems. Schovanec and Walton(1987c) recently completed the analysis of the dynamic steady–state propagation of two parallel mode III cracks in an infinite viscoelastic body. Also, Walton(1987b) has recently completed a study of the mode I analog of Walton(1987a). In both of these investigations a Barenblatt failure zone model was adopted. Of related interest are two additional papers by Schovanec and Walton(1987a,1987b) in which these same methods were applied to quasi–static mode I crack propagation in non–homogeneous viscoelastic material. It should also be noted that Knauss(1973) and Schapery(1975) applied the Barenblatt model to quasi–static viscoelastic crack growth in viscoelastic material and observed that whether or not a failure zone is incorporated into the model greatly affects the behavior of the ERR.

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[1] Private communication with Prof. W.G. Knauss and Prof. K. Kuo.

The present paper applies the above Barenblatt failure zone/ Riemann–Hilbert program to the transient mode III problem of a semi–infinite crack in an infinite, linear viscoelastic body that begins to propagate at a constant speed, v, due to the sudden application of tractions to the crack faces which then propagate with the moving crack tip. Thus the crack speed is constant for times $t>0$ and the crack face tractions are spatially constant, though time varying, relative to the moving crack tip. Clearly it would be desirable to allow nonsteady crack growth in order to study the initial crack acceleration phase. However, for dynamic crack propagation in viscoelastic material such an analysis seems quite difficult to carry out. Moreover, the primary issue addressed here is the determination of the influence that crack speed and viscoelastic constitutive properties have upon the rate of convergence to steady state of the work input to the failure zone. Consequently, it is felt that valuable insight into the asymptotic behavior of the more realistic model with unsteady crack growth can be obtained by studying the admittedly artificial transient problem in which the crack speed is kept constant. For the sake of mathematical completeness, a short time asymptotic analysis is also presented here.

The next section contains a derivation of the asymptotic singular stress field in front of the advancing crack tip in the absence of a Barenblatt failure zone. More specifically, a closed form expression for the Laplace transform of the stress intensity factor is constructed. The analysis is valid for a very general class of shear moduli, $\mu(t)$, and a very general class of time and spatially varying crack face tractions, thereby generalizing the results of Atkinson and List(1972).

In section 3, a Barenblatt failure zone is introduced and an expression for the work input to the failure zone is derived for general shear modulus $\mu(t)$ but for special crack face applied tractions and failure zone constitutive model. The failure zone constitutive model adopted here is certainly open to criticism on physical grounds. But again, for reasons discussed in section 3, it is felt that consideration of this admittedly artificial model provides valuable insight into the longtime asymptotic behavior of more realistic models under the assumption of small scale yielding. The final section contains the long and short time asymptotic analyses. The results are then illustrated through special cases and examples.

2. Problem Formulation and Stress Analysis.

The problem to be considered is that of a semi–infinite mode III crack that begins to propagate at a constant speed v in an infinite viscoelastic body due to the sudden application of crack face tractions that then travel with the crack. The governing field equations for the motion of a linear viscoelastic solid are

$$\rho\ddot{u}_i=\sigma_{ij,j}, \quad \epsilon_{ij}={}^1/_2(u_{i,j}+u_{j,i}), \quad \sigma_{ij}=2\mu^*d\epsilon_{ij}+\delta_{ij}\lambda^*d\epsilon_{kk}, \tag{2.1}$$

where σ_{ij}, ϵ_{ij}, and u_i denote the stress, strain, and displacement fields respectively. In (2.1), $\mu^*d\epsilon$ denotes the Riemann–Stieltjes convolution $\mu^*d\epsilon=\int_{-\infty}^{t}\mu(t-\tau)\,d\epsilon(\tau)$.

Since the deformation is assumed to be antiplane strain, $u_1=0$, $u_2=0$, and the only equation of motion not identically satisfied is $\mu^*d\Delta u_3=\rho\ddot{u}_3$, where Δ denotes the Laplacian operator.

A semi–infinite crack lying along the negative x_1–axis is assumed to begin to propagate at time $t=0$ with a constant speed v driven by loads $\sigma_{23}(x_1,0,t)=L_e\Lambda_e(\frac{x_1-vt}{a_e},t)$ which follow it, where $\Lambda_e(\ )$ is dimensionless while a_e and L_e have the dimensions of length and stress, respectively. Thus, while the crack speed is assumed to be constant, the driving load is allowed to be time varying. The corresponding initial–boundary value problem is

$$\rho\ddot{u}_3=\mu^*d\Delta u_3=\Delta\left[\mu(0)u_3(x_1,x_2,t)+\int_0^t u_3(x_1,x_2,\tau)\,\mu'(t-\tau)\,d\tau\right] \tag{2.2}$$

with initial conditions $u_3=0$, $\dot{u}_3=0$ at $t=0$

and boundary conditions
$$\begin{aligned} &\sigma_{23}(x_1,0,t)=L_e\Lambda_e(\tfrac{x_1-vt}{a_e},t) && x_1<vt \\ &u_3(x_1,0,t)=0 && x_1>vt \\ &\sigma_{23}(x_1,x_2,t)\to 0 && \text{as } x_1^2+x_2^2\to\infty. \end{aligned} \tag{2.3}$$

From (2.1) it can be seen that

$$\sigma_{23}(x_1,x_2,t)=\frac{\partial}{\partial x_2}(\mu^* du_3)=\frac{\partial}{\partial x_2}\left[\mu(0)u_3(x_1,x_2,t)+\int_0^t u_3(x_1,x_2,\tau)\mu'(t-\tau)\,d\tau\right]. \tag{2.4}$$

It is convenient to change from the fixed coordinates (x_1,x_2,t) to the moving coordinate system (x,y,t) given by $x=x_1-vt$, $y=x_2$ and to define $w(x,y,t)$ by $w(x,y,t)=w(x_1-vt,x_2,t)=u_3(x_1,x_2,t)$. In the moving coordinates, equation (2.2) becomes

$$\rho\left[\frac{\partial}{\partial t}-v\frac{\partial}{\partial x}\right]^2 w(x,y,t)=\Delta\left[\mu(0)w(x,y,t)+\int_0^t w(x_1-v\tau,x_2,\tau)\mu'(t-\tau)\,d\tau\right]$$
$$=\Delta\left[\mu(0)w(x,y,t)+\int_0^t w(x+vt-v\tau,y,\tau)\mu'(t-\tau)\,d\tau\right]. \tag{2.5}$$

Application of the Fourier transform, defined by $\hat{f}(p,y,t)=\int_{-\infty}^{\infty}e^{ipx}f(x,y,t)\,dx$, to equation (2.5) results in

$$\rho\left[\frac{\partial}{\partial t}+ivp\right]^2\hat{w}(p,y,t)=\left[(-ip)^2+\frac{\partial^2}{\partial y^2}\right]\left[\mu(0)\hat{w}(p,y,t)+\int_0^t\hat{w}(p,y,\tau)e^{-ivp(t-\tau)}\mu'(t-\tau)\,d\tau\right]. \tag{2.6}$$

Subsequent application of the Laplace transform $\bar{g}(p,y,s)=\int_0^{\infty}g(p,y,\tau)e^{-s\tau}\,d\tau$ to (2.6) yields the equation

$$\rho(s+ivp)^2\bar{\hat{w}}(p,y,s)=\left[\frac{\partial^2}{\partial y^2}-p^2\right]\left[\tilde{\mu}(s+ivp)\bar{\hat{w}}(p,y,s)\right] \tag{2.7}$$

in which $\tilde{\mu}(s)$ is the Carson transform of the shear modulus given by

$$\tilde{\mu}(s)=s\bar{\mu}(s)=\mu(0)+\int_0^{\infty}e^{-s\tau}\,d\mu.$$

Equation (2.7) can be rewritten as $\frac{\partial^2}{\partial y^2}\bar{\hat{w}}-\left[p^2+\frac{\rho}{\tilde{\mu}(s+ivp)}(s+ivp)^2\right]\bar{\hat{w}}=0$ which has the solution $\bar{\hat{w}}(p,y,s)=A(p,s)e^{-\beta(s,p)|y|}$, where $\beta(s,p)=\left[p^2+\frac{\rho}{\tilde{\mu}(s+ivp)}(s+ivp)^2\right]^{1/2}$ must be chosen so that Re $\beta\geq 0$.

In a similar manner, the Fourier and Laplace transforms may be applied to the constitutive equation (2.4) to produce $\bar{\hat{\sigma}}_{23}(p,y,s)=\tilde{\mu}(s+ivp)\frac{\partial}{\partial y}\bar{\hat{w}}(p,y,s)$. If one defines $\hat{f}^+(p)$ and $\hat{f}^-(p)$ by $\hat{f}^+(p)=\int_0^\infty e^{ipx}f(x)\,dx$ and $\hat{f}^-(p)=\int_{-\infty}^0 e^{ipx}f(x)\,dx$, then the boundary condition (2.3) transforms to

$$\bar{\hat{\sigma}}_{23}^{+}(p,0,s)+\bar{\hat{\sigma}}_{23}^{-}(p,0,s)=\tilde{\mu}(s+ivp)\frac{\partial}{\partial y}\bar{\hat{w}}(p,0,s)=-\beta(s,p)\tilde{\mu}(s+ivp)\bar{\hat{w}}(p,0,s). \tag{2.8}$$

From (2.3) it can be seen that $\bar{\hat{\sigma}}_{23}^{-}(p,0,s)=L_e a_e \bar{\hat{\Lambda}}_e(a_e p,s)\equiv g(p,s)$ and $\bar{\hat{w}}(p,0,s)=\bar{\hat{w}}^-(p,0,s)$. It is assumed a priori (and is easily verified a posteriori) that $\bar{\hat{w}}^-(p,0,s)$ and $\bar{\hat{\sigma}}_{23}^{+}(p,0,s)$ have analytic extensions $\bar{\hat{w}}^-(z,0,s)$ and $\bar{\hat{\sigma}}_{23}^{+}(z,0,s)$ for $\mathrm{Im}(z)<0$ and $\mathrm{Im}(z)>0$, respectively, which vanish as $|z|\to\infty$. Thus the transformed boundary condition (2.8) can be recast as the Riemann–Hilbert problem: find $F^+(z)$ analytic for $\mathrm{Im}(z)>0$ and $F^-(z)$ analytic for $\mathrm{Im}(z)<0$ such that

$\lim_{\mathrm{Im}(z)\to+\infty} F^+(z)=\lim_{\mathrm{Im}(z)\to-\infty} F^-(z)=0$ and on $\mathrm{Im}(z)=0$,

$$F^+(p)=T(p)F^-(p)-g(p,s) \quad \text{for } p\epsilon(-\infty,\infty) \tag{2.9}$$

where $F^+(z)=\bar{\hat{\sigma}}_{23}^{+}(z,0,s)$, $F^-(z)=\bar{\hat{w}}^-(z,0,s)$, and

$T(p)=-\tilde{\mu}(s+ivp)\left[p^2+\frac{\rho}{\tilde{\mu}(s+ivp)}(s+ivp)^2\right]^{1/2}$. For convenience, explicit reference to the s dependence of $T(p)$ and $F^\pm(z)$ is being suppressed. It suffices to note that all of the following analysis is valid for any complex s with $\mathrm{Re}(s)\geq 0$.

