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Homogenization for Semilinear Hyperbolic Systems with Oscillatory Data

THOMAS Y. HOU
Courant Institute

Abstract

The behavior of multi-dimensional discrete Boltzmann systems with highly oscillatory data is studied. Homogenized equations for the mean solutions are obtained. Uniform convergence of the oscillatory solutions of the discrete Boltzmann equations to the solutions of the corresponding homogenized equations is established. Moreover, we find that the weak limits of the oscillatory solutions for a model of Broadwell type are not continuous functions of the discrete velocities. Generalization of the above results to problems with multiple-scale initial data is also established.

1. Introduction

One of the essential problems in nonlinear partial differential equations is to understand how the behavior in the microscopic level affects the behavior in the macroscopic level. One approach is to find the corresponding effective or homogenized equations which determine the weak limits of the oscillatory solutions (see [3]). Such homogenization results are important both for the understanding of the nonlinear interaction between the high frequencies and for the study of the numerical approximation for problems with oscillatory solutions (see [7], [8], [9]).

We choose to study the nonlinear discrete Boltzmann equations in kinetic theory of discrete velocity. In such equations, high frequency components can be transformed into lower frequencies through nonlinear interaction, thus affecting the average of solutions. In this paper, we study the homogenization theory of the discrete Boltzmann equations in multi-dimensional space and with finitely many real-valued velocities (see equations (4.1)). We assume that the initial values are of the form $a(x, x/\varepsilon)$ with $a(x, y)$ *1-periodic* in each component of y. Our results show that the behavior of oscillatory solutions for a model of Broadwell type (see equations (2.1)) is very sensitive to the velocity coefficients. It depends on whether a certain ratio among the velocity components is a rational number or an irrational number. Furthermore we find that the form of homogenized equations depends on the velocity coefficients, and the weak limits of the oscillatory solutions are not continuous in the velocity components. This singular behavior for a model of Broadwell type is not shared by the simple Carleman model (see equations (3.1)).

Our study also shows that the structure of oscillatory solutions for a model of Broadwell type is quite stable (in the sense of Theorem 2.3) when we perturb velocity coefficients around irrational numbers. In this case, the resonance effect

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of u and v on w vanishes in the limit of $\varepsilon \to 0$. However, the behavior of oscillatory solutions for a model of Broadwell type becomes singular when perturbing around integer velocity coefficients. There is a strong interaction between the high frequency components of u and v, and the interaction in the u, v term would create oscillation of order O(1) on the w component even in the limit as $\varepsilon \to 0$. In [14], Tartar showed that for the Carleman model the weak limits of all powers of the initial oscillatory data will uniquely determine the weak limits of the oscillatory solutions at later time. We found that this is no longer true for a model of Broadwell type with integer-valued velocity coefficients.

The homogenization theory of the Carleman and Broadwell models with oscillatory initial data has been studied by McLaughlin, Papanicolaou and Tartar [11]. They proved that the oscillatory solutions of the Carleman and Broadwell models converge strongly in L^p-norm, $p < \infty$, to the solutions of the corresponding homogenized equations. By using certain ergodicity property of the oscillatory solutions and taking into account cancellations among high frequency components, we are able to obtain homogenization results for more general discrete Boltzmann equations. Moreover we establish uniform convergence of the oscillatory solutions to the solutions of the corresponding homogenized equations. This uniform convergence result is essential in the convergence analysis of particle methods for the discrete Boltzmann equations (see [7], [8], [9]).

The paper is organized as follows. In Section 2, we study a model of Broadwell type in detail and compare the homogenization results with those for the Carleman model. Section 3 contains homogenization results for the problems in which the initial data are of more than two scales. In Section 4, we extend the results of Section 2 to the discrete Boltzmann equations in multi-dimensional space with finitely many velocities.

2. Behavior of Oscillatory Solutions in the Model of Broadwell Type

The Broadwell model describes a three-dimensional model of rarefied gas in which particles travel with speed c in either direction along a coordinate axis (see [4]). If particles traveling in opposite directions collide, they are equally likely to move in each of the three coordinate directions after collision, with velocities of opposite sign. Other collisions can lead to an exchange of velocities. We denote by $N_1^+(x, y, z, t)$ the number density of particles with velocity $(c, 0, 0)$; a similar notation is used for N_1^-, $N_2^\pm$ and $N_3^\pm$. Then the resulting equations are

$$\frac{\partial N_1^+}{\partial t} + c\frac{\partial N_1^+}{\partial x} = \frac{1}{3}\sigma\left(N_2^+N_2^- + N_3^+N_3^- - 2N_1^+N_1^-\right),$$

$$\frac{\partial N_1^-}{\partial t} - c\frac{\partial N_1^-}{\partial x} = \frac{1}{3}\sigma\left(N_2^+N_2^- + N_3^+N_3^- - 2N_1^+N_1^-\right),$$

etc., where σ is the frequency of collision.

