Beam Breakup Instabilities in Linear Accelerators:
Transition, Phase Mixing and Nonlinear Focusing

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The temporal and spatial evolution of the cumulative beam breakup instability (BBU) is analyzed numerically using a continuum model. It is found that neither phase mixing nor linear transverse focusing is sufficient to render BBU stable in a long pulse machine, when external damping is absent. A sufficiently strong nonlinear octupole focusing field may limit the BBU growth, however.
BEAM BREAKUP INSTABILITIES IN LINEAR ACCELERATORS: TRANSITION, PHASE MIXING AND NONLINEAR FOCUSING

The control of beam breakup instabilities\(^1\) remains a major concern for accelerator builders. Various remedies have been examined\(^2\): lowering the quality factor \(Q\) of the deflecting modes, increasing the focal strength, phase mixing, nonlinear focusing, and detuning of the cavities, etc. All of these may readily be incorporated in the continuum models,\(^1-7\) from which asymptotic formulas have been obtained for certain cases. These formulas have frequently been used to compare with experimental observations and with other more elaborate theoretical works.\(^6,8\) At the moment, they are the only ones available to describe the transition of BBU from the cumulative type to the regenerative type.\(^7,9\)

In spite of the versatility in the continuum model, a direct numerical integration of the governing partial differential equations does not seem to have been published, although the evolution of BBU has been calculated in great detail by other means, in some limiting cases.\(^1,3,10\) Here, we analyze the evolution of the cumulative BBU by a direct numerical integration, intending to look into the effects of phase mixing and nonlinear focusing which defies analytic treatment. This work was also motivated by the recent observation\(^7\) that the asymptotic growth of the cumulative BBU is independent of the linear transverse focal strength. It is then of interest to determine in what way nonlinear focusing affects the temporal and spatial evolution of BBU.

To isolate the mechanism of phase mixing and of nonlinear focusing, we ignore the damping associated with finite \(Q\). Setting \(Q\) equal to infinity should not be construed as this damping effect being unimportant. It serves the additional purpose of providing a more stringent test for the numerical scheme. For without a finite \(Q\), the solution grows much more
rapidly, making the accuracy in the numerical integration more demanding. In any case, the effect of finite $Q$ enters only through the well-known decay factor $\exp\left(-\frac{\omega_0 t}{2Q}\right)$ in the amplitude of the solutions, where $\omega_0$ is the breakup mode frequency.

Our numerical results reveal the following. When $Q$ is infinite, the cumulative BBU evolves asymptotically (at a fixed distance from the injector) at a rate which is independent of the linear transverse focal strength.\textsuperscript{7} The excellent agreement with the analytic formula adds to our confidence in our numerical calculation. [In a separate test, the numerical solution agrees with the analytical results to within 20% over nine orders of magnitude variations in the beam’s transverse displacement.] When the analysis is extended to include a finite spread in the betatron frequency, we find that this finite spread has virtually no effect on the above-mentioned asymptotic behavior. While neither a linear focusing force nor a finite spread in the betatron frequency, by themselves, can provide long-term stabilization of BBU, a sufficiently strong nonlinear focusing force may limit the amplitude of the beam’s transverse displacement, even in the limit $Q \rightarrow \infty$.

For a continuous coasting beam, the evolution of the cumulative BBU is governed by the following normalized equations\textsuperscript{1,3}

\[
(\frac{\partial}{\partial T} + \frac{\partial}{\partial Z})^2 X + \Omega^2 X = A 
\]

(1)

\[
\frac{\partial^2 A}{\partial T^2} + A = 2\varepsilon X. 
\]

(2)

Here $T = \omega_0 t(>0)$, $Z = \omega_0 z/c(>0)$, $\Omega = k_B c/\omega_0$ where $k_B$ is the betatron wavenumber, $\varepsilon$ is the dimensionless coupling constant\textsuperscript{7} which is proportional to the beam current and shunt impedance but is independent of the focal
strength, $Q(Z,T)$ is a measure of the strength of the deflecting mode which drives the BBU and $X(Z,T)$ is the normalized transverse displacement of the beam. In all numerical calculations, we assume initial rest condition and homogeneous boundary conditions; only one non-trivial boundary condition is imposed at $Z=0$: $X(0,T) = 1$ for $0 < T < 0.2$ and is zero otherwise. Thus, $X$ is normalized to the initial displacement, imparted upon the beam at $Z=0$ for a short time (similar to a delta function excitation).\(^1,3-8\)