It is well known that the solution of (2.9) is

$$F^\pm(z)=X^\pm(z)\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{-g(\tau,s)}{X^+(\tau)}\frac{d\tau}{\tau-z}, \tag{2.10}$$

where $X^\pm(z)$ solves the homogeneous Riemann–Hilbert problem

$$X^+(p)=T(p)X^-(p) \quad \text{for } p\epsilon(-\infty,\infty). \tag{2.11}$$

To solve (2.11), it is convenient to factor $T(p)$ into the product $T(p)=T_1(p)T_2(p)T_3(p)$ in which $T_1(p)=-\tilde{\mu}(s+ivp)$, $T_2(p)=\left[(p-\frac{is}{v})^2\right]^{1/2}$, and

$T_3(p)=\left[\frac{p^2}{(p-\frac{is}{v})^2}-\frac{\rho v^2}{\tilde{\mu}(s+ivp)}\right]^{1/2}$. $X^{\pm}(z)$ may now be constructed as the product $X^{\pm}(z)=X_1^{\pm}(z)X_2^{\pm}(z)X_3^{\pm}(z)$ with each $X_i^{\pm}(z)$ satisfying the Riemann–Hilbert problem $X_i^{+}(p)=T_i(p)X_i^{-}(p)$.

What will ultimately be required is to solve the homogeneous Riemann–Hilbert problem (2.11) for each fixed s on a Bromwich path with $Re(s)>0$. This will be accomplished by first assuming s real and positive and then invoking an analytic continuation argument. Additionally, for the subsequent analysis the shear modulus will be assumed to be positive, continuously differentiable, non–increasing, convex and such that $\mu(\infty)=\lim_{t\to\infty}\mu(t)>0$. Convexity is sufficient but certainly not necessary to insure the validity of the following calculations and though theoretically overly restrictive, it holds for most of the customary models such as a standard linear solid or a power–law material. Moreover, it is worth noting that no explicit time decay rate for the shear modulus needs to be specified for the results to be valid. From the fact that $\mu(t)=0$ for $t<0$, it easily follows that $[\tilde{\mu}(s+ivz)]^{-1}$ is analytic for $Im(z)<0$. Therefore one may choose

$$X_1^{+}(z)=1 \text{ and } X_1^{-}(z)=-[\tilde{\mu}(s+ivp)]^{-1}. \tag{2.12}$$

Since the product $T_2(p)T_3(p)=\beta(s,p)$, the branches of $T_2(p)$ and $T_3(p)$ must be chosen so that their product satisfies the requirement that $Re\,\beta(s,p)\geq 0$. This condition can be met by choosing the branch of $z^{1/2}$ with branch cut along the negative real axis for both $T_2(p)$ and $T_3(p)$. (See Fig. 2.1.) Therefore $T_2(p)$ can be expressed as $T_2(p)=sgn(p)(p-\frac{is}{v})$ and $X_2^{\pm}(z)$ may be chosen to be

$$X_2^{+}(z)=\omega^{+}(z) \text{ and } X_2^{-}(z)=\omega^{-}(z)(z-\frac{is}{v})^{-1} \tag{2.13}$$

in which $\omega^{+}(z)=z^{1/2}$ with branch cut along the negative imaginary axis and $\omega^{-}(z)=z^{1/2}$ with branch cut along the positive imaginary axis. Finally, one may construct $X_3^{\pm}(z)$ by

$$X_3^{\pm}(z)=\exp(\Gamma^{\pm}(z)) \text{ where } \Gamma^{\pm}(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(T_3(\tau))}{\tau - z}\, d\tau. \tag{2.14}$$

To find a closed form expression for $X_3^{\pm}(z)$, it is necessary to determine the mapping properties of $T_3(z)$ for z in the half–plane $\mathrm{Im}(z)\leq 0$. If one first considers $\tilde{\mu}(s+ivz)=\tilde{\mu}(s-vq+ivp)$ on the horizontal lines $z=p+iq$, $q\leq 0$, $p\in(-\infty,\infty)$, then it follows easily from the stated assumptions on $\mu(t)$ that

(i) $\tilde{\mu}(0)=\mu(\infty)<\tilde{\mu}(s-vq)\leq \mathrm{Re}\ \tilde{\mu}(s-vq+ivp)\leq \mu(0)=\tilde{\mu}(s-vq\pm i\infty)$;

(ii) $\mathrm{Im}\ \tilde{\mu}(s-vq+ivp)=-\mathrm{Im}\ \tilde{\mu}(s-vq-ivp)$;

(iii) $\arg \tilde{\mu}(s-vq+ivp)\begin{cases}>0 & p>0\\ <0 & p<0\end{cases}$;

(iv) $\lim_{q\to-\infty} \tilde{\mu}(s-vq+ivp)=\mu(0)$ with $\tilde{\mu}(s-vq)$ converging monotonically to $\mu(0)$.

Therefore $\tilde{\mu}(s+ivz)$ maps these lines to the curves shown in Fig. 2.2a.

The linear fractional transformation $S(z)=\frac{z}{z-i(s/v)}$ maps the lines $z=p+iq$, $q\leq 0$, $p\in(-\infty,\infty)$ to the circles shown in Fig. 2.2b and $S(p+iq)$ exhibits the following properties:

(v) $0=S(0)\leq S(0+iq)\leq \mathrm{Re}\ S(p+iq)\leq S(\pm\infty+iq)=1$;

(vi) $\mathrm{Im}\ S(p+iq)=-\mathrm{Im}\ S(-p+iq)$;

(vii) $\arg S(p+iq)\begin{cases}>0 & p>0\\ <0 & p<0\end{cases}$;

(viii) $\lim_{q\to-\infty} S(p+iq)=1$ with $S(iq)$ converging monotonically to 1.

If the elastic shear wave speeds corresponding to infinite and zero time are defined by $c_*^2=\mu(\infty)/\rho$ and $c^2=\mu(0)/\rho$ then from (i)–(viii), it follows that $[T_3(z)]^2$ has the following properties:

(ix) $\mathrm{Im}\ T_3^2(p+iq)=-\mathrm{Im}\ T_3^2(-p+iq)$;

(x) $\arg T_3^2(p+iq)\begin{cases}>0 & p>0\\ <0 & p<0\end{cases}$;

(xi) $\lim_{q\to-\infty} T_3^2(p+iq)=1-(v/c)^2$;

(xii) $-(v/c_*)^2 < \dfrac{-\rho v^2}{\tilde{\mu}(s)} = T_3^{\,2}(0) \le T_3^{\,2}(iq) \le T_3^{\,2}(-i\infty) = 1-(v/c)^2$ where $T_3^{\,2}(iq)$ is monotonically increasing to $1-(v/c)^2$ as $q \to -\infty$.

Thus it can be seen that $T_3^{\,2}(z)$ maps the horizontal lines $z=p+iq$, $q \le 0$, $p \in (-\infty,\infty)$ to the curves shown in Fig. 2.2c. Furthermore, it can be seen from (xii) that $T_3(z)^2$ has a unique root $z_*=iq_*$ in the half–plane $\mathrm{Im}(z) \le 0$ for any positive real value of s.

If the branch cut for the square root defining $T_3(z)$ is chosen along the negative real axis then $T_3(z)$ is analytic for $\mathrm{Im}(z)<0$ except for the branch cut on a segment of the negative imaginary axis across which it has a jump discontinuity given by $T_3(\pm 0+iq) = \pm i \left| \dfrac{q^2}{(q-\frac{s}{v})^2} - \dfrac{\rho v^2}{\tilde{\mu}(s-vq)} \right|^{1/2}$ for $q_*<q \le 0$. It follows that $\log T_3(z)$ is analytic for $\mathrm{Im}(z)<0$ away from this line segment and has the jump discontinuity $\log(T_3(+0+iq)) - \log(T_3(-0+iq)) = i\pi$ across $z=iq$, $q_*<q\le 0$. One can then evaluate $\Gamma^{\pm}(z) = \dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{\infty} \dfrac{\log(T_3(\tau))}{\tau - z}\, d\tau$ as in Walton(1982) and conclude that

$$\Gamma^{\pm}(z) = (1/2)\left[\log(q_* i - z) - \log(-z) + \log(T_3(\infty)) + \begin{Bmatrix} 0 & \text{for } Im(z)>0 \\ -\log(T_3(z)) & \text{for } \mathrm{Im}(z)<0 \end{Bmatrix}\right.$$

Therefore (2.14) reduces to

$$X_3^{+}(z) = \{\omega^{+}(z-iq_*)[T_3(\infty)]^{1/2}\}/\omega^{+}(z), \tag{2.15}$$

$$X_3^{-}(z) = \{\omega^{-}(z-iq_*)[T_3(\infty)]^{1/2}\}/\omega^{-}(z)T_3(z).$$

From equations (2.12)–(2.15) it then follows that

$$X^{+}(z) = \omega^{+}(z-iq_*)[T_3(\infty)]^{1/2}, \tag{2.16}$$

$$X^{-}(z) = -\{\omega^{-}(z-iq_*)[T_3(\infty)]^{1/2}\}/[\tilde{\mu}(s+ivz)(z-i(s/v))T_3(z)].$$

Finally, for a specific load $\sigma_{23}(x,t) = L_e \Lambda_e(x/a_e, t)$ one can determine $F^{+}(z) = \bar{\hat{\sigma}}_{23}^{+}(z,0,s)$, $F^{-}(z) = \bar{\hat{w}}^{-}(z,0,s)$ from equations (2.10) and (2.16). The Laplace transform of the stress intensity factor (SIF) $\tilde{K}(s)$ can now be calculated as in Walton(1982). In particular, it is straightforward to show that

$$\bar{\sigma}_{23}(x,s) \sim \frac{\bar{K}(s)}{\sqrt{\pi}} x^{-1/2} \text{ as } x \to 0^{+}, \text{ where } \bar{K}(s) = -[T_3(\infty)]^{1/2} \frac{e^{\pi i/4}}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{\bar{\sigma}}^{-}(\tau,s)}{X^{+}(\tau)} d\tau \tag{2.17}$$

and $\hat{\bar{\sigma}}^{-}(\tau,s) = L_e a_e \hat{\bar{A}}_e(a_e \tau,s)$. Again, as in Walton(1982), (2.17) may be simplified to $\bar{K}(s) = \frac{-1}{\sqrt{\pi}} \int_{-\infty}^{0} L_e \sqrt{a_e} \bar{A}_e(x,s) |x|^{1/2} e^{-(xq_* a_e)} dx$, in which it should be emphasized that q_* is a function of s determined implicitly by $T_3(iq_*)=0$.