Here we consider the special case of one-dimensional motions in which the N's are independent of y, z, and furthermore $N_2^+ = N_2^- = N_3^+ = N_3^-$. Setting $N_1^+ = u(x, t)$, $N_1^- = v(x, t)$, $N_2^+ = w(x, t)$ and rescaling the variables so that $c = 1$, $\sigma = \frac{3}{2}$, we then obtain the 1-D Broadwell equations

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + uv - w^2 = 0,$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} + uv - w^2 = 0,$$

$$\frac{\partial w}{\partial t} - uv + w^2 = 0.$$

A lot of effort has been made in obtaining global solutions for the discrete Boltzmann equations. For the 1-D Broadwell model, Nishida and Mimura [12] first showed that a global solution exists when the initial values are small in some sense. Their result has been generalized to more general 1-D discrete Boltzmann equations by Crandall and Tartar [13], Cabannes [5], Illner [10] and Beale [2], among others.

To study how oscillatory solutions depend on velocity coefficients, we introduce an additional term $\alpha\, \partial w/\partial x$, $|\alpha| < 1$, in the last equation of the Broadwell model. We get

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + uv - w^2 = 0, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} + uv - w^2 = 0, \tag{2.1b}$$

$$\frac{\partial w}{\partial t} + \alpha \frac{\partial w}{\partial x} - uv + w^2 = 0. \tag{2.1c}$$

We refer to equations (2.1a–c) as a model of Broadwell type. Suppose in addition that initial values are given by

$$u(x,0) = u_0\left(x, \frac{x}{\varepsilon}\right), \quad v(x,0) = v_0\left(x, \frac{x}{\varepsilon}\right), \quad w(x,0) = w_0\left(x, \frac{x}{\varepsilon}\right), \tag{2.2}$$

where we assume that $u_0(x, y)$, $v_0(x, y)$ and $w_0(x, y)$ are 1-periodic in y.

We denote by u_ε, v_ε and w_ε the solutions of equations (2.1) and (2.2).

The behavior of solutions u_ε, v_ε and w_ε as $\varepsilon \to 0$ is very sensitive to the coefficient α. This is described by the following homogenization result.

Case I: $\alpha = m/n$ (m and n mutually prime); the homogenized equations for (2.1a–c) are

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U\int_0^1 V\,dy - \int_0^1 W^2\,dy = 0, \tag{2.3a}$$

$$\frac{\partial V}{\partial t} - \frac{\partial V}{\partial x} + V\int_0^1 U\,dy - \int_0^1 W^2\,dy = 0, \tag{2.3b}$$

$$\begin{aligned}&\frac{\partial W}{\partial t} + \alpha\frac{\partial W}{\partial x} + W^2\\ &\quad - \frac{1}{n}\int_0^n U\left(x, y + \left(\frac{m}{n} - 1\right)z, t\right)V\left(x, y + \left(\frac{m}{n} + 1\right)z, t\right)dz = 0.\end{aligned} \tag{2.3c}$$

If $m = 0$, then $n = 1$ in (2.3c).

Case II: α is an irrational number; the homogenized equations become

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U\int_0^1 V\,dy - \int_0^1 W^2\,dy = 0, \tag{2.4a}$$

$$\frac{\partial V}{\partial t} - \frac{\partial V}{\partial x} + V\int_0^1 U\,dy - \int_0^1 W^2\,dy = 0, \tag{2.4b}$$

$$\frac{\partial W}{\partial t} + \alpha\frac{\partial W}{\partial x} + W^2 - \left(\int_0^1 U(x, y, t)\,dy\right)\left(\int_0^1 V(x, y, t)\,dy\right) = 0. \tag{2.4c}$$

The initial data in both cases are given by

$$\begin{aligned}U(x, y, 0) = u_0(x, y), \quad V(x, y, 0) = v_0(x, y),\\ W(x, y, 0) = w_0(x, y).\end{aligned} \tag{2.5}$$

Here we have assumed that smooth and bounded global solutions of (2.3a–c) and (2.4a–c) exist up to time T.

THEOREM 2.1. *For smooth and bounded non-negative initial data, the solutions of* (2.1a–c) *and* (2.2) *converge to those of the corresponding homogenized equations strongly in the L^∞-norm,*

$$u_\varepsilon(x, t) - U\left(x, \frac{x - t}{\varepsilon}, t\right) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

$$v_\varepsilon(x, t) - V\left(x, \frac{x + t}{\varepsilon}, t\right) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

$$w_\varepsilon(x, t) - W\left(x, \frac{x - \alpha t}{\varepsilon}, t\right) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad 0 \leqq t \leqq T.$$

Remark 2.1. The homogenization result above can be generalized to the case when $u_0(x, y)$, $v_0(x, y)$ and $w_0(x, y)$ are periodic functions in y with arbitrary periods.

