A COMPARISON OF CONTROL VARIATES FOR QUEUEING NETWORK SIMULATION

THESIS

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Preface

The purpose of the research presented in this paper was to compare several control variates for queueing network simulation. During the literature search and review I noted the most promising internal and external control variates. I also included a new external control variate which was calculated using Bell Lab's Queueing Network Analyzer, a network decomposition algorithm employing two-moment approximations of the stochastic processes in the network.

The experiment was designed to be general enough to apply to many queueing networks. The variance ratios of each of the control variates provides a measure of their efficiency to reduce the variance of the parameter being estimated. The coverage probabilities of the confidence intervals formed about the controlled estimates of the mean responses provide a measure of the accuracy of the variance reduction technique. That is, a control variate that results in a confidence interval with poor coverage of the true parameter is not very useful even if the variance reduction is significant.

I wish to thank my thesis advisor, Major Joseph R. Litko, Ph.D., and Major Kenneth W. Bauer, Jr., Ph.D. for their guidance in this research effort. I also wish to thank Major Anthony P. Sharon for providing me with a copy of his thesis which was the foundation and inspiration of this work. Finally, I wish to thank my wife Jamie for her understanding and support during the many days and nights I was tied to my computer.

John J. Tomick

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Abstract

The purpose of this ~~study~~ thesis was to compare several control variates for queueing network simulation. The author's goal was to provide the simulation community with some guidance for selecting control variates that will lead to significant reductions in the variance of the estimated responses that do not introduce significant bias.

Both internal and external control variates were examined. The measures for comparing them were the variance ratios obtained for each control variate against each of the two response variables and the coverage of their respective 95% confidence intervals.

The two response variables selected for this research were the average sojourn time in the network and the probability that the number in the fourth queue exceeds twice the mean number in queue at steady-state. The internal controls included standardized routing controls and standardized work variables. The external controls included the average sojourn time in the network and the average number in queue at the fourth node. The external controls were further classified into two groups: analytic Jackson controls and analytic approximations. (KH)

The analytic Jackson control variates were found by decomposing the network and using the M/M/1 formulas. The analytic approximations were found using G/G/1 formulas, specifically those employed in Bell Laboratories' Queueing Network Analyzer. The observed values of the external control variates were found by using the following parameters observed during a run of the simulation model: the external arrival rate, the mean service times at each node, the squared coefficient of

variation of the service times at each node, and the probabilistic routing matrix. The "known" means of the external control variates were found using the values of the above parameters that were input to the simulation model.

The network of queues studied was an open network with Poisson external arrivals and exponential servers at the first three nodes. At the fourth node the service times were generated from the maximum entropy and the hyperexponential distributions characterized by two moments.

In general, the external control variates achieved smaller variance ratios than the internal control variates; however, the coverages of the confidence intervals about the controlled responses were worse. The range of the average variance ratios for the internal control variates was 0.886 to 0.949 with coverages from 0.898 to 0.929 for the 95% confidence intervals. The range of the average variance ratios for the external control variates was 0.494 to 0.774 with coverages from 0.576 to 0.775 for the 95% confidence intervals.

A COMPARISON OF CONTROL VARIATES FOR QUEUEING NETWORK SIMULATION

I. Introduction

The purpose of this paper is to report the results of the author's research comparing the effectiveness and the bias of several control variates for queueing network simulation. For this research, the author selected some of the most promising control variates found in the literature. Also, the author included a new external control technique that makes use of a software package which approximates the performance measures of a queueing network.

The inspiration and foundation for this research came from the work done by Sharon (1986). He examined the effectiveness of Jackson networks as control variates for queueing network simulation. A more complete discussion of Sharon's work can be found in the review of the literature presented in Chapter II.

The author's research makes a significant contribution to the experiential knowledge of control variates in two areas. First, most of the literature on experimental results of control variates report only the efficiency of the technique in terms of variance ratios, percent variance reduction, etc. This research effort also examined the bias introduced to the estimates through the use of control variates in terms of the estimated coverage of the 95% confidence intervals about the controlled responses. Second, this research is the first attempt to compare both internal and external control variates. The ultimate goal of this research was to provide the simulation community with some guidance for selecting control variates for use in queueing network simulation.

Following the literature review, the author discusses the methodology of his approach to the research in Chapter III. Then, the results of the research are presented in Chapter IV. Finally, the author's conclusions and recommendations are given in Chapter V.

The rest of this chapter is devoted to introducing the reader to some of the basic terms, concepts, and notation used in queueing theory and in simulation modeling, respectively. The reader who is familiar with these areas may skip to Chapter II.

Queueing Theory

Basic queueing theory involves customers arriving to a service center. The customers can represent people waiting in line for a bank teller, cars waiting in line at a toll booth, and so forth. Because queueing theory has such broad applicability, one will usually see the the more generic term *entity* used instead of *customer*.

A Single Queue. To describe a queueing system we need a description of the arrival process, a description of the service mechanism, and a queue discipline.

The arrival process can be deterministic; that is, the time between arrivals of entities is a constant. It can also be a random process described by a probability density function. Traditionally, the arrival rate (the mean number of entities arriving to a node per unit time) is denoted by the Greek letter lambda (λ). The mean interarrival time, or mean time between arrivals, is simply the reciprocal of the arrival rate.

Similarly, the service mechanism may be deterministic or probabilistic. Furthermore, the service center may have more than one identical server. Traditionally, the service rate (the mean number of entities that can be serviced per unit time) is denoted by the Greek letter mu (μ). Likewise, the mean service time, or mean time between service completions, is the reciprocal of the service rate.

The traffic intensity at a node is defined to be the ratio of the arrival rate to the product of the service rate and the number of identical servers, which is denoted by the Greek letter rho (ρ). Mathematically, $\rho = \lambda/m\mu$, where m denotes the number of identical servers. If the traffic intensity is greater than one, then entities are arriving to the queue faster than they can be serviced. Therefore, the queue grows infinitely long (i.e. the queue is unstable) unless the calling population is finite or the queue is capacitated with blocking or balking. When a queue has a finite capacity with blocking, then the servers that feed the queue stop servicing entities until there is room for them in the following queue. On the other hand, a queue filled to capacity that allows balking causes entities arriving to the queue to be routed to another queue or to leave the system entirely. The research presented in this paper examined a network of queues with infinite capacities at equilibrium (or steady state), which implies a traffic intensity less than unity.

The traffic intensity is also referred to as the utilization factor for the service center, since it represents the fraction of the server's capacity that is being utilized on the average by arriving entities.

Finally, the queue discipline indicates how to select the next entity waiting for service. The most common queue discipline is *first in, first out* (FIFO), which is also referred to as *first come, first served* (FCFS). However, one might also specify *last in, first out* (LIFO) as the queue discipline, or set up some sort of priority selection based upon the type of entity.

A Network of Queues. Networks of queues are useful models for describing many real-world systems, such as computer time-sharing processes, communication systems, transportation systems, and assembly-line operations.

A queueing network is composed of nodes representing a system's service mechanisms and directed arcs between the nodes representing the flow of entities through the system. The individual nodes are conventionally labeled by the distribution of the interarrival times, the distribution of the service times, and the number of identical servers. For example, an M/M/1 queue has exponentially distributed interarrival times and service times with one server. The "M" stands for Markovian (memoryless), and the exponential distribution is the only continuous distribution with the Markovian property. In general, a queue with many independent input processes that are not necessarily Markovian, a non-Markovian service-time distribution, and many servers is denoted by GI/G/m.

There are two broad classifications of queueing networks: open, and closed. An open queueing network allows for external arrivals to the nodes (referred to as exogenous arrivals). In general, entities may enter or leave the network from any node. Alternatively, a closed network has a fixed number of entities cycling through it; no additional entities arrive to the network, and none of the entities in the network ever leave.

Jackson Networks. A Jackson network is an analytically tractable queueing network model. Although, it is a fairly restrictive model, it has found many uses and is a good approximation to many real-world systems. In a Jackson network, all external arrivals are independent Poisson processes (that is, their interarrival times are independent and exponentially distributed), all servers have exponentially distributed service times, the queue discipline is FIFO, the queue capacity at each node is infinite, the routing between nodes can be probabilistic but not conditional, and the time to travel between two nodes is zero.

Burke's theorem states that "the steady-state output of a stable M/M/m queue with input parameter λ and service-time parameter μ for each of the m channels is in fact a Poisson process at the same rate λ" (Kleinrock, 1975:149). Therefore, given a network of M/M/m queues, the input process to any node (i) in the network is a Poisson process with parameter λ_i, which is a mixture of the output processes of nodes feeding into it plus the external arrival process. And, as Jackson discovered "each node ... in the network behaves as if it were an independent M/M/m system with a Poisson input rate λ_i" (Kleinrock, 1975:150). This allows a network to be decomposed into independent M/M/m queues whose performance measures can be solved for analytically.

If the queueing network can be represented as a Jackson network, then the performance measures (such as the average waiting time at a particular queue, the utilization of a particular server, or the average time an entity spends in the system) can be solved for analytically. If we drop the subscript (i) for readability, then the steady-state results for the M/M/m queue at any node can be solved for using the following equations:

$$\rho = \lambda/m\mu \tag{1.1}$$

$$P_0 = \left[\sum_{n=0}^{m-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^m}{m!}\cdot\frac{1}{(1-\rho)}\right]^{-1} \tag{1.2}$$

$$P_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} P_0, & \text{if } 0 \le n \le m \\[2ex] \dfrac{(\lambda/\mu)^n}{m!\,m^{n-m}} P_0, & \text{if } n \ge m \end{cases} \tag{1.3}$$

$$L_q = \frac{P_0(\lambda/\mu)^m\rho}{m!(1-\rho)^2} \tag{1.4}$$

$$W_q = L_q/\lambda \tag{1.5}$$

$$L = L_q + (\lambda/\mu) \tag{1.6}$$

$$W = W_q + (1/\mu) \tag{1.7}$$

where

λ = arrival rate of entities to the node
m = number of identical servers
μ = service rate of servers at the node
ρ = traffic intensity or server utilization
P_0 = probability that there are 0 entities in the system
P_n = probability that there are n entities in the system
L = expected number of entities in the system
L_q = expected queue length (excludes entities in service)
W = expected time in system
W_q = expected waiting time (excludes service time)

However, there are many systems for which the assumptions of the Jackson network are grossly violated. This leads to a queueing model that is analytically intractable. The performance measures of such networks can be solved for by methods which use approximations, or by simulating the model. The reader is referred to Chapter II for the author's review of analytical approaches to solving queueing networks using approximations.

Simulation Modeling

Simulation is an experimental technique for analyzing complex systems which are analytically intractable and usually involve stochastic processes (indexed collections of random variables). As such, the output from a simulation experiment is a random variable; and therefore, the measured response is only an estimate of the true parameter of interest. The variance of the estimate is a measure of its precision; and, in most cases, variance reduction techniques provide a means of obtaining more precise estimates with minimal cost in terms of computer resources.

Variance Reduction Techniques. In a recent survey of variance reduction techniques (VRTs), Wilson classified "... all VRTs into two major categories--correlation methods and importance methods" (Wilson, 1984:280). In the paper he discusses three correlation methods (common random numbers, antithetic variates, and control variates) and four importance methods (importance sampling, conditional Monte Carlo, stratified sampling, and systematic sampling).

The basic distinction between the two categories is the underlying principle of the techniques. Correlation methods increase the efficiency of a simulation by making use of the linear correlations among simulation responses, and importance methods use prior knowledge of the input domain to achieve a variance reduction. For a more complete discussion of these techniques the reader is referred to Kleijnen (1974), Law and Kelton (1982), and Wilson (1984).

Sharon's research and the research presented in this paper use control variates to achieve a variance reduction in the estimates of the performance measures of interest. A control variate must have a known expectation and be correlated with the response.

Methods for Obtaining Control Variates. Law and Kelton describe three general methods for obtaining control variates. The first method uses the correlation between the input random variable(s) and the output random variable(s). Since these kinds of control variates must be generated during the simulation they are called *internal* or *concomitant* control variates.

A second method involves the simulation of a similar system that is analytically tractable. Using common random numbers, the corresponding output of the second simulation becomes the control variate for the system under study. These kinds of control variates are called *external* control variates. Note that this method assumes that there is a significant correlation between the results of the two systems through the use of common random numbers.

The third method makes use of the situations when there are several unbiased estimators of the mean of the performance measure of interest. In such cases a new controlled estimator can be formed as a convex combination of the existing estimators (Law and Kelton, 1982:358-359).

II. Literature Review

The following discussion is a review of the literature that is applicable to the author's research into the following two areas: (1) control variates for queueing network simulation, and (2) two-moment approximations to performance measures of the GI/G/m queue. The discussion is presented in a topical format as outlined below.

A. Control Variates
    1. Justification for New Research
    2. Theory of Control Variates
    3. Results of Previous Research

B. Approximations to Point Processes
    1. Need for Approximations
    2. Two-Moment Approximations
    3. The Queueing Network Analyzer

Control Variates

In the first chapter, the author introduced variance reduction techniques which are used to provide more precise estimates of the response from a simulation experiment. This research compared and contrasted several control variates for their efficiency in reducing the variance of the estimated responses and for the amount of bias that may have been introduced to the estimates.

Justification for New Research. As mentioned previously, there are two major categories of variance reduction techniques--correlation methods and importance methods. Of all the techniques, "... the method of control variates is one of the most promising" (Lavenberg and Welch, 1981:322).

Correlation Methods. The correlation methods include common random numbers, antithetic variates, and control variates. A brief discussion of the uses and drawbacks of these correlation methods are presented below.

Common Random Numbers. The method of common random numbers can be used when comparing two or more alternative system designs, or when designing an experiment for a response surface model. This method assumes the existence of a positive correlation between the random number streams driving the simulation and the measured response(s).

However, in complex simulation models, especially queueing network models, the correlation tends to be very weak. In some experiments it may even be negative resulting in a variance increase. Furthermore, the synchronization of common random number streams may be difficult if not impossible for some simulation experiments.

Antithetic Variates. The method of antithetic variates is applicable to the simulation of a single system. This method tries to induce a negative correlation between pairs of runs of the simulation model to achieve a variance reduction. According to Law and Kelton, the method "... dates back at least to 1956 with the paper of Hammersley and Morton in the context of Monte Carlo simulation" (Law and Kelton, 1982:354). More recently, Schruben and Margolin (1978) demonstrated the effectiveness of antithetic variates in conjunction with a 2^k factorial experimental design.

But, this method suffers from the same drawbacks as does the method of common random numbers--the correlation may be weak or opposite in sign to that desired, and the synchronization of random number streams may be a problem. Also, there are some other assumptions that must be met to use Schruben and Margolin's assignment rule. As with common random numbers, the method of antithetic variates seems to be more successful when used with Monte Carlo simulation experiments.

Control Variates. The method of control variates is applicable to any simulation experiment involving stochastic processes with known means. A practitioner may simultaneously collect observations for several internal controls and select those having a significant correlation with the response variable(s). Furthermore, the practitioner can calculate the magnitude of the variance reduction without further simulation runs. If the practitioner had used common random numbers or antithetic variates and wanted to know by how much the variances of the estimates were reduced, it would require additional simulation runs using independent random number streams.

However, control variates have some drawbacks as well. The traditional type of external control variates require additional simulation runs and, the resulting controlled estimator may be biased.

Importance Methods. The importance methods include several sampling techniques, which have not found much popularity among simulation practitioners. Furthermore, Pritsker concluded that the importance methods require further refinement before they can be applied to complex simulation experiments (Pritsker, 1986:749).

Theory of Control Variates. The fundamental idea behind the method of control variates is to select a random variable with a known expectation that is highly correlated with the response variable.

Univariate Simulation Response with a Single Control. Let Y be an unbiased estimator of the parameter of interest θ; that is, the expectation of Y, denoted E(Y), equals θ. Let C be another random variable with known expectation μ_c that is highly correlated with Y. Then, for any constant b (known as the control coefficient), the controlled estimator Y(b), given by Eq (2.1), is unbiased for θ.

$$Y(b) = Y - b(C - \mu_c) \tag{2.1}$$

The variance of Y(b) is given by

$$Var[Y(b)] = Var(Y) + b^2Var(C) - 2bCov(Y,C) \tag{2.2}$$

and a variance reduction will be realized if

$$2bCov(Y,C) > b^2Var(C) \tag{2.3}$$

That is, if Eq (2.3) is satisfied, then the controlled estimator will have a smaller variance than the uncontrolled estimator. With a little calculus it is easy to show from Eq (2.2) that Y(b) has minimum variance when b is set equal to the optimal control coefficient, β, which is given by

$$\beta = Cov(Y,C)/Var(C) \tag{2.4}$$

Substituting Eq (2.4) into Eq (2.1) leads to the optimal controlled estimator $Y(\beta)$, which is given by

$$Y(\beta) = Y - [Cov(Y,C)/Var(C)]\cdot(C - \mu_c) \tag{2.5}$$

Anderson (1984) provides a proof that the variance of the controlled estimator which is given by

$$\mathrm{Var}[Y(\beta)] = (1 - \rho_{YC}^2)\cdot \mathrm{Var}(Y) \tag{2.6}$$

where ρ_{YC}^2 is the square of the correlation coefficient between the response variable Y and and the control variate C. Because the correlation coefficient in Eq (2.6) is squared, the sign of the correlation does not matter; only the size of the correlation is important. As $|\rho_{YC}| \to 1$, the correlation between Y and C becomes more significant, and the size of the variance reduction increases.

Let θ, the parameter of interest, be denoted by μ_Y. Then, we know that the average of the uncontrolled observations Y_i is an unbiased point estimator of μ_Y. Furthermore, the average of the controlled observations $Y_i(\beta)$ is also an unbiased estimator of μ_Y. This is represented mathematically as follows:

$$\bar{Y}(\beta) = (1/K) \sum_{i=1}^{K} Y_i(\beta) \tag{2.7}$$

where K is the sample size and

$$Y_i(\beta) = Y_i - \beta(C_i - \mu_C) \tag{2.8}$$

In practice, Cov(Y,C) and Var(C) are unknown; and therefore, β is unknown and must be estimated. Following Bauer (1987), the intuitive approach to estimating β is to replace the right-hand side of Eq (2.4) with the appropriate sample statistics, which yields the least-squares solution. Under the assumption of joint normality between Y and C, the least squares solution is also the maximum likelihood solution (Bauer, 1987:6). Then, β can be estimated by

$$\hat{\beta} = \sum_{i=1}^{K}(Y_i - \bar{Y})(C_i - \bar{C}) / \sum_{i=1}^{K}(C_i - \bar{C})^2 \tag{2.9}$$

where

$$\bar{Y} = \sum_{i=1}^{K} Y_i / K \tag{2.10}$$

and

$$\bar{C} = \sum_{i=1}^{K} C_i / K \tag{2.11}$$

Furthermore, a point estimate of μ_Y is given by

$$\bar{Y}(\hat{\beta}) = \sum_{i=1}^{K} Y_i(\hat{\beta}) / K \tag{2.12}$$

or

$$\bar{Y}(\hat{\beta}) = \bar{Y} - \hat{\beta}(\bar{C} - \mu_c) \tag{2.13}$$

And, the variance of the point estimator is given by

$$\hat{Var}[\bar{Y}(\hat{\beta})] = \hat{Var}[Y(\hat{\beta})]/K \tag{2.14}$$

where

$$\hat{Var}[Y(\hat{\beta})] = (1 - \hat{\rho}^2_{Yc}) \cdot \hat{Var}(Y) \tag{2.15}$$

Bauer (1987) provides the derivation of the interval estimate under the assumption that Y and C are jointly normal random variates.

The resulting $100(1-\alpha)\%$ confidence interval is given by the following equation

$$\bar{Y}(\hat{\beta}) \pm t_{K-2}(1-\alpha/2)\cdot\{\hat{Var}[\bar{Y}(\hat{\beta})]\cdot s_{11}\}^{1/2} \tag{2.16}$$

where

$$s_{11} = \sum_{i=1}^{K}(C_i - \mu_c)^2/K \sum_{i=1}^{K}(C_i - \bar{C})^2 \tag{2.17}$$

and $t_{K-2}(1-\alpha/2)$ is the $100(1-\alpha/2)$ percentile of Student's t-distribution with (K-2) degrees of freedom.

Since β must be estimated, we expect to achieve a smaller variance reduction than that which could have been obtained had we known the optimal control coefficient. Lavenberg, Moeller and Welch (1982) quantified this loss by what is known as the loss factor (LF). It is defined as "the ratio of the variance of the estimator of μ_Y when the optimal control coefficient is not known to the variance of the estimator when the coefficient is known" (Bauer, 1987:9). Bauer provides the derivation of the loss factor, which reduces to

$$LF = (K-2)/(K-Q-2) \tag{2.18}$$

where

Q = the number of controls (for the univariate case Q=1)

This "loss factor acts as a multiplier to the minimum variance ratio (MVR)" (Bauer, 1987:10,14) given by

$$MVR = Var[Y(\beta)]/Var(Y) \tag{2.19}$$

The MVR represents the possible variance reduction when the optimal control coefficient is known. Multiplying Eq (2.18) and Eq (2.19) together leads to the variance ratio (VR). The VR represents the possible variance reduction when β is not known.

$$VR = LF \cdot MVR \tag{2.20}$$

<u>Univariate Simulation Response with Multiple Controls</u>. Kleijnen (1974) addresses the extensions to multiple control variates. Also, Bauer (1987) provides a summary of "the development presented by Lavenberg and Welch (1981) for simulation output analysis based on independent replications, batch means, and regenerative analysis" (Bauer, 1987:11).

Let Y be the univariate response with variance σ_Y^2, $\underset{\sim}{C}$ be the (QX1) vector of controls, $\underset{\sim}{\sigma}_{cY}$ be the (QX1) vector of covariances between Y and $\underset{\sim}{C}$, and $\underset{\sim}{\Sigma}_c$ be the (QXQ) covariance matrix of the controls. Then, rewriting Eq (2.13) with multiple controls leads to

$$\bar{Y}(\hat{\underset{\sim}{\beta}}) = \bar{Y} - \hat{\underset{\sim}{\beta}}^T(\bar{\underset{\sim}{C}} - \underset{\sim}{\mu}_c) \tag{2.21}$$

where $\hat{\underset{\sim}{\beta}}$, $\bar{\underset{\sim}{C}}$, and $\underset{\sim}{\mu}_c$ are (QX1) vectors. The vector of optimal control coefficients, is then given by

$$\underset{\sim}{\beta} = \underset{\sim}{\Sigma}_c^{-1}\underset{\sim}{\sigma}_{cY} \tag{2.22}$$

Since the covariance matrices are usually unknown, β can be estimated by substituting the sample analogs of $\underset{\sim}{\Sigma}_c$ and $\underset{\sim}{\sigma}_{cY}$ into Eq (2.22). This leads to the following equation:

$$\hat{\underset{\sim}{\beta}} = \underset{\sim}{S}_c^{-1}\underset{\sim}{S}_{cy} \tag{2.23}$$

where $\underset{\sim}{S}_c^{-1}$ is the inverse of the (QXQ) sample covariance matrix of the controls, and $\underset{\sim}{S}_{cy}$ is the (QX1) vector of sample covariances between the univariate response and the vector of controls.

Under the assumption that Y and $\underset{\sim}{C}$ have the joint multivariate normal distribution

$$\begin{bmatrix} Y \\ \underset{\sim}{C} \end{bmatrix} \sim N_{Q+1}\left(\begin{bmatrix} \underset{\sim}{\mu}_y \\ \underset{\sim}{\mu}_c \end{bmatrix}, \begin{bmatrix} \underset{\sim}{\sigma}_y^2 & \underset{\sim}{\sigma}_{yc} \\ \underset{\sim}{\sigma}_{cy} & \underset{\sim}{\Sigma}_c \end{bmatrix}\right) \tag{2.24}$$

$Y(\underset{\sim}{\beta})$ is unbiased for μ_y, and an exact $100(1-\alpha)\%$ confidence interval is given by

$$\bar{Y}(\hat{\underset{\sim}{\beta}}) \pm t_{K-Q-1}(1-\alpha/2)D\cdot S_{y\cdot c} \tag{2.25}$$

where

$$D^2 = K^{-1} + (K-1)^{-1}(\bar{\underset{\sim}{C}} - \underset{\sim}{\mu}_c)^T\underset{\sim}{S}_c^{-1}(\bar{\underset{\sim}{C}} - \underset{\sim}{\mu}_c) \tag{2.26}$$

$$S_{y\cdot c}^2 = (K-Q-1)^{-1}(K-1)(S_y^2 - \underset{\sim}{S}_{cy}^T\underset{\sim}{S}_c^{-1}\underset{\sim}{S}_{cy}) \tag{2.27}$$

$t_{K-Q-1}(1-\alpha/2)$ is the $100(1-\alpha/2)$ percentile of Student's t-distribution with (K-Q-1) degrees of freedom, and S_y^2 is the sample variance of Y (Bauer and others, 1988:3-4). Experimental results have shown that the assumption of joint multivariate normality is robust (Bauer, 1988).

Multiple Simulation Responses with Multiple Controls. Bauer, Venkatraman and Wilson (1987) provide an outline of the theoretical formulas for the case when there are P response variables and Q control variables. In terms of the notation, the univariate response $\underset{\sim}{Y}$ becomes a (PX1) vector of response variables, $\underset{\sim}{\beta}$ becomes a (PXQ) matrix of control coefficients, and the scaler sample standard deviation $\underset{\sim}{S}_y$ becomes the sample covariance matrix of the response vector. Under the assumption that $\underset{\sim}{Y}$ and $\underset{\sim}{C}$ have the joint multivariate normal distribution

$$\begin{bmatrix} \underset{\sim}{Y} \\ \underset{\sim}{C} \end{bmatrix} \sim N_{Q+1}\left(\begin{bmatrix} \underset{\sim}{\mu}_y \\ \underset{\sim}{\mu}_c \end{bmatrix}, \begin{bmatrix} \underset{\sim}{\Sigma}_y & \underset{\sim}{\Sigma}_{yc} \\ \underset{\sim}{\Sigma}_{cy} & \underset{\sim}{\Sigma}_c \end{bmatrix}\right) \tag{2.28}$$

$\bar{\underset{\sim}{Y}}(\hat{\underset{\sim}{\beta}})$ is an unbiased estimator of $\underset{\sim}{\mu}_y$, and an exact 100(1-α)% confidence ellipsoid for $\underset{\sim}{\mu}_y$ is given by

$$[\bar{\underset{\sim}{Y}}(\hat{\underset{\sim}{\beta}})-\underset{\sim}{\mu}_y]^T \underset{\sim}{S}_{y\cdot c}{}^{-1}[\bar{\underset{\sim}{Y}}(\hat{\underset{\sim}{\beta}})-\underset{\sim}{\mu}_y] \leq P(K-Q-1)(K-P-Q)^{-1}D\cdot F(1-\alpha;P,K-P-Q) \tag{2.29}$$

where

$$D^2 = K^{-1} + (K-1)^{-1}(\bar{\underset{\sim}{C}} - \underset{\sim}{\mu}_c)^T \underset{\sim}{S}_c{}^{-1}(\bar{\underset{\sim}{C}} - \underset{\sim}{\mu}_c) \tag{2.30}$$

$$\underset{\sim}{S}_{y\cdot c}{}^2 = (K-Q-1)^{-1}(K-1)(\underset{\sim}{S}_y - \underset{\sim}{S}_{yc}\underset{\sim}{S}_c{}^{-1}\underset{\sim}{S}_{cy}) \tag{2.31}$$

and $F(1-\alpha;m_1,m_2)$ is the 100(1-α) percentile of the F-distribution with m_1 and m_2 degrees of freedom (Bauer and others, 1987:335).