Equations (2.10) and (2.16) can be shown to hold for each fixed complex s such that the Re(s)>0. Therefore these expressions (with slight modification) are valid on a Bromwich path and numerical Laplace inversion may be performed. These details are found in Herrmann and Walton(1988).

3. Calculation of the Energy Release Rate.

The Energy Release Rate (ERR) will now be calculated based upon the assumption that a Barenblatt type failure zone exists at the crack–tip. Specifically, it is assumed that two loads are acting on the crack–faces: the applied (external) tractions denoted $\sigma_e^-(x,t)=L_e\Lambda_e(\frac{x}{a_e},t)$ and the cohesive (failure) stresses $\sigma_f^-(x,t)=-L_f\Lambda_f(w(x,t),\frac{x}{a_f},t)$ acting in a failure zone of length a_f immediately behind the crack–tip. Two essential features of the Barenblatt model are that $a_f<<a_e$ and that $K_e+K_f=0$ where K_e and K_f are the SIF's corresponding to σ_e^- and σ_f^-, respectively. The effect of the failure zone is to cancel the singular stresses ahead of the crack–tip and thereby produce a cusp shaped crack profile behind the tip. The resulting mathematical problem is: given $\sigma_e^-(x,t)$, find the "response" stress in the failure zone and crack face displacement w(x,t) that cancel the stress singularity due to $\sigma_e^-(x,t)$ while maintaining a constant crack speed. The goal is then to compute the time evolution of the energy release rate as described below.

The ERR, G(t) (defined to be the energy flux into the crack tip per unit crack advance) is given by $G(t)=\frac{1}{v}\int_{vt-a_f}^{vt}\sigma_f^-(x_1-vt,t)\dot{u}_3(x_1-vt,0,t)dx_1$, which in the moving coordinates becomes

$$G(t)=\frac{1}{v}\int_{-a_f}^{0}\sigma_f^-(x,t)\left[\frac{\partial}{\partial t}-v\frac{\partial}{\partial x}\right]w(x,0,t)dx. \tag{3.1}$$

Deriving a closed form expression for (3.1) valid for arbitrary applied tractions $\sigma_e^-(x,t)$ and failure zone constitutive law

$$\sigma_f^-(x,t)=-L_f\Lambda_f(w(x,t),\frac{x}{a_f},t) \tag{3.2}$$

requires solving an exceedingly complicated non–linear boundary value problem. The constitutive law (3.2) models the failure zone response as that of a nonlinear elastic spring, accounting for inhomogeneity and aging. In general the failure zone length, a_f, is dynamically changing and determined by the equation $K_e+K_f=0$. In the absence of any generally accepted, physically motivated form for $\Lambda_f(\cdot,\cdot,\cdot)$, various ad hoc, artificial models have been introduced and studied in the literature.

In Schapery (1975), it is argued that taking $\sigma_f^-(x,t)$ constant for $-a_f<x<0$ with $a_f=a_f(v,t,\sigma_e^-(x,t))$ is a reasonable approximation to reality, at least provided a_f is very much smaller than all significant physical length scales such as crack length and the distance from the crack tip to the boundary. More recently, Goleniewski(1988) studied this same failure zone model for the corresponding dynamic, steady–state problem for a Maxwell fluid and piecewise constant applied tractions $\sigma_e^-(x,t)$. However, the methods of Goleniewski(1988) do not seem to generalize to the transient problem, even for a Maxwell fluid Due to the complicated non–linear coupling between w(x,t) and a_f, deriving a convenient expression for G(t) from which qualitative properties can be inferred when σ_f is assumed constant seems unlikely.

In contrast, for the dynamic steady–state problem it was shown in Walton(1987a) that a simple closed form expression for G is obtained for the special class of loads

$$\sigma_e^-(x)=L_e\exp(x/a_e) \quad\text{and}\quad \sigma_f^-(x)=-L_f\exp(x/a_f) \quad\text{for } -\infty<x<0. \tag{3.3}$$

It was argued there that for $a_f<<a_e$, the fact that $\sigma_f^-(x,t)$ does not have compact support should have a relatively minor effect on the results provided the essential requirements for the Barenblatt model are still satisfied: $a_f<<a_e$ and $K_e+K_f=0$. Furthermore, the two cases (1) L_f constant with $a_f=a_f(v)$ and (2) a_f constant with $L_f=L_f(v)$ were compared quantitatively. It was found that, except for very high crack speeds, the two cases produce nearly identical G vs v curves. In light of these

considerations it is likely that valuable insight into the combined influence that crack speed, inertia and viscoelastic properties have upon the rate of convergence to steady–state of the transient G(t) can be obtained from generalizing the analysis of Walton(1987a) to the transient problem addressed here, the unphysical nature of the failure zone constitutive law (3.3) notwithstanding.

Henceforth, the forms (3.3) will be assumed for σ_e^- and σ_f^-. Moreover, time dependent tractions $\sigma_e^-(x,t)$ will be permitted by taking

$$\sigma_e^-(x,t)=L_e l_e(t)\exp(x/a_e), \quad -\infty<x<0, \tag{3.4}$$

where $l_e(t)$ is a dimensionless function of time. As with the steady–state analysis in Walton(1987a), consideration of the two models, (1) L_f constant with $a_f=a_f(v,t)$ and (2) a_f constant with $L_f=L_f(v,t)$, arises naturally. While case (1) is the more physically compelling, it is mathematically much more complicated owing to the nonlinear manner in which the initially unknown function $a_f(v,t)$ occurs in the problem. In contrast, case (2) is clearly unphysical in a dynamic analysis (e.g. it suggests that information travels with an infinite speed of propagation in the failure zone) but, as shown below, admits an elegant closed form expression for G(t) with the aid of which qualitative and quantitative properties can be easily studied. In light of the steady state results, it is likely that, except for very high crack speeds, the two cases should exhibit similar long time asymptotic behavior whenever $a_f<<a_e$. The analysis of case (2) is presented here and that of case (1) is the subject of a future paper.

With $\sigma_e^-(x,t)$ given by (3.4) and $\sigma_f^-(x,t)$ by

$$\sigma_f^-(x,t)=L_f l_f(t)\exp(x/a_f), \quad -\infty<x<0, \tag{3.5}$$

the appropriate definition for G(t) becomes

$$G(t)=\frac{1}{v}\int_{-\infty}^{0} \sigma_f^-(x,t)\left[\frac{\partial}{\partial t}-v\frac{\partial}{\partial x}\right]w(x,0,t)dx. \tag{3.6}$$

In (3.5) the explicit dependence of $l_f(t)$ upon v has been temporarily suppressed. As shown in Walton (1987a), it is straight forward to extend the analysis to treat more general loads of the form $\sigma^{-}(x,t)=L(t)\int_0^\infty e^{rx/a}\,dh(r)$ where h(r) is any signed measure for which the integral makes sense. However, for the sake of brevity that development is not included here.

Incorporating (3.5) into (3.6) there results

$$G(t)=I(t)+W(t) \quad \text{with} \tag{3.7}$$

$$I(t)=-L_f l_f(t)\frac{1}{v}\int_{-\infty}^{0} e^{(x/a_f)}\frac{\partial}{\partial t}w(x,0,t)\,dx \quad \text{and} \tag{3.8}$$

$$W(t)=L_f l_f(t)\int_{-\infty}^{0} e^{(x/a_f)}\frac{\partial}{\partial x}w(x,0,t)\,dx. \tag{3.9}$$

I(t) and W(t) in themselves do not have any direct physical significance but the decomposition (3.7) does provide some insight into the behavior of G(t). In particular, it is shown below that $\lim_{t\to 0} G(t)=\lim_{t\to 0} I(t)$ and $\lim_{t\to\infty} G(t)=\lim_{t\to\infty} W(t)$ whereas $\lim_{t\to 0} W(t)=\lim_{t\to\infty} I(t)=0$. Thus the short time behavior of G(t) is governed primarily by I(t) and the long time behavior by W(t). Moreover, in anticipation of comparison with future results from the analysis of case (1) $l_f(t)\equiv 1$ and $a_f=a_f(v,t)$, both G(t) and W(t) are studied in detail here.

Motivation for the separate consideration of W(t) derives from the following observations. In Schapery(1975), it is argued that it is physically more compelling to build a viscoelastic fracture criterion upon the work done to the trailing ligament in the failure zone, $\Gamma(t)$, rather than upon G(t), the total energy flux to the failure zone per unit crack advance. Of course this concept only has meaning for a failure zone with finite length. For any failure zone constitutive model, one may write

G(t) as in (3.7) with $W(t)=\int_{-a_f}^{0} \sigma_f^-(x,t)\frac{\partial}{\partial x}w(x,0,t)dx$. For the particular constitutive model $\sigma_f^-(x,t)=-L_f$, $-a_f<x<0$ with $a_f=a_f(v,t)$, studied by Schapery, $\Gamma(t)=W(t)$. It then seems reasonable to study W(t) as a suitable generalization of $\Gamma(t)$ to the constitutive model (3.5) for case (1) $l_f(t)\equiv 1$, $a_f=a_f(v,t)$. Expecting case (1) and case (2), a_f constant, $l_f=l_f(v,t)$, to exhibit similar behavior, at least as $t\to\infty$, a separate analysis of W(t) given by (3.9) is included here.