Remark 2.2. In the case when $\alpha = 0$, our homogenization result is the same as that obtained in [11]. The only difference is that we obtain uniform convergence instead of convergence in L^p-norm, $p < \infty$, of [11].

Remark 2.3. The local existence result of the homogenized equations (2.3) or (2.4) can be obtained by classical analysis for smooth and bounded non-negative initial data. The global existence result for the homogenized equations then follow by combining the known global existence results for (2.1)–(2.2) (e.g. [10]) with Theorem 2.1. Therefore the value of T in Theorem 2.1 is arbitrarily large.

LEMMA 2.1. *Let $f(x)$, $g(x, y) \in C^1$. Assume further that $g(x, y)$ is n-periodic in y and satisfies the relation $\int_0^n g(x, y)\, dy = 0$. Then for any constants a and b, we have*

$$\left| \frac{1}{b-a} \int_a^b f(x) g\left(x, \frac{x}{\varepsilon}\right) dx \right| \leqq C\varepsilon.$$

Proof: Express $g(x, x/\varepsilon)$ as

$$g\left(x, \frac{x}{\varepsilon}\right) = \frac{d}{dx} \int_a^x g\left(x, \frac{s}{\varepsilon}\right) ds - \int_a^x \frac{\partial g}{\partial x}\left(x, \frac{s}{\varepsilon}\right) ds. \tag{2.6}$$

Since, for any real number d,

$$\int_d^{d+n} \frac{\partial g}{\partial x}(x, y)\, dy = \frac{\partial}{\partial x} \int_d^{d+n} g(x, y)\, dy \equiv 0,$$

we conclude that

$$\left| \int_a^b g\left(x, \frac{s}{\varepsilon}\right) ds \right| \leqq C_1 \varepsilon, \qquad \left| \int_a^x \frac{\partial g}{\partial x}\left(x, \frac{s}{\varepsilon}\right) ds \right| \leqq C_2 \varepsilon.$$

From this, we deduce that

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x) g\left(x, \frac{x}{\varepsilon}\right) dx \right| \\ &\quad = \frac{1}{|b-a|} \left| f(b) \int_a^b g\left(b, \frac{s}{\varepsilon}\right) ds - \int_a^b \left(\int_a^x g\left(x, \frac{s}{\varepsilon}\right) ds \right) \frac{df(x)}{dx}\, dx \right. \\ &\qquad\qquad \left. - \int_a^b f(x) \left(\int_a^x \frac{\partial g}{\partial x}\left(x, \frac{s}{\varepsilon}\right) ds \right) dx \right| \leqq C\varepsilon. \end{aligned}$$

This completes the proof of the lemma.

LEMMA 2.2. *Suppose that $f(x, y, z)$ is continuous. Moreover, we assume that f is 1-periodic in y and z. If γ_2/γ_1 is an irrational number, then we have*

$$\left|\frac{1}{b-a}\int_a^b\left(f\left(x, \frac{x_1+\gamma_1 x}{\varepsilon}, \frac{x_2+\gamma_2 x}{\varepsilon}\right) - \int_0^1\int_0^1 f(x, y, z)\, dy\, dz\right) dx\right| \leqq C(\varepsilon) \to 0,$$

as $\varepsilon \to 0$ for any constants a, b, x_1 and x_2.

Proof: We assume for simplicity that f has been normalized so that

$$\int_0^1\int_0^1 f(x, y, z)\, dy\, dz = 0.$$

We first prove the lemma in the case when $f(x, y, z)$ is independent of the first variable x. By change of variables, we can further reduce the problem to showing that

$$\left|\frac{1}{T}\int_0^T f(x, x_{1,2} + \lambda x)\, dx\right| \to 0 \quad \text{as} \quad T \to \infty,$$

where $\lambda = \gamma_2/\gamma_1$ is an irrational number, $T = \gamma_1(b-a)/\varepsilon$, $x_{1,2} = (x_2 - \lambda x_1)/\varepsilon$.