The advantage of the above approach over selecting separate controls for each response is the capability to form a joint confidence region for the response vector, rather than being limited to univariate confidence intervals.

Results of Previous Research. The following discussion of the experimental results found in the literature is presented in chronological order.

Review of Cheng (1978). Cheng (1978) provided interpretations of statistically well-known formulas used in ordinary regression analysis to control variates in simulation under the assumption of normality.

The reader should note that there is an error in Equation (3) of Cheng's article. The correct equation, which appeared in Cheng and Feast (1980), reads as follows:

$$\hat{\alpha} = \bar{Y} - \beta^{T}(\bar{X} - \mu) \tag{2.32}$$

Review of Cheng and Feast (1980). Cheng and Feast (1980) made the statement that "practically all control variables suggested in the literature are of the form where their mean μ is known, but ... [the covariance matrix] is not" (Cheng and Feast, 1980:51). However, more recent literature follows their suggestion of using standardized sums for control variables rather than using sample means or straight sums. These standardized controls are of the form

$$\underset{\sim}{C} = \sum_{j=1}^{N} \underset{\sim}{Y}_j/(N)^{1/2} \tag{2.33}$$

If the $\underset{\sim}{Y}_j$'s are m-independent ($\underset{\sim}{Y}_i$ and $\underset{\sim}{Y}_{i+j}$ are independent if $j \geq m$), with zero mean and $E(\underset{\sim}{Y}_i\underset{\sim}{Y}_i^{T})$ finite, then in the limit $\underset{\sim}{C}$ is normally distributed with mean zero and covariance matrix

$$\underset{\sim}{\Sigma} = E(\underset{\sim}{Y}_1\underset{\sim}{Y}_1^{T} + \underset{\sim}{Y}_1\underset{\sim}{Y}_2^{T} + \underset{\sim}{Y}_2\underset{\sim}{Y}_1^{T} + \ldots + \underset{\sim}{Y}_1\underset{\sim}{Y}_m^{T} + \underset{\sim}{Y}_m\underset{\sim}{Y}_1^{T}) \tag{2.34}$$

(Cheng and Feast, 1980:52)

Review of Lavenberg, Moeller, and Welch (1982). Lavenberg, Moeller, and Welch experimented with three internal control variates in closed queueing networks with more than one class of entities, where a generic class is denoted by the letter (d). Briefly, the control variates used were work variables (the sum of service times for type d entities per type d event), flow variables (the fraction of type d events at node i), and service-time variables (the sample service times for type d entities at node i).

They reported achieving the largest variance reductions using work variables and limited their research accordingly. They also reported that as the server utilization increased so did the size of the variance reduction in waiting time. The range of the estimated variance ratios using six work variables for controls was from 0.30 to 0.85. These results translate to minimum variance ratios in the range of 0.16 to 0.77, which are obtained by dividing the variance ratios by the theoretical loss factor.

Review of Wilson and Pritsker (1984). Wilson and Pritsker experimented with poststratified sampling and standardized work variables as variance reduction techniques adapted to the estimation methods of replication analysis (independent replications of a simulation) and regenerative analysis (independent cycles within a simulation). The standardized work variables are given by Eq (2.36), and poststratified sampling refers to an importance technique which groups the responses into strata according to a stratification variate. Wilson and Pritsker reported the following reductions in the variance of the point-estimators and in the width of the 90% confidence intervals that can be obtained with each procedure for several closed and mixed machine-repair systems:

Table 2.1. Experimental Results of Wilson and Pritsker (1984)

Variance Reduction Technique	Variance Reduction	Confidence-Interval Width Reduction
Poststratification	10 - 40%	1 - 20%
Work Variables	20 - 90%	10 - 70%

Review of Sharon (1986). Sharon "investigated two types of Jackson control variates, external and analytic, for estimating the utilization factors and waiting times in three different queueing networks" (Sharon, 1986:35). He obtained control variates for each network using three different service time distributions (exponential, Weibull, and uniform) and two different traffic intensities (0.5 and 0.9), which resulted in eighteen experiments.

External Jackson Control Variates. To obtain external control variates he used a Jackson network approximation to each of the more general networks he studied. He substituted the exponential distribution for the service time distributions with the identical mean service times used in the original model. Of course, this required a second simulation for each of the three networks. Then, the output of these second simulations were contrasted with the results derived analytically to regress out some of the variance of the estimates obtained from the simulation of the original model.

For estimating the means of the two response variables at any given node, the controlled observations were of the form given by Eq (2.8). For example, in terms of estimating the mean waiting time at any node in the network

$$W_i(b) = W_i - b(C_i - \mu_c) \qquad (2.35)$$

where

$W_i(b)$ = a controlled observation of the average waiting time
W_i = an uncontrolled observation of the average waiting time
b = the control coefficient
C_i = an observation of the average waiting time from a second simulation model of the approximating Jackson Network
μ_c = the mean waiting time, which is the analytic solution of an M/M/m queue given by Eq (1.1), using the mean arrival rates, mean service rates, and probabilistic routing structure that was input to the simulation model

The controlled observations of the utilization factors were defined in a similar manner.

Analytic Jackson Control Variates. What Sharon terms an *analytic* Jackson control variate is "an amalgam of the internal and external approaches" (Sharon, 1986:18). Instead of using the input random variables directly as control variates, Sharon substituted the known means and the observed averages of the input random variables into the steady-state equations for an M/M/m queue to derive the desired control variates. The input random variables included the arrival rates, the service rates, and the probabilistic routing structure.

Looking back to Eq (2.35), only the definition of C_i changes. It is now the analytic solution of an M/M/m queue using the average arrival rates, average service rates, and the probabilistic routing observed from the original simulation.

Unfortunately, as Sharon indicated, there are two drawbacks to using his analytic Jackson control variates. First, he was able to obtain sample values of the arrival rates, the service rates, and probabilistic routing at the high setting of the traffic intensity (0.9) such that the derived traffic intensity using these observed values was greater than unity (Sharon, 1986:32).

The second drawback of the analytic Jackson control variate is that the use of "the observed mean arrival rates and service rates in the Jackson model equations will result in a biased control variate" (Sharon, 1986:32-33). The severity of this bias was left to future research.

Results. Sharon's results show favorable variance reductions in the estimates of the server utilization factors in the range of 68 to 99 percent. The higher variance reductions were achieved at the lower setting of the traffic intensity (0.5). However, the Jackson analytic control variates for the waiting times produced little or no variance reduction, and in some cases, they produced variance increases in the estimates of the waiting times (Sharon, 1986:73-75).

Review of Bauer, Venkatraman and Wilson (1987). Bauer, Venkatraman and Wilson report a new control variate estimator which makes use of the cases when the covariance matrix of the controls is known. For the experiment, they selected the standardized work variables given by Wilson and Pritsker (1984) and the standardized routing variables defined by Bauer (1987). Both types of internal control variates mentioned above have known means and known covariance matrices.

Standardized Work Variables. If we assume that the service time process at node j is given by the independent and identically distributed (IID) sequence $\{U_i(j): i \geq 1\}$, $j = 1, \ldots, g$, and we define f_j to be the number of service times completed at node j in the period [0,t], then a standardized work variable for node j is

$$W_j = (f_j)^{1/2}(f\omega_j)^{-1}\sum_{i=1}^{f_j}[U_i(j) - \mu_j]/\sigma_j \tag{2.36}$$

where ω_j is the frequency with which an entity visits node j and f is the sum of the f_j's (Bauer and others, 1987:337).

Standardized Routing Variables. Define $N_j(t)$ to be the number of entities exiting from node j in the time interval [0,t]. Define p_{jk} to be the probability that an entity exiting from node j will go to node k, and define the indicator variable I_{ijk} as follows:

$$I_{ijk} = \begin{cases} 1 & \text{if the ith entity leaving node j goes to node k} \\ 0 & \text{otherwise} \end{cases}$$

Then a standardized routing variable for node j is given by

$$R_j = \sum_{i=1}^{N_j(t)}(I_{ijk} - p_{jk})/[N_j(t)\cdot(1 - p_{jk})p_{jk}]^{1/2} \tag{2.37}$$

Results. The new control variate estimator yielded a confidence-region that is somewhat larger than the confidence region obtained using the usual controlled confidence-region estimator. But, it also demonstrated more reliable coverage properties. However, the $100(1-\alpha)\%$ confidence ellipsoid for μ_Y is only approximate. Further research is being conducted to develop a more refined estimator of the covariance matrix of the controlled response vector and an improved confidence-region estimator for μ_Y.

Approximations to Point Processes

The author investigated the efficiency of applying two-moment approximations to control variates in a manner similar to Sharon's analytical controls. In effect, this technique uses approximations of performance measures of the exact network as external control variates. In the case of Sharon's analytical controls, his technique uses exact performance measures of an approximating network as external control variates.

Need for Approximations. In terms of queueing networks the arrival and departure processes are point processes. If the arrival processes are renewal processes, then the congestion measures of the individual queues and the entire network can be solved for analytically. In most cases the departure process of a queue is not a renewal process. And since the departure process of one queue becomes an arrival process to the next queue in the network, then the congestion measures cannot be solved for using exact analytic methods.

Whitt (1982) investigated simple approximations for stochastic point processes. He considered point processes on the positive real line for which the "total number of points is infinite but the number of points in any bounded interval is finite" (Whitt, 1982:129). Following Whitt (1982), let

S_n = the position of the nth point from the origin, $n \geq 0$, and $S_0 = 0$

$X_n = S_n - S_{n-1}$, $n > 1$ (the time interval between successive points)

$N(t) = \max\{n \geq 0: S_n \leq t\}$, $t \geq 0$ (the counting process recording the number of points in the interval $(0,t]$

Then, "the stochastic processes $\{S_n\}$, $\{X_n\}$, and $\{N(t)\}$ are three different representations of the same point process" (Whitt, 1982:129).

Let the points in the point process represent the occurrence of a certain event, then $N(t)$ represents the number of times the event occurred in the time interval $(0,t]$. If the time intervals between each occurrence of the event are independent and identically distributed, then $\{N(t)\}$ is called a renewal counting process. The most common renewal counting process is the Poisson process. Furthermore, $\{X_n\}$ is called a renewal process, and if $\{N(t)\}$ is a Poisson process with rate λ, then the X_n's are distributed exponentially with mean $1/\lambda$.

However, if the point process $\{N(t)\}$ is not a renewal counting process, then the interval sequence $\{X_n\}$ is not a renewal process; and therefore, the model becomes analytically intractable. Under such circumstances most practitioners will simulate the model. However, another approach is to approximate the point process by a renewal process and then solve the model analytically.

<u>Two-Moment Approximations</u>. Whitt (1982) refers to several authors who

> ... suggest approximating all the flows (point processes) in a network of queues by renewal processes characterized by two parameters. It was discovered that one parameter (representing the rate of the process) is usually not good enough, but two parameters (representing the rate and the variability) often are sufficient (Whitt, 1982:126).

Review of Whitt (1982). Whitt describes ways to approximate a single point process by a renewal process in two steps: "first, properties of the point process are used to specify a few moments of the interval between renewals; then a convenient distribution is fit to these moments" (Whitt, 1982:125). There are other appropriate parameters, but the parameters that he has focused on are "the moments of the renewal interval in the approximating renewal process" (Whitt, 1982:126). In the paper he outlines two methods for specifying the first few moments of the renewal interval--the stationary-interval method and the asymptotic method. Briefly,

> the stationary-interval method equates the moments of the renewal interval with the moments of the stationary interval in the point process to be approximated. The asymptotic method, in an attempt to account for the dependence among successive intervals, determines the moments of the renewal interval by matching the asymptotic behavior of the moments of the sums of successive intervals (Whitt, 1982:125).

Review of Whitt (1984). Later, in 1984, Whitt published methods for approximating the departure process of a single-server queue. This result is significant because the departure process of one queue becomes the arrival process to the next queue in a network.

Whitt discusses how to use the two methods above for approximating the departure process. An interesting note is that

> the asymptotic-method approximation for the departure process is just the arrival process, provided that the arrival process is in the class of approximating processes, e.g. a renewal process. Otherwise, the approximating process for [the departure process] obtained by the asymptotic method is the same as the approximating process for [the arrival process] (Whitt, 1984:502).

Finally, Whitt and his associates

> indicated three ways the approximations might be improved: (1) using the third moment, (2) using the lag-1 correlation ..., and (3) developing a hybrid procedure ... However [they] examined the last two methods and did not find an improvement (Whitt, 1984:516).

<u>Review of Albin and Kai (1986)</u>. Albin and Kai studied two queues in series "to identify a renewal process to approximate the departure process of a $\Sigma GI_i/M/1$ queue" (Albin and Kai, 1986:132). The arrival process to the first queue was a superposition of independent stationary renewal processes. Each queue had a single server with exponentially distributed service times, an infinite queue capacity, and FIFO queue discipline (Albin and Kai, 1986:130). A hybrid method of two basic methods (the Poisson and the asymptotic) led to an average absolute error in hybrid approximations of the expected number in the second queue of 6% compared to the 22-41% error in the basic methods (Albin and Kai, 1986:131). The Queueing Network Analyzer, discussed later in this chapter, uses the stationary-interval method to identify renewal-process approximations for departure processes. Albin and Kai's hybrid method applies to queues with exponentially distributed service times.

The squared coefficient of variation for approximating the renewal departure intervals using the hybrid method is given by

$$c_d^2 = \omega c_a^2 + (1 - \omega) c_p^2 \qquad (2.38)$$

where the squared coefficient of variation for approximating the renewal departure intervals using the asymptotic method (c_a^2) and the squared coefficient of variation for approximating the renewal departure intervals using the Poisson method (c_p^2) are given by

$$c_a^2 = \sum_{i=1}^{n} (\lambda_i/\lambda)c_i^2 \tag{2.39}$$

$$c_p^2 = 1 \tag{2.40}$$

(λ is the total arrival rate) and the weighting coefficient $\omega \equiv \omega(\eta^*,\rho_1,\rho_2)$ is given by

$$\omega(\eta^*,\rho_1,\rho_2) = [A + B\eta^*(1 - \rho_2)^2 + C/(1 - \rho_1)^2]^{-1} \tag{2.41}$$

The symbols in Eq (2.41) are defined as follows:

The coefficients (A,B,C) equal

(1.0, 2.0, 0.05) for calculating the expected number of entities at the second node,

(1.0, 1.0, 0.05) for calculating the standard deviation of the number of entities at the second node, and

(1.7, 2.3, 0.04) for calculating the probability of an entity being delayed at the second node;

ρ_1 and ρ_2 are the traffic intensities at the respective nodes; and

$\eta^* = \left[\sum_{i=1}^{n} (\lambda_i/\lambda)^2\right]^{-1}$, is the effective number of component arrival processes.

The weighting function is the result of an experimental design involving 27 combinations of ρ_1, ρ_2, and η^*, where the constants A, B, and C were identified using multiple linear regression. "Different weighting functions are needed for different congestion measures because of the properties of the basic methods" (Albin and Kai, 1986:138).

This hybrid method works well when the squared coefficient of variation of each of the interval renewal processes input to the first queue are in the range (0,9] (Albin and Kai, 1986: 138).

Review of Kimura (1986). Kimura reported on a two-moment approximation that yields better results than those achieved by the Queueing Network Analyzer (which is described later in this chapter). Let EW(GI/G/m) denote the steady-state mean waiting time (until beginning service) in the GI/G/m queue. The approximation formula he gives in his paper is

$$EW(GI/G/m) \approx (c_a^2+c_s^2)\Big/\left[\frac{1 - c_a^2}{EW(D/M/m)} + \frac{1 - c_s^2}{EW(M/D/m)} + \frac{2(c_a^2+c_s^2-1)}{EW(M/M/m)}\right] \quad (2.42)$$

where c_a^2 and c_s^2 are the coefficients of variation of the interarrival times and service times respectively; ρ is the traffic intensity; and EW(D/M/m), EW(M/D/m) and EW(M/M/m) are the steady-state mean waiting times in the respective queueing systems. Kimura's approximation formula given by Eq (2.42) "is a weighted harmonic mean of the expected waiting times for the D/M/m, M/D/m and M/M/m queues and it is exact for these queueing systems" (Kimura, 1986:751).

Note that EW(M/M/m) is equivalent to W_q given by Eq (1.5) in the first chapter. Because of some numerical difficulties in solving for the exact analytic solutions of EW(D/M/m) and EW(M/D/m), Kimura suggests using the following approximations:

$$EW(D/M/m) \approx (1/2)[1-4C(m,\rho)]\cdot\exp[-2(1-\rho)/3\rho]\cdot EW(M/M/m) \quad (2.43)$$

$$EW(M/D/m) \approx (1/2)[1+C(m,\rho)]\cdot EW(M/M/m) \quad (2.44)$$

where

$$C(m,\rho) = (1-\rho)(m-1)[(4+5m)^{1/2} - 2]/(16m\rho) \quad (2.45)$$

Kimura found "that these approximations are fairly accurate unless ρ is close to zero" (Kimura, 1986:761).

Substituting the approximations given by Eqs (2.43) through (2.45) into Eq (2.42) yields a simpler approximation that is more tractable. This simpler approximation is given by

$$EW(GI/G/m) \approx (1/2)(c_a^2 + c_s^2)k \cdot EW(M/M/m) \quad (2.46)$$

where the correction factor $k \equiv k(GI/G/m)$ is defined by

$$k(GI/G/m) = \left[\frac{1 - c_a^2}{k(D/M/m)} + \frac{1 - c_s^2}{k(M/D/m)} + c_a^2 + c_s^2 - 1\right]^{-1} \quad (2.47)$$

and

$$k(D/M/m) = \max\{[1-4C(m,\rho)]\cdot\exp[-2(1-\rho)/3\rho],\ 10^{-6}\} \quad (2.48)$$

$$k(M/D/m) = 1 + C(m,\rho) \quad (2.49)$$

The maximum in Eq (2.48) is used to avoid dividing by zero or meaningless approximations with negative values (Kimura, 1986:761).

<u>The Queueing Network Analyzer</u>. The Queueing Network Analyzer (QNA) is a commercially available "software package developed at Bell Laboratories to calculate approximate congestion measures for a network of queues" (Whitt, 1983a:2779). The first version of QNA operates under the following assumptions:

> *Assumption 1.* The network is *open* rather than closed. Customers come from outside, receive service at one or more nodes, and eventually leave the system.
>
> *Assumption 2.* There are *no capacity contraints*. There is no limit on the number of customers that can be in the

entire network and each service facility has unlimited waiting space.

> *Assumption 3.* There can be *any number of servers* at each node. They are identical independent servers, each serving one customer at a time.
>
> *Assumption 4.* Customers are selected for service at each facility according to the *first come, first-served* discipline.
>
> *Assumption 5.* There can be *any number of customer classes*, but customers cannot change classes. Moreover, much of the analysis in QNA is done for the aggregate or typical customer.
>
> *Assumption 6. Customers can be created* or combined at the nodes, e.g. an arrival can cause more than one departure.
>
> (Whitt, 1983a:2781-2782)

QNA uses two parameters to characterize the arrival process and the service times--one to describe the rate and the other to describe the variability (Whitt, 1983a:2782).

Required Inputs. QNA allows several different formats for entering the necessary information. In general, the information that must be supplied is as follows: (1) the number of nodes in the network, (2) the number of servers at each node, (3) the external arrival rate to each node, (4) the variability parameter of the external arrival process to each node, (5) the mean service time at each node, (6) the squared coefficient of variation of the service-time distribution at each node, and (7) the Markovian routing of entities within the network.

QNA Outputs. QNA will provide the steady-state congestion measures for each node in the network. The main congestion measure is the mean waiting time (before beginning service), but QNA also generates an entire probability distribution for the waiting time. QNA will also provide the probability that the server is busy at an arbitrary time, the expected number of entities in the node, the probability that an entity is delayed, and the conditional delay given that the server is busy (Whitt, 1983a:2802-2807).

QNA will also calculate the approximate congestion measures for the network as a whole. It provides congestion measures representing the system view (e.g. throughput and number of entities in the network) and congestion measures representing the customer view (e.g. number of nodes visited and response times) (Whitt, 1983a:2807).

Performance of QNA. Whitt (1983b) describes the performance of QNA and compares the congestion measures to those obtained through simulation and the standard Markovian algorithm (which is represented by the M/M/m equations given in Chapter I). He tested the performance of QNA on a variety of queueing networks from a single GI/G/1 queue to a packet-switched communication-network model. The results of Whitt's study demonstrated the importance of the variability parameter used in QNA to estimate the congestion measures of networks that do not satisfy the assumptions of the Jackson network. Furthermore, when the Jackson network assumptions are satisfied, then the approximations used in QNA yield the exact measures.

III. Methodology

The objective of this study was to compare several control variates that have shown promising results for queueing network simulation. This was done with the ultimate goal in mind of providing the simulation community with some guidance for selecting control variates that will lead to significant reductions in the variance of the estimated responses while not introducing bias to the estimates.

The major obstacle to achieving the stated goal was to design an experiment that is general enough so that the results are applicable to a wide range of queueing network models. And, at the same time, the size of the experiment must be manageable so that it can be completed in the alloted time.

Description of the Queueing Network

The first step in keeping with the above considerations was to select a queueing network. The author decided to select a single network that is small in terms of the number of nodes but complex enough to incorporate many aspects of queueing networks in general.

The basic form of the network was based upon the third network that was used by Sharon (1986). It is an open network which consists of four nodes, each of which has a single server. Entities arrive from outside the network to the first service center (or node) according to a Poisson process with arrival rate $\lambda = 1$. At each service center, the queue capacity is infinite, and the queue discipline is FIFO. From the first node entities can go to either node 2 or node 3 with given probabilities. Entities from nodes 2 and 3 proceed to node 4. Finally, entities leaving node 4 may loop back to the first node or may exit the system. The basic structure of the queueing network is illustrated below in Figure 3.1.

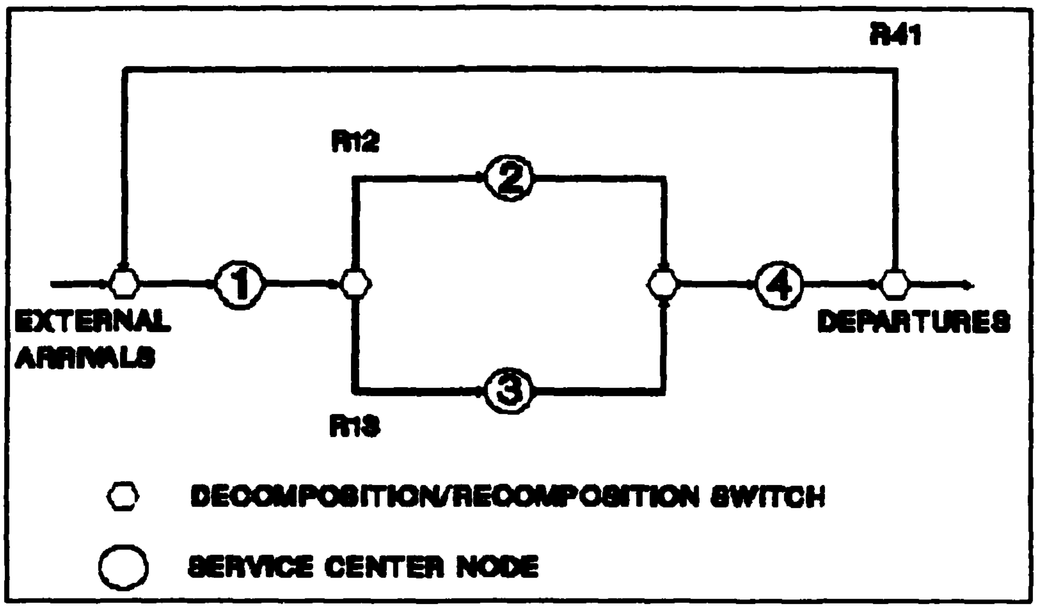

Figure 3.1. Diagram of Experimental Queueing Network

Furthermore, the service times at the first three nodes were distributed exponentially. This decision was made to simplify the experiment, rather than introduce another variable to the experimental design. In effect, this does not limit the applicability of the results because the superposition of independent arrival processes approaches a Poisson process in the limit no matter how they were originally distributed.

However, the service time distribution at the fourth node was varied to obtain the desired variability in the service times, since an exponential distribution is limited to a coefficient of variation equal to one. The coefficient of variation is a measure of the variability of a distribution and is given by the square root of the variance divided by the mean. The maximum entropy distribution was chosen to generate service times with a coefficient of variation less than one and the hyperexponential distribution was chosen to generate service times with a coefficient of variation greater than one. There is more detail about these distributions later in this chapter.

The Response Variables

Next, two response variables were selected for the analysis. One of them was the average sojourn time for an entity to pass through the network, which can be approximated by finding the solution to the approximating Jackson Network. It can also be solved by finding the approximate solution to the exact network using QNA.

The other response variable selected was a quantile at the fourth node representing the probability that the number in queue exceeds some threshhold value. The threshhold value chosen was twice the mean number in queue at equilibrium. For practical purposes, QNA was used to find the threshhold value by taking the next highest integer of the following result: $2 \times (EN - \rho)$, where EN is the expected number at the fourth node, and ρ is the associated traffic intensity.

Quantiles are not easily estimated, even though knowledge of them may prove to be important. To illustrate, consider a communication network where the queues represent buffers for incoming messages. And, suppose that one proposes to determine the capacity of the buffers in this network by finding the mean number in queue at equilibrium for a system with infinite capacity and simply doubling the result. The user of the network may then want to know how often the capacity of the buffers are exceeded to determine whether or not the buffers' capacities should be increased.

The Control Variables

Next, examples of both types of control variates, internal and external, were selected. The author picked two different types of internal control variates (standardized routing controls and standardized work variables) and two different types of external control variates (analytic Jackson controls and analytic approximations).

The Internal Control Variates. Two standardized routing controls, one for each of the two probabilistic branchings in the network, were selected for this research. For further reference, let R_{13} denote the routing control for the proportion of entities that took the path from node 1 to node 3 (as opposed to node 1 to node 2), and let R_{41} denote the routing control for the proportion of entities that took the path from node 4 back to node 1. These standardized routing controls are given by Eq (2.37).

In addition to the two routing controls, four standardized work variables, one for each of the service nodes, were selected. The four work variables, denoted by W_1, W_2, W_3, and W_4, are given by Eq (2.36).

The External Control Variates. Also, two external control variates under each of the two different approaches alluded to earlier were selcted for this research. These external controls were the steady-state expected sojourn time for the entire network and the steady-state expected waiting time in the fourth queue.

The analytic Jackson control variates and the analytic approximations were found using Bell Laboratories' Queueing Network Analyzer (QNA). The following input parameters to the simulation model were used to generate the "known" means of the external control variates: the external arrival rate, the mean service times at each node, the squared coefficient of variation of the service times at each node, and the probabilistic routing matrix. Under the assumption that the squared coefficient of variation of the service times are all equal to one yields the M/M/1 (or Jackson Network) results. On the other hand, using the input squared coefficient of variation for the service-times at the fourth node (the only one that violates the above asumption) yields the G/G/1 (or approximate) results.

The observed values of the external control variates were found using the observed values of these same parameters as inputs to QNA. The input files to QNA for generating the "known" means at the first design point are provided in Appendix B, and the "known" means of the external control variates at all design points are summarized in a table in Appendix C.