I(t) and W(t) can be constructed by separately evaluating the integral expressions in (3.8) and (3.9) and multiplying the results by $l_f(t)$, which is determined from the Barenblatt assumption that $K_e+K_f=0$.

Consider first the integral

$$g(t)=\frac{1}{v}\int_{-\infty}^{0} e^{(x/a_f)}\left[\frac{\partial}{\partial t}-v\frac{\partial}{\partial x}\right]w(x,0,t)dx. \tag{3.10}$$

If one notes that the inverse Fourier transform of $H(-x)e^{(x/a_f)}$ is $\frac{i}{2\pi}(p+i/a_f)^{-1}$ and applies Parseval's relation to (3.10) and then applies the Laplace transform to the resulting expression, the Laplace transform $\bar{g}(s)$ is found to be

$\bar{g}(s)=\frac{-1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{s}{v}+ip\right]\bar{\hat{w}}^-(p,0,s)\frac{dp}{p+i/a_f}$. Since $\bar{\hat{w}}^-(p,0,s)=F^-(p)$ has an analytic extension $F^-(z)$ for $Im(z)<0$, the integral can be evaluated by residues, whence it follows that

$$\bar{g}(s)=\left[\frac{s}{v}+\frac{1}{a_f}\right]F^-(-i/a_f). \tag{3.11}$$

It remains to evaluate $F^-(-i/a_f)$. To this end, one begins by first noting that (2.8) can be rewritten as

$$F^-(p)=\bar{\hat{\sigma}}(p)/T(p), \tag{3.12}$$

$$\text{where } \bar{\hat{\sigma}}(p)=\bar{\hat{\sigma}}^-(p)+\bar{\hat{\sigma}}^+(p). \tag{3.13}$$

For the Barenblatt model $\sigma^-(x)=\sigma_e^-(x)+\sigma_f^-(x)$ and consequently

$\bar{\hat{\sigma}}^-(p)=\bar{\hat{\sigma}}_e^-(p)+\bar{\hat{\sigma}}_f^-(p)$. Also $\bar{\hat{\sigma}}^+(p)$ in (3.13) can be determined by application of the Plemelj formula to (2.10) thereby obtaining

$\bar{\hat{\sigma}}^+(p)=F^+(p)=-\frac{1}{2}g(p)+\frac{1}{2\pi i}X^+(p)\int_{-\infty}^{\infty}\frac{-g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau-p}$. This can be rewritten using

$$\bar{\hat{\sigma}}^+(p)=\bar{\hat{\sigma}}_e^+(p)+\bar{\hat{\sigma}}_f^+(p) \tag{3.14}$$

wherein $\bar{\hat{\sigma}}_e^+(p)$ and $\bar{\hat{\sigma}}_f^+(p)$ are given by

$$\bar{\hat{\sigma}}_\alpha^+(p)=-\frac{1}{2}\bar{\hat{\sigma}}_\alpha^-(p)+\frac{1}{2\pi i}X^+(p)\int_{-\infty}^{\infty}\frac{-\bar{\hat{\sigma}}_\alpha^-(\tau)}{X^+(\tau)}\frac{d\tau}{\tau-p}. \tag{3.15}$$

From (3.4) it can be seen that $\bar{\hat{\sigma}}_e^-(p)=\frac{L_e\bar{l}_e(s)}{i(p-(i/a_e))}$ and therefore $\bar{\hat{\sigma}}_e^-(z)$ is meromorphic with pole $z=i/a_e$. Since $X^+(z)$ is analytic for $\mathrm{Im}(z)>0$, (3.15) can be calculated by residues with the result

$$\bar{\hat{\sigma}}_e^+(p)=-\bar{\hat{\sigma}}_e^-(p)\left[1-\frac{X^+(p)}{X^+(i/a_e)}\right]. \tag{3.16}$$

Similarly, $\bar{\hat{\sigma}}_f^+(p)=-\bar{\hat{\sigma}}_f^-(p)\left[1-\frac{X^+(p)}{X^+(i/a_f)}\right]$ which combined with (3.14) and (3.16) yields

$$\bar{\hat{\sigma}}(p)=\frac{X^+(p)}{i}\left[\frac{L_e\bar{l}_e(s)}{(p-(i/a_e))X^+(i/a_e)}-\frac{L_f\bar{l}_f(s)}{(p-(i/a_f))X^+(i/a_f)}\right]. \tag{3.17}$$

Equation (3.17) can now be simplified by the Barenblatt hypothesis $K_e+K_f=0$. For the special loadings given by (3.4) and (3.5), equation (2.17) yields

$$\bar{K}_e(s)=-[T_3(\infty)]^{1/2}e^{\pi i/4}\frac{L_e\bar{l}_e(s)}{X^+(i/a_e)} \text{ and } \bar{K}_f(s)=-[T_3(\infty)]^{1/2}e^{\pi i/4}\frac{-L_f\bar{l}_f(s)}{X^+(i/a_f)},$$

which under the Barenblatt hypothesis results in the identity

$$\frac{L_e\bar{l}_e(s)}{X^+(i/a_e)}=\frac{L_f\bar{l}_f(s)}{X^+(i/a_f)}. \tag{3.18}$$

Equations (3.17), (3.18), (3.12) and the observation that $T(p)=X^{+}(p)/X^{-}(p)$ then yield

$$F^{-}(-i/a_f)=\frac{a_f L_e \bar{l}_e(s)}{2}\frac{(a_e-a_f)}{(a_e+a_f)}\frac{X^{-}(-i/a_f)}{X^{+}(i/a_e)}. \tag{3.19}$$

One may now substitute (3.18) and (3.19) into (3.11) thereby obtaining

$$\bar{g}(s)=\frac{L_e \bar{l}_e(s)}{2}\left[\frac{s}{v}+\frac{1}{a_f}\right]\frac{a_f(a_e-a_f)}{(a_e+a_f)}\frac{X^{-}(-i/a_f)}{X^{+}(i/a_e)},$$

which when combined with the result (2.16) derived previously for $X^{\pm}(z)$ produces

$$\bar{g}(s)=\frac{-L_e \bar{l}_e(s)}{2}\frac{(a_e-a_f)}{(a_e+a_f)}\sqrt{a_e a_f}\left|\frac{1+a_f q_*}{1-a_e q_*}\right|^{1/2}\frac{1}{\tilde{\mu}(s+v/a_f)}\left|\frac{1}{(1+sa_f/v)^2}-\frac{\rho v^2}{\tilde{\mu}(s+v/a_f)}\right|^{-1/2}.$$

In order to present the results in a nondimensional form, it is necessary to introduce certain parameters. First, a nondimensional shear modulus is defined by $\mu(t)=\mu_\infty m(t/\tau)$ where $\mu_\infty=\lim_{t\to\infty}\mu(t)$ and thus $\lim_{t\to\infty}m(t)=1$. Also, the nondimensional parameters γ, ϵ, α, and β are defined by $\gamma=v/c_*$, $\epsilon=a_f/a_e$, $\alpha=c_*\tau/a_e$, and $\beta=-a_e q_*$ where $0<\gamma<c/c_*$, $\epsilon<<1$, $\alpha>0$, and $\beta>0$.

The Laplace transform $\bar{g}(s)$ can now be rewritten in nondimensional form as

$$\bar{g}(s)=\frac{-L_e a_e \bar{l}_e(s)\sqrt{\epsilon}}{2\gamma\mu_\infty}\frac{(1-\epsilon)}{(1+\epsilon)}\left|\frac{1-\epsilon\beta}{1+\beta}\right|^{1/2}\frac{1}{\tilde{m}(\tau s+\alpha\gamma/\epsilon)}\left|\frac{(\alpha/\epsilon)^2}{(\tau s+\alpha\gamma/\epsilon)^2}-\frac{1}{\tilde{m}(\tau s+\alpha\gamma/\epsilon)}\right|^{-1/2}. \tag{3.20}$$

Also note that $\beta=\beta(s,\gamma)$ is the unique root of the equation

$$\tilde{m}(\tau s+\alpha\gamma\beta)=\gamma^2(1+(\tau s/\alpha\gamma\beta))^2.$$

Attention will now be turned to $l_f(t)$. It follows from (2.16) and (3.18) that

$$\bar{l}_f(s)=\bar{l}_e(s)\frac{L_e}{L_f\sqrt{\epsilon}}\left|\frac{1+\epsilon\beta}{1+\beta}\right|^{1/2}. \tag{3.21}$$

Finally, utilizing (3.7)–(3.10), (3.20), and (3.21) one may write the ERR as

$$G(t)=-L_f l_f(t)g(t)=\frac{L_e^2 a_e}{2\mu_\infty}l(t)g_1(t), \tag{3.22}$$

where the Laplace transforms of $l(t)$ and $g(t)$ are given by

$$\bar{l}(s)=\bar{l}_e(s)\left|\frac{1+\epsilon\beta}{1+\beta}\right|^{1/2}, \tag{3.23}$$

$$\bar{g}_1(s)=\frac{\bar{l}_e(s)}{\gamma}\frac{(1-\epsilon)}{(1+\epsilon)}\left|\frac{1-\epsilon\beta}{1+\beta}\right|^{1/2}\frac{1}{\tilde{m}(\tau s+\alpha\gamma/\epsilon)}\left|\frac{(\alpha/\epsilon)^2}{(\tau s+\alpha\gamma/\epsilon)^2}-\frac{1}{\tilde{m}(\tau s+\alpha\gamma/\epsilon)}\right|^{-1/2}. \tag{3.24}$$

Unfortunately, it is not possible in general to invert (3.23) and (3.24) analytically. Therefore to obtain a better understanding of $G(t)$ either an asymptotic analysis based on (3.22)–(3.24) may be performed which is valid for any shear modulus $\mu(t)$ or numerical inversion of the Laplace transforms (3.23) and (3.24) may be obtained for a specific $\mu(t)$. The asymptotic analysis and the consideration of special cases are addressed in the next section. Numerical inversion of (3.23) and (3.24) for the standard linear solid and power–law models can be found in Herrmann and Walton(1988).