We need only to show this for integer-valued T. Note that

$$\begin{aligned} \frac{1}{N}\int_0^N f(x, x_{1,2}+\lambda x)\, dx &= \frac{1}{N}\sum_{n=0}^{N-1}\int_n^{n+1} f(x, x_{1,2}+\lambda x)\, dx \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 f(x, x_{1,2}+n\lambda+\lambda x)\, dx. \end{aligned} \tag{2.7}$$

Define

$$F(y) \equiv \int_0^1 f(x, y+\lambda x)\, dx.$$

Then $F(y)$ is a 1-periodic continuous function. Applying the well-known theorem of equipartition modulo 1 (Bohl-Serpinskii-Weyl) of ergodic theory (see [1]), we obtain

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} F(x_{1,2}+n\lambda) - \int_0^1 F(y)\, dy\right| \to 0 \quad \text{as} \quad N \to \infty, \tag{2.8}$$

independently of the value of $x_{1,2}$. By interchanging the order of integration, we get

$$\int_0^1 F(y)\, dy = \int_0^1\int_0^1 f(y, z)\, dy\, dz = 0.$$

Thus (2.7) and (2.8) yield

$$\frac{1}{b-a}\int_a^b \left(f\left(\frac{x_1+\gamma_1 x}{\varepsilon}, \frac{x_2+\gamma_2 x}{\varepsilon} \right) dx \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{2.9}$$

For the case where f depends on its first variable, we have, in light of (2.6),

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \left(f\left(x, \frac{x_1+\gamma_1 x}{\varepsilon}, \frac{x_2+\gamma_2 x}{\varepsilon} \right) dx \\ &\quad = \frac{1}{b-a}\int_a^b f\left(b, \frac{x_1+\gamma_1 s}{\varepsilon}, \frac{x_2+\gamma_2 s}{\varepsilon} \right) ds \\ &\quad - \frac{1}{b-a}\int_a^b \left(\int_a^x \frac{\partial f}{\partial x}\left(x, \frac{x_1+\gamma_1 s}{\varepsilon}, \frac{x_2+\gamma_2 s}{\varepsilon} \right) ds \right) dx. \end{aligned} \tag{2.10}$$

Note that $\partial f(x, y, z)/\partial x$ is 1-periodic in the y and z variables, and satisfies

$$\int_0^1 \int_0^1 \frac{\partial f}{\partial x}(x, y, z)\, dy\, dz = \frac{\partial}{\partial x}\int_0^1 \int_0^1 f(x, y, z)\, dy\, dz = 0,$$

by the assumption on $f(x, y, z)$. Applying (2.9) to the integrals with respect to ds on the right-hand side of (2.10), we obtain

$$\frac{1}{b-a}\int_a^b \left(f\left(x, \frac{x_1+\gamma_1 x}{\varepsilon}, \frac{x_2+\gamma_2 x}{\varepsilon} \right) dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

This completes the proof of Lemma 2.2.

Proof of Theorem 2.1: We only give the proof for the case when α is an irrational number. The case when $\alpha = m/n$ follows similarly by using Lemma 2.1. Subtracting equation (2.4c) from equation (2.1c) and integrating the resulting equation along their characteristics from 0 to t, we obtain

$$\begin{aligned} &w_\varepsilon(x, t) - W\left(x, \frac{x-\alpha t}{\varepsilon}, t \right) \\ &\quad = -\int_0^t \left(w_\varepsilon(x-\alpha(t-s), s)^2 - W\left(x-\alpha(t-s), \frac{x-\alpha t}{\varepsilon}, s \right)^2 \right) ds \\ &\quad + T(x, t) + \int_0^t u_\varepsilon(x-\alpha(t-s), s) v_\varepsilon(x-\alpha(t-s), s)\, ds \\ &\quad - \int_0^t U\left(x-\alpha(t-s), \frac{(x-\alpha t)+(\alpha-1)s}{\varepsilon}, s \right) \\ &\quad \cdot V\left(x-\alpha(t-s), \frac{(x-\alpha t)+(\alpha+1)s}{\varepsilon}, s \right) ds, \end{aligned} \tag{2.11}$$

where $T(x, t)$ is defined by

$$T(x,t) = \int_0^t U\left(x - \alpha(t-s), \frac{x-\alpha t}{\varepsilon} + \frac{(\alpha-1)s}{\varepsilon}, s\right)$$

$$\cdot V\left(x - \alpha(t-s), \frac{x-\alpha t}{\varepsilon} + \frac{(\alpha+1)s}{\varepsilon}, s\right) ds$$

$$- \int_0^t \left(\int_0^1 U(x - \alpha(t-s), z, s)\, dz\right)\left(\int_0^1 V(x-\alpha(t-s), y, s)\, dy\right) ds.$$

For fixed x and t, define $\lambda \equiv (\alpha+1)/(\alpha-1)$ and

$$f(s, y, z) = U(x - \alpha(t-s), y, s)V(x - \alpha(t-s), z, s),$$

$$x_1 = x_2 = x - \alpha t, \quad \gamma_1 = \alpha - 1, \quad \gamma_2 = \alpha + 1.$$

Then the $T(x, t)$ term in (2.11) becomes

$$\int_0^t \left(f\left(s, \frac{x_1 + \gamma_1 s}{\varepsilon}, \frac{x_2 + \gamma_2 s}{\varepsilon}\right) - \int_0^1\int_0^1 f(s, y, z)\, dy\, dz\right) ds,$$

which is bounded uniformly by $C(\varepsilon)$ by Lemma 2.2.