Selecting a Service-Time Distribution

In almost all of the literature examined, there has not been any rationale specified for selecting a particular distributional form for the interarrival times or service times. Rather, most researchers select a few of the more commonly used distributions and include the type of distribution as a variable in their research. However, in this case, in keeping with the goal of maintaining enough generality, the selection of a specific distribution might bias the results. Therefore, the maximum entropy distribution was selected for this research because it is the least biased.

However, because of the difficulties associated with generating random numbers according to the maximum entropy distribution, the author decided to use it for generating service times at the fourth service center only. As indicated earlier, the fourth service center is of particular interest because of the selection of the quantile representing the probability that the number in queue at the fourth service center exceeds twice the expected number under steady-state conditions.

Unfortunately, the author discovered that the maximum entropy distribution could not be used to generate a nonnegative random variate with a coefficient of variation (c_v) greater than one. (The author did not explore whether or not the maximum entropy distribution can have a coefficient of variation greater than one over the real numbers.) Instead, the hyperexponential distribution was selected for such cases. This selection is also suggested from graphical representations of the distributions. At the lower setting of the coefficient of variation (0.5) the maximum entropy distribution looks like a normal distribution that was truncated at the origin (negative times are impossible). As the coefficient of variation increases to one the graph looks exponential. In fact, when the coefficient of variation equals one the maximum entropy distribution reduces to the exponential distribution. The hyperexponential distribution with coefficient of variation greater than one takes on an exponential shape but with a longer tail.

Therefore, the maximum entropy distribution was used to generate random variates with a coefficient of variation less than one and the hyperexponential distribution was used to generate random variates with a coefficient of variation greater than one. The functional forms of the density functions for these two distributions are given below:

Maximum Entropy: $f(x) = \exp(-1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2)$ (3.1)

Hyperexponential: $f(x) = c_1/\beta_1 \cdot \exp(-x/\beta_1) + c_2/\beta_2 \cdot \exp(-x/\beta_2)$ (3.2)

Generating Random Variates Using Selected Distributions

The author used the simulation programming language SLAM II to code up the network model. SLAM II has built-in functions to generate random variates according to several distributions. However, it does not have any built-in functions for generating the maximum entropy or hyperexponential distributions. Therefore, the author had to build FORTRAN subroutines using well-known techniques to generate the random variates desired. The SLAM II Network code and FORTRAN subroutines are provided for the reader in Appendix A.

Maximum Entropy Distribution. The development of the parameters (λ_0, λ_1, and λ_2) for the maximum entropy distribution given the first two moments (μ_1 and μ_2) is no trivial matter. All of the automated search procedures tried have not been successful, but a program using MINOS is currently being pursued. However, the author was able to find parameters through a somewhat manual search process using a routine to numerically evaluate the appropriate integrals to reproduce the desired moments to within 4 parts in 10,000. Given the density function f(x) as defined in Eq (3.1) the three integrals are defined below:

$$\int_0^\infty f(x)\,dx = 1 \quad \text{(definition of a density function)} \tag{3.3}$$

$$\int_0^\infty x \cdot f(x)\,dx = \mu_1 \quad \text{(definition of the first moment)} \tag{3.4}$$

$$\int_0^\infty x^2 \cdot f(x)\,dx = \mu_2 \quad \text{(definition of the second moment)} \tag{3.5}$$

However, having found the parameters still leaves the problem of generating random variates according to the maximum entropy distribution with those parameters. To generate the random variates according to the desired maximum entropy distribution the author decided to use the acceptance-rejection method (Law and Kelton, 1982:250-252).

The efficiency of the acceptance-rejection method is determined by the area between the density fuctions of the majorizing distribution and the original distribution of interest. As the area decreases the efficiency of the method increases. If f(x) and g(x) are density functions of two different distributions, g(x) majorizes f(x) if and only if $g(x) \geq f(x)$ at every x where f(x) is defined.

Using MathCAD the author was able to visually fit a majorizing distribution to each maximum entropy distribution. The majorizing distributions were developed using the density function of a Normal distribution that was truncated at the origin. To insure that they truly majorized the maximum entropy distribution they were checked by creating a table of values in MathCAD and adjusting them as necessary. Figures 3.2 through 3.5 are graphical representations of the density functions for the four maximum entropy distributions that were used in this research with their respective majorizing distributions.

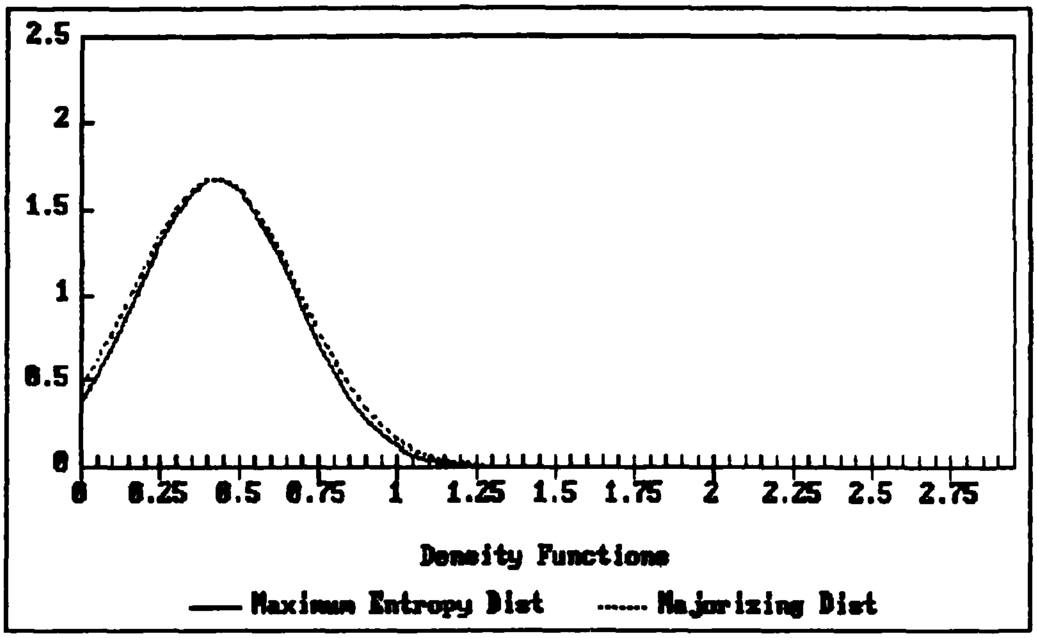

Figure 3.2. Service-Time Distribution at Node #4 with $\mu_1 = 0.45$ and $\mu_2 = 0.253125$

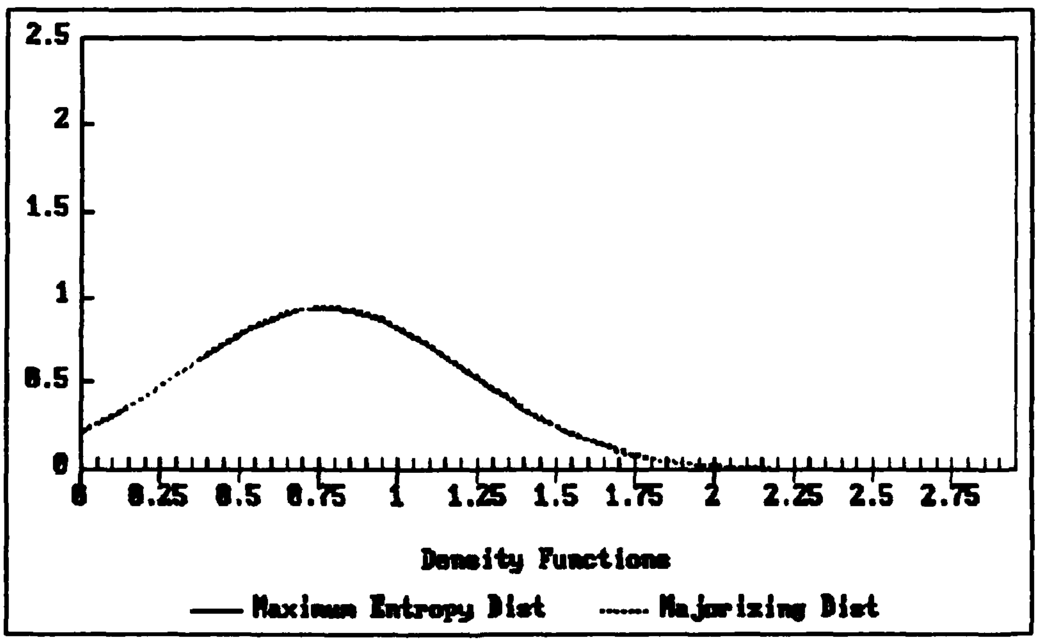

Figure 3.3. Service-Time Distribution at Node #4 with $\mu_1 = 0.81$ and $\mu_2 = 0.820125$

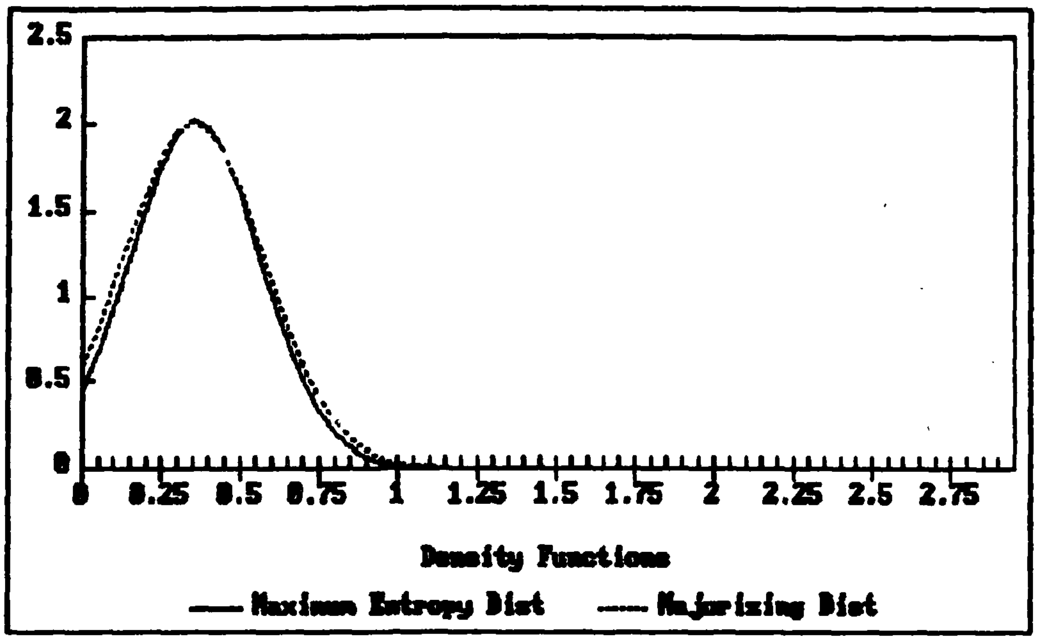

Figure 3.4. Service-Time Distribution at Node #4
with μ_1 = 0.375 and μ_2 = 0.1757812

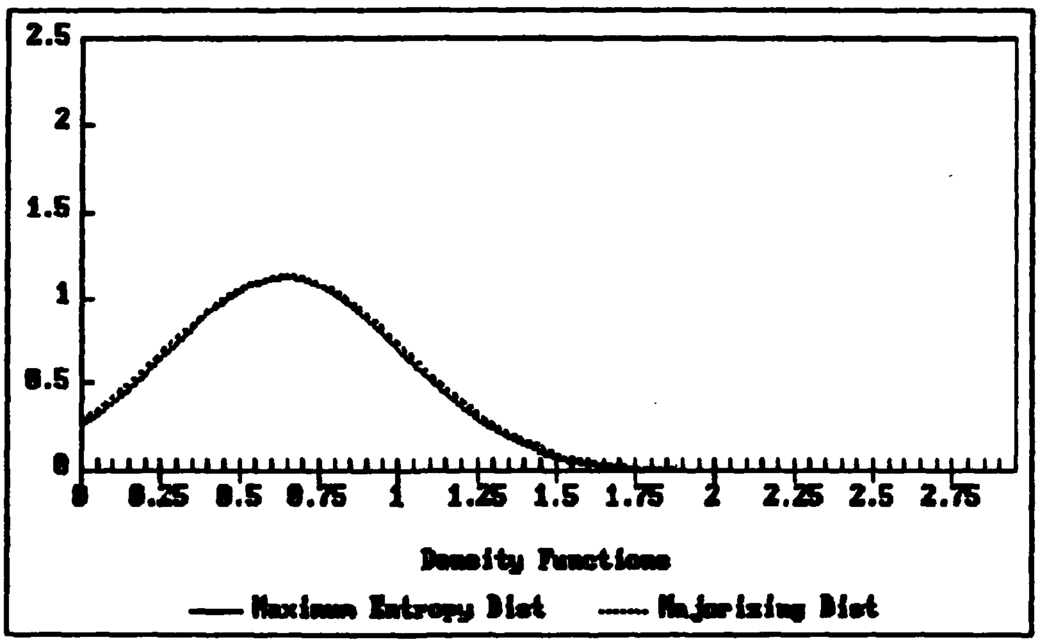

Figure 3.5. Service-Time Distribution at Node #4
with μ_1 = 0.675 and μ_2 = 0.5695312

Hyperexponential Distribution. Given the first two moments, the parameters of the hyperexponential distribution can be solved for directly from the four equations given below.

$$c_1\beta_1 + c_2\beta_2 = \mu_1 \text{ (first moment)} \tag{3.6}$$

$$2c_1\beta_1^2 + 2c_2\beta_2^2 = \mu_2 \text{ (second moment)} \tag{3.7}$$

$$c_1 + c_2 = 1 \tag{3.8}$$

$$c_1\beta_1 = c_2\beta_2 \tag{3.9}$$

Eqs (3.6) and (3.7) were derived from the definitions of the first and second moments, respectively. Eq (3.8) was derived from the fact that the integral of any density function equals one. And, Eq (3.9) is added to provide a unique solution. Simultaneously solving Eqs (3.6) through (3.9) in terms of the first two moments leads to the following results:

$$c_1 = (0.25 - 0.5\mu_1^2/\mu_2)^{1/2} + 0.5 \tag{3.10}$$

$$c_2 = 1 - c_1 \tag{3.11}$$

$$\beta_1 = 0.5\mu_1/c_1 \tag{3.12}$$

$$\beta_2 = 0.5\mu_1/c_2 \tag{3.13}$$

A hyperexponential distribution that satisifies Eq (3.9) is said to be balanced. The reader is referred to pages 139-147 of Kleinrock (1975) for more information.

To generate random variates according to the hyperexponential distribution the author selected the composition method (Law and Kelton, 1982:247-249). Figures 3.6 through 3.9 are graphical representations of the density functions of the four hyperexponential distributions that were used in this research.

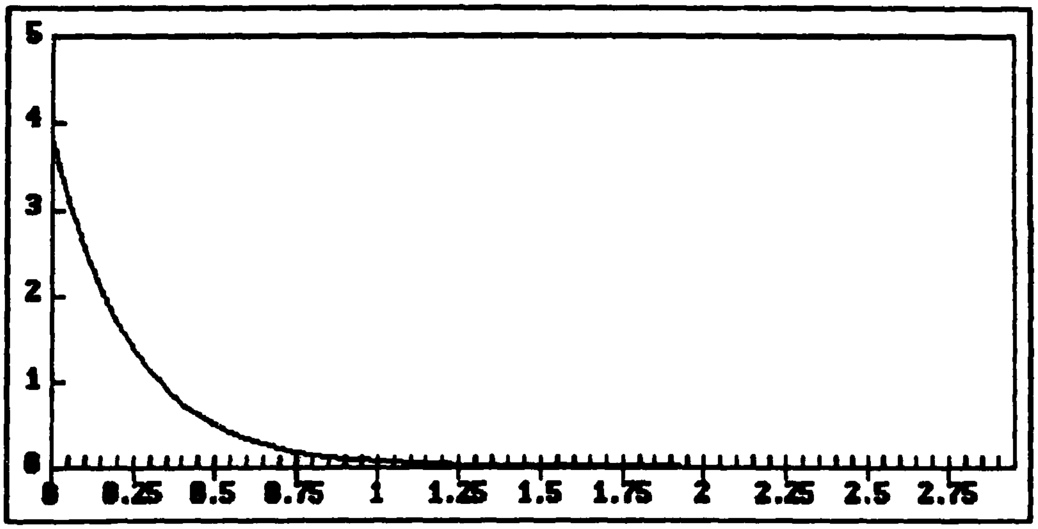

Figure 3.6. Service-Time Distribution at Node #4 with μ_1 = 0.45 and μ_2 = 1.468125

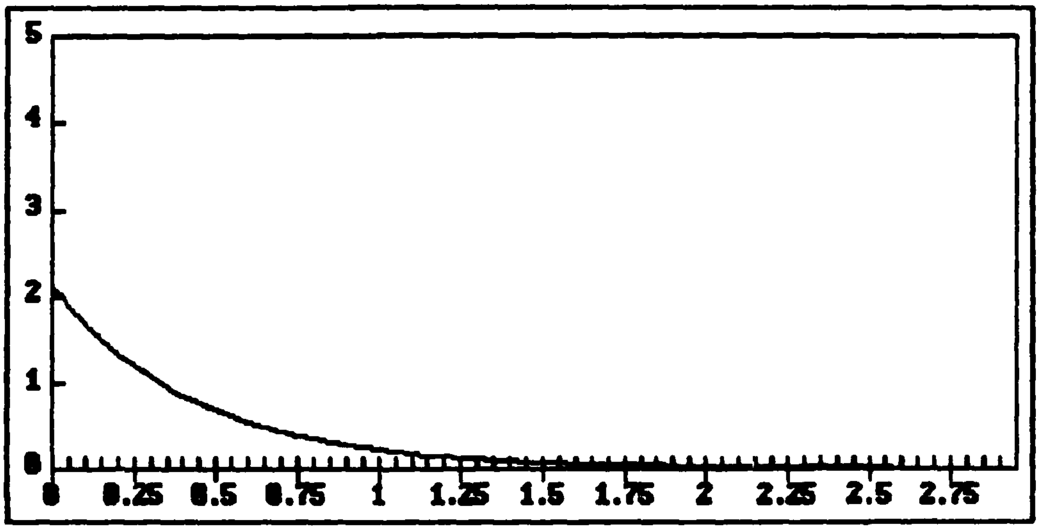

Figure 3.7. Service-Time Distribution at Node #4 with μ_1 = 0.81 and μ_2 = 4.756725

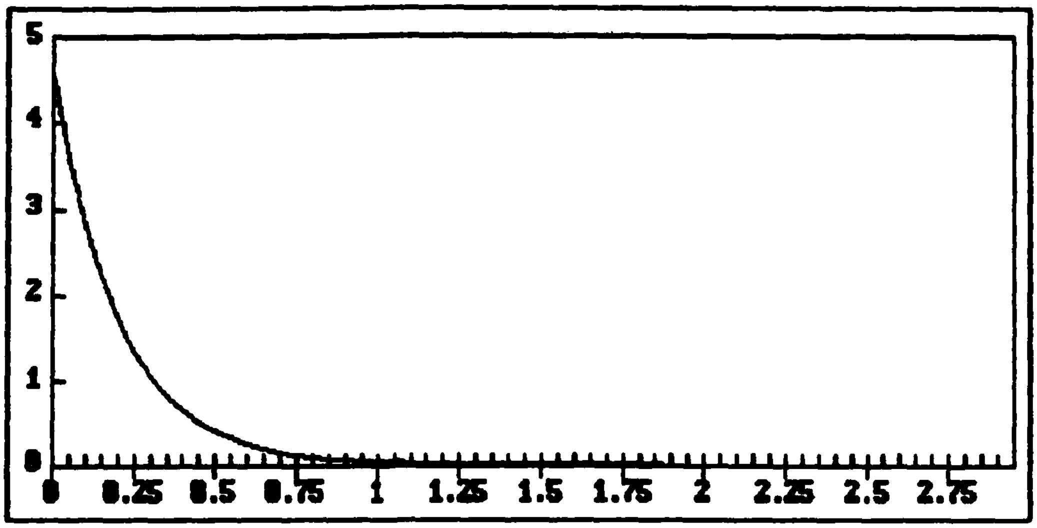

Figure 3.8. Service-Time Distribution at Node #4 with μ_1 = 0.375 and μ_2 = 1.0195313

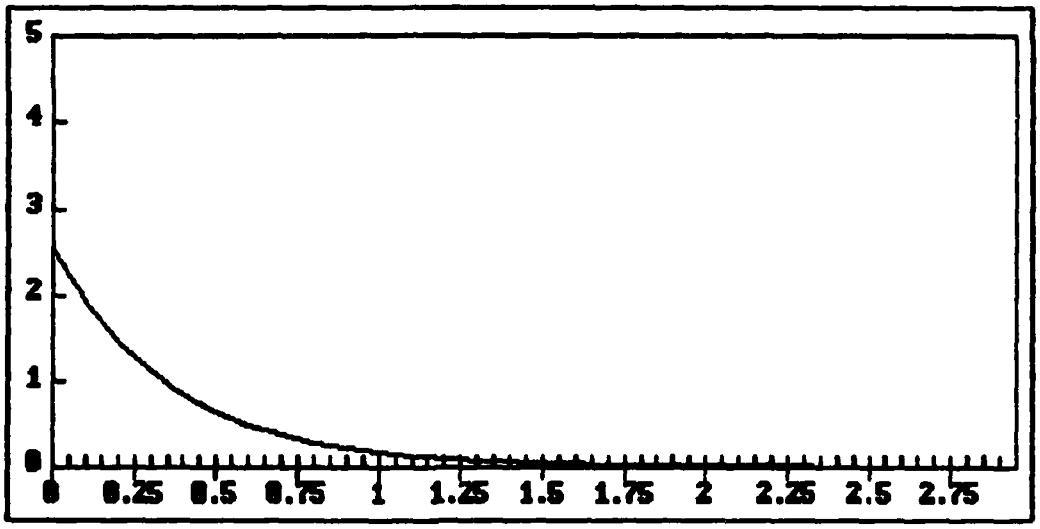

Figure 3.9. Service-Time Distribution at Node #4 with μ_1 = 0.675 and μ_2 = 3.3032813

Lag-1 Correlation of the Interdeparture Times

Prior to any experimentation with the network described earlier, the author experimented with a single-server queue to test for the strength of the lag-1 correlation of the interdeparture times of a G/G/1 queue. The queue capacity was infinite and the queue discipline was FIFO. Furthermore, the maximum entropy and hyperexponential distributions were used to generate the interarrival times and the service times of the entities. The author assumed a first-order autoregressive time series model for the autocorrelation function of the departure process.

Experimental Design. Also, the author used a 2^3 factorial design to explore the effect of the traffic intensity and the variability of the interarrival-time and service-time distributions on the first-order autoregressive parameter. The levels of the three factors (traffic intensity, coefficient of variation of the interarrival times, and coefficient of variation of the service times) in the experiment are summarized in the table below.

Table 3.1. Levels of Factors in 2^3 Experiment of a G/G/1 Queue

Parameter	Low Value (-1)	High Value (+1)
ρ	0.5	0.9
c_a	0.5	2.5
c_s	0.5	2.5

To eliminate the initialization bias, the author discarded the first five thousand observations of the interdeparture times. This decision was made based upon a visual interpretation of a time-persistent plot of the average waiting time in the queue. Then, the next one thousand observations of the interdeparture times were used to fit a first-order autoregressive time-series model.

Experimental Results. Using PROC ARIMA under SAS the resulting first-order autoregressive parameter ranged from -0.117 to +0.078. Although these extreme values proved to be statistically significant (most of the intervening values did not), they are not practically significant.

This result is significant to the development of analytic approximations for estimating the performance measures of a network. That is, since the arrival process to a queue in a network is the superposition of an external arrival process and the departure processes of other queues in the network, then this results validates the assumption that the interarrival times are independent. Otherwise, the approximation formulas would have to incorporate the dependency between the interarrival times of the entities to the queues in the network. Furthermore, the author's findings provide the reason why Whitt (1984) did not find any improvement when he incorporated the lag-1 correlation of the interdeparture times in his two-moment approximations.

Experimental Design

The experimental design for comparing the control variates over the network illustrated in Figure 3.1 was a 2^4 factorial design. The four factors selected were the traffic intensity of the network, the coefficient of variation of the service times at the fourth node, and the two routing variables (r_{13} and r_{41}). The levels of these four factors are given in the table below.

Table 3.2. Levels of Factors in 2^4 Experiment of the Queueing Network

Parameter	Low Value (-1)	High Value (+1)
ρ	0.5	0.9
c_4	0.5	2.5
r_{13}	0.2	0.4
r_{41}	0.1	0.25

Given the external arrival rate to the first node (which was set equal to one) and the routing matrix, the effective arrival rates to each of the four nodes in the network can be solved for from the traffic rate equations given by

$$\lambda_j = \lambda_{0j} + \sum_{i=1}^{n} \lambda_i r_{ij} \tag{3.14}$$

where

λ_j = the effective arrival rate to node j
λ_{0j} = the external arrival rate to node j
λ_i = the effective arrival rate to node i
r_{ij} = the routing probability from node i to node j
n = the number of nodes in the network

Next the mean service time (τ) for each node (j) can be solved for by

$$\tau_j = \rho/\lambda_j \tag{3.15}$$

Finally, the first two moments of the service-time distribution at the fourth node can be solved for by

$$\mu_1 = \tau_4 \tag{3.16}$$

$$\mu_2 = \mu_1^2(1 + c_s^2) \tag{3.17}$$

The values of the input parameters to the simulation model at each of the design points are summarized in Table 3.3.

Table 3.3. Values of the Input Parameters to the Queueing Network Simulation Model

Design	--Factor Settings--				-----Mean Service Times------				
Point	ρ	c_s	r_{13}	r_{41}	τ_1	τ_2	τ_3	$\tau_4(\mu_1)$	μ_2
1	0.5	0.5	0.4	0.10	0.450	0.75000	1.1250	0.450	0.2531250
2	0.9	0.5	0.4	0.10	0.810	1.35000	2.0250	0.810	0.8201250
3	0.5	2.5	0.4	0.10	0.450	0.75000	1.1250	0.450	1.4681250
4	0.9	2.5	0.4	0.10	0.810	1.35000	2.0250	0.810	4.7567250
5	0.5	0.5	0.2	0.10	0.450	0.56250	2.2500	0.450	0.2531250
6	0.9	0.5	0.2	0.10	0.810	1.01250	4.0500	0.810	0.8201250
7	0.5	2.5	0.2	0.10	0.450	0.56250	2.2500	0.450	1.4681250
8	0.9	2.5	0.2	0.10	0.810	1.01250	4.0500	0.810	4.7567250
9	0.5	0.5	0.4	0.25	0.375	0.67500	0.9375	0.375	0.1757812
10	0.9	0.5	0.4	0.25	0.675	1.12500	1.6875	0.675	0.5695312
11	0.5	2.5	0.4	0.25	0.375	0.67500	0.9375	0.375	1.0195313
12	0.9	2.5	0.4	0.25	0.675	1.12500	1.6875	0.675	3.3032813
13	0.5	0.5	0.2	0.25	0.375	0.46875	1.8750	0.375	0.1757812
14	0.9	0.5	0.2	0.25	0.675	0.84375	3.3750	0.675	0.5695312
15	0.5	2.5	0.2	0.25	0.375	0.46875	1.8750	0.375	1.0195313
16	0.9	2.5	0.2	0.25	0.675	0.84375	3.3750	0.675	3.3032813

Note that at the design points where $c_s = 0.5$, the service-time distribution at the fourth node was the maximum entropy distribution; and, at those design points where $c_s = 2.5$, the service-time distribution at the fourth node was the hyperexponential distribution.