4. Special Cases and Asymptotic Solutions.

Asymptotic expansions for the ERR, G(t), as $t\to 0$ and as $t\to\infty$ can now be constructed from (3.22)–(3.24). More specifically, asymptotic expansions for $l(t)$ and $g_l(t)$ can be constructed separately and multiplied together. Moreover, it can be seen that $l(t)=l_e(t)*l_l(t)$ where $\bar{l}_l(s)=\left|\frac{1+\epsilon\beta}{1+\beta}\right|^{1/2}$. Unless indicated to the contrary, henceforth attention will be limited to the special case $l_e(t)=1$, i.e., the external tractions driving the crack will be assumed to be time independent. In that case one clearly sees that $\bar{l}(s)=s^{-1}\bar{l}_l(s)$. More general time dependence for $l_e(t)$ is easily incorporated into $l(t)$ by a simple convolution of $l_e(t)$ and $l_l(t)$.

For short time approximation, it is necessary to determine the asymptotic behavior of $\bar{l}(s)$ and $\bar{g}_l(s)$ as $s\to\infty$. To this end, it is required to study the behavior of $\tilde{m}(\tau s+\alpha\gamma/\epsilon)$ and β as $s\to\infty$. From the definition of the Carson transform $\tilde{m}(s)$, it is easily seen that $\tilde{m}(\infty)=m(0)=\frac{\mu(0)}{\mu_\infty}=\left[\frac{c}{c_*}\right]^2$. Furthermore, since $\beta=\beta(s,\gamma)$ satisfies the equation

$$\tilde{m}(\tau s+\alpha\gamma\beta)=\gamma^2(1+(\tau s/\alpha\gamma\beta))^2 \tag{4.1}$$

it can be shown (see appendix) that

$$\beta=\frac{\tau s}{\alpha[(c/c_*)-\gamma]}+o(s) \text{ as } s\to\infty. \tag{4.2}$$

Therefore if $\bar{l}(s)$ and $\bar{g}_l(s)$ are written as asymptotic series in powers of $\frac{1}{s}$, it is found that as $s\to\infty$

$$\bar{l}(s)=\sqrt{\epsilon}\left[\frac{1}{s}+\frac{(1-\epsilon)}{\epsilon}\frac{\alpha}{2}[(c/c_*)-\gamma]\frac{1}{\tau s^2}+o(s^{-2})\right], \tag{4.3}$$

$$\bar{g}_l(s)=\frac{1}{\gamma}\frac{(1-\epsilon)}{(1+\epsilon)}\frac{\sqrt{\epsilon}}{\sqrt{m(0)}}\left[\frac{1}{s}-\left[\frac{(1+\epsilon)}{\epsilon}\frac{\alpha}{2}[(c/c_*)-\gamma]+\frac{m'(0)}{2\,m(0)}\right]\frac{1}{\tau s^2}+o(s^{-2})\right].$$

Hence, if one assumes that $l(t)$ and $g_l(t)$ have Maclaurin expansions in a neighborhood of $t=0$ then from (4.3) and standard asymptotic results for the

Laplace transform,

$$l(t)=\sqrt{\epsilon}\left[1+\frac{(1-\epsilon)}{\epsilon}\frac{\alpha}{2}[(c/c_*)-\gamma](t/\tau)+o(t)\right], \tag{4.4}$$

$$g_1(t)=\frac{1}{\gamma}\frac{(1-\epsilon)}{(1+\epsilon)}\frac{\sqrt{\epsilon}}{\sqrt{m(0)}}\left[1-\left[\frac{(1+\epsilon)}{\epsilon}\frac{\alpha}{2}[(c/c_*)-\gamma]+\frac{m'(0)}{2m(0)}\right](t/\tau)\right]+o(t) \text{ as } t\to 0. \tag{4.5}$$

From (4.4) and (4.5) it follows after algebraic simplification that

$$G(t)=\frac{L_e^2 a_e}{2\mu_o}\frac{(1-\epsilon)}{(1+\epsilon)}\frac{c}{v}\,\epsilon\left[1-a(t/\tau)+o(t)\right] \text{ as } t\to 0, \text{ where } a=\frac{\tau}{a_e}(c-v)+\frac{m'(0)}{2m(0)}. \tag{4.6}$$

It should be remembered that (4.4) *and* (4.5) (*and hence* (4.6) also) are not uniformly valid expansions as $\epsilon\to 0$. The case $\epsilon=0$ is a singular limit treated separately later. It should also be remarked that near $t=0$, $G(t)$ is governed by the glassy properties except for the inclusion of the *term* $m'(0)$ which incorporates the effect of the initial rate of stress relaxation. Since $m(0)\geq 1$ and $m'(0)\leq 0$, one sees that the sign of $G'(0)$ depends upon the crack speed and material properties through the relation

$$G'(0)>0\ \left\{G'(0)<0\right\} \text{ if and only if } a_e\frac{|\mu'(0)|}{2\mu(0)}>[c-v]\ \left\{a_e\frac{|\mu'(0)|}{2\mu(0)}<[c-v]\right\}.$$

In particular for fast enough crack speeds, $G'(0)$ will be positive, i.e. $G(t)$ will initially increase with time. However, for any given crack speed if the combination $a_e\frac{|\mu'(0)|}{2\mu(0)}$ is small enough, then $G(t)$ will initially decrease with time.

To determine long–time asymptotic solutions as $t\to\infty$, it is necessary to find first $\lim_{t\to\infty} G(t)=\frac{L_e^2 a_e}{2\mu_\infty}\lim_{s\to 0} s\bar{l}(s)\lim_{s\to 0} s\bar{g}_1(s)$. The expression $\tilde{m}(\tau s+\alpha\gamma/\epsilon)$ has the limit $\tilde{m}(\alpha\gamma/\epsilon)$ as $s\to 0$ and β, as shown in the appendix, has different asymptotic limits as $s\to 0$ depending on whether (i) $0<v<c_*$, (ii) $v=c_*$, or (iii) $c_*<v<c$, namely

$$\beta=\frac{\tau s}{\alpha[1-\gamma]}+o(s) \quad \text{for } 0<v<c_*,$$

$$\beta=\frac{\sqrt{2\tau}}{\alpha\gamma}\left[-\int_0^\infty r m'(r)\,dr\right]^{1/2} s^{1/2}+o(s^{1/2}) \quad \text{for } v=c_*,$$

$\beta=\beta_o+o(1)$ for $c_*<v<c$, where β_o is defined implicitly through the equation

$\gamma^2-\left[\frac{c}{c_*}\right]^2=\int_0^\infty e^{-\alpha\gamma\beta_o r}m'(r)\,dr$. To avoid separate cases in displaying subsequent formulas it is convenient to define $\beta_o=0$ for $0<v\leq c_*$.

Again from standard asymptotic arguments for the Laplace transform

$$\lim_{t\to\infty} G(t)=\frac{L_e{}^2 a_e}{2\mu_\infty}\frac{(1-\epsilon)}{(1+\epsilon)}\frac{\left|1-(\epsilon\beta_o)^2\right|^{1/2}}{(1+\beta_o)}\frac{1}{\tilde{m}(\alpha\gamma/\epsilon)}\left|1-\frac{\gamma^2}{\tilde{m}(\alpha\gamma/\epsilon)}\right|^{-1/2}, \tag{4.7}$$

which is precisely the steady-state solution found in Walton(1987a).

It is interesting to consider separately the asymptotic behavior of $L_f l_f(t)$ for a general time dependence $L_e l_e(t)$ in $\sigma_e^-(x,t)$. One notes that as $t\to 0$, $L_f l_f(0)=\lim_{s\to\infty} sL_f\bar{l}_f(s)=L_e l_e(0)$. Thus initially $L_f l_f(t)$ and $L_e l_e(t)$ are equal for any crack speed and $\epsilon>0$. In steady-state, however,

$L_f l_f(\infty)=\lim_{s\to 0} sL_f\bar{l}_f(s)=L_e l_e(\infty)\left[\frac{(1/\epsilon)+\beta_o}{1+\beta_o}\right]^{1/2}$. For $0<\gamma\leq 1$, $\beta_o=0$, which implies that $L_f l_f(\infty)=(1/\sqrt{\epsilon})L_e l_e(\infty)$ which is the same as found for quasi-static crack propagation. For $1<\gamma<c/c_*$, β_o is a monotone increasing function of γ with $\lim_{\gamma\to c/c_*}\beta_o(\gamma)=\infty$. Thus $\frac{L_f l_f(\infty)}{L_e l_e(\infty)}$ is a monotone decreasing function of γ with $\lim_{\gamma\to c/c_*} L_f(\infty)=L_e(\infty)$. For the special case of $L_e l_e(t)$ constant, one obviously uses $L_e l_e(0)=L_e l_e(\infty)=L_e$ in the above formulas.

It is instructive to consider several special cases. Setting $\epsilon=0$ corresponds to a model with no failure zone. Allowing $\epsilon\to 0$ in (3.23) and (3.24) yields $\bar{l}(s)=\bar{l}_e(s)/\sqrt{1+\beta}$ and $\bar{g}_1(s)=\bar{l}_e(s)\frac{\mu_\infty}{\mu(0)}\frac{1}{\sqrt{1+\beta}}\left[1-(v/c)^2\right]^{-1/2}$. It then follows easily from (3.18) and (3.22) that $G_{nf}(t)$, i.e. $G(t)$ based on the singular stress field without inclusion of a failure zone, becomes

$$G_{nf}(t)=[K(t,\gamma)]^2\left[\frac{1}{2\mu_o}\right]\left[1-(v/c)^2\right]^{-1/2}. \tag{4.8}$$

Thus, $G_{nf}(t)$ is simply the product of the square of the SIF, $K(t,\gamma)^2$, with a time constant function of crack speed and the glassy material properties.