Define

$$M = \sup_{0 \le t \le T} \left\{ \sup_{x,y} \{ |u_\varepsilon(x,t)|, |U(x,y,t)|, \cdots, |W(x,y,t)| \} \right\}.$$

We deduce from (2.11) that

$$\left| w_\varepsilon(x,t) - W\left(x, \frac{x-\alpha t}{\varepsilon}, t\right)\right| \le 2M \int_0^t G(x - \alpha(t-s), s)\, ds + C(\varepsilon), \tag{2.12}$$

where $G(x, t)$ is defined by

$$\begin{aligned} G(x,t) = &\left|u_\varepsilon(x,t) - U\left(x, \frac{x-t}{\varepsilon}, t\right)\right| + \left|v_\varepsilon(x,t) - V\left(x, \frac{x+t}{\varepsilon}, t\right)\right| \\ &+ \left|w_\varepsilon(x,t) - W\left(x, \frac{x-\alpha t}{\varepsilon}, t\right)\right|. \end{aligned} \tag{2.13}$$

Similarly, we obtain

$$\left|u_\varepsilon(x,t) - U\left(x, \frac{x-t}{\varepsilon}, t\right)\right| \leqq 2M\int_0^t G(x-t+s, s)\,ds + C(\varepsilon), \tag{2.14}$$

$$\left|v_\varepsilon(x,t) - V\left(x, \frac{x+t}{\varepsilon}, t\right)\right| \leqq 2M\int_0^t G(x+t-s, s)\,ds + C(\varepsilon). \tag{2.15}$$

Adding (2.12), (2.14) and (2.15) yields

$$\begin{aligned} G(x,t) \leqq 2M\int_0^t (G(x-t+s, s) & \\ &+G(x+t-s,s) \\ &+G(x-\alpha(t-s),s))\,ds + 3C(\varepsilon). \end{aligned} \tag{2.16}$$

Define

$$E(t) = \sup_x \{G(x,t)\}.$$

It follows immediately from (2.16) that

$$E(t) \leqq 6M\int_0^t E(s)\,ds + 3C(\varepsilon). \tag{2.17}$$

Application of the Gronwall inequality to (2.17) then proves the theorem.

Suppose that w_0 is y independent. The oscillation of u and v will create oscillations on w at later time. The homogenized equation (2.3c) indicates that w remains oscillatory as $\varepsilon \to 0$ if α is rational. However, if α is an irrational number, (2.4c) implies that $W(x, y, t)$ is y independent. Thus we expect that there is some kind of singularity in the high-order powers of solutions. Since the equations are nonlinear, such a singularity would affect the local average of solutions. This is described by the following theorem.

THEOREM 2.2. *Let $\alpha_0 = m/n$. Assume $w(x,0) = w_0(x)$ and*

$$\frac{\partial}{\partial y}\int_0^n u_0\left(x, y + \left(\frac{m}{n} - 1\right)z\right)v_0\left(x, y + \left(\frac{m}{n} + 1\right)z\right)dz \neq 0.$$

Then at least one of the following limits does not hold as $\alpha \to \alpha_0$:

$$\int_0^1 U_\alpha\,dy \to \int_0^1 U_{\alpha_0}\,dy, \quad \int_0^1 V_\alpha\,dy \to \int_0^1 V_{\alpha_0}\,dy, \quad \int_0^1 W_\alpha\,dy \to \int_0^1 W_{\alpha_0}\,dy,$$

where U_α, V_α and W_α are solutions of equations (2.3) *or* (2.4).

Proof: It can be shown that

$$\int_0^1 \frac{1}{n}\int_0^n U_{\alpha_0}\left(x, y + \left(\frac{m}{n} - 1\right)z, t\right) V_{\alpha_0}\left(x, y + \left(\frac{m}{n} + 1\right)z, t\right) dz\, dy$$

$$= \left(\int_0^1 U_{\alpha_0}\, dy\right)\left(\int_0^1 V_{\alpha_0}\, dy\right).$$