The parameters for generating the hyperexponential distribution were derived in a FORTRAN subroutine in the model using Eqs (3.10) through (3.13). But, the parameters for generating the maximum entropy distribution were derived outside of the model. These parameters were then input to the model and are given in Table 3.4 below.

Table 3.4. Parameters for Generating Random Variates According to the Maximum Entropy Distribution

-----Moments-----		-----Parameters of the Density Functions-----					
		Maximum Entropy Dist			Majorizing Dist		
μ_1	μ_2	λ_0	λ_1	λ_2	c	μ	σ^2
0.450	0.2531250	-0.028146	-7.000	8.1980	1.12	0.425	0.07
0.810	0.8201250	0.557130	-3.882	2.5265	1.06	0.770	0.20
0.375	0.1757812	-0.210450	-8.400	11.8050	1.13	0.350	0.05
0.675	0.5695312	0.375020	-4.659	3.6385	1.10	0.640	0.15

Collecting Data

Now that we have selected a queueing network, the response variables, the control variables, the interarrival-time and service-time distributions, and an experimental design, we need to decide how much data to collect and how to generate the data. The first question, "how much data to collect," is related to estimating the bias introduced to the controlled estimators by forming confidence intervals about them and counting what percentage of the confidence intervals cover the "known" value of the response. The second question, "how to generate the data," refers to choosing a technique that will yield unbiased independent observations of the uncontrolled estimators.

Determining How Much Data to Collect. The author decided to use twenty observations (that is twenty runs of the simulation model) to obtain each estimate of the mean controlled responses. Then, about the estimated controlled mean response a single 95% confidence interval was formed. Now, to obtain a reasonable estimate of the actual coverage percentages of the confidence interval the author decided that this required a minimum of two hundred confidence intervals. Therefore, four thousand runs (twenty runs per confidence interval X two hundred confidence intervals) were required to determine the bias introduced to the controlled responses at each of the sixteen design points.

Determining How to Generate the Data. Knowing that we need four thousand runs of the simulation model at each of the sixteen design points poses some problems in terms of the computer time required to make all these runs. The two major considerations here are eliminating the initialization bias and deciding whether to use the method of independent replications or batch means; and, both of these considerations are related.

Independent Replications vs Batch Means. The method of independent replications will yield independent observations since each run of the simulation model uses different random numbers. However, the transient period of the simulation must be discarded for each run. On the other hand, the transient period is discarded only once for the batch means approach, but the observations may be autocorrelated. The practitioner must then determine the appropriate batch size necessary to pass some statistical tests of independence. Any good text on simulation modeling should have more information about the batch means approach; a few of them are Kleijnen (1974), Law and Kelton (1982), and Pritsker (1986).

In general, when one is faced with making so many runs of a simulation model the batch means approach is preferred to independent replications because of the savings in computer time. But, the effect of the batch means approach upon the correlation between the response variable(s) and the control variable(s) has not been studied (or, at least, the author has not found any literature reporting such a study). In many situations there is a lag time in the correlation between the response variable(s) and the control variable(s).

For example, suppose for our queueing network we select the mean sojourn time for the response variable and the mean service time at the first node for the control variable. Then, when the observed average service time at the first node is high compared to its known mean, we would expect the average sojourn time to be higher than its true mean. But, we would also expect that the average sojourn time is not affected imediately by some longer than average service times at the first node. Therefore, when using the batch means approach, if the batch size is not large enough to capture the correlation between the response variable(s) and the control variable(s) we may not achieve a variance reduction, or bias may be introduced to the controlled estimates.

For these reasons the author decided to use independent replications instead of the batch means approach.

Eliminating the Initialization Bias. Since we are interested in steady-state results, the transient (or warm-up) period of the simulation must be discarded. The author made some trial runs of the simulation and plotted a time-persistent average of the sojourn time. The author made a visual determination that the transient period was essentially over after about three thousand time units when the simulation was started at empty and idle conditions. But, this would be too costly in terms of computer time to discard this much data for each run of the model. Therefore, the author decided to start the simulation model essentially at steady state by initializing the number of entities in each queue to the expected number as given by QNA. Then, in order to provide some randomization, the observations generated during the first one hundred time units were discarded.

Weighting the Observations

Since observations were collected on so many different variables (two response variables, six internal control variables, and four external control variables) it was more convenient to make each simulation run of the same fixed length (which was one thousand time units) rather than try to obtain the same number of observations for each variable. That is, each observation simulation run of the variables under study (resulting from one run of the simulation) are averages of the individual observations within a simulation run over one thousand time units; and, although the time interval is constant, the number of observations during the fixed time interval (within a simulation) varies from one run to the next. Therefore, to obtain unbiased estimates of discrete performance variables (such as the mean sojourn time) we need to weight the observations from each simulation run accordingly.

For example, each run of the simulation yields one value of the average sojourn time, which itself is an average of all the individual sojourn times of the entities completing the network during the fixed time interval of one thousand time units. Then, the uncontrolled estimate of the mean sojourn time is a weighted average of twenty averages, and the weights were derived from the number of individual observations that comprised each of the twenty averages respectively. The weighted average of any random variable X is given by

$$\bar{X} = \sum_{i=1}^{K} w_i X_i / \sum_{i=1}^{K} w_i \qquad (3.18)$$

where K is the number of observations (simulation runs) and w_i is the weighting coefficient on the ith observation of X. The weighting coefficients are given by

$$w_i = K \cdot n_i \; / \sum_{i=1}^{K} n_i \qquad (3.19)$$

where n_i is the number of individual observations (within a simulation) that make up the ith observation of X. Similarly, the weighted sample variance of X is given by

$$s^2 = \left[K \sum_{i=1}^{K} w_i X_i^{\,2} - \left[\sum_{i=1}^{K} w_i X_i \right]^2 \right] / [K(K-1)] \qquad (3.20)$$

<u>Statistics Used to Compare Control Variates</u>

Finally, we need to specify the measures to be used to compare the results of the different control variables on the response variables.

First of all, the control variates were compared on the basis of efficiency (or the size of the variance reduction) as given by the variance ratio (VR). Recall from Chapter II that the VR is the ratio of the variance of the controlled estimator when the optimal control coefficient is unknown (and must be estimated) to the variance of the uncontrolled estimator.

Secondly, the potential bias introduced to the controlled estimates was measured by the percentage of the total number of confidence intervals about the controlled estimates that covered the grand mean of the 4000 uncontrolled observations.

The Experimental Procedure

In this section, the author traces his steps in running the programs and generating the data. Recall that four thousand runs of the simulation model were made at each of the sixteen design points and that each run of the simulation model was eleven hundred time units in length.

Because of CPU time limits, one thousand runs of the simulation model were submitted at a time; therefore, four submissions were required to complete a single design point. Each of these job submissions used approximately one hour of CPU time on a VAX 8650 computer and generated five output files. The four replications of these output files for each design point were then appended to one another.

The contents of these five output files are as follows: (1) JACKSON.IN contained the records of the parameters necessary for input to QNA to solve for the observed values of the two external control variates under the assumption that all the service variability parameters equalled one; (2) QNA.IN was similar to JACKSON.IN except that the observed values of the service variability parameters were included; (3) RESPONSE.OUT contained the number of entities completing the network and the two response variables; (4) ROUTING.OUT contained the two standardized routing control variables with their respective number of observations; and (5) WORK.OUT contained the total number of service completions and the four standardized work variables. The reader may examine the FORTRAN code that generated these files in Appendix A.

Next, the author used a slightly modified version of the QNA software was to produce the two output files JACKSON.OUT and QNA.OUT which contained the observations on the external control variates generated from their respective input files. (The author's modifications to QNA were to suppress the normal detailed output and to report the values of the two variables of interest only).

The five files with the ".OUT" extension (along with the two files M16.OUT and G16.OUT which contained the "known" means of the external control variates) then became the input files to the program called CONTROL. The files M16.OUT and G16.OUT are given in table format in Appendix C, and the FORTRAN code for the program CONTROL is provided in Appendix A. The program CONTROL was written to calculate the uncontrolled as well as the controlled estimates of the means of the two response variables against each of the ten control variates. It also provided the variance ratios and 95% confidence limits about the controlled responses. Each output record was based on twenty records of input data (representing twenty runs of the simulation) and from here is referred to as a macro-replication; and, there were two hundred macro-replications at each design point. The output data (macro-replications) were stored in four files: (1) SOJOURN.VR which contained the uncontrolled estimates, the controlled estimates, and the variance ratios for the response variable sojourn time; (2) SOJOURN.CI which contained the 95% confidence limits for the response variable sojourn time; (3) QUANTILE.VR which contained the uncontrolled estimates, the controlled estimates, and the variance ratios for the response variable representing the probability that the number in queue exceeded twice the expected number; and (4) QUANTILE.CI which contained the respective 95% confidence limits.

Finally, the two SOJOURN files and the two QUANTILE files were separately input to the program RESULTS which calculated the minimum, mean, and maximum values of the uncontrolled estimates, the controlled estimates, and the variance ratios over the two hundred macro-replications. RESULTS also counted the number of times the grand uncontrolled estimate of the mean response fell within the 95% confidence limits of the two hundred macro-replications and reports the percentage that do so as an estimate of the actual coverage of the confidence intervals. The output files produced by the program RESULTS are provided in Appendix D. Also, the FORTRAN code for the program RESULTS is provided in Appendix A.

IV. Results

The results of the experimentation with control variates on the open queueing network described in Chapter III are given in this chapter. The variance ratios and confidence interval coverages are reported for ten control variates against two response variables. The output files from the program RESULTS are provided in Appendix D.

Table 4.1 lists the grand means of the uncontrolled simulation response of sojourn time and the analytic solutions provided by the M/M/1 and G/G/1 formulas. Note that the analytic solutions were used as the "known" means of the external control variate of sojourn time as given in Appendix C.

Since the M/M/1 analytic results assume that the squared coefficient of variation of the service times at the fourth node is equal to one, we would expect a difference between the results obtained with the M/M/1 formulas and the results obtained from the simulation model. However, it is interesting to note the significant differences between the simulation results and those obtained using the G/G/1 approximations. These differences are given in terms of percentages in Table 4.1 below.

The differences between the estimated mean sojourn times for an entity to complete the network were more pronounced as the traffic intensity of the network increased; and likewise, these differences increased as the coefficient of variation of the service times at the fourth node increased. These findings are summarized in Table 4.2 below.

Table 4.1. Analytic Results vs. Simulation Results
For Estimating Mean Sojourn Time in Network

Design Point	------Analytic Results----- M/M/1	G/G/1	Simulation Results	% Delta*
1	0.400000E+01	0.381177E+01	0.381331E+01	0.04
2	0.360000E+02	0.329330E+02	0.292945E+02	-11.05
3	0.400000E+01	0.531762E+01	0.521644E+01	-1.90
4	0.360000E+02	0.574688E+02	0.457255E+02	-20.43
5	0.400000E+01	0.381173E+01	0.381449E+01	0.07
6	0.360000E+02	0.329329E+02	0.287763E+02	-12.62
7	0.400000E+01	0.531792E+01	0.521470E+01	-1.94
8	0.360000E+02	0.574698E+02	0.451113E+02	-21.50
9	0.417391E+01	0.398269E+01	0.398874E+01	0.15
10	0.360000E+02	0.327806E+02	0.299427E+02	-8.66
11	0.417391E+01	0.551237E+01	0.530441E+01	-3.77
12	0.360000E+02	0.585327E+02	0.470301E+02	-19.65
13	0.400000E+01	0.380878E+01	0.381095E+01	0.06
14	0.360000E+02	0.327797E+02	0.292389E+02	-10.80
15	0.400000E+01	0.533846E+01	0.512254E+01	-4.04
16	0.360000E+02	0.585391E+02	0.456687E+02	-21.99

*% Delta is the percent difference between the G/G/1 results and the simulation results.

Table 4.2. Effect of ρ and c_s on % Delta
Between Analytic G/G/1 and Simulation Results

ρ	c_s	Average % Delta
0.5	0.5	0.08
0.5	2.5	-2.91
0.9	0.5	-10.78
0.9	2.5	-20.89

Because the differences increase as the traffic intensity of the network increases and as the variability of the service times at the fourth node increases, this may indicate one of three things, or possibly a combination of the three. First, the G/G/1 approximations used in QNA may be less accurate as the traffic intensity increases and as the coefficient of variation of the service times increases. Or, there may be significant initialization bias in the simulation results at the higher traffic intensity and the higher variability of the service times. Finally, the fixed time interval of the simulation may have been too short to provide good estimates at the higher traffic intensity and higher variability of the service times.

The author's first suspicion that the G/G/1 approximations used in QNA might be a significant contributor to these differences seems to be justified in light of the findings of Whitt (1983b). As to the initialization bias, the author started the simulation with the expected number in each queue. However, these numbers were provided by QNA; and therefore, under the assumption that QNA's results were less accurate as the traffic intensity and service variability increased an initial bias was probably introduced. Although further experimentation with the simulation model is required to prove any of these suspicions, knowing these differences exist proved helpful in interpreting the resulting variance ratios and coverages of the confidence intervals.

The average variance ratios achieved using each control variate against the first response variable, sojourn time, are given in Table 4.3. At the bottom of each column in Table 4.3 appears the grand average variance ratio for each control variate across the sixteen design points. Recall that the smaller the variance ratio, the greater the variance reduction. Table 4.4 provides the average variance ratios when the first two factors (traffic intensity and service variability) are held constant.

Table 4.3. Variance Ratios Achieved With Sojourn Time as Response Variable

	Internal Controls						External Controls			
	Routing		Work Variables				M/M/1		G/G/1	
Pt	R_{13}	R_{41}	W_1	W_2	W_3	W_4	SOJ	W_q	SOJ	W_q
1	0.937	0.868	0.955	0.949	0.949	0.955	0.418	0.768	0.385	0.751
2	0.942	0.907	0.944	0.946	0.951	0.944	0.697	0.645	0.698	0.642
3	0.941	0.925	0.956	0.951	0.954	0.956	0.468	0.386	0.251	0.317
4	0.942	0.933	0.947	0.942	0.943	0.948	0.662	0.615	0.601	0.611
5	0.870	0.886	0.957	0.957	0.954	0.957	0.378	0.797	0.355	0.783
6	0.941	0.917	0.960	0.947	0.951	0.960	0.771	0.650	0.772	0.644
7	0.928	0.914	0.950	0.951	0.949	0.950	0.508	0.422	0.279	0.369
8	0.940	0.946	0.951	0.948	0.939	0.952	0.704	0.620	0.609	0.617
9	0.950	0.762	0.939	0.948	0.943	0.939	0.355	0.662	0.334	0.643
10	0.951	0.859	0.943	0.951	0.948	0.943	0.708	0.604	0.710	0.602
11	0.946	0.874	0.947	0.949	0.945	0.947	0.450	0.416	0.266	0.387
12	0.950	0.917	0.949	0.953	0.946	0.949	0.689	0.672	0.671	0.673
13	0.878	0.777	0.946	0.949	0.942	0.946	0.331	0.712	0.315	0.700
14	0.929	0.858	0.952	0.945	0.948	0.952	0.748	0.633	0.750	0.627
15	0.935	0.899	0.939	0.958	0.948	0.939	0.448	0.404	0.253	0.379
16	0.942	0.929	0.945	0.944	0.951	0.945	0.726	0.624	0.655	0.624
Avg	0.933	0.886	0.949	0.949	0.948	0.949	0.566	0.602	0.494	0.586

Table 4.4. Average Variance Ratios for the First Response Across Traffic Intensity and Service Variability

Control	Variance Ratios at Factor Levels (ρ, c_s)			
Variate	(0.5,0.5)	(0.9,0.5)	(0.5,2.5)	(0.9,2.5)
R_{13}	0.909	0.941	0.938	0.943
R_{41}	0.823	0.885	0.903	0.931
W_1	0.949	0.950	0.948	0.948
W_2	0.951	0.947	0.952	0.947
W_3	0.947	0.949	0.949	0.945
W_4	0.949	0.950	0.948	0.948
S(M/M/1)	0.371	0.731	0.469	0.695
W(M/M/1)	0.735	0.633	0.407	0.633
S(G/G/1)	0.347	0.733	0.262	0.634
W(G/G/1)	0.719	0.629	0.363	0.631

The results listed in Table 4.3 show that among the internal control variates R_{41} achieved the smallest variance ratio and the others were roughly equal. However, the external control variates achieved significantly greater variance ratios that any individual internal control variate. Also, among the external control variates, those based upon the G/G/1 formulas achieved smaller variance ratios than those based upon the M/M/1 formulas.

Furthermore, one can see from Table 4.4 that except for R_{41} the average variance ratios of the internal control variates are not greatly affected by changes in the traffic intensity or service variability. However, the average variance ratios for the external control variates are greatly affected by both the traffic intensity and service variability.

In a similar manner, the average variance ratios achieved using each control variate against the second response variable, the quantile representing the probability of the number in the fourth queue exceeding twice the expected number, are given in Table 4.5. Likewise, Table 4.6 provides the average variance ratios when the first two factors (traffic intensity and service variability) are held constant.

The same general observations hold true for the results of the variance ratios achieved against the second response variable as for the first response variable. Additionally, in comparing the variance ratios between the two response variables, the variance ratios are smaller for the first response variable.

Next, Tables 4.7 through 4.10 are the respective analogs to Tables 4.3 through 4.6 with the confidence interval coverages as the statistic for comparison in place of the variance ratios.

Table 4.5. Variance Ratios Achieved
With Quantile at 4th Node as Response Variable

	Internal Controls						External Controls			
	Routing		Work Variables				M/M/1		G/G/1	
Pt	R_{13}	R_{41}	W_1	W_2	W_3	W_4	SOJ	W_q	SOJ	W_q
1	0.939	0.909	0.950	0.946	0.952	0.951	0.705	0.443	0.702	0.421
2	0.945	0.918	0.946	0.944	0.948	0.946	0.918	0.723	0.920	0.721
3	0.946	0.943	0.952	0.951	0.951	0.952	0.580	0.268	0.263	0.233
4	0.951	0.956	0.945	0.947	0.941	0.945	0.837	0.752	0.775	0.748
5	0.947	0.904	0.950	0.953	0.951	0.950	0.779	0.446	0.797	0.418
6	0.951	0.939	0.953	0.950	0.948	0.953	0.940	0.735	0.941	0.732
7	0.943	0.935	0.945	0.948	0.952	0.945	0.669	0.270	0.310	0.243
8	0.944	0.934	0.957	0.951	0.948	0.957	0.864	0.747	0.775	0.746
9	0.943	0.838	0.948	0.952	0.942	0.948	0.612	0.396	0.624	0.374
10	0.953	0.914	0.951	0.955	0.953	0.951	0.900	0.713	0.901	0.711
11	0.947	0.934	0.945	0.946	0.946	0.945	0.611	0.271	0.272	0.239
12	0.946	0.935	0.948	0.950	0.943	0.948	0.849	0.743	0.797	0.740
13	0.948	0.840	0.946	0.944	0.947	0.946	0.691	0.382	0.712	0.359
14	0.946	0.905	0.953	0.949	0.956	0.953	0.923	0.704	0.924	0.700
15	0.947	0.948	0.950	0.955	0.947	0.950	0.653	0.256	0.300	0.238
16	0.947	0.949	0.952	0.947	0.948	0.952	0.852	0.712	0.773	0.707
Avg	0.946	0.919	0.949	0.949	0.948	0.949	0.774	0.535	0.674	0.521

Table 4.6. Average Variance Ratios for the Second Response
Across Traffic Intensity and Service Variability

Control Variate	Variance Ratios at Factor Levels (ρ, c_s) (0.5,0.5)	(0.9,0.5)	(0.5,2.5)	(0.9,2.5)
R_{13}	0.944	0.949	0.946	0.947
R_{41}	0.873	0.919	0.940	0.944
W_1	0.949	0.951	0.948	0.951
W_2	0.949	0.949	0.950	0.949
W_3	0.948	0.951	0.949	0.945
W_4	0.949	0.951	0.948	0.951
S(M/M/1)	0.697	0.920	0.628	0.850
W(M/M/1)	0.417	0.719	0.266	0.739
S(G/G/1)	0.709	0.922	0.286	0.780
W(G/G/1)	0.393	0.716	0.238	0.735

Table 4.7. Coverage Percentages of 95% Confidence Interval
About Controlled Estimate of Mean Sojourn Time

	Internal Controls						External Controls			
	Routing		Work Variables				M/M/1		G/G/1	
Pt	R_{13}	R_{41}	W_1	W_2	W_3	W_4	SOJ	W_q	SOJ	W_q
1	0.930	0.885	0.920	0.925	0.935	0.920	0.550	0.830	0.545	0.810
2	0.935	0.955	0.940	0.935	0.930	0.940	0.800	0.695	0.800	0.700
3	0.945	0.945	0.950	0.950	0.945	0.950	0.570	0.510	0.365	0.430
4	0.930	0.935	0.910	0.900	0.915	0.910	0.615	0.640	0.580	0.625
5	0.910	0.925	0.935	0.960	0.955	0.935	0.495	0.815	0.475	0.820
6	0.920	0.925	0.945	0.940	0.935	0.945	0.810	0.745	0.810	0.720
7	0.920	0.895	0.930	0.925	0.905	0.925	0.610	0.545	0.420	0.510
8	0.945	0.955	0.945	0.930	0.950	0.945	0.655	0.690	0.590	0.695
9	0.915	0.765	0.895	0.930	0.905	0.895	0.465	0.695	0.440	0.710
10	0.910	0.895	0.890	0.925	0.895	0.890	0.785	0.685	0.780	0.660
11	0.935	0.905	0.930	0.935	0.935	0.930	0.580	0.580	0.465	0.525
12	0.915	0.905	0.925	0.935	0.920	0.925	0.680	0.725	0.685	0.720
13	0.880	0.800	0.895	0.910	0.915	0.895	0.430	0.765	0.435	0.740
14	0.945	0.900	0.945	0.930	0.935	0.945	0.815	0.745	0.815	0.745
15	0.930	0.925	0.915	0.925	0.925	0.915	0.565	0.515	0.385	0.495
16	0.915	0.905	0.895	0.915	0.905	0.890	0.655	0.650	0.630	0.665
Avg	0.924	0.901	0.923	0.929	0.925	0.922	0.630	0.677	0.576	0.661

Table 4.8. Average Coverages for the First Response
Across Traffic Intensity and Service Variabiliy

Control	-Variance Ratios at Factor Levels (ρ,c_s)--			
Variate	(0.5,0.5)	(0.9,0.5)	(0.5,2.5)	(0.9,2.5)
R_{13}	0.909	0.928	0.933	0.926
R_{41}	0.844	0.919	0.917	0.925
W_1	0.911	0.930	0.931	0.919
W_2	0.931	0.933	0.934	0.920
W_3	0.928	0.924	0.927	0.923
W_4	0.911	0.930	0.930	0.918
S(M/M/1)	0.485	0.803	0.581	0.651
W(M/M/1)	0.776	0.717	0.538	0.676
S(G/G/1)	0.474	0.801	0.409	0.621
W(G/G/1)	0.770	0.706	0.490	0.676

Table 4.9. Coverage Percentages of 95% Confidence Interval
About Controlled Estimate of Fourth Node Quantile

	Internal Controls						External Controls			
	Routing		Work Variables				M/M/1		G/G/1	
Pt	R_{13}	R_{41}	W_1	W_2	W_3	W_4	SOJ	W_q	SOJ	W_q
1	0.905	0.890	0.900	0.910	0.895	0.900	0.750	0.570	0.770	0.560
2	0.940	0.930	0.930	0.920	0.935	0.930	0.915	0.765	0.915	0.760
3	0.940	0.930	0.955	0.940	0.930	0.955	0.660	0.435	0.365	0.375
4	0.850	0.865	0.855	0.840	0.845	0.855	0.735	0.685	0.685	0.675
5	0.910	0.885	0.910	0.905	0.905	0.910	0.780	0.545	0.795	0.510
6	0.925	0.935	0.930	0.925	0.930	0.930	0.915	0.800	0.915	0.800
7	0.940	0.930	0.925	0.930	0.915	0.925	0.750	0.485	0.485	0.435
8	0.870	0.865	0.865	0.860	0.865	0.865	0.755	0.720	0.670	0.715
9	0.930	0.850	0.910	0.940	0.925	0.910	0.690	0.495	0.690	0.465
10	0.935	0.920	0.930	0.940	0.925	0.930	0.930	0.760	0.930	0.750
11	0.965	0.960	0.935	0.955	0.930	0.935	0.710	0.470	0.485	0.430
12	0.875	0.860	0.880	0.885	0.875	0.880	0.755	0.730	0.745	0.730
13	0.915	0.830	0.905	0.900	0.905	0.905	0.705	0.510	0.715	0.505
14	0.960	0.925	0.935	0.920	0.940	0.935	0.915	0.765	0.915	0.760
15	0.935	0.950	0.905	0.925	0.945	0.905	0.730	0.470	0.500	0.445
16	0.850	0.850	0.855	0.850	0.830	0.855	0.705	0.670	0.655	0.655
Avg	0.915	0.898	0.908	0.909	0.906	0.908	0.775	0.617	0.702	0.598

Table 4.10. Average Coverages for the Second Response
Across Traffic Intensity and Service Variabiliy

Control Variate	-Variance Ratios at Factor Levels (ρ, c_s)--			
	(0.5,0.5)	(0.9,0.5)	(0.5,2.5)	(0.9,2.5)
R_{13}	0.915	0.940	0.945	0.861
R_{41}	0.864	0.928	0.943	0.860
W_1	0.906	0.931	0.930	0.864
W_2	0.914	0.926	0.938	0.859
W_3	0.907	0.933	0.930	0.854
W_4	0.906	0.931	0.930	0.864
S(M/M/1)	0.731	0.919	0.713	0.737
W(M/M/1)	0.530	0.773	0.465	0.701
S(G/G/1)	0.743	0.919	0.459	0.689
W(G/G/1)	0.510	0.767	0.421	0.694

A general statement that can be made about the coverage percentages of the 95% confidence intervals about the controlled mean responses is that the coverage worsens as the variance ratio decreases. This result is not satisfactory, because we would like to be able to achieve large variance reductions (i.e. small variance ratios) and not bias the controlled responses (i.e. good coverages of the confidence intervals).

Finally, another way of examining the same results is presented in Tables 4.11 through 4.14. Here, these four tables give the average effects for the four factors. An effect, from regression analysis, is simply the average value of the response when a factor is at its high setting minus the average value of the response when the same factor is at its low setting. The way to interpret these tables is that the factors with the larger effects in magnitude have a greater impact on the statistic (variance ratios or coverages of confidence intervals).

From Tables 4.11 through 4.14 one can see that the four factors have a relatively small effect upon the results using the internal control variates; however, two factors (the traffic intensity and the service variability) have a relatively large effect upon the results using the external control variates.