If one assumes that $K(t,\gamma)$ has an asymptotic expansion in powers of $t^{1/2}$ as $t\to0$, then it can be shown from (2.16), (3.18), and (4.2) that $K(t,\gamma)=-L_e\sqrt{(a_e\alpha/\pi)}[(c/c_*)-\gamma]\sqrt{t/\tau}+o(\sqrt{t})$ as $t\to0$. Substitution into (4.8) then produces

$$G_{nf}(t)=\frac{L_e^2 a_e}{2\pi\mu_o}\left[\frac{c-v}{c+v}\right]^{1/2}(tc/a_e)+o(t) \text{ as } t\to0. \tag{4.9}$$

The steady–state limit when $t\to\infty$ is found to be

$$G_{nf}(\infty)=\frac{L_e^2 a_e}{2\mu_o}\frac{1}{1+\beta_o}[1-(v/c)^2]^{-1/2}. \tag{4.10}$$

It should be remarked that the right hand side of (4.10) can be shown to vanish as $v\to c$.

The special case of an elastic material is obtained from the general solution (3.22)–(3.24) by letting $\tau\to\infty$. There results

$$G(t)=\frac{L_e^2 a_e}{2\mu}l(t)g_1(t), \tag{4.11}$$

$$\bar{l}(s)=\bar{l}_e(s)\left|\frac{1+\epsilon\beta}{1+\beta}\right|^{1/2},$$

$$\bar{g}_1(s)=\bar{l}_e(s)\frac{(1-\epsilon)}{(1+\epsilon)}[sa_f/v+1]\left|\frac{1-\epsilon\beta}{1+\beta}\right|^{1/2}[1-(v/c+sa_f/c)^2]^{-1/2} \text{ where } \beta=\frac{sa_e}{c-v}.$$

The quasi–static limit is obtained from (3.23) and (3.24) by letting $c_*\to\infty$ ($\rho\to0$). It follows that $\beta=0$ and hence that

$$\bar{l}(s)=\bar{l}_e(s), \tag{4.12}$$

$$\bar{g}_1(s)=\bar{l}_e(s)\frac{(1-\epsilon)}{(1+\epsilon)}\frac{1}{\tilde{m}(\tau s+v\tau/a_f)}(1+sa_f/v).$$

In particular, one sees that $L_f l_f(t)=L_e l_e(t)/\sqrt{\epsilon}$ and thus that $L_f l_f(t)$ is crack speed independent. Moreover, consideration of (4.12) reveals that $g_1(t)$ in the quasi–static case has a Dirac delta function singularity at $t=0$.

Several comments on these asymptotic results are in order. First, it should be observed that whether or not the model assumes a failure zone dramatically affects both qualitatively and quantitatively the behavior of the ERR as a function of time, crack speed, and material properties. For example, from (3.20) it is easily shown that for any t, if $\epsilon>0$, $\lim_{\gamma\to 0} G(t)=\infty$ whereas if $\epsilon=0$, $\lim_{\gamma\to 0} G_{nf}(t)=\frac{K(t,0)^2}{2\mu_o}$, where $\bar{K}(s,0)=\frac{-L_e\sqrt{a_e}\,\bar{l}_e(s)}{\sqrt{1+\beta(s,0)}}$ and from (4.1) $\beta(s,0)=\frac{\tau s}{\alpha}[\tilde{m}(\tau s)]^{1/2}$. However, in the steady-state limit (4.7), G approaches a finite limit as the crack speed vanishes. both with and without a failure zone. Specifically, it is easily seen that for the steady-state limit

$$\lim_{\gamma\to 0} G(\infty)=\frac{L_e^2 a_e}{2}\frac{1-\epsilon}{1+\epsilon}\frac{1}{\mu_\infty} \quad \text{for } \epsilon>0$$

$$=\frac{L_e^2 a_e}{2}\frac{1}{\mu_o} \qquad \text{for } \epsilon=0.$$

Thus G becomes infinite as $\gamma\to 0$ except for $\epsilon=0$ (no failure zone) or under steady-state conditions.

The reason for this behavior is found through consideration of the crack face particle velocity $\dot{u}_3(x_1,0,t)=\frac{\partial w}{\partial t}(x,0,t)-v\frac{\partial w}{\partial x}(x,0,t)$. A consequence of the assumption that there is an initial jump discontinuity in the applied crack face tractions is that $\frac{\partial w}{\partial t}(x,0,t)$ does not vanish as $v\to 0$. Thus, from (3.6) it follows that $\lim_{v\to 0} G(t)=\infty$. In contrast, when $\epsilon=0$, one sees from (4.8) that G is merely a product of $K(t)^2$ and a simple function of crack speed and glassy material properties that is independent of the crack face particle velocity and that remains bounded as $v\to 0$. Moreover, in steady-state, G is given by $G=-\int_{-\infty}^{0}\sigma_f^-(x)\frac{\partial w}{\partial x}(x,0)\,dx$ and thus remains bounded as

$v \to 0$.

Other differences between the $\epsilon=0$ and $\epsilon>0$ cases are evident in the short time behavior of G(t). In particular, from (4.6) it is easily seen that when $\epsilon>0$,

$\lim_{t\to 0} G(t)=\frac{L_e^2 a_e}{2\mu_o}\frac{1-\epsilon}{1+\epsilon}\frac{c}{v}\ \epsilon>0$ whereas, for $\epsilon=0$, it follows from (4.9) that $\lim_{t\to 0} G(t)=0$.

The singular behavior for G(t,v) as $v\to 0$ in the dynamic solution and the delta function contribution at t=0 in the quasi–static solution are both due to the integral I(t) in (3.7) which contains the relative crack face particle velocity $\frac{\partial}{\partial t}w(x,0,t)$. As shown below, W(t) exhibits none of the singular behavior seen in G(t).

An expression for W(t) is easily constructed from simple modifications of (3.10), (3.11), and (3.22). Specifically, one sees that

$$W(t)=\frac{L_e^2 a_e}{2\mu_\infty} I(t)g_2(t) \tag{4.13}$$

where $\bar{I}(s)$ is given by (3.20) and

$$\bar{g}_2(s)=\bar{I}_e(s)\frac{(1-\epsilon)}{(1+\epsilon)}\left|\frac{1-\epsilon\beta}{1+\beta}\right|^{1/2}\frac{1}{\tilde{m}(\tau s+\alpha\gamma/\epsilon)}\left|1-\frac{(\gamma+s\tau\epsilon/\alpha)^2}{\tilde{m}(\tau s+\alpha\gamma/\epsilon)}\right|^{-1/2}.$$

It is trivial to show that the limit of W(t) as $\epsilon\to 0$ (no failure zone) is $G_{nf}(t)$ given in (4.8). The other special cases considered above are easily constructed for W(t). In particular, for elastic material $\bar{I}(s)$ is still given by (4.11) whereas $\bar{g}_1(s)$ is replaced by $\bar{g}_2(s)$ given by

$$\bar{g}_2(s)=\bar{I}_e(s)\frac{(1-\epsilon)}{(1+\epsilon)}\left|\frac{1-\epsilon\beta}{1+\beta}\right|^{1/2}[1-(v/c + sa_f/c)^2]^{-1/2},\ \text{with } \beta=\frac{sa_e}{c-v}, \text{ as before.} \tag{4.14}$$

The quasi–static limit of W(t), $W_{qs}(t)$, $(c_*\to\infty)$ yields as before $\bar{I}(s)=\bar{I}_e(s)$ and has $\bar{g}_1(s)$ replaced by $\bar{g}_2(s)=\bar{I}_e(s)\frac{(1-\epsilon)}{(1+\epsilon)}\frac{1}{\tilde{m}(\tau s+v\tau/a_f)}$. In particular, if $I_e(t)\equiv 1$, i.e. the applied tractions are time independent, then $\bar{l}_e(s)=\frac{1}{s}$ and it follows that $g_2(t)=\frac{(1-\epsilon)}{(1+\epsilon)}\left[\frac{1}{m(0)}+\int_0^{(t/\tau)} e^{-(rv\tau/a_f)} c'(r)\, dr\right]$, where c(t) is the non–dimensional compliance corresponding to m(t) defined by $\tilde{m}(s)=1/\tilde{c}(s)$. Thus,

$$W_{qs}(t)=\frac{L_e^2 a_e}{2\mu_\infty}\frac{(1-\epsilon)}{(1+\epsilon)}\left[\frac{1}{m(0)}+\int_0^{(t/\tau)} e^{-(rv\tau/a_f)} c'(r)\, dr\right]. \tag{4.15}$$

It should be noted that $\lim_{v\to 0} W_{qs}(t)=L_e^2 a_e D(t/\tau)$, where $D(t/\tau)$ is the creep compliance corresponding to the shear modulus $\mu(t/\tau)$.

Short time asymptotic expansions for W(t) can be constructed as before for G(t). In particular from (4.4) and (4.13) it follows (assuming $l_e(t)\equiv 1$) that

$$W(t)=\frac{L_e^2 a_e}{2\mu_o}\frac{(1-\epsilon)}{(1+\epsilon)}\frac{ct}{a_e}+o(t) \text{ as } t\to 0+. \tag{4.16}$$

A similar asymptotic expansion of (4.14) reveals that the short time response for W(t) in (4.16) is unchanged for an elastic material. Obviously though, the asymptotic series for elastic and viscoelastic materials differ in their higher order terms. Moreover, as noted previously for G(t), the asymptotic expansions for l(t) and $g_2(t)$ are not uniformly valid as $\epsilon\to 0$. Thus, the limit of (4.16) as $\epsilon\to 0$ does not yield the small t expansion (4.9) for $G_{nf}(t)$. It should also be noted that the first term in the asymptotic expansion (4.16) is independent of crack speed. Thus, to leading order as $t\to 0$, W(t) for a moving crack is the same as that for a stationary crack. However, the higher order terms are crack speed dependent and influence W(t) for all but asymptotically small times t.

It is interesting to compare the dynamic result (4.16) with the quasi–static result (4.15). In particular, the dynamic analysis shows that $\lim_{t\to 0} W(t)=0$, whereas the quasi–static result has $\lim_{t\to 0} W_{qs}(t)=\frac{L_e^2 a_e}{2\mu_o}\frac{(1-\epsilon)}{(1+\epsilon)}$. Thus, for short times the quasi–static and dynamic analyses yield quite different results for all crack speeds. This is in marked contrast to the steady–state limits which show good agreement for all crack speeds up to nearly (and sometimes beyond) the equilibrium shear wave speed. The short time asymptotic analysis reflects the obvious fact that the initial transient response for the quasi–static case should differ from that occurring when the effect of material inertia is included.