Let $U^{(m)} = \int_0^1 U^m\, dy$. For irrational α, we first integrate equation (2.3c) corresponding to α_0 and equation (2.4c) corresponding to α in y from 0 to 1 and then integrate from 0 to t along their characteristics, respectively. The difference of the resulting equations gives

$$\begin{aligned} & W_\alpha^{(1)}(x,t) - W_{\alpha_0}^{(1)}(x,t) + \int_0^1 \left(w_0(x - \alpha_0 t, y) - w_0(x - \alpha t, y)\right) dy \\ & \qquad + \int_0^t \left(U_{\alpha_0}^{(1)} V_{\alpha_0}^{(1)} - U_\alpha^{(1)} V_\alpha^{(1)}\right) ds \\ & = -\int_0^t \left(W_\alpha^{(2)}(x - \alpha(t-s), s) - W_{\alpha_0}^{(2)}(x - \alpha_0(t-s), s)\right) ds. \end{aligned} \tag{2.18}$$

Since $W_\alpha(x, y, 0)$ is y-independent, $W_\alpha(x, y, t)$ is independent of y for irrational α according to equation (2.4c). Thus $W_\alpha^{(2)} = (W_\alpha^{(1)})^2$. On the other hand, the assumption on u_0, v_0 and equation (2.3c) imply that $W_{\alpha_0}(x, y, t)$ is a nontrivial function of y at least in some interval $[0, t_1]$ with $t_1 > 0$. Thus we have

$$\begin{aligned} \int_0^{t_1}\left(W_\alpha^{(2)} - W_{\alpha_0}^{(2)}\right) ds &= \int_0^{t_1}\left(\left(W_\alpha^{(1)}\right)^2 - \left(W_{\alpha_0}^{(1)}\right)^2\right) ds \\ & \quad - \int_0^{t_1}\left(W_{\alpha_0}^{(2)} - \left(W_{\alpha_0}^{(1)}\right)^2\right) ds. \end{aligned} \tag{2.19}$$

Since $W_{\alpha_0}(x, y, t)$ is not independent of y, Schwartz's inequality yields

$$W_{\alpha_0}^{(2)} - \left(W_{\alpha_0}^{(1)}\right)^2 > 0.$$

Therefore the last term on the right of (2.19) is a non-zero function independent of α. Consequently,

$$U_\alpha^{(1)} \to U_{\alpha_0}^{(1)}, \quad V_\alpha^{(1)} \to V_{\alpha_0}^{(1)}, \quad W_\alpha^{(1)} \to W_{\alpha_0}^{(1)} \quad \text{as} \quad \alpha \to \alpha_0,$$

would contradict (2.18) and (2.19). This completes the proof of Theorem 2.2.

The situation for α_0 irrational is quite different. Weinan E and I can show that the solution is 'structurally stable' near α_0 in the following sense.

THEOREM 2.3. *Suppose that solutions $U_\alpha(x, \cdot, t)$ and $V_\alpha(x, \cdot, t)$ of (2.3) or (2.4) belong to C^2. If α_0 is an irrational number, then solutions U_α, V_α and W_α of (2.3) or (2.4) are continuous functions of α at α_0. Moreover, for any positive integer m, we have*

$$\lim_{\alpha \to \alpha_0} \int_0^1 U_\alpha(x, y, t)^m \, dy = \int_0^1 U_{\alpha_0}(x, y, t)^m \, dy,$$

$$\lim_{\alpha \to \alpha_0} \int_0^1 V_\alpha(x, y, t)^m \, dy = \int_0^1 V_{\alpha_0}(x, y, t)^m \, dy,$$

$$\lim_{\alpha \to \alpha_0} \int_0^1 W_\alpha(x, y, t)^m \, dy = \int_0^1 W_{\alpha_0}(x, y, t)^m \, dy.$$

Proof: Case 1: $\alpha \to \alpha_0$ and $\alpha = m/n$. We assume that $0 < |\alpha_0| < 1$, and that m and n are mutually prime with $n > 0$. Since α_0 is an irrational number, clearly we have $\lim_{\alpha \to \alpha_0} n(\alpha) = \infty$.

Define the function $H(x, y, t)$ by

$$H(x, y, t) \equiv \frac{1}{n} \int_0^n U_\alpha\left(x, y + \left(\frac{m}{n} - 1\right)z, t\right) V_\alpha\left(x, y + \left(\frac{m}{n} + 1\right)z, t\right) dz$$
$$- \left(\int_0^1 U_\alpha(x, y, t)\, dy\right)\left(\int_0^1 V_\alpha(x, y, t)\, dy\right).$$

We first show that $\lim_{n \to \infty} H(x, y, t) = 0$. Since $U_\alpha(x, y, t)$ and $V_\alpha(x, y, t)$ are 1-periodic functions in y, we can expand U_α and V_α by their Fourier series. We get

$$H = \frac{1}{n} \int_0^n \left(\sum_{k \neq 0} a_k \exp\{2\pi i k(y + (m/n - 1)z)\}\right) \tag{2.20}$$
$$\cdot \left(\sum_{l \neq 0} b_l \exp\{2\pi i l(y + (m/n + 1)z)\}\right) dz.$$