Table 4.11. Average Effects of the Four Factors on the Variance Ratios of the First Response

Control Variate	Effects of the Four Factors: Traffic Intensity	C_s of 4th Server	r_{13} Prob.	r_{41} Prob.
R_{13}	0.009	0.008	0.012	0.003
R_{41}	0.023	0.031	-0.005	-0.026
W_1	0.000	-0.001	-0.001	-0.004
W_2	-0.002	0.000	-0.001	0.000
W_3	0.000	-0.001	0.000	-0.001
W_4	0.000	-0.001	-0.001	-0.004
S(M/M/1)	0.147	0.016	-0.010	-0.009
W(M/M/1)	0.031	-0.082	-0.006	-0.011
S(G/G/1)	0.189	-0.046	-0.005	0.000
W(G/G/1)	0.044	-0.088	-0.007	-0.006

Table 4.12. Average Effects of the Four Factors on the Variance Ratios of the Second Response

Control Variate	Effects of the Four Factors: Traffic Intensity	C_s of 4th Server	r_{13} Prob.	r_{41} Prob.
R_{13}	0.001	0.000	0.000	0.001
R_{41}	0.012	0.023	0.000	-0.011
W_1	0.001	0.000	-0.001	0.000
W_2	0.000	0.000	0.000	0.001
W_3	0.000	-0.001	-0.001	-0.001
W_4	0.001	0.000	-0.001	0.000
S(M/M/1)	0.111	-0.035	-0.022	-0.013
W(M/M/1)	0.194	-0.033	0.004	-0.013
S(G/G/1)	0.177	-0.141	-0.017	-0.011
W(G/G/1)	0.205	-0.034	0.003	-0.012

Table 4.13. Average Effects of the Four Factors on the Coverages of the First Response

Control Variate	Traffic Intensity	C_s of 4th Server	r_{13} Prob.	r_{41} Prob.
	-------Effects of the Four Factors--------			
R_{13}	0.003	0.006	0.003	-0.006
R_{41}	0.021	0.020	-0.003	-0.026
W_1	0.002	0.002	-0.003	-0.012
W_2	-0.003	-0.003	0.000	-0.004
W_3	-0.002	0.000	-0.003	-0.008
W_4	0.002	0.002	-0.002	-0.012
S(M/M/1)	0.097	-0.014	0.001	-0.008
W(M/M/1)	0.020	-0.070	-0.007	-0.007
S(G/G/1)	0.135	-0.061	0.006	0.003
W(G/G/1)	0.031	-0.077	-0.013	-0.003

Table 4.14. Average Effects of the Four Factors on the Coverages of the Second Response

Control Variate	Traffic Intensity	C_s of 4th Server	r_{13} Prob.	r_{41} Prob.
	-------Effects of the Four Factors--------			
R_{13}	-0.015	-0.012	0.002	0.005
R_{41}	-0.005	0.003	0.002	-0.005
W_1	-0.010	-0.011	0.004	-0.001
W_2	-0.017	-0.011	0.007	0.005
W_3	-0.013	-0.014	0.002	0.003
W_4	-0.010	-0.011	0.004	-0.001
S(M/M/1)	0.053	-0.050	-0.007	-0.008
W(M/M/1)	0.120	-0.034	-0.003	-0.008
S(G/G/1)	0.102	-0.128	-0.004	0.002
W(G/G/1)	0.132	-0.041	-0.005	-0.006

V. Conclusions

The purpose of this study was to compare several control variates for queueing network simulation. The author's goal was to provide the simulation community with some guidance for selecting control variates that will lead to significant reductions in the variance of the estimated responses that do not introduce significant bias. The conclusions of this research are important because they add to the body of knowledge a simulation practitioner can draw upon when selecting a variance reduction technique. Also, these results indicate that further research in this area is warranted.

The results of this research are consistent with previous findings demonstrating the potential variance reduction that can be obtained when using control variates. However, this research went beyond most of the previous efforts in measuring the bias introduced to the controlled estimates by way of the percentage of confidence intervals about the controlled responses that covered the grand averages of the uncontrolled responses.

Another novel addition to this research was the use of the maximum entropy distribution to generate service times, which is the least-biased distribution when only the first two moments of the distribution are known. However, finding the parameters to the maximum entropy distribution was more difficult than expected, and generating random variates according to the maximum entropy distribution was an involved process. The author recommends that future research efforts look at comparing the Erlang distribution (which is much easier to generate) to the maximum entropy distribution for generating random variates with a coefficient of variation less than one. The hyperexponential distribution, however, is the recommended distribution for generating random variates with a coefficient of variation greater than one when only the first two moments of the distribution are known.

The use of the Queueing Network Analyzer (QNA) to provide external control variates was also a new approach. However, the results showed that QNA could yield estimates of the congestion measures as much as twenty percent in error to the simulation estimates, which might have been a potential source of bias. Future research is also needed to determine exactly how these differences can be reduced.

The way to reduce the errors in the estimates of the congestion measures of QNA is to use better approximations. One suggestion is to use the approximation for the expected waiting time in the GI/G/m queue given by Kimura (1986), since he reported achieving better results as compared to those achieved using QNA (the reader is referred to Chapter II). The author pursued this suggestion by computing the expected waiting times at the fourth node over the sixteen design points using Kimura's formula. However, the largest percentage change from QNA's result was only -1.29%; therefore, using Kimura's formula in place of the one in QNA would not have helped in this case. These results are presented for the reader in Table 5.1.

In general, the external control variates achieved smaller variance ratios than the internal control variates; however, the coverages of the confidence intervals about the controlled responses were worse. The range of the average variance ratios for the internal control variates was 0.886 to 0.949 with coverages from 0.898 to 0.929

Table 5.1. Waiting Times at Fourth Node
Computed Using QNA's and Kimura's Approximations

DESIGN POINT	---EXPECTED WAITING TIMES--- QNA'S	KIMURA'S	% CHANGE FROM QNA
1	0.281197E+00	0.281135E+00	0.022
2	0.455585E+01	0.455531E+01	0.012
3	0.163162E+01	0.163416E+01	-0.156
4	0.264290E+02	0.264305E+02	-0.006
5	0.281160E+00	0.281053E+00	0.038
6	0.455572E+01	0.455566E+01	0.001
7	0.163188E+01	0.163623E+01	-0.267
8	0.264300E+02	0.264320E+02	-0.008
9	0.234173E+00	0.233935E+00	0.102
10	0.379484E+01	0.379464E+01	0.005
11	0.136079E+01	0.137057E+01	-0.719
12	0.220362E+02	0.220441E+02	-0.036
13	0.234015E+00	0.233590E+00	0.182
14	0.379416E+01	0.379390E+01	0.007
15	0.136189E+01	0.137946E+01	-1.290
16	0.220409E+02	0.220515E+02	-0.048

for the 95% confidence intervals. The range of the average variance ratios for the external control variates was 0.494 to 0.774 with coverages from 0.576 to 0.775 for the 95% confidence intervals.

More specifically, smaller variance ratios were achieved against the first response variable, the average sojourn time for the network, than for the second response variable, the probability of the number in the fourth queue exceeding twice the expected number. This was expected since one would expect the control variates chosen to have a greater correlation with the first response variable in general.

Also, the standardized routing controls achieved smaller variance ratios than the standardized work variables. Particularly, any route which feeds entities back through a portion of the network (thereby increasing the congestion) is a good candidate for an internal control variable.

When comparing the results of the internal control variates to the external control variates one should keep in mind that the standardized internal controls are generally independent of one another. Therefore, multiple internal controls could be used to achieve even larger variance reductions provided the number of replications is sufficient to overcome the loss factor. It is also important to recognize that the external control variates tend to introduce bias to the controlled estimates. Another area for future research would be into reducing this bias since the potential for variance reductions is so promising. Some potential ways to reduce the bias are to use jacknife estimators or some other estimators that do not shrink the confidence interval width as much.

Furthermore, future efforts should look into the effect of the batch means approach on the control variate technique. The batch means approach provides a means of reducing the cost of obtaining several thousand observations when trying to estimate the coverage percentages of the confidence intervals. However, exactly how the batch size affects the correlation between the response variable(s) and the control variable(s) is not known; and therefore, the author used the independent replications approach.

An important observation to note is that the standardized internal controls were less influenced by the changes in the factor settings than the external controls, which in turn yielded more consistent results over the experimental design. In particular, the internal controls are robust for the traffic intensity.

The author recommends using internal control variates since they demonstrated better coverage; however, multiple controls will be necessary to achieve a significant reduction. The use of external controls is very promising since a single control can achieve a very significant variance reduction; however, further research should be done to determine ways to reduce the bias they introduce to the estimated responses.

When selecting internal control variates for use in a queueing network simulation the author recommends the standardized routing controls over the standardized work variables, especially when there are probabilistic branches in the network that have a significant effect on the response(s). For example, in the author's research the entities completing the fourth node could either exit the system or be fed back to the first node. Any deviation from the expected number of entities being fed back has a significant impact upon the estimate of the average sojourn time in the network; and therefore, the routing control R_{41} achieved a significantly greater variance reduction than the other internal control variates.

On a final note, when comparing the control variates the smaller variance ratios indicate a larger variance reduction. The increased number of runs that would be required to achieve the same variance reduction without the control variate(s) can be calculated by rounding the result given by Eq (5.1) to the next highest integer.

$$K_{ADDITIONAL} = K \cdot [(1/VR) - 1] \qquad (5.1)$$

For example, if we made twenty runs of the simulation and achieved a variance ratio of 0.90, then we would need 3 additional runs. However, if we achieved a variance ratio of 0.50, then we would need 20 additional runs (for a total of 40 runs) to achieve the same amount of variance reduction without control variates. Thus, significant savings in computer time can be realized through the use of control variates.

Appendix A: Computer Source Code

SLAM II Network Code

```
GEN,QUEUEING NETWORK,JOHN TOMICK,10/20/88,1000,N,N,Y/Y,N,N,72;
LIMITS,4,2,500;
INTLC,XX(1)=0.5,XX(2)=0.5,XX(3)=0.45,XX(4)=0.253125;
INTLC,XX(6)=0.45,XX(7)=0.75,XX(8)=1.125,XX(9)=100;
INTLC,XX(10)=-0.028146,XX(11)=-7.0,XX(12)=8.198,XX(13)=1.12;
INTLC,XX(14)=0.425,XX(15)=0.07,XX(20)=0.4,XX(21)=0.1,XX(22)=1.;
NETWORK;
      CREATE,EXPON(1.,1),0,1;
      EVENT,1;                         COUNT EXTERNAL ARRIVALS
QUE1  ASSIGN,ATRIB(2)=EXPON(XX(6),2);
      EVENT,2;                         COUNT ENTITIES ENTERING QUEUE 1
      QUEUE(1),1;
        ACTIVITY/1,ATRIB(2);           SERVER #1
      EVENT,3;                         WORK VARIABLE #1
      GOON,1;
        ACTIVITY,,XX(20),QUE3;
        ACTIVITY;
QUE2  ASSIGN,ATRIB(2)=EXPON(XX(7),3);
      QUEUE(2),1;
        ACTIVITY/2,ATRIB(2);           SERVER #2
      EVENT,4;                         WORK VARIABLE #2
        ACTIVITY,,,QUE4;
QUE3  ASSIGN,ATRIB(2)=EXPON(XX(8),4);
      EVENT,5;                         COUNT ENTITIES ENTERING NODE 3
      QUEUE(3),1;
        ACTIVITY/3,ATRIB(2);           SERVER #3
      EVENT,6;                         WORK VARIABLE #3
QUE4  ASSIGN,ATRIB(2)=USERF(1);
      EVENT,7;                         COUNT ENTITIES ENTERING NODE 4
      QUEUE(4),1;
        ACTIVITY/4,ATRIB(2);           SERVER #4
      EVENT,8;                         WORK VARIABLE #4
      GOON,1;
        ACTIVITY,,XX(21),LOOP;
        ACTIVITY;
      EVENT,10;                        COLLECT SOJOURN TIME
      TERMINATE;
LOOP  EVENT,9;                         COUNT ENTITIES LOOPING BACK
      ACTIVITY,,,QUE1;
      ENDNETWORK;
INIT,0,1100;
FIN;
```

```
C       XX() VARIABLE DEFINITIONS
C
C       XX(1)  = TRAFFIC INTENSITY AT EACH NODE IN NETWORK
C       XX(2)  = COEF OF VAR OF SERVICE-TIMES AT 4TH NODE
C       XX(3)  = 1ST MOMENT OF SERVICE-TIMES AT 4TH NODE
C       XX(4)  = 2ND MOMENT OF SERVICE-TIMES AT 4TH NODE
C       XX(5)  = STANDARD DEV OF SERVICE-TIMES AT 4TH NODE
C
C       XX(6)  = MEAN OF SERVICE TIMES AT NODE 1
C       XX(7)  = MEAN OF SERVICE TIMES AT NODE 2
C       XX(8)  = MEAN OF SERVICE TIMES AT NODE 3
C
C       XX(9)  = TIME TO BEGIN COLLECTING STATISTICS
C
C       XX(10) = LAMBDA0 OF MAX ENTROPY DISTRIBUTION
C       XX(11) = LAMBDA1 OF MAX ENTROPY DISTRIBUTION
C       XX(12) = LAMBDA2 OF MAX ENTROPY DISTRIBUTION
C       XX(13) = COEFFICIENT 'C' IN ACCEPTANCE-REJECTION METHOD
C       XX(14) = MEAN OF MAJORIZING DISTRIBUTION
C       XX(15) = VARIANCE OF MAJORIZING DISTRIBUTION
C
C       XX(16) = COEFFICIENT 'C1' OF HYPEREXPONENTIAL DISTRIBUTION
C       XX(17) = MEAN OF FIRST EXPONENTIAL DISTRIBUTION
C       XX(18) = COEFFICIENT 'C2' OF HYPEREXPONENTIAL DISTRIBUTION
C       XX(19) = MEAN OF SECOND EXPONENTIAL DISTRIBUTION
C
C       XX(20) = PROBABILITY OF AN ENTITY GOING FROM NODE 1 TO NODE 3
C       XX(21) = PROBABILITY OF AN ENTITY GOING FROM NODE 4 TO NODE 1
C       XX(22) = TWICE MEAN NUMBER IN QUEUE #4 (FOR QUANTILE ESTIMATION)
C
C       XX(23) = NUMBER OF ENTITIES COMPLETING NETWORK
C       XX(24) = SUM OF SOJOURN TIMES
C       XX(25) = AVERAGE SOJOURN TIME
C
C       XX(26) = FLAG INDICATING NNQ(4) > XX(22) (0=FALSE,1=TRUE)
C       XX(27) = LAST TIME WHEN XX(26) WAS SET TO 1
C       XX(28) = SUM OF TIME INTERVALS WHEN XX(26)=1
C       XX(29) = PROPORTION OF TIME WHEN XX(26)=1
C
C       XX(30) = NUMBER OF EXTERNAL ARRIVALS TO NODE 1
C       XX(31) = EXTERNAL ARRIVAL RATE TO NODE 1
C
C       XX(32) = NUMBER OF ARRIVALS TO NODE 1
C       XX(33) = NUMBER OF ARRIVALS TO NODE 3
C       XX(34) = STANDARDIZED ROUTING VARIABLE (1,3)
C       XX(35) = NUMBER OF ARRIVALS TO NODE 4
C       XX(36) = NUMBER OF ARRIVALS TO NODE 1 FROM NODE 4
C       XX(37) = STANDARDIZED ROUTING VARIABLE (4,1)
C
C       XX(38) = NOT USED
C
```

```
C       XX(39) = TOTAL NUMBER OF SERVICE COMPLETIONS AT ALL NODES
C       XX(40) = NUMBER OF SERVICE COMPLETIONS AT NODE 1
C       XX(41) = INTERMEDIATE SUM FOR WORK VARIABLE #1
C       XX(42) = STANDARDIZED WORK VARIABLE #1
C       XX(43) = NUMBER OF SERVICE COMPLETIONS AT NODE 2
C       XX(44) = INTERMEDIATE SUM FOR WORK VARIABLE #2
C       XX(45) = STANDARDIZED WORK VARIABLE #2
C       XX(46) = NUMBER OF SERVICE COMPLETIONS AT NODE 3
C       XX(47) = INTERMEDIATE SUM FOR WORK VARIABLE #3
C       XX(48) = STANDARDIZED WORK VARIABLE #3
C       XX(49) = NUMBER OF SERVICE COMPLETIONS AT NODE 4
C       XX(50) = INTERMEDIATE SUM FOR WORK VARIABLE #4
C       XX(51) = STANDARDIZED WORK VARIABLE #4
C
C       XX(52) = SUM OF SERVICE TIMES AT NODE #1
C       XX(53) = SUM OF SQUARED SERVICE TIMES AT NODE #1
C       XX(54) = SQUARED COEF OF VAR OF SERVICE TIMES AT NODE #1
C       XX(55) = SUM OF SERVICE TIMES AT NODE #2
C       XX(56) = SUM OF SQUARED SERVICE TIMES AT NODE #2
C       XX(57) = SQUARED COEF OF VAR OF SERVICE TIMES AT NODE #2
C       XX(58) = SUM OF SERVICE TIMES AT NODE #3
C       XX(59) = SUM OF SQUARED SERVICE TIMES AT NODE #3
C       XX(60) = SQUARED COEF OF VAR OF SERVICE TIMES AT NODE #3
C       XX(61) = SUM OF SERVICE TIMES AT NODE #4
C       XX(62) = SUM OF SQUARED SERVICE TIMES AT NODE #4
C       XX(63) = SQUARED COEF OF VAR OF SERVICE TIMES AT NODE #4
C
      PROGRAM MAIN
      DIMENSION NSET(10000)
      COMMON/SCOM1/ATRIB(100),DD(100),DDL(100),DTNOW,II,MFA,MSTOP,NCLNR
     1,NCRDR,NPRNT,NNRUN,NNSET,NTAPE,SS(100),SSL(100),TNEXT,TNOW,XX(100)
      COMMON QSET(10000)
      EQUIVALENCE(NSET(1),QSET(1))
      NNSET=10000
      NCRDR=5
      NPRNT=6
      NTAPE=7
      NPLOT=2
      OPEN(UNIT=1,FILE='QNA.IN',STATUS='NEW')
      OPEN(UNIT=2,FILE='JACKSON.IN',STATUS='NEW')
      OPEN(UNIT=3,FILE='RESPONSE.OUT',STATUS='NEW')
      OPEN(UNIT=4,FILE='ROUTING.OUT',STATUS='NEW')
      OPEN(UNIT=8,FILE='WORK.OUT',STATUS='NEW')
      II = 1000
      WRITE (1,1) II
      WRITE (2,1) II
      CALL SLAM
      STOP
    1 FORMAT (3X,I6)
      END
C
C
C
```

```
      SUBROUTINE INTLC
      COMMON/SCOM1/ATRIB(100),DD(100),DDL(100),DTNOW,II,MFA,MSTOP,NCLNR
     1,NCRDR,NPRNT,NNRUN,NNSET,NTAPE,SS(100),SSL(100),TNEXT,TNOW,XX(100)
      XX(5) = XX(4) - XX(3)*XX(3)
      IF (XX(2).GT.1.) THEN
         XX(16) = 0.5 + SQRT(0.25 - 0.5*XX(3)*XX(3)/XX(4))
         XX(17) = 0.5*XX(3)/XX(16)
         XX(18) = 1.0 - XX(16)
         XX(19) = 0.5*XX(3)/XX(18)
      END IF
      DO 10 I=23, 63
         XX(I) = 0.0
   10 CONTINUE
      RETURN
      END
C
C
C
      FUNCTION USERF(I)
      COMMON/SCOM1/ATRIB(100),DD(100),DDL(100),DTNOW,II,MFA,MSTOP,NCLNR
     1,NCRDR,NPRNT,NNRUN,NNSET,NTAPE,SS(100),SSL(100),TNEXT,TNOW,XX(100)
      INTEGER ACCEPT
      REAL F, G, PI, SDEV, U, Y
      PARAMETER (PI = 3.1415927)
C
C     SELECT DISTRIBUTION FOR SERVICE TIMES
C
      IF (XX(2).LT.1.) THEN
C
C     GENERATE MAX ENTROPY DISTRIBUTION USING ACCEPT-REJECT
C
         ACCEPT = 0
         SDEV = SQRT(XX(15))
   10    IF (ACCEPT.EQ.0) THEN
            Y = RNORM(XX(14),SDEV,9)
            IF (Y.GE.0.) THEN
               U = DRAND(10)
               F = EXP(-1. - XX(10) - XX(11)*Y - XX(12)*Y**2)
               G = (XX(13)/SQRT(2*XX(15)*PI))*
     &              EXP(-(1/(2.*XX(15)))*(Y-XX(14))**2)
               IF (U.LE.F/G) ACCEPT=1
            END IF
            GO TO 10
         END IF
         USERF = Y
      ELSE
C
C     GENERATE HYPEREXPONENTIAL DISTRIBUTION USING COMPOSITION
C
         U = DRAND(9)
         IF (U.LE.XX(16)) THEN
            USERF = EXPON(XX(17),10)
         ELSE
```

```
            USERF = EXPON(XX(19),10)
          END IF
        END IF
        RETURN
        END
C
C
C
        SUBROUTINE EVENT(I)
        COMMON/SCOM1/ATRIB(100),DD(100),DDL(100),DTNOW,II,MFA,MSTOP,NCLNR
       1,NCRDR,NPRNT,NNRUN,NNSET,NTAPE,SS(100),SSL(100),TNEXT,TNOW,XX(100)
        IF (TNOW.LE.XX(9)) RETURN
        GO TO (1, 2, 3, 4, 5, 6, 7, 8, 9, 10), I
C
C       COUNT EXTERNAL ARRIVALS TO NODE 1
C
      1 XX(30) = XX(30) + 1.
        RETURN
C
C       COUNT ENTITIES ARRIVING TO NODE 1
C
      2 XX(32) = XX(32) + 1.
        RETURN
C
C       COLLECT WORK VARIABLE #1 DATA
C
      3 XX(40) = XX(40) + 1.
        XX(41) = (ATRIB(2) - XX(6))/XX(6)
        XX(52) = XX(52) + ATRIB(2)
        XX(53) = XX(53) + ATRIB(2)*ATRIB(2)
        RETURN
C
C       COLLECT WORK VARIABLE #2 DATA
C
      4 XX(43) = XX(43) + 1.
        XX(44) = (ATRIB(2) - XX(7))/XX(7)
        XX(55) = XX(55) + ATRIB(2)
        XX(56) = XX(56) + ATRIB(2)*ATRIB(2)
        RETURN
C
C       COUNT ENTITIES ARRIVING TO NODE 3
C
      5 XX(33) = XX(33) + 1.
        RETURN
C
C       COLLECT WORK VARIABLE #3 DATA
C
      6 XX(46) = XX(46) + 1.
        XX(47) = (ATRIB(2) - XX(8))/XX(8)
        XX(58) = XX(58) + ATRIB(2)
        XX(59) = XX(59) + ATRIB(2)*ATRIB(2)
        RETURN
C
```

```
C     COUNT ENTITIES ARRIVING TO NODE 4, AND
C     SET FLAG IF NNQ(4) > XX(22), STORE TNOW
C
    7 XX(35) = XX(35) + 1.
      IF ((NNQ(4)+1.GT.XX(22)).AND.(XX(26).EQ.0.)) THEN
        XX(26) = 1.
        XX(27) = TNOW
      END IF
      RETURN
C
C     COLLECT WORK VARIABLE #4 DATA, AND
C     COLLECT TIME INTERVAL NNQ(4) > XX(22)
C
    8 XX(49) = XX(49) + 1.
      XX(50) = (ATRIB(2) - XX(3))/XX(5)
      XX(61) = XX(61) + ATRIB(2)
      XX(62) = XX(62) + ATRIB(2)*ATRIB(2)
      IF ((NNQ(4).LT.XX(22)).AND.(XX(26).EQ.1.)) THEN
        XX(26) = 0.
        XX(28) = XX(28) + TNOW - XX(27)
      END IF
      RETURN
C
C     COUNT ENTITIES LOOPING BACK TO NODE 1
C
    9 XX(36) = XX(36) + 1.
      RETURN
C
C     COLLECT DATA FOR AVERAGE SOJOURN TIME
C
   10 XX(23) = XX(23) + 1.
      XX(24) = XX(24) + TNOW - ATRIB(1)
      RETURN
      END
C
C
C
      SUBROUTINE OTPUT
      COMMON/SCOM1/ATRIB(100),DD(100),DDL(100),DTNOW,II,MFA,MSTOP,NCLNR
     1,NCRDR,NPRNT,NNRUN,NNSET,NTAPE,SS(100),SSL(100),TNEXT,TNOW,XX(100)
      INTEGER METHOD, NNODES, OPTION(5), TYPE
      REAL RATE(4), ROUTE(4,4), SERVICE(4)
C
C     CALCULATE AVERAGE SOJOURN TIME
C
      XX(25) = XX(24)/XX(23)
C
C     CALCULATE PROPORTION OF TIME NNQ(4) EXCEEDED
C     TWICE THE MEAN NUMBER IN QUEUE FROM QNA
C
      IF ((NNQ(4).LT.XX(22)).AND.(XX(26).EQ.1.)) THEN
        XX(26) = 0.
        XX(28) = XX(28) + TNOW - XX(27)
```

```
      END IF
      XX(29) = XX(28)/1000
C
C     CALCULATE OBSERVED EXTERNAL ARRIVAL RATE
C
      XX(31) = XX(30)/1000
C
C     CALCULATE STANDARDIZED ROUTING CONTROL VARIABLES
C
      XX(34) = (XX(33) - XX(32)*XX(20))/SQRT(XX(32)*(1.-XX(20))*XX(20))
      XX(37) = (XX(36) - XX(35)*XX(21))/SQRT(XX(35)*(1.-XX(21))*XX(21))
C
C     CALCULATE STANDARDIZED WORK VARIABLES
C
      XX(39) = XX(40) + XX(43) + XX(46) + XX(49)
      XX(42) = 3*SQRT(XX(40))/XX(39)*XX(41)
      XX(45) = 3*SQRT(XX(43))/(XX(39)*(1.-XX(33)/XX(32)))*XX(44)
      XX(48) = 3*SQRT(XX(46))/(XX(39)*XX(33)/XX(32))*XX(47)
      XX(51) = 3*SQRT(XX(49))/XX(39)*XX(41)
C
C     CALCULATE OBSERVED MEAN SERVICE TIMES
C
      SERVICE(1) = XX(52)/XX(40)
      SERVICE(2) = XX(55)/XX(43)
      SERVICE(3) = XX(58)/XX(46)
      SERVICE(4) = XX(61)/XX(49)
C
C     CALCULATE SQUARED COEFFICIENT OF VARIATION OF SERVICE TIMES
C
      XX(54) = ((XX(40)**2)*XX(53) - XX(40)*(XX(52)**2))/
     &((XX(40)-1)*XX(52)**2)
      XX(57) = ((XX(43)**2)*XX(56) - XX(43)*(XX(55)**2))/
     &((XX(43)-1)*(XX(55)**2))
      XX(60) = ((XX(46)**2)*XX(59) - XX(46)*(XX(58)**2))/
     &((XX(46)-1)*(XX(58)**2))
      XX(63) = ((XX(49)**2)*XX(62) - XX(49)*(XX(61)**2))/
     &((XX(49)-1)*(XX(61)**2))
C
C     WRITE TO 'QNA.IN'
C
      METHOD = 3
      NNODES = 4
      OPTION(1) = 1
      OPTION(2) = 2
      OPTION(3) = 0
      OPTION(4) = -1
      OPTION(5) = 1
      TYPE = 1
      ROUTE(1,1) = 0.
      ROUTE(1,2) = 1. - XX(33)/XX(32)
      ROUTE(1,3) = XX(33)/XX(32)
      ROUTE(1,4) = 0.
      ROUTE(2,1) = 0.
```

```
      ROUTE(2,2) = 0.
      ROUTE(2,3) = 0.
      ROUTE(2,4) = 1.
      ROUTE(3,1) = 0.
      ROUTE(3,2) = 0.
      ROUTE(3,3) = 0.
      ROUTE(3,4) = 1.
      ROUTE(4,1) = XX(36)/XX(35)
      ROUTE(4,2) = 0.
      ROUTE(4,3) = 0.
      ROUTE(4,4) = 0.
      RATE(1) = XX(31)
      RATE(2) = 0.
      RATE(3) = 0.
      RATE(4) = 0.
      WRITE (1,1) METHOD
      WRITE (1,2) NNODES, TYPE
      WRITE (1,3) OPTION(1), OPTION(2), OPTION(3), OPTION(4), OPTION(5)
      DO 10 I=1,4
        WRITE (1,4) ROUTE(I,1), ROUTE(I,2), ROUTE(I,3), ROUTE(I,4)
   10 CONTINUE
      WRITE (1,4) RATE(1), RATE(2), RATE(3), RATE(4)
      WRITE (1,4) SERVICE(1), SERVICE(2), SERVICE(3), SERVICE(4)
      WRITE (1,4) XX(54), XX(57), XX(60), XX(63)
C
C     WRITE TO 'JACKSON.IN'
C
      OPTION(1) = 1
      OPTION(2) = 0
      OPTION(3) = 0
      OPTION(4) = -1
      OPTION(5) = 1
      WRITE (2,1) METHOD
      WRITE (2,2) NNODES, TYPE
      WRITE (2,3) OPTION(1), OPTION(2), OPTION(3), OPTION(4), OPTION(5)
      DO 20 I=1,4
        WRITE (2,4) ROUTE(I,1), ROUTE(I,2), ROUTE(I,3), ROUTE(I,4)
   20 CONTINUE
      WRITE (2,4) RATE(1), RATE(2), RATE(3), RATE(4)
      WRITE (2,4) SERVICE(1), SERVICE(2), SERVICE(3), SERVICE(4)
C
C     WRITE TO 'RESPONSE.OUT'
C     NUMBER OF OBSERVATIONS, AVE SOJOURN TIME, AND P(NNQ(4)>2*EN)
C
      WRITE (3,5) XX(23), XX(25), XX(29)
C
C     WRITE TO 'ROUTING.OUT'
C     ROUTING CONTROLS R13, R14
C     WITH RESPECTIVE NUMBER OF OBSERVATIONS ON EACH
C
      WRITE (4,4) XX(32), XX(34), XX(35), XX(37)
C
C     WRITE TO 'WORK.OUT'
```

```
C     WORK VARIABLES W1, W2, W3, W4
C     WITH NUMBER OF OBSERVATIONS
C
      WRITE (8,6) XX(39), XX(42), XX(45), XX(48), XX(51)
      RETURN
    1 FORMAT(2X,I2)
    2 FORMAT(2(2X,I2))
    3 FORMAT(5(2X,I2))
    4 FORMAT(4(2X,E13.6))
    5 FORMAT(3(2X,E13.6))
    6 FORMAT(5(2X,E13.6))
      END
```