Attention will now be directed toward describing the manner in which W(t) converges to the steady–state limit as $t \to \infty$. It is useful for comparison purposes to consider the question first for elastic material. Again it is assumed that $l_e(t) \equiv 1$, or equivalently $\bar{l}_e(s)=s^{-1}$. Thus,

$$W_{EL}(t)=\frac{L_e^2 a_e}{2\mu} l(t) g_2(t) \tag{4.17}$$

with $\bar{l}(s)$ and $\bar{g}_2(s)$ given by (4.11) and (4.14), respectively. It is instructive to consider first the elastic case with no failure zone. In that case $\bar{l}(s)=\frac{1}{s}\frac{1}{\sqrt{1+\beta}}$ and $\bar{g}_2(s)=\frac{1}{s}\frac{1}{\sqrt{1+\beta}}[1-(v/c)^2]^{-1/2}$ with $\beta=\frac{sa_e}{c-v}$. It follows easily that $l(t)=\mathrm{Erf}(\sqrt{t/b})$ where $b=a_e/(c-v)$ and hence that $W(t)=\frac{L_e^2 a_e}{2\mu}[1-(v/c)^2]^{-1/2}[\mathrm{Erf}(\sqrt{t/b})]^2$. It is easily seen that

$$W(t)=W(\infty) + O(e^{-(t/b)}) \text{ as } t \to \infty. \tag{4.26}$$

Thus W(t) converges exponentially to its steady–state limit with exponential order $e^{-(t/b)}$. It should be noted that $b \to \infty$ as $v \to c$ which implies that the exponential decay rate vanishes as the crack speed approaches the shear wave speed. The case of an elastic material with a failure zone is slightly more complicated but can be treated in a similar fashion. One has $\bar{l}(s)=\frac{1}{s}\left|\frac{1+\epsilon\beta}{1+\beta}\right|^{1/2}$ and $\bar{g}_2(s)=\frac{1}{s}\frac{(1-\epsilon)}{(1+\epsilon)}[a\beta^2+(1+a)\beta+1]^{-1/2}[1-(v/c)^2]^{-1/2}$ with $a=\epsilon\left[\frac{c-v}{c+v}\right]$. A simple calculation then yields that $W_{EL}(t)$ is given by (4.17) with

$l(t)=\sqrt{\epsilon}I_o(d_2t)e^{-d_1t} + \frac{1}{b\sqrt{\epsilon}}\int_0^t I_o(d_2r)e^{-d_1r}\,dr$ and

$g_2(t)=\frac{1}{b\sqrt{a}}\frac{(1-\epsilon)}{(1+\epsilon)}[1-(v/c)^2]^{-1/2}\int_0^t I_o(c_2r)e^{-c_1r}\,dr$ where $c_1=\frac{a+1}{2ab}$, $c_2=\frac{1-a}{2ab}$, $d_1=\frac{\epsilon+1}{2\epsilon b}$, $d_2=\frac{1-\epsilon}{2\epsilon b}$, and $I_o(t)$ is the modified Bessel function of the first kind of order zero defined by $I_o(t)=\sum_{r=0}^{\infty}\frac{(t/2)^{2r}}{(r!)^2}$. Since $\epsilon<1$ and $a<1$, one can show that $l(t)$ and $g_2(t)$ decay exponentially to their steady-state limits at the order $e^{-(t/b)}$ with $b=a_e/(c-v)$ as in the no failure zone case. Therefore (4.18) is valid also for an elastic material with a failure zone.

It is readily apparent that for a viscoelastic material the situation is considerably more complicated due to the combined influence of material inertia and viscoelastic stress relaxation upon convergence to steady state. A general property of the Laplace transform $F(s)$ of a function $f(t)$ is that $f(t)$ decays exponentially in time, say $f(t)\sim e^{-at}$ with $a>0$, if and only if $F(s)$ is analytic in the halfplane $Re(s)>-a$. The expressions (3.23) and (3.24) for $\bar{l}(s)$ and $\bar{g}_1(s)$ were derived for real positive s. Determining the largest value of a, a_{max}, for which $s\bar{l}(s)$ and $s\bar{g}_1(s)$ are analytic in the halfplane $Re(s)>-a$ is a difficult task that clearly depends upon the particular details of the transform $\tilde{m}(\tau s+\alpha\gamma/\epsilon)$. From the previously stated properties assumed for $m(t)$ it follows that $[\tilde{m}(\tau s+\alpha\gamma/\epsilon)]^{-1}$ is analytic for $Re(s)>-\frac{\alpha\gamma}{\tau\epsilon}$. If $m(t)$ is a power-law in t, then $\tilde{m}(\tau s+\alpha\gamma/\epsilon)$ has $s=-\frac{\alpha\gamma}{\tau\epsilon}$ as a branch point and a_{max} can be no larger than that. As a second example, for a standard linear solid with $m(t)=1+\eta e^{-t}$, $[\tilde{m}(\tau s+\alpha\gamma/\epsilon)]^{-1}$ is analytic for $Re(s)>-[(\eta+1)^{-1}+\alpha\gamma/\epsilon]/\tau$.

The heart of the matter lies in determining the analyticity properties of

$\beta(s,\gamma)$, which is defined implicitly through equation (4.1). Whether or not $\beta(s,\gamma)$ is analytic in some halfplane $\mathrm{Re}(s)>-a$, $a>0$, depends upon the particular way $m(t)$ decays to its equilibrium value as $t\to\infty$. This necessitates a case by case analysis for different forms of $m(t)$. We content ourselves here with illustrating the differences that exist between materials with exponentially decaying modulus, such as a standard linear solid, and those for which $m(t)$ decays as a power of t to its equilibrium value, such as a simple power–law material with $m(t)=1+\eta(1+t)^{-n}$, $n>0$.

An important observation to be made for power–law material is that $G(t)$ cannot have exponential decay to its steady–state limit when $\gamma<1$, i.e. $v<c_*$. The reason for this is that $\beta(s,\gamma)$ is not analytic at $s=0$ as can be seen from the following argument. It was remarked earlier that $\beta(0,\gamma)=0$ whenever $\gamma<1$, in particular $\lim_{s\to 0+} \beta(s,\gamma)/s=\frac{\tau}{\alpha(1-\gamma)}$. Moreover, for power–law material $\tilde{m}(s)$ has $s=0$ as a branch point. If $\beta(s,\gamma)$ were analytic at $s=0$ then the right hand side of equation (4.1) would also be analytic there. However, the left hand side of (4.1) has $s=0$ as a branch point. This contradiction proves the claim.

On the other hand, for $1<\gamma<m(0)^{1/2}$ (i.e. $c_*<v<c$), since $\beta(0,\gamma)>0$, equation (4.1) defines $\beta(s,\gamma)$ as an analytic function in a neighborhood of $s=0$. In particular, $\beta(s,\gamma)$ is analytic in a halfplane $\mathrm{Re}(s)>-a$, $a>0$. However, finding the largest such a is difficult. For elastic material, the rate of exponential convergence of $G(t)$ to its steady–state limit corresponds to the negative real value of s for which $\beta(s,\gamma)=-1$. For viscoelastic materials, again a case by case analysis will be required to determine precisely where the singularity of $s\bar{l}(s)$ or $s\bar{g}_1(s)$ with largest real part will occur.

The remaining case $\gamma=1$ (i.e. $v=c_*$) is easily handled. Indeed, for a general material, not just power–law or a standard linear solid, equation (4.1) admits no solutions for $s<0$. Thus $\beta(s,1)$ is not analytic in any halfplane $Re(s)>-a$, $a>0$ and $G(t)$ cannot converge exponentially to its steady–state limit. Furthermore, it is not difficult to see from equation (4.1) that the left endpoint of the largest s–interval containing zero for which an admissible solution $\beta(s,\gamma)$ exists converges to zero as γ approaches 1 from above.

These observations made for power–law material whenever $1\leq\gamma<m(0)^{1/2}$ ($c_*<v<c$) are equally valid for a standard linear solid. A departure in behavior occurs for $0<\gamma<1$. If $m(t)=1+\eta e^{-t}$, then $\tilde{m}(s)=\frac{1+(1+\eta)s}{1+s}$ and (4.1) can be rewritten as

$$\frac{1+(1+\eta)z}{1+z}=\gamma^2\left[\frac{z}{z-\tau s}\right]^2, \tag{4.19}$$

where $z=\tau s+\alpha\gamma\beta(s,\gamma)$. When $s>0$, a root z must be sought that is greater than τs, whereas $z<\tau s$ is required when $s<0$ is suitably near zero. An examination of the graphs of the functions on either side of equation (4.19) quickly reveals that admissible solutions exist for any $s>0$ and $s<0$ suitably near zero. Moreover, no solution exists for $\gamma=1$ and the left hand endpoint of the largest s interval containing zero on which a solution exists tends to zero as $s\to0-$. One concludes from this that for a standard linear solid, $G(t)$ converges exponentially to its steady–state limit for $0<\gamma<1$ and $1<\gamma<m(0)^{1/2}$ but not for $\gamma=1$. Moreover, the rate of exponential convergence, i.e. a_{max}, tends to zero as $\gamma\to1\pm$. It is also easy to see from (4.1) that a_{max} must vanish as $\gamma\to m(0)^{1/2}$.

5. Conclusions

The principal contributions of this paper to the study of transient mode III crack propagation are the solutions for the displacement, stress, and stress intensity factor for general loadings and general shear moduli, the inclusion of a failure zone into the model and the calculation of the total energy flux into the failure zone, G(t). It was then observed that significant qualitative and quantitative differences exist in the behavior of G(t) as a function of time, crack speed, and material properties between a model incorporating a failure zone and one which does not.