Shown in Fig. 1 is the spatial distribution of $X(Z,T)$ at four different times, obtained from a fourth order numerical scheme in space and time. We set $\varepsilon = 0.000413$ and $\Omega = 0.14$. The dotted curves in this figure are obtained from the asymptotic solution\(^7\):

$$
|X(Z,T)| = B \frac{Z^{1/4}}{(T-Z)^{3/4}} \exp \left\{ \frac{2\varepsilon(1-T-Z)}{\Omega} \right\}
$$

where $\tau = \varepsilon(T-Z)/(\Omega^2 Z) > 0$, $B = C/[\text{Max}(1,\tau/2.81)]^{1/12}$,

$$p(\tau) = \begin{cases} 
1.1573 \left(1 - \frac{0.21}{\tau^{2/3}}\right) & \text{for } \tau \geq 5 \\
1 - 0.0556\tau + 0.004\tau^2 & \text{for } \tau \leq 5
\end{cases} \quad (4a,b)$$

and $C$ is a constant to be determined by a "one-point fit" with the numerical solution. Equation (3) describes the transition from the strong focusing solution\(^8\) [$p(\tau) \to 1$, Eq. (4b)] to the weak focusing solution\(^1\) [Eq. (4a)] as $\tau$ increases.\(^7\) This transition occurs when $\tau = 0.5$. The excellent agreement, over a wide range of $\tau$, between the numerical solution and the analytic theory shown in Fig. 1 may be regarded as a validation for both the numerical scheme and the analytic result that $X$ evolves at a later time as if the linear focusing force were absent.\(^7\) The numerical data shows
that the spatial variation of $X$ is biased toward the "slow beam-cyclotron mode".  

The effect of phase mixing due to a spread in the betatron frequency may be examined by pretending that the electron beam consists of $N$ groups of particles, the $i$th group bearing its individual betatron frequency $\Omega_i$. Thus, we replace Eq. (1) with $N$ equations

$$
(\partial^2/\partial T^2 + \partial^2/\partial Y^2)X_i^2 + \Omega_i^2X_i = A, \quad i = 1, \ldots, N
$$

and the quantity $X$ in the right-hand side of Eq. (2) is replaced by $\langle X \rangle = G_1X_1 + G_2X_2 + \ldots + G_NX_N$ where $G_i$ represents the fraction occupied by the $i$th group ($G_1 + G_2 + \ldots + G_N = 1$). In the "spread mass" model, $\Omega_i = \Omega(i/N)^{1/2}$ and $G_i = 6i(N - i)/N(N^2 - 1)$. The solution $\langle X \rangle$ is shown in Fig. 2 for $\epsilon = 0.000413$, $\Omega = 0.14$, $N = 8$. A virtually identical figure was obtained when we increased $N$ to 16. From this figure, we see that a finite spread in the betatron frequency has no effect on the long-time growth of the cumulative BBU. This is due to the fact that the cumulative BBU grows asymptotically at a rate independent of the linear focusing force, and a finite spread in the betatron wavelength cannot alter this long-time behavior.

A nonlinear octupole focusing force may be handled also by solving Eqs. (1) and (2), except that the $\Omega^2X$ term in Eq. (1) is now replaced by $AX^3$ in a simplified model. The evolution of $X$ is shown in Fig. 3 for the case $A = 0.0392$. Here, we see that the amplitude of $X$ is limited at large $T$, in marked contrast to the cases of linear focusing and phase mixing. It is possible that the change in the phase relationship between $A$ and $X$ due to nonlinearity might have contributed to the stabilization shown in Fig. 3, but this issue remains to be examined in the future.
In summary, an accurate numerical scheme was devised to analyze the classic model of cumulative BBU. It is suggested that this line of attack could be fruitfully applied to a great variety of problems in a rather straightforward manner. Interesting areas such as phase mixing and nonlinear focusing were analyzed in this initial study, and some unexpected features reported.

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Fig. 1 Evolution of $X(Z,T)$ and comparison with the analytic theory (dotted curves).
Fig. 2 Temporal evolution of $\langle X \rangle$ at $Z = 40$, including the effect of phase mixing.
Fig. 3 Evolution and saturation of $X(Z,T)$ in the presence of nonlinear octupole focusing.
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