FORTRAN Code for Program CONTROL

```
      PROGRAM CONTROL
C
C     THIS PROGRAM CALCULATES THE VARIANCE RATIO (VR) AND
C     THE 95% CONFIDENCE LIMITS (LCL) AND (UCL) ON THE
C     CONTROLLED RESPONSES.
C
C     THE INPUT DATA IS SUPPLIED BY SEVEN FILES:
C         (1) RESPONSE.OUT CONTAINS:
C                 NUMBER OF OBSERVATIONS OF SOJOURN TIME,
C                 SOJOURN TIME (UNCONTROLLED RESPONSE), AND
C                 P(NNQ(4)>2*EN) (UNCONTROLLED RESPONSE).
C         (2) ROUTING.OUT CONTAINS:
C                 NUMBER OF OBSERVATIONS ON R13,
C                 R13 (STANDARDIZED ROUTING CONTROL),
C                 NUMBER OF OBSERVATIONS ON R41, AND
C                 R41 (STANDARDIZED ROUTING CONTROL).
C         (3) WORK.OUT CONTAINS:
C                 TOTAL NUMBER OF SERVICE COMPLETIONS, AND
C                 W1, W2, W3, W4 (STANDARDIZED WORK VARIABLES).
C         (4) JACKSON.OUT CONTAINS EXTERNAL CONTROL VARIATES
C             USING SHARON'S (1986) METHOD AND M/M/1 FORMULAS:
C                 SOJOURN TIME, AND
C                 WAITING TIME IN 4TH QUEUE.
C         (5) QNA.OUT CONTAINS EXTERNAL CONTROL VARIATES
C             USING SHARON'S (1986) METHOD AND G/G/1 FORMULAS:
C                 SOJOURN TIME, AND
C                 WAITING TIME IN 4TH QUEUE.
C         (6) M16.OUT CONTAINS THE MEANS OF THE EXTERNAL
C             CONTROL VARIATES USING M/M/1 FORMULAS.
C         (7) G16.OUT CONTAINS THE MEANS OF THE EXTERNAL
C             CONTROL VARIATES USING G/G/1 FORMULAS.
C
C     THE OUTPUT DATA IS WRITTEN TO FOUR FILES (SOJOURN.VR,
C     SOUJOURN.CI, QUANTILE.VR AND QUANTILE.CI) IN THE
C     FOLLOWING FORMAT:
C         THE *.VR FILES CONTAIN:
C            MEAN OF UNCONTROLLED RESPONSE
C            MEAN OF CONTROLLED RESPONSE
C            VARIANCE RATIO
C         THE *.CI FILES CONTAIN:
C            95% LOWER CONFIDENCE LIMIT
C            95% UPPER CONFIDENCE LIMIT
C
C     EACH RECORD IN THE OUTPUT FILES ARE THE RESULT OF
C     TWENTY INPUT RECORDS (I.E. TWENTY SIMULATION RUNS).
C     EVERY TENTH RECORD IN THE OUTPUT FILES REPRESENTS A
C     REPLICATION OF THE RESULTS WITH ONE CONTROL VARIATE.
C
C     INPUT VARIABLES
C        Y(I,K) = TWO RESPONSE VARIABLES (20 OBS. EA.)
C        C(J,K) = TEN CONTROL VARIABLES (20 OBS. EA.)
```

```
C         GG1(2,16) = MEANS OF ANALYTIC CONTROLS USING QNA (G/G/1)
C         MM1(2,16) = MEANS OF ANALYTIC CONTROLS USING QNA (M/M/1)
C         OBSY(K) = NUMBER OF OBSERVATIONS IN SIMULATION MODEL
C                   OF THE SOJOURN TIME TO YIELD 'KTH' OBSERVATION
C                   OF AVERAGE SOJOURN TIME
C         OBSR(J,K) = NUMBER OF OBSERVATIONS IN SIMULATION MODEL
C                     OF THE ROUTE TAKEN TO COMPUTE THE 'KTH' OBSERVATION
C                     OF THE ROUTING CONTROL VARIABLES R13 AND R41
C         OBSW(K) = TOTAL NUMBER OF SERVICE COMPLETIONS FOR THE
C                   'KTH' OBSERVATION OF THE FOUR WORK VARIABLES
C
      REAL Y(2,20), C(10,20), GG1(2,16), MM1(2,16)
      REAL OBSY(20), OBSR(2,20), OBSW(20)
C
C     OUTPUT VARIABLES
C        YBAR(I) = GRAND UNCONTROLLED MEAN OF THE TWO RESPONSES
C        YCBAR(I,J) = GRAND CONTROLLED MEAN OF THE TWO RESPONSES
C                     USING THE 'JTH' CONTROL
C        VR(I,J) = VARIANCE RATIO OF THE 'ITH' RESPONSE USING THE
C                  'JTH' CONTROL
C        LCL(I,J) = 95% LOWER CONFIDENCE LIMIT FOR THE CONTROLLED
C                   MEAN OF THE 'ITH' RESPONSE USING THE 'JTH' CONTROL
C        UCL(I,J) = 95% UPPER CONFIDENCE LIMIT ...
C
      REAL YBAR(2), YCBAR(2,10), VR(2,10), LCL(2,10), UCL(2,10)
C
C     INTERMEDIATE VARIABLES USED IN CALCULATIONS
C        BHAT(I,J) = ESTIMATED OPTIMAL CONTROL COEFFICIENT
C                    FOR THE 'ITH' RESPONSE AND 'JTH' CONTROL
C        CBAR(J) = MEAN OF THE 'JTH' CONTROL VARIABLE
C        COVYC(I,J) = COVARIANCE BETWEEN 'ITH' RESPONSE AND
C                     'JTH' CONTROL
C        D2(J) = D SQUARED STATISTIC OF 'JTH' CONTROL
C        I,J,K = ITERATION VARIABLES/INDEX VARIABLES
C        MU(J) = KNOWN MEAN OF 'JTH' CONTROL
C        POINT = EXPERIMENTAL DESIGN POINT (INDEX VARIABLE)
C        REP = ITERATION VARIABLE (REPLICATION)
C        REPS = TOTAL NUMBER OF REPLICATIONS OF EXPERIMENT, WHERE
C               AN EXPERIMENT IS 20 OBSERVATIONS OF THE SIMULATION
C               MODEL UNDER THE SAME INTIAL CONDITIONS
C        RHOYC(I,J) = PEARSON'S PRODUCT-MOMENT CORRELATION STATISTIC
C                     BETWEEN 'ITH' RESPONSE AND 'JTH' CONTROL
C        S2C(J) = SAMPLE VARIANCE OF 'JTH' CONTROL
C        S2Y(I) = SAMPLE VARIANCE OF 'ITH' RESPONSE
C        S2YC(I,J) = S SQUARED (Y DOT C) STATISTIC OF 'ITH'
C                    RESPONSE AND 'JTH' CONTROL
C        SUMC(J) = SUM OF OBSERVATIONS ON 'JTH' CONTROL
C        SUMC2(J) = SUM OF SQUARED OBSERVATIONS ON 'JTH' CONTROL
C        SUMY(I) = SUM OF OBSERVATIONS ON 'ITH' RESPONSE
C        SUMY2(I) = SUM OF SQUARED OBSERVATIONS ON 'ITH' RESPONSE
C        TOTOBS = TOTAL NUMBER OF OBSERVATIONS
C        TOTOBSR(I) = TOTAL NUMBER OF SIMULATION OBSERVATIONS TO
C                     PRODUCE THE 20 OBSERVATIONS OF THE TWO ROUTING
```

```
C                      CONTROLS
C        TOTOBSW = TOTAL NUMBER OF SIMULATION OBSERVATIONS OF
C                  SERVICE COMPLETIONS USED IN COMPUTING THE 20
C                  OBSERVATIONS ON THE FOUR WORK VARIABLES
C        TOTOBSY = TOTAL NUMBER OF SIMULATION OBSERVATIONS TO PRODUCE
C                  THE 20 OBSERVATIONS OF AVERAGE SOJOURN TIME
C        WC(J,K) = WEIGHT OF 'KTH' OBSERVATION ON 'JTH' CONTROL
C        WY(I,K) = WEIGHT OF 'KTH' OBSERVATION ON 'ITH' RESPONSE
C        ZERO = INITIALIZATION VARIABLE (= 0.0)
C
      REAL BHAT(2,10), CBAR(10), COVYC(2,10), D2(10)
      REAL MU(10), RHOYC(2,10), S2C(10), S2Y(2), S2YC(2,10)
      REAL SUMC(10), SUMC2(10), SUMY(2), SUMY2(2)
      REAL TOTOBSR(2), TOTOBSY, TOTOBSW
      REAL WC(10,20), WY(2,20), ZERO
      INTEGER I, J, K, POINT, REP, REPS, TOTOBS
C
C     INTERACTIVE USER INPUT
C
      WRITE (5,*) 'ENTER TOTAL NUMBER OF OBSERVATIONS'
      READ (5,*) TOTOBS
      REPS = TOTOBS/20
      WRITE (5,*) 'ENTER THE DESIGN POINT NUMBER (1 TO 16)'
      READ (5,*) POINT
C
C     READ ANALYTIC RESULTS
C
      OPEN (UNIT=1,FILE='G16.OUT',STATUS='OLD')
      OPEN (UNIT=2,FILE='M16.OUT',STATUS='OLD')
      DO 10 I=1,16
         READ (1,1) GG1(1,I), GG1(2,I)
         READ (2,1) MM1(1,I), MM1(2,I)
   10 CONTINUE
    1 FORMAT (2(2X,E13.6))
      CLOSE (1)
      CLOSE (2)
C
C     INITIALIZE KNOWN MEANS OF CONTROLS
C
      ZERO = 0.0E+00
      DO 20 I=1,6
         MU(I) = ZERO
   20 CONTINUE
      MU(7) = MM1(1,POINT)
      MU(8) = MM1(2,POINT)
      MU(9) = GG1(1,POINT)
      MU(10) = GG1(2,POINT)
C
C     EXPERIMENTAL DATA FILES FOR INPUT
C
      OPEN (UNIT=1,FILE='RESPONSE.OUT',STATUS='OLD')
      OPEN (UNIT=2,FILE='ROUTING.OUT',STATUS='OLD')
      OPEN (UNIT=3,FILE='WORK.OUT',STATUS='OLD')
```

```
      OPEN (UNIT=4,FILE='JACKSON.OUT',STATUS='OLD')
      OPEN (UNIT=8,FILE='QNA.OUT',STATUS='OLD')
C
C     OUTPUT DATA FILES
C
      OPEN (UNIT=9,FILE='SOJOURN.VR',STATUS='NEW')
      OPEN (UNIT=10,FILE='SOJOURN.CI',STATUS='NEW')
      OPEN (UNIT=11,FILE='QUANTIL?.VR',STATUS='NEW')
      OPEN (UNIT=12,FILE='QUANTILE.CI',STATUS='NEW')
C
      DO 1000 REP=1,REPS
C
C        INITIALIZE SUMMATION VARIABLES
C
         TOTOBSY = ZERO
         TOTOBSW = ZERO
         TOTOBSR(1) = ZERO
         TOTOBSR(2) = ZERO
C
C        READ EXPERIMENTAL DATA
C
         DO 30 J=1,20
            READ (1,2) OBSY(J), Y(1,J), Y(2,J)
            TOTOBSY = TOTOBSY + OBSY(J)
            READ (2,3) OBSR(1,J), C(1,J), OBSR(2,J), C(2,J)
            TOTOBSR(1) = TOTOBSR(1) + OBSR(1,J)
            TOTOBSR(2) = TOTOBSR(2) + OBSR(2,J)
            READ (3,4) OBSW(J), (C(I,J), I=3,6)
            TOTOBSW = TOTOBSW + OBSW(J)
            READ (4,5) C(7,J), C(8,J)
            READ (8,5) C(9,J), C(10,J)
   30    CONTINUE
    2    FORMAT (3(2X,E13.6))
    3    FORMAT (4(2X,E13.6))
    4    FORMAT (5(2X,E13.6))
    5    FORMAT (2(2X,E13.6))
C
C        INITIALIZE VARIABLES FOR SAMPLE MEANS,
C        VARIANCES, AND COVARIANCES
C
         DO 31 I=1,2
            YBAR(I) = ZERO
            SUMY(I) = ZERO
            SUMY2(I) = ZERO
            S2Y(I) = ZERO
   31    CONTINUE
         DO 32 I=1,10
            CBAR(I) = ZERO
            SUMC(I) = ZERO
            SUMC2(I) = ZERO
            S2C(I) = ZERO
            COVYC(1,I) = ZERO
            COVYC(2,I) = ZERO
```

```
   32    CONTINUE
C
C        CALCULATE WEIGHTS OF OBSERVATIONS
C
         DO 35 J=1,20
            WY(1,J) = 20.0*OBSY(J)/TOTOBSY
            WY(2,J) = 1.0
            WC(1,J) = 20.0*OBSR(1,J)/TOTOBSR(1)
            WC(2,J) = 20.0*OBSR(2,J)/TOTOBSR(2)
            DO 33 I=3,6
               WC(I,J) = 20.0*OBSW(J)/TOTOBSW
   33       CONTINUE
            DO 34 I=7,10
               WC(I,J) = 20.0*OBSY(J)/TOTOBSY
   34       CONTINUE
   35    CONTINUE
C
C        CALCULATE WEIGHTED SUMS
C
         DO 42 J=1,20
            DO 40 I=1,2
               SUMY(I) = SUMY(I) + WY(I,J)*Y(I,J)
               SUMY2(I) = SUMY2(I) + WY(I,J)*Y(I,J)**2
   40       CONTINUE
            DO 41 I=1,10
               SUMC(I) = SUMC(I) + WC(I,J)*C(I,J)
               SUMC2(I) = SUMC2(I) + WC(I,J)*C(I,J)**2
   41       CONTINUE
   42    CONTINUE
C
C        CALCULATE WEIGHTED AVERAGES AND WEIGHTED SAMPLE VARIANCES
C
         DO 50 I=1,2
            YBAR(I) = SUMY(I)/20.0
            S2Y(I) = (20.0*SUMY2(I) - SUMY(I)**2)/380.0
   50    CONTINUE
         DO 51 I=1,10
            CBAR(I) = SUMC(I)/20.0
            S2C(I) = (20.0*SUMC2(I) - SUMC(I)**2)/380.0
   51    CONTINUE
C
C        CALCULATE SAMPLE COVARIANCES
C
         DO 62 I=1,2
            DO 61 J=1,20
               DO 60 K=1,10
                  COVYC(I,K) = COVYC(I,K) + (C(K,J) - CBAR(K))*
     &(Y(I,J) - YBAR(I))/19
   60          CONTINUE
   61       CONTINUE
   62    CONTINUE
C
C        ESTIMATE OPTIMAL CONTROL COEFFICIENT AND
```

```
C        PEARSON'S PRODUCT-MOMENT CORRELATION COEFFICIENT
C
         DO 71 J=1,10
            DO 70 I=1,2
               BHAT(I,J) = COVYC(I,J)/S2C(J)
               RHOYC(I,J) = COVYC(I,J)/SQRT(S2C(J)*S2Y(I))
   70       CONTINUE
   71    CONTINUE
C
C        CALCULATE CONTROLLED MEANS OF RESPONSE VARIABLES
C
         DO 73 J=1,10
            DO 72 I=1,2
               YCBAR(I,J) = YBAR(I) + BHAT(I,J)*(CBAR(J)-MU(J))
   72       CONTINUE
   73    CONTINUE
C
C        CALCULATE STATISTICS 'D2' AND 'S2YC' TO ESTIMATE
C        VARIANCE OF CONTROLLED MEANS
C
C        VAR[YCBAR(I,J)] = D2(I,J)*S2YC(I,J)
C
         DO 81 J=1,10
            D2(J) = 0.05+(1/19)*(CBAR(J)-MU(J))**2/S2C(J)
            DO 80 I=1,2
               S2YC(I,J) = (19/18)*S2Y(I)*(1.0 - RHOYC(I,J)**2)
   80       CONTINUE
   81    CONTINUE
C
C        ESTIMATE VARIANCE RATIOS AND 95% CONFIDENCE LIMITS
C
         DO 91 J=1,10
            DO 90 I=1,2
                VR(I,J) = (D2(J)*S2YC(I,J))/(S2Y(I)/20)
               LCL(I,J) = YCBAR(I,J) - 2.101*SQRT(D2(J)*S2YC(I,J))
               UCL(I,J) = YCBAR(I,J) + 2.101*SQRT(D2(J)*S2YC(I,J))
   90       CONTINUE
   91    CONTINUE
C
C        OUTPUT RESULTS
C
         DO 100 J=1,10
            WRITE (9,101) YBAR(1), YCBAR(1,J), VR(1,J)
            WRITE (10,102) LCL(1,J), UCL(1,J)
            WRITE (11,101) YBAR(2), YCBAR(2,J), VR(2,J)
            WRITE (12,102) LCL(2,J), UCL(2,J)
  100    CONTINUE
  101    FORMAT (3(2X,E13.6))
  102    FORMAT (2(2X,E13.6))
 1000 CONTINUE
      END
```

FORTRAN Code for Program RESULTS

```
      PROGRAM RESULTS
C
C     THIS PROGRAM FINDS THE MINIMUM, MAXIMUM AND AVERAGE OF THE THREE
C     VARIABLES (YBAR, YCBAR, AND VR) FROM THE OUTPUT FILES OF THE
C     PROGRAM 'CONTROL'.  IT ALSO COMPUTES THE COVERAGE OF THE 95%
C     CONFIDENCE INTERVAL AGAINST THE UNCONTROLLED RESPONSE (YBAR).
C
C     INPUT VARIABLES
C        Y(1,J) = UNCONTROLLED MEAN OF THE RESPONSE VARIABLE
C        Y(2,J) = CONTROLLED MEANS OF THE RESPONSE VARIABLE
C                 USING 'JTH' CONTROL
C        Y(3,J) = VARIANCE RATIO OBTAINED USING 'JTH' CONTROL
C        LCL(J) = 95% LOWER CONFIDENCE LIMIT ON CONTROLLED MEAN OF
C                 THE RESPONSE VARIABLE USING THE 'JTH' CONTROL
C        UCL(J) = 95% UPPER CONFIDENCE LIMIT ...
C
      REAL Y(3,10), LCL(10), UCL(10)
C
C     OUTPUT VARIABLES
C        MIN(I,J) = MINIMUM VALUE OF 'ITH' INPUT USING 'JTH' CONTROL
C        MAX(I,J) = MAXIMUM VALUE OF 'ITH' INPUT USING 'JTH' CONTROL
C        MEAN(I,J) = MEAN VALUE OF 'ITH' INPUT USING 'JTH' CONTROL
C        COVER(J) = COVERAGE PROBABILITIES OF 95% CONFIDENCE INTERVAL
C                   OF THE CONTROLLED RESPONSE COMPARED TO THE
C                   GRAND UNCONTROLLED MEAN
C
      REAL MIN(3,10), MAX(3,10), MEAN(3,10), COVER(10)
C
C     INTERMEDIATE VARIABLES USED IN CALCULATIONS
C        COUNT(J) = NUMBER OF TIMES THE MEAN RESPONSE FALLS WITHIN
C                   THE CONFIDENCE INTERVALS
C        INFILE = INPUT FILE TYPE
C        I,J = ITERATION/INDEX VARIABLES
C        REP = ITERATION VARIABLE (REPLICATION)
C        REPS = TOTAL NUMBER OF REPLICATIONS OF THE EXPERIMENT
C        L(J) = LABEL OF 'JTH' CONTROL
C        N(J) = LABEL OF 'ITH' STATISTIC
C
      INTEGER COUNT(10), INFILE, I, J, REP, REPS
      CHARACTER*14 L(10)
      CHARACTER*14 N(3)
C
C     INTIALIZE MINIMUM VALUES TO A HIGH NUMBER
C
      DATA (MIN(1,J), J=1,10) /10*1.0E+10/
      DATA (MIN(2,J), J=1,10) /10*1.0E+10/
      DATA (MIN(3,J), J=1,10) /10*1.0E+10/
C
C     INITIALIZE LABELS
C
      L(1) = 'ROUTING(1,3)  '
```

```
      L(2) = 'ROUTING(1,4)   '
      L(3) = 'WORK(1)        '
      L(4) = 'WORK(2)        '
      L(5) = 'WORK(3)        '
      L(6) = 'WORK(4)        '
      L(7) = 'SOJOURN(M/M/1)'
      L(8) = 'WAIT4(M/M/1)  '
      L(9) = 'SOJOURN(G/G/1)'
      L(10) = 'WAIT4(G/G/1)  '
      N(1) = 'YBAR          '
      N(2) = 'YBAR(BHAT)    '
      N(3) = 'VARIANCE RATIO'
C
C     INTERACTIVE USER INPUT
C
      WRITE (5,*) 'ENTER THE NUMBER OF REPLICATIONS'
      READ (5,*) REPS
      WRITE (5,*) 'INDICATE INPUT FILES (1=SOJOURN, 2=QUANTILE)'
      READ (5,*) INFILE
C
C     OPEN APPROPRIATE INPUT DATA FILES (CREATED BY PROGRAM 'CVR')
C
      IF (INFILE.EQ.1) THEN
         OPEN (UNIT=1,FILE='SOJOURN.VR',STATUS='OLD')
         OPEN (UNIT=2,FILE='SOJOURN.CI',STATUS='OLD')
      ELSE
         OPEN (UNIT=1,FILE='QUANTILE.VR',STATUS='OLD')
         OPEN (UNIT=2,FILE='QUANTILE.CI',STATUS='OLD')
      END IF
C
C     OPEN OUTPUT DATA FILE
C
      OPEN (UNIT=3,FILE='RESULTS.DAT',STATUS='NEW')
C
      DO 30 REP=1,REPS
C
C        READ INPUT VARIABLES FROM *.VR FILE
C
         DO 10 J=1,10
            READ (1,1) (Y(I,J), I=1,3)
   10    CONTINUE
C
C        COMPUTE MIN, MAX, AND MEAN OF THE INPUT VARIABLES
C
         DO 21 I=1,3
            DO 20 J=1,10
               IF (Y(I,J).LT.MIN(I,J)) MIN(I,J) = Y(I,J)
               IF (Y(I,J).GT.MAX(I,J)) MAX(I,J) = Y(I,J)
               MEAN(I,J) = MEAN(I,J) + Y(I,J)/REPS
   20       CONTINUE
   21    CONTINUE
   30 CONTINUE
C
```

```
C     READ CONFIDENCE LIMITS FROM *.CI FILE
C
      DO 60 REP=1,REPS
         DO 40 J=1,10
            READ (2,2) LCL(J), UCL(J)
   40    CONTINUE
C
C        CALCULATE COVERAGE ESTIMATES OF CONFIDENCE INTERVALS
C
         DO 50 J=1,10
            IF ((LCL(J).LE.MEAN(1,J)).AND.(MEAN(1,J).LE.UCL(J))) THEN
               COUNT(J) = COUNT(J) + 1
            END IF
   50    CONTINUE
   60 CONTINUE
C
      DO 70 J=1,10
         COVER(J) = 1.0*COUNT(J)/REPS
   70 CONTINUE
C
C     OUTPUT RESULTS
C
      IF (INFILE.EQ.1) THEN
         WRITE (3,3)
      ELSE
         WRITE (3,4)
      END IF
      WRITE (3,8)
      WRITE (3,6)
      WRITE (3,7)
      WRITE (3,8)
      DO 80 J=1,10
         WRITE (3,5) L(J)
         WRITE (3,9) N(1), MIN(1,J), MEAN(1,J), MAX(1,J), COVER(J)
         WRITE (3,11) N(2), MIN(2,J), MEAN(2,J), MAX(2,J)
         WRITE (3,11) N(3), MIN(3,J), MEAN(3,J), MAX(3,J)
         WRITE (3,8)
   80 CONTINUE
      STOP
    1 FORMAT (3(2X,E13.6))
    2 FORMAT (2(2X,E13.6))
    3 FORMAT (15X,'CONTROL VARIATE ANALYSIS ON RESPONSE:  SOJOURN TIME')
    4 FORMAT (10X,'CONTROL VARIATE ANALYSIS ON RESPONSE:  ',
     &        'PROB(NNQ(4) > 2*E[N])')
    5 FORMAT (1X,'USING CONTROL VARIABLE:  ',A14)
    6 FORMAT (3X,'     NAME            MINIMUM         AVERAGE     ',
     &        '   MAXIMUM     COVERAGE')
    7 FORMAT (3X,'--------------  -------------  -------------  ',
     &        '-------------  --------')
    8 FORMAT (1X)
    9 FORMAT (3X,A14,3(2X,E13.6),3X,F6.4)
   11 FORMAT (3X,A14,3(2X,E13.6))
      END
```

Appendix B: Sample Input Files to QNA

M/M/1 Input File at First Design Point

```
1                                (indicates 1 network)
3                                (indicates single customer class)
4  1                             (4 nodes, nonstandard input)
1  0  0 -1  0                    (input options)
0.0  0.6  0.4  0.0               (1st row of routing matrix)
0.0  0.0  0.0  1.0               (2nd row of routing matrix)
0.0  0.0  0.0  1.0               (3rd row of routing matrix)
0.1  0.0  0.0  0.0               (4th row of routing matrix)
1.0  0.0  0.0  0.0               (external arrival rates)
0.450  0.750  1.125  0.450       (mean service times)
```

G/G/1 Input File at First Design Point

```
1                                (indicates 1 network)
3                                (indicates single customer class)
4  1                             (4 nodes, nonstandard input)
1  2  0 -1  0                    (input options)
0.0  0.6  0.4  0.0               (1st row of routing matrix)
0.0  0.0  0.0  1.0               (2nd row of routing matrix)
0.0  0.0  0.0  1.0               (3rd row of routing matrix)
0.1  0.0  0.0  0.0               (4th row of routing matrix)
1.0  0.0  0.0  0.0               (external arrival rates)
0.450  0.750  1.125  0.450       (mean service times)
1.000  1.000  1.000  0.250       (service variability parameters)
```

Appendix C: Known Means of External Control Variates

DESIGN POINT	--ANALYTIC JACKSON (M/M/1)- SOJOURN	WAIT(4)	-ANALYTIC APPROX. (G/G/1)-- SOJOURN	WAIT(4)
1	0.400000E+01	0.450000E+00	0.381177E+01	0.281197E+00
2	0.360000E+02	0.729000E+01	0.329330E+02	0.455585E+01
3	0.400000E+01	0.450000E+00	0.531762E+01	0.163162E+01
4	0.360000E+02	0.729000E+01	0.574688E+02	0.264290E+02
5	0.400000E+01	0.450000E+00	0.381173E+01	0.281160E+00
6	0.360000E+02	0.729000E+01	0.329329E+02	0.455572E+01
7	0.400000E+01	0.450000E+00	0.531792E+01	0.163188E+01
8	0.360000E+02	0.729000E+01	0.574698E+02	0.264300E+02
9	0.417391E+01	0.375000E+00	0.398269E+01	0.234173E+00
10	0.360000E+02	0.607500E+01	0.327806E+02	0.379484E+01
11	0.417391E+01	0.375000E+00	0.551237E+01	0.136079E+01
12	0.360000E+02	0.607500E+01	0.585327E+02	0.220362E+02
13	0.400000E+01	0.375000E+00	0.380878E+01	0.234015E+00
14	0.360000E+02	0.607500E+01	0.327797E+02	0.379416E+01
15	0.400000E+01	0.375000E+00	0.533846E+01	0.136189E+01
16	0.360000E+02	0.607500E+01	0.585391E+02	0.220409E+02

Note: SOJOURN refers to the expected sojourn time of an entity to complete the entire network. WAIT(4) refers to the expected waiting time in the queue at the fourth node in the network.