The question of the rate of convergence of G(t) to its steady–state limit as $t \to \infty$ was also investigated. It was observed that this rate of convergence depends in a complicated way upon the rate of stress relaxation and crack speed. In particular, for a standard linear solid in which stresses relax exponentially fast, G(t) converges exponentially fast for all crack speeds except the equilibrium shear wave speed. Thus for crack speeds near the equilibrium shear wave speed, it is expected that steady–state conditions would set in more slowly than for crack speeds above or below it. Also the exponential rate of convergence is lost at the glassy shear wave speed. In contrast, for power–law material, an exponential rate of convergence of G(t) does not occur for any crack speeds less than or equal to the equilibrium shear wave speed whereas for speeds between the equilibrium and glassy shear wave speeds G(t) does converge to steady state at an exponential rate.

Appendix

The behavior of β as $s\to\infty$ is determined by equation (4.1). If one lets $h=\beta/s>0$ then (4.1) becomes

$$\tilde{m}(s[\tau+\alpha\gamma h])=\gamma^2(1+(\tau/\alpha\gamma h))^2. \tag{A.1}$$

The left-hand side $\tilde{m}(s[\tau+\alpha\gamma h])=m(0)+\int_0^\infty e^{-(\tau+\alpha\gamma h)sr}m'(r)\,dr$ has the limit $m(0)=\left[\frac{c}{c_*}\right]^2>\gamma^2$ as $s\to\infty$. Therefore (A.1) will be satisfied only if $h=h_\infty+o(1)$ where $h_\infty>0$. As $s\to\infty$, (A.1) becomes $\left[\frac{c}{c_*}\right]^2=\gamma^2(1+(\tau/\alpha\gamma h_\infty))^2$. It is easily seen that $h_\infty=\frac{\tau}{\alpha\gamma[(c/c_*)-\gamma]}$ and thus $\beta=\frac{\tau s}{\alpha\gamma[(c/c_*)-\gamma]}+o(s)$ as $s\to\infty$.

To determine the behavior of β as $s\to 0$, one must again consider equation (4.1). Note that the left-hand side of (4.1) is

$$\tilde{m}(\tau s+\alpha\gamma\beta)=m(0)+\int_0^\infty e^{-(\tau s+\alpha\gamma\beta)r}m'(r)\,dr\geq 1. \tag{A.2}$$

For $0<v<c_*$, the limit as $s\to 0$ of the right-hand side of (4.1), $\gamma^2(1+\tau s/\alpha\gamma\beta)^2$, will satisfy (A.2) only if $\beta=\beta_1 s+o(s)$ as $s\to 0$. Therefore as $s\to 0$, (4.1) becomes

$$1=\gamma^2(1+\tau/\alpha\gamma\beta_1)^2. \tag{A.3}$$

The solution of (A.3) for β_1 is easily seen to be $\beta_1=\frac{\tau}{\alpha\gamma[1-\gamma]}$.

For $c_*<v<c$, the right-hand side of (4.1) satisfies

$$\gamma^2(1+\tau s/\alpha\gamma\beta)^2\geq\gamma^2>1. \tag{A.4}$$

If $\beta\to 0$ as $s\to 0$ then it can be seen from (A.2) that the limit of $\tilde{m}(\tau s+\alpha\gamma\beta)=1$ as $s\to 0$ which contradicts (A.4). Thus $\beta=\beta_o+o(1)$, $\beta_o>0$ as $s\to 0$ where β_o satisfies

$$\gamma^2-(c/c_*)^2=\int_0^\infty e^{-\alpha\gamma\beta_o r}m'(r)\,dr.$$

For $v=c_*$, it can be shown that to satisfy (4.1), $\beta\to 0$ and $s/\beta\to 0$ as $s\to 0$. Therefore consider $\beta=\beta_1 s^\eta+o(s^\eta)$, $0<\eta<1$, as $s\to 0$. If one substitutes this into (4.1) and simplifies the equation, one obtains

$-\alpha\gamma\beta_1 s^{\eta}\int_0^{\infty} rm'(r)\,dr + o(s^{\eta}) = \frac{2\tau s^{1-\eta}}{\alpha\gamma\beta_1} + o(s^{1-\eta})$ as $s\to 0$, assuming that $\int_0^{\infty} rm'(r)\,dr$ exists. If this integral does not exist then one must do the asymptotics for the particular shear modulus needed. Since the coefficient of each side is non–zero, $\eta=1/2$ and $\beta_1=\frac{\sqrt{2\tau}}{\alpha\gamma}\left[-\int_0^{\infty} rm'(r)\,dr\right]^{-1/2}$. In summary, $\beta=\frac{\tau s}{\alpha[1-\gamma]} + o(s)$ for $0<v<c_*$, $\beta=\frac{\sqrt{2\tau}}{\alpha\gamma}\left[-\int_0^{\infty} rm'(r)\,dr\right]^{-1/2} s^{1/2} + o(s^{1/2})$ for $v=c_*$, and $\beta=\beta_o+o(1)$ for $c_*<v<c$.

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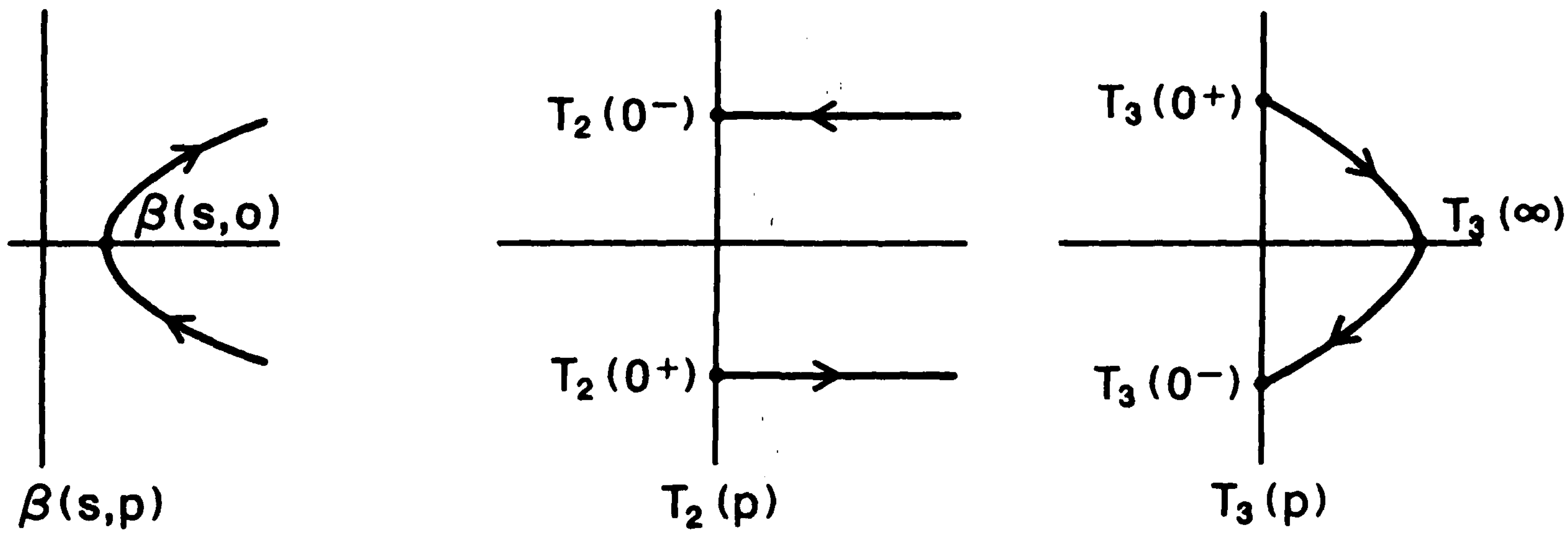
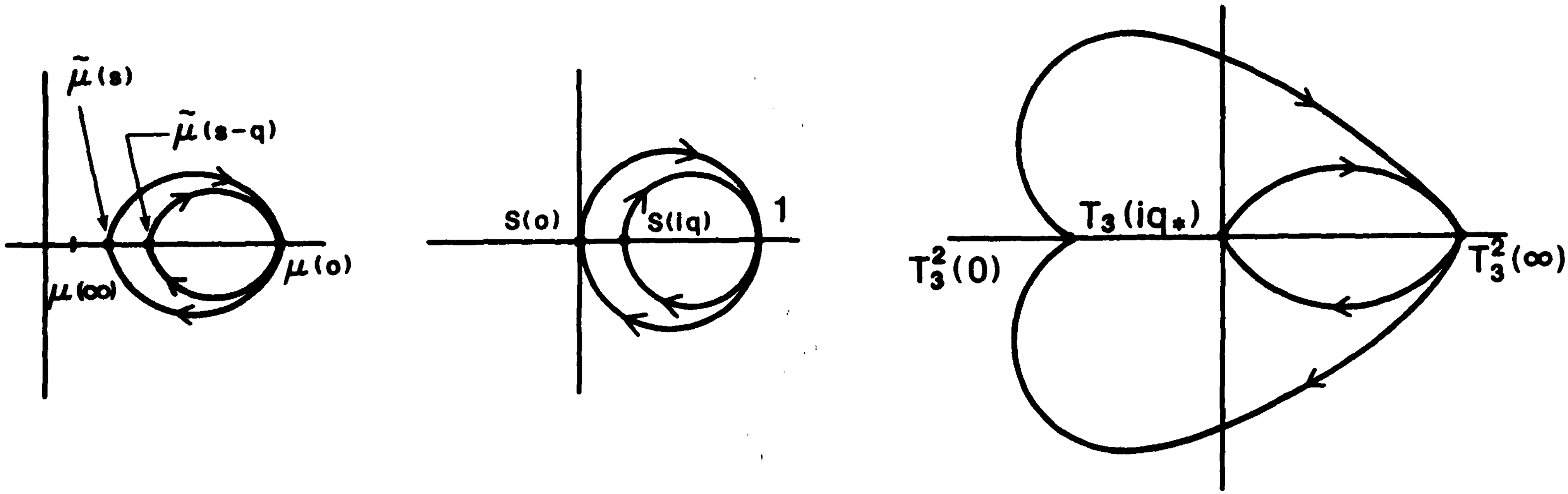


Fig. 2.1



(a) $\tilde{\mu}(s-q+ivp)$ (b) $S(p+iq)$ (c) $T_3^2(p+iq)$

Fig. 2.2