It can be shown that $\partial_y^2 U_\alpha$ is bounded independently of α. Therefore, we have $|a_k| \leqq c(1/k^2)$ and

$$\left| \sum_{|k| > n/3} a_k \exp\{2\pi i k(y + (m/n - 1)z)\} \right| \leqq C/n.$$

Similarly, we have

$$\left| \sum_{|l| > n/3} b_l \exp\{2\pi i l(y + (m/n + 1)z)\} \right| \leqq C/n.$$

On the other hand, note that, for $0 < |k| \leqq \frac{1}{3}n$, $0 < |l| \leqq \frac{1}{3}n$,

$$k(m/n - 1) + l(m/n + 1) \neq 0. \tag{2.21}$$

Suppose otherwise. We then get $m/n = (k-l)/(k+l)$. But $|k+l| \leqq \frac{2}{3}n < n$, $|m| < n$, which contradicts the assumption that m and n are mutually prime.

As a result of (2.21), we can easily show by interchanging the order of integration and summation that

$$\left|\frac{1}{n}\int_0^n\left(\sum_{0<|k|\leqq n/3} a_k \exp\{2\pi i k(y+(m/n-1)z)\}\right)\right.$$
$$\left.\cdot\left(\sum_{0<|l|\leqq n/3} b_l \exp\{2\pi i l(y+(m/n+1)z)\}\right) dz\right| = 0.$$

Therefore we have proven that $\lim_{n\to\infty} H(x,y,t) = 0$ by showing that

$$|H(x,y,t)| \leqq C/n, \tag{2.22}$$

where C is independent of α.

Integrate equation (2.4c) corresponding to α_0 and equation (2.3c) corresponding to α from 0 to t along their characteristics, respectively. The difference of the resulting equations gives

$$\begin{aligned} W_\alpha(x,y,t) - W_{\alpha_0}(x,y,t) &= w_0(x-\alpha t, y) - w_0(x-\alpha_0 t, y) \\ &+ \int_0^t\left(\int_0^1 U_\alpha(x-\alpha(t-s),y,s)\,dy\right)\left(\int_0^1 V_\alpha(x-\alpha(t-s),y,s)\,dy\right) ds \\ &- \int_0^t\left(\int_0^1 U_{\alpha_0}(x-\alpha_0(t-s),y,s)\,dy\right)\left(\int_0^1 V_{\alpha_0}(x-\alpha_0(t-s),y,s)\,dy\right) ds \\ &+ \int_0^t H(x-\alpha(t-s),y,s)\,ds \\ &+ \int_0^t\left(W_{\alpha_0}(x-\alpha_0(t-s),y,s)^2 - W_\alpha(x-\alpha(t-s),y,s)^2\right) ds. \end{aligned} \tag{2.23}$$

Define

$$G(x,t) \equiv \|U_\alpha(x,\cdot,t) - U_{\alpha_0}(x,\cdot,t)\|_{L^\infty} + \|V_\alpha(x,\cdot,t) - V_{\alpha_0}(x,\cdot,t)\|_{L^\infty}$$
$$+ \|W_\alpha(x,\cdot,t) - W_{\alpha_0}(x,\cdot,t)\|_{L^\infty}.$$

Since U_α, V_α and W_α are bounded and u_0, v_0 and w_0 have bounded partial derivatives, we can show that

$$\begin{aligned}(2.24)\qquad |W_\alpha(x,y,t) - W_{\alpha_0}(x,y,t)| &\leqq 2M\int_0^t G(x-\alpha(t-s),t)\,ds \\ &\quad + C/n + C|\alpha-\alpha_0|.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(2.25)\qquad &|U_\alpha(x,y,t) - U_{\alpha_0}(x,y,t)| \\ &\leqq 2M\int_0^t G(x-(t-s),t)\,ds + C/n + C|\alpha-\alpha_0|,\end{aligned}$$

$$\begin{aligned}(2.26)\qquad &|V_\alpha(x,y,t) - V_{\alpha_0}(x,y,t)| \\ &\leqq 2M\int_0^t G(x+(t-s),t)\,ds + C/n + C|\alpha-\alpha_0|.\end{aligned}$$

Define $E(t) = \sup_x G(x,t)$. Adding (2.24), (2.25) and (2.26) yields

$$(2.27)\qquad E(t) \leqq 6M\int_0^t E(s)\,ds + 3C/n + 3C|\alpha-\alpha_0|.$$

The Gronwall inequality then implies the theorem for Case 1.

Case 2: α is an irrational number. In this case the homogenized equations corresponding to α and α_0 are of the same form (2.4). The proof is identical to the second step in the Case 1 beginning from (2.22). Hence the proof of Theorem 2.3 is complete.