Appendix D: Tables of Results

The output files of the program RESULTS are provided as tables in this appendix. Tables D.1 through D.16 summarize the results of the ten control variates against the response variable sojourn time at the sixteen design points. Similarly, Tables D.17 through D.32 summarize the results of the ten control variates against the response variable of the probability that the number in queue exceeds twice the expected number. These files list the minimum, maximum, and mean values of the uncontrolled response, denoted by YBAR, first. Then for each control variate the minimum, maximum, and mean values of the controlled response, denoted by YBAR(BHAT), and the variance ratio are listed. Also, the percentage of confidence intervals about the controlled response that cover the mean value of YBAR are given for each control variate.

Table D.1. Control Variate Results Against Sojourn Time at the First Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.369443E+01	0.381331E+01	0.396763E+01	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.369317E+01	0.381367E+01	0.399705E+01	0.9300
VARIANCE RATIO	0.648974E+00	0.936955E+00	0.999987E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.366235E+01	0.380847E+01	0.399742E+01	0.8850
VARIANCE RATIO	0.424270E+00	0.867965E+00	0.999990E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.367848E+01	0.381138E+01	0.393465E+01	0.9200
VARIANCE RATIO	0.621534E+00	0.954747E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.361416E+01	0.381202E+01	0.396768E+01	0.9250
VARIANCE RATIO	0.491398E+00	0.948515E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.367779E+01	0.381282E+01	0.396515E+01	0.9350
VARIANCE RATIO	0.631606E+00	0.949113E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.367833E+01	0.381140E+01	0.393472E+01	0.9200
VARIANCE RATIO	0.621459E+00	0.954740E+00	0.999999E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.360603E+01	0.383068E+01	0.412233E+01	0.5500
VARIANCE RATIO	0.125763E+00	0.418483E+00	0.817554E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.365523E+01	0.381422E+01	0.402782E+01	0.8300
VARIANCE RATIO	0.317536E+00	0.768369E+00	0.999978E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.361327E+01	0.382999E+01	0.412578E+01	0.5450
VARIANCE RATIO	0.859748E-01	0.384818E+00	0.774845E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.365208E+01	0.381421E+01	0.401822E+01	0.8100
VARIANCE RATIO	0.288632E+00	0.751309E+00	0.999933E+00	

Table D.2. Control Variate Results Against Sojourn Time at the Second Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.253301E+02	0.292945E+02	0.340209E+02	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.253703E+02	0.293262E+02	0.343614E+02	0.9350
VARIANCE RATIO	0.629330E+00	0.941911E+00	0.999992E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.252341E+02	0.292181E+02	0.344292E+02	0.9550
VARIANCE RATIO	0.611720E+00	0.906871E+00	0.999988E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.250874E+02	0.292620E+02	0.351617E+02	0.9400
VARIANCE RATIO	0.470587E+00	0.943858E+00	0.999997E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.240332E+02	0.293179E+02	0.371701E+02	0.9350
VARIANCE RATIO	0.673789E+00	0.946194E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.253199E+02	0.293383E+02	0.339248E+02	0.9300
VARIANCE RATIO	0.608466E+00	0.950724E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.250668E+02	0.292583E+02	0.350784E+02	0.9400
VARIANCE RATIO	0.472088E+00	0.944131E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.260485E+02	0.307690E+02	0.381936E+02	0.8000
VARIANCE RATIO	0.152546E+00	0.697212E+00	0.100000E+01	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.247132E+02	0.305386E+02	0.365465E+02	0.6950
VARIANCE RATIO	0.157891E+00	0.644684E+00	0.999635E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.260450E+02	0.307653E+02	0.383204E+02	0.8000
VARIANCE RATIO	0.150624E+00	0.698383E+00	0.100000E+01	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.247680E+02	0.305323E+02	0.365467E+02	0.7000
VARIANCE RATIO	0.159307E+00	0.641734E+00	0.995649E+00	

Table D.3. Control Variate Results Against Sojourn Time at the Third Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.477141E+01	0.521644E+01	0.585939E+01	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.473141E+01	0.521239E+01	0.580107E+01	0.9450
VARIANCE RATIO	0.590194E+00	0.940776E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.478012E+01	0.521678E+01	0.590412E+01	0.9450
VARIANCE RATIO	0.537425E+00	0.925474E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.483237E+01	0.521699E+01	0.582824E+01	0.9500
VARIANCE RATIO	0.416803E+00	0.955932E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.477196E+01	0.521219E+01	0.617023E+01	0.9500
VARIANCE RATIO	0.446203E+00	0.950714E+00	0.999980E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.475023E+01	0.521933E+01	0.599340E+01	0.9450
VARIANCE RATIO	0.599295E+00	0.953930E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.483239E+01	0.521705E+01	0.582838E+01	0.9500
VARIANCE RATIO	0.416409E+00	0.955908E+00	0.999997E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.463239E+01	0.528310E+01	0.655387E+01	0.5700
VARIANCE RATIO	0.125285E+00	0.468408E+00	0.890346E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.455333E+01	0.527696E+01	0.645269E+01	0.5100
VARIANCE RATIO	0.100617E+00	0.386469E+00	0.791341E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.451417E+01	0.527574E+01	0.645615E+01	0.3650
VARIANCE RATIO	0.620310E-01	0.250908E+00	0.667591E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.439451E+01	0.526400E+01	0.635990E+01	0.4300
VARIANCE RATIO	0.699815E-01	0.316967E+00	0.703822E+00	

Table D.4. Control Variate Results Against Sojourn Time at the Fourth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.371266E+02	0.457255E+02	0.555158E+02	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.378508E+02	0.457554E+02	0.548320E+02	0.9300
VARIANCE RATIO	0.486904E+00	0.942278E+00	0.999998E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.373728E+02	0.457098E+02	0.554014E+02	0.9350
VARIANCE RATIO	0.531825E+00	0.932978E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.375918E+02	0.458027E+02	0.623515E+02	0.9100
VARIANCE RATIO	0.652300E+00	0.947219E+00	0.999990E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.344602E+02	0.456777E+02	0.573404E+02	0.9000
VARIANCE RATIO	0.567247E+00	0.941918E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.349138E+02	0.457327E+02	0.552863E+02	0.9150
VARIANCE RATIO	0.574240E+00	0.943468E+00	0.999986E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.376215E+02	0.457768E+02	0.618071E+02	0.9100
VARIANCE RATIO	0.651065E+00	0.947981E+00	0.999998E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.376422E+02	0.498143E+02	0.691257E+02	0.6150
VARIANCE RATIO	0.160514E+00	0.662006E+00	0.999999E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.380585E+02	0.492114E+02	0.671139E+02	0.6400
VARIANCE RATIO	0.101392E+00	0.615236E+00	0.999963E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.376549E+02	0.498277E+02	0.694260E+02	0.5800
VARIANCE RATIO	0.731372E-01	0.600644E+00	0.999975E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.382182E+02	0.492173E+02	0.668756E+02	0.6250
VARIANCE RATIO	0.971600E-01	0.610729E+00	0.999952E+00	

Table D.5. Control Variate Results Against Sojourn Time at the Fifth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.368183E+01	0.381449E+01	0.400045E+01	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.363501E+01	0.381666E+01	0.401037E+01	0.9100
VARIANCE RATIO	0.407602E+00	0.869664E+00	0.999999E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.358449E+01	0.381251E+01	0.406761E+01	0.9250
VARIANCE RATIO	0.388817E+00	0.885916E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.363304E+01	0.381513E+01	0.401350E+01	0.9350
VARIANCE RATIO	0.732037E+00	0.957379E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.367582E+01	0.381078E+01	0.402608E+01	0.9600
VARIANCE RATIO	0.769947E+00	0.957897E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.358585E+01	0.381085E+01	0.398226E+01	0.9550
VARIANCE RATIO	0.707264E+00	0.954170E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.363303E+01	0.381514E+01	0.401370E+01	0.9350
VARIANCE RATIO	0.731882E+00	0.957361E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.361066E+01	0.384197E+01	0.414274E+01	0.4950
VARIANCE RATIO	0.139045E+00	0.377817E+00	0.881662E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.364710E+01	0.382226E+01	0.408382E+01	0.8150
VARIANCE RATIO	0.335014E+00	0.797228E+00	0.999920E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.361933E+01	0.384037E+01	0.412813E+01	0.4750
VARIANCE RATIO	0.921910E-01	0.354609E+00	0.767348E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.364605E+01	0.382218E+01	0.407732E+01	0.8200
VARIANCE RATIO	0.347844E+00	0.783402E+00	0.999992E+00	

Table D.6. Control Variate Results Against Sojourn Time at the Sixth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.248461E+02	0.287763E+02	0.332896E+02	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.249434E+02	0.287970E+02	0.341161E+02	0.9200
VARIANCE RATIO	0.650989E+00	0.941351E+00	0.999999E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.235258E+02	0.287802E+02	0.348140E+02	0.9250
VARIANCE RATIO	0.541487E+00	0.917009E+00	0.999993E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.251509E+02	0.288086E+02	0.346363E+02	0.9450
VARIANCE RATIO	0.663926E+00	0.959623E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.245202E+02	0.287082E+02	0.334179E+02	0.9400
VARIANCE RATIO	0.437341E+00	0.947478E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.243634E+02	0.288033E+02	0.370763E+02	0.9350
VARIANCE RATIO	0.508923E+00	0.951356E+00	0.999982E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.251519E+02	0.288058E+02	0.346073E+02	0.9450
VARIANCE RATIO	0.665197E+00	0.959802E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.257135E+02	0.302904E+02	0.382519E+02	0.8100
VARIANCE RATIO	0.236069E+00	0.771186E+00	0.999962E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.241961E+02	0.300718E+02	0.380983E+02	0.7450
VARIANCE RATIO	0.131538E+00	0.649533E+00	0.999407E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.257132E+02	0.302892E+02	0.383264E+02	0.8100
VARIANCE RATIO	0.245796E+00	0.771530E+00	0.999961E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.243035E+02	0.300624E+02	0.380028E+02	0.7200
VARIANCE RATIO	0.125844E+00	0.644162E+00	0.999675E+00	

Table D.7. Control Variate Results Against Sojourn Time at the Seventh Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.473118E+01	0.521470E+01	0.582294E+01	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.475897E+01	0.521184E+01	0.592922E+01	0.9200
VARIANCE RATIO	0.572002E+00	0.927714E+00	0.999996E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.469524E+01	0.521139E+01	0.585067E+01	0.8950
VARIANCE RATIO	0.608010E+00	0.914174E+00	0.999992E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.469958E+01	0.521112E+01	0.604592E+01	0.9300
VARIANCE RATIO	0.551904E+00	0.950072E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.473291E+01	0.521587E+01	0.576332E+01	0.9250
VARIANCE RATIO	0.474959E+00	0.950610E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.476462E+01	0.520558E+01	0.563329E+01	0.9050
VARIANCE RATIO	0.634175E+00	0.948817E+00	0.999996E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.469965E+01	0.521116E+01	0.604597E+01	0.9250
VARIANCE RATIO	0.551983E+00	0.950057E+00	0.999999E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.464045E+01	0.527604E+01	0.620366E+01	0.6100
VARIANCE RATIO	0.968237E-01	0.507894E+00	0.951437E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.456468E+01	0.526524E+01	0.639802E+01	0.5450
VARIANCE RATIO	0.111213E+00	0.422013E+00	0.837698E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.450120E+01	0.526341E+01	0.640132E+01	0.4200
VARIANCE RATIO	0.863734E-01	0.279231E+00	0.658191E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.450152E+01	0.524690E+01	0.646256E+01	0.5100
VARIANCE RATIO	0.934624E-01	0.369244E+00	0.861650E+00	

Table D.8. Control Variate Results Against Sojourn Time at the Eight Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.368068E+02	0.451113E+02	0.556231E+02	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.367612E+02	0.451561E+02	0.568654E+02	0.9450
VARIANCE RATIO	0.557175E+00	0.939957E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.356995E+02	0.451467E+02	0.553535E+02	0.9550
VARIANCE RATIO	0.640790E+00	0.945581E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.370118E+02	0.451506E+02	0.563468E+02	0.9450
VARIANCE RATIO	0.615958E+00	0.951195E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.369133E+02	0.452114E+02	0.562391E+02	0.9300
VARIANCE RATIO	0.492479E+00	0.947571E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.369315E+02	0.452633E+02	0.572890E+02	0.9500
VARIANCE RATIO	0.552045E+00	0.939136E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.370132E+02	0.451325E+02	0.562732E+02	0.9450
VARIANCE RATIO	0.613340E+00	0.952044E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.363551E+02	0.495517E+02	0.659250E+02	0.6550
VARIANCE RATIO	0.167355E+00	0.703501E+00	0.998401E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.371424E+02	0.483798E+02	0.614534E+02	0.6900
VARIANCE RATIO	0.105631E+00	0.620355E+00	0.999962E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.363782E+02	0.494878E+02	0.635396E+02	0.5900
VARIANCE RATIO	0.100811E+00	0.608829E+00	0.999998E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.372278E+02	0.483759E+02	0.623317E+02	0.6950
VARIANCE RATIO	0.104318E+00	0.617487E+00	0.999963E+00	

Table D.9. Control Variate Results Against Sojourn Time at the Ninth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.380592E+01	0.398874E+01	0.415140E+01	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.382653E+01	0.398936E+01	0.415606E+01	0.9150
VARIANCE RATIO	0.650857E+00	0.949540E+00	0.999997E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.375724E+01	0.399063E+01	0.420255E+01	0.7650
VARIANCE RATIO	0.376625E+00	0.761644E+00	0.998550E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.380441E+01	0.398905E+01	0.417934E+01	0.8950
VARIANCE RATIO	0.513316E+00	0.938565E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.379973E+01	0.398678E+01	0.416616E+01	0.9300
VARIANCE RATIO	0.587106E+00	0.948372E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.375108E+01	0.398441E+01	0.418557E+01	0.9050
VARIANCE RATIO	0.634476E+00	0.943080E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.380442E+01	0.398906E+01	0.417932E+01	0.8950
VARIANCE RATIO	0.513602E+00	0.938575E+00	0.999999E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.371078E+01	0.401222E+01	0.427863E+01	0.4650
VARIANCE RATIO	0.113236E+00	0.355449E+00	0.821358E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.370819E+01	0.399823E+01	0.424922E+01	0.6950
VARIANCE RATIO	0.205460E+00	0.662065E+00	0.999135E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.371430E+01	0.401247E+01	0.428437E+01	0.4400
VARIANCE RATIO	0.114329E+00	0.334087E+00	0.783583E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.372064E+01	0.399890E+01	0.426272E+01	0.7100
VARIANCE RATIO	0.238847E+00	0.643083E+00	0.997044E+00	

Table D.10. Control Variate Results Against Sojourn Time at the Tenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.246467E+02	0.299427E+02	0.353747E+02	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.247080E+02	0.300389E+02	0.361594E+02	0.9100
VARIANCE RATIO	0.639244E+00	0.951075E+00	0.999996E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.241610E+02	0.299488E+02	0.357995E+02	0.8950
VARIANCE RATIO	0.293134E+00	0.859462E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.245217E+02	0.299633E+02	0.363288E+02	0.8900
VARIANCE RATIO	0.609402E+00	0.943036E+00	0.999994E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.236332E+02	0.299989E+02	0.357056E+02	0.9250
VARIANCE RATIO	0.490976E+00	0.951623E+00	0.999996E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.247234E+02	0.299672E+02	0.363240E+02	0.8950
VARIANCE RATIO	0.635769E+00	0.948150E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.245203E+02	0.299608E+02	0.363192E+02	0.8900
VARIANCE RATIO	0.609347E+00	0.943254E+00	0.999999E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.256705E+02	0.314889E+02	0.411867E+02	0.7850
VARIANCE RATIO	0.273576E+00	0.708201E+00	0.100000E+01	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.241567E+02	0.314347E+02	0.384832E+02	0.6850
VARIANCE RATIO	0.206467E+00	0.604245E+00	0.999674E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.256697E+02	0.314904E+02	0.413126E+02	0.7800
VARIANCE RATIO	0.277785E+00	0.709799E+00	0.999999E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.241370E+02	0.314233E+02	0.384162E+02	0.6600
VARIANCE RATIO	0.200296E+00	0.602308E+00	0.999712E+00	

Table D.11. Control Variate Results Against Sojourn Time at the Eleventh Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.490559E+01	0.530441E+01	0.566416E+01	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.490509E+01	0.530760E+01	0.566931E+01	0.9350
VARIANCE RATIO	0.572129E+00	0.946071E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.491179E+01	0.531109E+01	0.575136E+01	0.9050
VARIANCE RATIO	0.421668E+00	0.873746E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.490478E+01	0.530375E+01	0.572119E+01	0.9300
VARIANCE RATIO	0.558922E+00	0.946952E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.440455E+01	0.529597E+01	0.582324E+01	0.9350
VARIANCE RATIO	0.645532E+00	0.949085E+00	0.999978E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.490411E+01	0.530130E+01	0.623082E+01	0.9350
VARIANCE RATIO	0.516269E+00	0.944746E+00	0.999989E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.490473E+01	0.530378E+01	0.572085E+01	0.9300
VARIANCE RATIO	0.557668E+00	0.946925E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.478901E+01	0.536703E+01	0.608927E+01	0.5800
VARIANCE RATIO	0.136079E+00	0.450433E+00	0.902546E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.466496E+01	0.535122E+01	0.598138E+01	0.5800
VARIANCE RATIO	0.936413E-01	0.415536E+00	0.979822E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.461028E+01	0.535548E+01	0.608413E+01	0.4650
VARIANCE RATIO	0.653325E-01	0.266464E+00	0.702594E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.462090E+01	0.533501E+01	0.596814E+01	0.5250
VARIANCE RATIO	0.131066E+00	0.386874E+00	0.930679E+00	

Table D.12. Control Variate Results Against Sojourn Time at the Twelfth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.385956E+02	0.470301E+02	0.590437E+02	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.385054E+02	0.472231E+02	0.595735E+02	0.9150
VARIANCE RATIO	0.678291E+00	0.949828E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.375837E+02	0.468042E+02	0.589178E+02	0.9050
VARIANCE RATIO	0.546805E+00	0.917021E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.387353E+02	0.471866E+02	0.688168E+02	0.9250
VARIANCE RATIO	0.600185E+00	0.949178E+00	0.999992E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.385828E+02	0.470423E+02	0.596938E+02	0.9350
VARIANCE RATIO	0.615101E+00	0.953366E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.389498E+02	0.471159E+02	0.588525E+02	0.9200
VARIANCE RATIO	0.386973E+00	0.945840E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.387146E+02	0.471739E+02	0.679613E+02	0.9250
VARIANCE RATIO	0.595894E+00	0.949470E+00	0.999991E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.393682E+02	0.505069E+02	0.698632E+02	0.6800
VARIANCE RATIO	0.135096E+00	0.689193E+00	0.999557E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.386547E+02	0.497819E+02	0.698244E+02	0.7250
VARIANCE RATIO	0.135505E+00	0.671610E+00	0.998631E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.393751E+02	0.502915E+02	0.699491E+02	0.6850
VARIANCE RATIO	0.129665E+00	0.670991E+00	0.999958E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.390337E+02	0.497711E+02	0.699372E+02	0.7200
VARIANCE RATIO	0.143626E+00	0.672896E+00	0.998564E+00	

Table D.13. Control Variate Results Against Sojourn Time at the Thirteenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.362597E+01	0.381095E+01	0.398981E+01	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.362032E+01	0.380911E+01	0.400185E+01	0.8800
VARIANCE RATIO	0.335364E+00	0.877644E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.359593E+01	0.381070E+01	0.406907E+01	0.8000
VARIANCE RATIO	0.212558E+00	0.776582E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.362653E+01	0.381054E+01	0.398310E+01	0.8950
VARIANCE RATIO	0.487505E+00	0.945541E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.362480E+01	0.381052E+01	0.398868E+01	0.9100
VARIANCE RATIO	0.385303E+00	0.949118E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.361801E+01	0.380606E+01	0.399455E+01	0.9150
VARIANCE RATIO	0.476760E+00	0.942032E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.362652E+01	0.381054E+01	0.398302E+01	0.8950
VARIANCE RATIO	0.487568E+00	0.945545E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.354936E+01	0.383049E+01	0.414142E+01	0.4300
VARIANCE RATIO	0.611835E-01	0.331030E+00	0.881636E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.358877E+01	0.381797E+01	0.403301E+01	0.7650
VARIANCE RATIO	0.179257E+00	0.712198E+00	0.999670E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.354646E+01	0.382954E+01	0.413612E+01	0.4350
VARIANCE RATIO	0.643630E-01	0.315746E+00	0.865978E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.356530E+01	0.381773E+01	0.404205E+01	0.7400
VARIANCE RATIO	0.191999E+00	0.699534E+00	0.999399E+00	

Table D.14. Control Variate Results Against Sojourn Time at the Fourteenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.250717E+02	0.292389E+02	0.348594E+02	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.244769E+02	0.292682E+02	0.348573E+02	0.9450
VARIANCE RATIO	0.417618E+00	0.928636E+00	0.999999E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.245819E+02	0.293296E+02	0.367279E+02	0.9000
VARIANCE RATIO	0.443243E+00	0.858234E+00	0.999979E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.252217E+02	0.292557E+02	0.361015E+02	0.9450
VARIANCE RATIO	0.506835E+00	0.952300E+00	0.999996E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.231941E+02	0.292391E+02	0.376504E+02	0.9300
VARIANCE RATIO	0.702847E+00	0.945412E+00	0.999991E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.257546E+02	0.293223E+02	0.348498E+02	0.9350
VARIANCE RATIO	0.562071E+00	0.948348E+00	0.999996E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.252241E+02	0.292533E+02	0.360774E+02	0.9450
VARIANCE RATIO	0.510453E+00	0.952444E+00	0.999997E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.246920E+02	0.307517E+02	0.384403E+02	0.8150
VARIANCE RATIO	0.173840E+00	0.748431E+00	0.999981E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.224575E+02	0.306417E+02	0.385982E+02	0.7450
VARIANCE RATIO	0.205944E+00	0.632572E+00	0.995176E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.244717E+02	0.307440E+02	0.383845E+02	0.8150
VARIANCE RATIO	0.156665E+00	0.750094E+00	0.999974E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.219894E+02	0.306433E+02	0.385125E+02	0.7450
VARIANCE RATIO	0.195685E+00	0.626817E+00	0.993579E+00	

Table D.15. Control Variate Results Against Sojourn Time at the Fifteenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.470558E+01	0.512254E+01	0.545352E+01	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.470723E+01	0.512507E+01	0.545784E+01	0.9300
VARIANCE RATIO	0.539615E+00	0.935454E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.464156E+01	0.512207E+01	0.551342E+01	0.9250
VARIANCE RATIO	0.435000E+00	0.899192E+00	0.999996E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.460887E+01	0.511759E+01	0.548528E+01	0.9150
VARIANCE RATIO	0.491643E+00	0.939493E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.445197E+01	0.511944E+01	0.548606E+01	0.9250
VARIANCE RATIO	0.695184E+00	0.957703E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.463500E+01	0.512162E+01	0.546696E+01	0.9250
VARIANCE RATIO	0.615519E+00	0.947813E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.460917E+01	0.511760E+01	0.548530E+01	0.9150
VARIANCE RATIO	0.491485E+00	0.939497E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.446228E+01	0.517711E+01	0.578296E+01	0.5650
VARIANCE RATIO	0.123806E+00	0.448358E+00	0.937828E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.455079E+01	0.516825E+01	0.569372E+01	0.5150
VARIANCE RATIO	0.123214E+00	0.404259E+00	0.765727E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.442982E+01	0.517591E+01	0.579209E+01	0.3850
VARIANCE RATIO	0.328271E-01	0.253222E+00	0.724730E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.450089E+01	0.516114E+01	0.572054E+01	0.4950
VARIANCE RATIO	0.938772E-01	0.379051E+00	0.778870E+00	

Table D.16. Control Variate Results Against Sojourn Time at the Sixteenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.368060E+02	0.456687E+02	0.572155E+02	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.367545E+02	0.455973E+02	0.581674E+02	0.9150
VARIANCE RATIO	0.569230E+00	0.941763E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.361866E+02	0.455062E+02	0.576799E+02	0.9050
VARIANCE RATIO	0.459856E+00	0.928508E+00	0.999994E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.371077E+02	0.457249E+02	0.600144E+02	0.8950
VARIANCE RATIO	0.608909E+00	0.944781E+00	0.999996E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.370771E+02	0.457057E+02	0.576533E+02	0.9150
VARIANCE RATIO	0.646092E+00	0.944484E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.370829E+02	0.456703E+02	0.576092E+02	0.9050
VARIANCE RATIO	0.525826E+00	0.951012E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.371067E+02	0.457137E+02	0.598441E+02	0.8900
VARIANCE RATIO	0.608657E+00	0.945294E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.374931E+02	0.491399E+02	0.681191E+02	0.6550
VARIANCE RATIO	0.214510E+00	0.725930E+00	0.999967E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.322618E+02	0.484143E+02	0.565718E+02	0.6500
VARIANCE RATIO	0.200033E-03	0.623759E+00	0.999869E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.374865E+02	0.491428E+02	0.688984E+02	0.6300
VARIANCE RATIO	0.700837E-02	0.655286E+00	0.999997E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.318773E+02	0.483783E+02	0.663283E+02	0.6650
VARIANCE RATIO	0.200033E-03	0.624382E+00	0.999870E+00	

Table D.17. Control Variate Results Against Fourth Node Quantile at the First Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.108982E+00	0.121346E+00	0.133839E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.107888E+00	0.121325E+00	0.133859E+00	0.9050
VARIANCE RATIO	0.584686E+00	0.939386E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.106945E+00	0.121218E+00	0.133643E+00	0.8900
VARIANCE RATIO	0.332811E+00	0.909225E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.108926E+00	0.121482E+00	0.134308E+00	0.9000
VARIANCE RATIO	0.697780E+00	0.950873E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.108073E+00	0.121293E+00	0.133982E+00	0.9100
VARIANCE RATIO	0.621506E+00	0.946486E+00	0.999983E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.106275E+00	0.121373E+00	0.134094E+00	0.8950
VARIANCE RATIO	0.565127E+00	0.951841E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.108926E+00	0.121483E+00	0.134307E+00	0.9000
VARIANCE RATIO	0.698097E+00	0.950868E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.106004E+00	0.122063E+00	0.137733E+00	0.7500
VARIANCE RATIO	0.282425E+00	0.704738E+00	0.999972E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.102571E+00	0.121814E+00	0.141884E+00	0.5700
VARIANCE RATIO	0.159547E+00	0.443010E+00	0.837345E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.106520E+00	0.121954E+00	0.138474E+00	0.7700
VARIANCE RATIO	0.283256E+00	0.720013E+00	0.999996E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.102915E+00	0.121783E+00	0.143014E+00	0.5600
VARIANCE RATIO	0.140546E+00	0.421196E+00	0.832287E+00	

Table D.18. Control Variate Results Against Fourth Node Quantile at the Second Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.800609E-01	0.122987E+00	0.183393E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.796589E-01	0.122874E+00	0.188708E+00	0.9400
VARIANCE RATIO	0.665956E+00	0.944736E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.690142E-01	0.122216E+00	0.187343E+00	0.9300
VARIANCE RATIO	0.505389E+00	0.918495E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.713555E-01	0.123318E+00	0.183295E+00	0.9300
VARIANCE RATIO	0.548735E+00	0.946070E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.708987E-01	0.122474E+00	0.188564E+00	0.9200
VARIANCE RATIO	0.520072E+00	0.944244E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.810515E-01	0.123990E+00	0.187336E+00	0.9350
VARIANCE RATIO	0.711601E+00	0.947854E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.715930E-01	0.123326E+00	0.183302E+00	0.9300
VARIANCE RATIO	0.550165E+00	0.945971E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.767694E-01	0.127122E+00	0.197354E+00	0.9150
VARIANCE RATIO	0.381408E+00	0.918162E+00	0.999999E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.803986E-01	0.134137E+00	0.213651E+00	0.7650
VARIANCE RATIO	0.247330E+00	0.723061E+00	0.999982E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.766782E-01	0.126980E+00	0.196664E+00	0.9150
VARIANCE RATIO	0.381622E+00	0.919540E+00	0.999998E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.800893E-01	0.134095E+00	0.215977E+00	0.7600
VARIANCE RATIO	0.251796E+00	0.721281E+00	0.999996E+00	

Table D.19. Control Variate Results Against Fourth Node Quantile at the Third Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.117299E+00	0.141142E+00	0.165164E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.114428E+00	0.140852E+00	0.167499E+00	0.9400
VARIANCE RATIO	0.647251E+00	0.946466E+00	0.999999E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.117293E+00	0.141151E+00	0.168151E+00	0.9300
VARIANCE RATIO	0.641702E+00	0.942667E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.119165E+00	0.141217E+00	0.167973E+00	0.9550
VARIANCE RATIO	0.464607E+00	0.951705E+00	0.999995E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.116956E+00	0.140959E+00	0.169834E+00	0.9400
VARIANCE RATIO	0.532538E+00	0.951487E+00	0.999959E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.115201E+00	0.141266E+00	0.167604E+00	0.9300
VARIANCE RATIO	0.555494E+00	0.950901E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.119136E+00	0.141222E+00	0.167956E+00	0.9550
VARIANCE RATIO	0.464179E+00	0.951710E+00	0.999992E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.109585E+00	0.144851E+00	0.193024E+00	0.6600
VARIANCE RATIO	0.152195E+00	0.579707E+00	0.992372E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.939711E-01	0.145167E+00	0.193472E+00	0.4350
VARIANCE RATIO	0.809851E-01	0.268138E+00	0.677336E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.933476E-01	0.144652E+00	0.191033E+00	0.3650
VARIANCE RATIO	0.796989E-01	0.262950E+00	0.661056E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.835182E-01	0.144021E+00	0.187470E+00	0.3750
VARIANCE RATIO	0.822466E-01	0.232744E+00	0.625153E+00	

Table D.20. Control Variate Results Against Fourth Node Quantile at the Fourth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.738233E-02	0.613441E-01	0.141887E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.758367E-02	0.614113E-01	0.136472E+00	0.8500
VARIANCE RATIO	0.637302E+00	0.951017E+00	0.999995E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.732101E-02	0.607389E-01	0.147175E+00	0.8650
VARIANCE RATIO	0.623753E+00	0.956300E+00	0.999985E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.486476E-02	0.615487E-01	0.163884E+00	0.8550
VARIANCE RATIO	0.339997E+00	0.944999E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	-0.128170E-01	0.613136E-01	0.144628E+00	0.8400
VARIANCE RATIO	0.485296E+00	0.946526E+00	0.999993E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.530243E-02	0.617015E-01	0.148124E+00	0.8450
VARIANCE RATIO	0.617724E+00	0.940743E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.486544E-02	0.614900E-01	0.163384E+00	0.8550
VARIANCE RATIO	0.339796E+00	0.945119E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.826074E-02	0.754763E-01	0.201013E+00	0.7350
VARIANCE RATIO	0.421055E-01	0.837181E+00	0.999949E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.747168E-02	0.771357E-01	0.201252E+00	0.6850
VARIANCE RATIO	0.162178E-01	0.751506E+00	0.999986E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.829340E-02	0.777838E-01	0.200373E+00	0.6850
VARIANCE RATIO	0.356739E-01	0.774744E+00	0.999996E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.769022E-02	0.771427E-01	0.197136E+00	0.6750
VARIANCE RATIO	0.162676E-01	0.747599E+00	0.100000E+01	

Table D.21. Control Variate Results Against Fourth Node Quantile at the Fifth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.111325E+00	0.121418E+00	0.132414E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.111788E+00	0.121434E+00	0.131716E+00	0.9100
VARIANCE RATIO	0.581385E+00	0.947085E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.105001E+00	0.121351E+00	0.132163E+00	0.8850
VARIANCE RATIO	0.578932E+00	0.903681E+00	0.999983E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.109514E+00	0.121498E+00	0.133965E+00	0.9100
VARIANCE RATIO	0.702469E+00	0.949806E+00	0.999995E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.107856E+00	0.121264E+00	0.134142E+00	0.9050
VARIANCE RATIO	0.673564E+00	0.953347E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.105744E+00	0.121409E+00	0.134251E+00	0.9050
VARIANCE RATIO	0.627567E+00	0.951125E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.109515E+00	0.121498E+00	0.133964E+00	0.9100
VARIANCE RATIO	0.702539E+00	0.949797E+00	0.999996E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.110182E+00	0.122408E+00	0.136964E+00	0.7800
VARIANCE RATIO	0.303391E+00	0.779417E+00	0.999997E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.105685E+00	0.122206E+00	0.140539E+00	0.5450
VARIANCE RATIO	0.130080E+00	0.445908E+00	0.925947E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.110200E+00	0.122268E+00	0.137154E+00	0.7950
VARIANCE RATIO	0.323205E+00	0.796578E+00	0.999815E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.105442E+00	0.122140E+00	0.139733E+00	0.5100
VARIANCE RATIO	0.127179E+00	0.418037E+00	0.908348E+00	

Table D.22. Control Variate Results Against Fourth Node Quantile at the Sixth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.622885E-01	0.108684E+00	0.168433E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.620325E-01	0.108875E+00	0.170645E+00	0.9250
VARIANCE RATIO	0.513468E+00	0.950657E+00	0.999997E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.676376E-01	0.108407E+00	0.169864E+00	0.9350
VARIANCE RATIO	0.583689E+00	0.939187E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.594641E-01	0.109773E+00	0.172394E+00	0.9300
VARIANCE RATIO	0.612138E+00	0.953165E+00	0.999997E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.622200E-01	0.108051E+00	0.172440E+00	0.9250
VARIANCE RATIO	0.593514E+00	0.949890E+00	0.999988E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.587167E-01	0.108695E+00	0.180395E+00	0.9300
VARIANCE RATIO	0.468669E+00	0.948411E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.595041E-01	0.109778E+00	0.172424E+00	0.9300
VARIANCE RATIO	0.612476E+00	0.953100E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.580289E-01	0.110269E+00	0.177221E+00	0.9150
VARIANCE RATIO	0.454583E+00	0.940221E+00	0.999979E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.651024E-01	0.120729E+00	0.223535E+00	0.8000
VARIANCE RATIO	0.264705E+00	0.735405E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.579146E-01	0.110227E+00	0.176488E+00	0.9150
VARIANCE RATIO	0.470616E+00	0.940779E+00	0.999979E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.647745E-01	0.120575E+00	0.222433E+00	0.8000
VARIANCE RATIO	0.264302E+00	0.731573E+00	0.100000E+01	

Table D.23. Control Variate Results Against Fourth Node Quantile at the Seventh Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.112569E+00	0.140501E+00	0.185156E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.113335E+00	0.140270E+00	0.190232E+00	0.9400
VARIANCE RATIO	0.547357E+00	0.942611E+00	0.999999E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.109996E+00	0.140422E+00	0.186807E+00	0.9300
VARIANCE RATIO	0.570730E+00	0.934657E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.112687E+00	0.140580E+00	0.201817E+00	0.9250
VARIANCE RATIO	0.607985E+00	0.944603E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.112672E+00	0.140510E+00	0.183614E+00	0.9300
VARIANCE RATIO	0.611688E+00	0.948069E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.110439E+00	0.140202E+00	0.171872E+00	0.9150
VARIANCE RATIO	0.660412E+00	0.951762E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.112691E+00	0.140583E+00	0.201832E+00	0.9250
VARIANCE RATIO	0.608191E+00	0.944580E+00	0.999999E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.108940E+00	0.143605E+00	0.204380E+00	0.7500
VARIANCE RATIO	0.234856E+00	0.669272E+00	0.997503E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.987313E-01	0.143796E+00	0.217044E+00	0.4850
VARIANCE RATIO	0.848460E-01	0.270045E+00	0.888704E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.984601E-01	0.143404E+00	0.216561E+00	0.4850
VARIANCE RATIO	0.797107E-01	0.309553E+00	0.768329E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.943801E-01	0.142356E+00	0.221662E+00	0.4350
VARIANCE RATIO	0.624765E-01	0.243331E+00	0.717937E+00	

Table D.24. Control Variate Results Against Fourth Node Quantile at the Eight Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.145248E-01	0.645567E-01	0.177051E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.152320E-01	0.641725E-01	0.177135E+00	0.8700
VARIANCE RATIO	0.709700E+00	0.943849E+00	0.999999E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.137351E-01	0.651024E-01	0.180345E+00	0.8650
VARIANCE RATIO	0.570944E+00	0.933924E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	-0.429269E-02	0.647223E-01	0.178368E+00	0.8650
VARIANCE RATIO	0.702579E+00	0.956512E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.955922E-02	0.649550E-01	0.183485E+00	0.8600
VARIANCE RATIO	0.388151E+00	0.950948E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.145098E-01	0.656369E-01	0.170770E+00	0.8650
VARIANCE RATIO	0.389760E+00	0.948317E+00	0.999971E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	-0.441317E-02	0.646924E-01	0.178622E+00	0.8650
VARIANCE RATIO	0.711096E+00	0.956903E+00	0.999995E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.123770E-01	0.800651E-01	0.190834E+00	0.7550
VARIANCE RATIO	0.256196E+00	0.863874E+00	0.999977E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.139049E-01	0.804204E-01	0.191781E+00	0.7200
VARIANCE RATIO	0.909681E-01	0.747265E+00	0.999910E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.135561E-01	0.829255E-01	0.198735E+00	0.6700
VARIANCE RATIO	0.196644E+00	0.775468E+00	0.999973E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.139048E-01	0.804041E-01	0.192440E+00	0.7150
VARIANCE RATIO	0.116133E+00	0.746031E+00	0.100000E+01	

Table D.25. Control Variate Results Against Fourth Node Quantile at the Ninth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.111008E+00	0.122395E+00	0.132180E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.109439E+00	0.122366E+00	0.133052E+00	0.9300
VARIANCE RATIO	0.497627E+00	0.942552E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.109022E+00	0.122632E+00	0.134885E+00	0.8500
VARIANCE RATIO	0.433837E+00	0.838388E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.110767E+00	0.122497E+00	0.134761E+00	0.9100
VARIANCE RATIO	0.715594E+00	0.948371E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.109774E+00	0.122348E+00	0.131130E+00	0.9400
VARIANCE RATIO	0.471377E+00	0.951846E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.111225E+00	0.122125E+00	0.132695E+00	0.9250
VARIANCE RATIO	0.687864E+00	0.941987E+00	0.999993E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.110766E+00	0.122497E+00	0.134753E+00	0.9100
VARIANCE RATIO	0.715583E+00	0.948372E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.106708E+00	0.123562E+00	0.137148E+00	0.6900
VARIANCE RATIO	0.227410E+00	0.611888E+00	0.999926E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.101738E+00	0.123172E+00	0.139209E+00	0.4950
VARIANCE RATIO	0.104222E+00	0.395685E+00	0.840172E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.106979E+00	0.123520E+00	0.135986E+00	0.6900
VARIANCE RATIO	0.223015E+00	0.624305E+00	0.999225E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.101454E+00	0.123210E+00	0.138830E+00	0.4650
VARIANCE RATIO	0.796724E-01	0.374487E+00	0.844993E+00	

Table D.26. Control Variate Results Against Fourth Node Quantile at the Tenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.685479E-01	0.112705E+00	0.177696E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.645306E-01	0.112575E+00	0.179238E+00	0.9350
VARIANCE RATIO	0.639193E+00	0.953142E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.657353E-01	0.112503E+00	0.175888E+00	0.9200
VARIANCE RATIO	0.473448E+00	0.913959E+00	0.999992E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.648471E-01	0.114017E+00	0.177704E+00	0.9300
VARIANCE RATIO	0.669707E+00	0.951018E+00	0.999992E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.665513E-01	0.112739E+00	0.177008E+00	0.9400
VARIANCE RATIO	0.703021E+00	0.955436E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.546768E-01	0.112561E+00	0.182294E+00	0.9250
VARIANCE RATIO	0.678082E+00	0.953411E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.647503E-01	0.114022E+00	0.177704E+00	0.9300
VARIANCE RATIO	0.668615E+00	0.951008E+00	0.999991E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.668401E-01	0.116957E+00	0.206801E+00	0.9300
VARIANCE RATIO	0.358874E+00	0.899939E+00	0.999996E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.561665E-01	0.125670E+00	0.225853E+00	0.7600
VARIANCE RATIO	0.161109E+00	0.713087E+00	0.999976E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.668441E-01	0.116895E+00	0.206502E+00	0.9300
VARIANCE RATIO	0.364805E+00	0.901445E+00	0.999997E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.535490E-01	0.125568E+00	0.224347E+00	0.7500
VARIANCE RATIO	0.160630E+00	0.711324E+00	0.999846E+00	

Table D.27. Control Variate Results Against Fourth Node Quantile at the Eleventh Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.109092E+00	0.133791E+00	0.156356E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.109037E+00	0.133896E+00	0.153300E+00	0.9650
VARIANCE RATIO	0.717994E+00	0.946812E+00	0.999999E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.109303E+00	0.134058E+00	0.156681E+00	0.9600
VARIANCE RATIO	0.583121E+00	0.933723E+00	0.999992E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.105861E+00	0.133454E+00	0.160838E+00	0.9350
VARIANCE RATIO	0.595487E+00	0.944692E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.858128E-01	0.133426E+00	0.164153E+00	0.9550
VARIANCE RATIO	0.565806E+00	0.946036E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.110183E+00	0.133773E+00	0.194814E+00	0.9300
VARIANCE RATIO	0.583655E+00	0.946190E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.105856E+00	0.133455E+00	0.160815E+00	0.9350
VARIANCE RATIO	0.595146E+00	0.944685E+00	0.100000E+01	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.103230E+00	0.137180E+00	0.179016E+00	0.7100
VARIANCE RATIO	0.153193E+00	0.610513E+00	0.995094E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.918101E-01	0.137244E+00	0.180314E+00	0.4700
VARIANCE RATIO	0.740851E-01	0.271448E+00	0.791968E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.906200E-01	0.137089E+00	0.179633E+00	0.4850
VARIANCE RATIO	0.659187E-01	0.271682E+00	0.650611E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.878249E-01	0.135907E+00	0.178790E+00	0.4300
VARIANCE RATIO	0.654438E-01	0.238751E+00	0.696407E+00	

Table D.28. Control Variate Results Against Fourth Node Quantile at the Twelfth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.183013E-01	0.654637E-01	0.151707E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.186208E-01	0.660864E-01	0.162674E+00	0.8750
VARIANCE RATIO	0.438063E+00	0.946059E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.180504E-01	0.647238E-01	0.154406E+00	0.8600
VARIANCE RATIO	0.631840E+00	0.934873E+00	0.999966E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.136727E-01	0.662149E-01	0.208152E+00	0.8800
VARIANCE RATIO	0.479767E+00	0.947534E+00	0.999994E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.140515E-01	0.661724E-01	0.142004E+00	0.8850
VARIANCE RATIO	0.471836E+00	0.950482E+00	0.999997E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.154062E-01	0.661085E-01	0.163343E+00	0.8750
VARIANCE RATIO	0.412782E+00	0.942611E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.135963E-01	0.662051E-01	0.208216E+00	0.8800
VARIANCE RATIO	0.471930E+00	0.947586E+00	0.999998E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.150423E-01	0.789136E-01	0.199755E+00	0.7550
VARIANCE RATIO	0.282376E+00	0.849428E+00	0.999994E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.134631E-01	0.809273E-01	0.199835E+00	0.7300
VARIANCE RATIO	0.814486E-01	0.742618E+00	0.999995E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.150765E-01	0.804845E-01	0.202885E+00	0.7450
VARIANCE RATIO	0.107620E+00	0.796594E+00	0.999999E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.148620E-01	0.810003E-01	0.200335E+00	0.7300
VARIANCE RATIO	0.813528E-01	0.739736E+00	0.100000E+01	

Table D.29. Control Variate Results Against Fourth Node Quantile at the Thirteenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.112617E+00	0.123006E+00	0.133489E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.111830E+00	0.122966E+00	0.133523E+00	0.9150
VARIANCE RATIO	0.608003E+00	0.948123E+00	0.100000E+01	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.112125E+00	0.123066E+00	0.135463E+00	0.8300
VARIANCE RATIO	0.395735E+00	0.839989E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.112958E+00	0.123095E+00	0.139967E+00	0.9050
VARIANCE RATIO	0.619763E+00	0.945904E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.111766E+00	0.122816E+00	0.136052E+00	0.9000
VARIANCE RATIO	0.725878E+00	0.944308E+00	0.999997E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.113695E+00	0.123076E+00	0.140833E+00	0.9050
VARIANCE RATIO	0.475461E+00	0.947033E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.112956E+00	0.123095E+00	0.139963E+00	0.9050
VARIANCE RATIO	0.620128E+00	0.945910E+00	0.999998E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.111110E+00	0.123937E+00	0.138478E+00	0.7050
VARIANCE RATIO	0.165284E+00	0.690984E+00	0.998524E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.109029E+00	0.123774E+00	0.142480E+00	0.5100
VARIANCE RATIO	0.124364E+00	0.382093E+00	0.806462E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.110603E+00	0.123842E+00	0.138925E+00	0.7150
VARIANCE RATIO	0.217436E+00	0.712158E+00	0.995266E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.108567E+00	0.123744E+00	0.143333E+00	0.5050
VARIANCE RATIO	0.109572E+00	0.358926E+00	0.782032E+00	

Table D.30. Control Variate Results Against Fourth Node Quantile at the Fourteenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.635066E-01	0.115893E+00	0.172982E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.631326E-01	0.115665E+00	0.174980E+00	0.9600
VARIANCE RATIO	0.428457E+00	0.946227E+00	0.999992E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.634904E-01	0.116603E+00	0.207008E+00	0.9250
VARIANCE RATIO	0.531987E+00	0.904762E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.639428E-01	0.115812E+00	0.187707E+00	0.9350
VARIANCE RATIO	0.568546E+00	0.952948E+00	0.999991E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.627988E-01	0.116558E+00	0.206533E+00	0.9200
VARIANCE RATIO	0.626489E+00	0.948541E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.654682E-01	0.117183E+00	0.173880E+00	0.9400
VARIANCE RATIO	0.629324E+00	0.956051E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.639314E-01	0.115819E+00	0.187862E+00	0.9350
VARIANCE RATIO	0.568238E+00	0.952959E+00	0.999996E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.605557E-01	0.119998E+00	0.195994E+00	0.9150
VARIANCE RATIO	0.371597E+00	0.922926E+00	0.100000E+01	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.609148E-01	0.129821E+00	0.220835E+00	0.7650
VARIANCE RATIO	0.224505E+00	0.704063E+00	0.998094E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.606823E-01	0.119896E+00	0.195955E+00	0.9150
VARIANCE RATIO	0.369715E+00	0.923695E+00	0.999985E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.577421E-01	0.129756E+00	0.221314E+00	0.7600
VARIANCE RATIO	0.226006E+00	0.700010E+00	0.997459E+00	

Table D.31. Control Variate Results Against Fourth Node Quantile at the Fifteenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.107831E+00	0.132848E+00	0.155592E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.108732E+00	0.132963E+00	0.155574E+00	0.9350
VARIANCE RATIO	0.709745E+00	0.946663E+00	0.999999E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.106303E+00	0.132818E+00	0.155641E+00	0.9500
VARIANCE RATIO	0.524224E+00	0.947768E+00	0.999997E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.108570E+00	0.132973E+00	0.156596E+00	0.9050
VARIANCE RATIO	0.751773E+00	0.949830E+00	0.999998E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.811531E-01	0.132490E+00	0.161033E+00	0.9250
VARIANCE RATIO	0.660564E+00	0.955374E+00	0.999989E+00	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.999523E-01	0.132687E+00	0.158323E+00	0.9450
VARIANCE RATIO	0.687302E+00	0.946703E+00	0.999967E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.108579E+00	0.132973E+00	0.156592E+00	0.9050
VARIANCE RATIO	0.751612E+00	0.949824E+00	0.999999E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.972244E-01	0.135528E+00	0.165982E+00	0.7300
VARIANCE RATIO	0.254447E+00	0.652769E+00	0.997814E+00	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	0.882594E-01	0.135996E+00	0.173967E+00	0.4700
VARIANCE RATIO	0.586143E-01	0.255615E+00	0.607032E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.884096E-01	0.136073E+00	0.176498E+00	0.5000
VARIANCE RATIO	0.845022E-01	0.300370E+00	0.808305E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	0.895580E-01	0.135318E+00	0.173735E+00	0.4450
VARIANCE RATIO	0.640602E-01	0.237888E+00	0.605448E+00	

Table D.32. Control Variate Results Against Fourth Node Quantile at the Sixteenth Design Point

NAME	MINIMUM	AVERAGE	MAXIMUM	COVERAGE
YBAR	0.136025E-01	0.637900E-01	0.164999E+00	
USING CONTROL VARIABLE: ROUTING(1,3)				
YBAR(BHAT)	0.140895E-01	0.637519E-01	0.163840E+00	0.8500
VARIANCE RATIO	0.546540E+00	0.946557E+00	0.999997E+00	
USING CONTROL VARIABLE: ROUTING(1,4)				
YBAR(BHAT)	0.133346E-01	0.632153E-01	0.165482E+00	0.8500
VARIANCE RATIO	0.588743E+00	0.949309E+00	0.999999E+00	
USING CONTROL VARIABLE: WORK(1)				
YBAR(BHAT)	0.134627E-01	0.639593E-01	0.192337E+00	0.8550
VARIANCE RATIO	0.661003E+00	0.952177E+00	0.999996E+00	
USING CONTROL VARIABLE: WORK(2)				
YBAR(BHAT)	0.132382E-01	0.635254E-01	0.180854E+00	0.8500
VARIANCE RATIO	0.549468E+00	0.947156E+00	0.100000E+01	
USING CONTROL VARIABLE: WORK(3)				
YBAR(BHAT)	0.911578E-02	0.640720E-01	0.165083E+00	0.8300
VARIANCE RATIO	0.369341E+00	0.948231E+00	0.999989E+00	
USING CONTROL VARIABLE: WORK(4)				
YBAR(BHAT)	0.134616E-01	0.639375E-01	0.192181E+00	0.8550
VARIANCE RATIO	0.663000E+00	0.952478E+00	0.999970E+00	
USING CONTROL VARIABLE: SOJOURN(M/M/1)				
YBAR(BHAT)	0.120263E-01	0.783109E-01	0.226699E+00	0.7050
VARIANCE RATIO	0.200864E+00	0.852099E+00	0.100000E+01	
USING CONTROL VARIABLE: WAIT4(M/M/1)				
YBAR(BHAT)	-0.242423E-02	0.800065E-01	0.234492E+00	0.6700
VARIANCE RATIO	0.362815E-01	0.711689E+00	0.999992E+00	
USING CONTROL VARIABLE: SOJOURN(G/G/1)				
YBAR(BHAT)	0.121130E-01	0.811965E-01	0.245617E+00	0.6550
VARIANCE RATIO	0.366392E-01	0.773461E+00	0.999999E+00	
USING CONTROL VARIABLE: WAIT4(G/G/1)				
YBAR(BHAT)	-0.367256E-02	0.798992E-01	0.228922E+00	0.6550
VARIANCE RATIO	0.362530E-01	0.706804E+00	0.999991E+00	

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18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

Simulation, Open Network of Queues, Variance Reduction Techniques, Control Variates

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This investigation compared several control variates (both internal and external) on the basis of the variance ratios and actual confidence interval coverages against two response variables for an open network of queues. The results showed that although the external controls achieved larger variance reductions, the respective confidence intervals about the controlled responses had poor coverage. Furthermore, the internal controls proved to be robust with respect to the factors in the experiment (including the traffic intensity of the network).

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT

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21. ABSTRACT SECURITY CLASSIFICATION

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