The proof of Theorem 2.3 contains the following result.

THEOREM 2.4. *If $\alpha_0 = m/n$ and $n \gg 1$, then there exists $\delta(n) > 0$ such that, for $|\alpha-\alpha_0| < \delta(n)$,*

$$\left|\int_0^1 U_\alpha(x,y,t)^k\,dy - \int_0^1 U_{\alpha_0}(x,y,t)^k\,dy\right| \leqq O\left(\frac{1}{n}\right) + \Delta(\alpha-\alpha_0).$$

Similar expressions hold for V and W. Here $\Delta(\delta) \to 0$ as $\delta \to 0$, and k is any positive integer.

The following theorem tells us how the structure of oscillatory initial data affects the higher-order powers of solutions in the future. For the Carleman model, Tartar [14] has shown that the weak limits of $u_\epsilon(x,t)^m$ are uniquely determined by the weak limits of $u_\epsilon(x,0)^k$ for all k. For a model of Broadwell type, we show that this is true only if α is an irrational number.

THEOREM 2.5. *Case I: α is an irrational number. Assume $\{U_a, V_a, W_a\}$ and $\{U_b, V_b, W_b\}$ are two different solutions of system (2.4). Then we have*

$$\int_0^1 U_a(x,y,t)^m\,dy = \int_0^1 U_b(x,y,t)^m\,dy, \tag{2.28a}$$

$$\int_0^1 V_a(x,y,t)^m\,dy = \int_0^1 V_b(x,y,t)^m\,dy, \tag{2.28b}$$

$$\int_0^1 W_a(x,y,t)^m\,dy = \int_0^1 W_b(x,y,t)^m\,dy, \tag{2.28c}$$

for all integer $m \geqq 1$ and $0 < t \leqq T$ provided that (2.28a), (2.28b) *and* (2.28c) *are valid at $t = 0$ for all integer $m \geqq 1$.*

Case II: α is a rational number. Let $\{U_a, V_a, W_a\}$ and $\{U_b, V_b, W_b\}$ be two different solutions of system (2.3). *Then* (2.28a), (2.28b) *and* (2.28c) *may not be true in general for all integer $m \geqq 1$ and $0 < t \leqq T$ even if* (2.28a), (2.28b) *and* (2.28c) *are valid at $t = 0$ for all integer $m \geqq 1$.*

Proof: Case I: We follow closely the proof given by Tartar in [14]. Throughout the proof we shall use the notation $U^{(m)}(x,t) = \int_0^1 U(x,y,t)^m\,dy$. For irrational α, one could easily derive equations for $U_a^{(m)}$, $V_a^{(m)}$ and $W_a^{(m)}$ as follows:

$$\frac{\partial U_a^{(m)}}{\partial t} + \frac{\partial U_a^{(m)}}{\partial x} + mU_a^{(m)}V_a^{(1)} - mU_a^{(m-1)}W_a^{(2)} = 0, \tag{2.29a}$$

$$\frac{\partial V_a^{(m)}}{\partial t} - \frac{\partial V_a^{(m)}}{\partial x} + mV_a^{(m)}U_a^{(1)} - mV_a^{(m-1)}W_a^{(2)} = 0, \tag{2.29b}$$

$$\frac{\partial W_a^{(m)}}{\partial t} + \alpha\frac{\partial W_a^{(m)}}{\partial x} + mW_a^{(m+1)} - mU_a^{(1)}V_a^{(1)}W_a^{(m-1)} = 0; \tag{2.29c}$$

$U_b^{(m)}$, $V_b^{(m)}$ and $W_b^{(m)}$ satisfy the same equations (2.29a–c).

From the global existence results [12], [13] and [2], we know that there exists a constant M_0 such that solutions $U_a, V_a, \cdots, W_b$ are bounded by M_0. Define

$$\begin{aligned}\delta_m(t) = \sup_{\substack{0\leqq s\leqq t\\ 1\leqq p\leqq m}} M_0^{m-p}\max\big(&\|U_a^{(p)}(\cdot,s) - U_b^{(p)}(\cdot,s)\|_{L^\infty},\\ &\|V_a^{(p)}(\cdot,s) - V_b^{(p)}(\cdot,s)\|_{L^\infty},\\ &\|W_a^{(p)}(\cdot,s) - W_b^{(p)}(\cdot,s)\|_{L^\infty}\big).\end{aligned} \tag{2.30}$$

We obtain from (2.29c)

$$\begin{aligned}(2.31)\qquad &\left(\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x}\right)\left(W_a^{(m)} - W_b^{(m)}\right)\\ &= -m\left(W_a^{(m+1)} - W_b^{(m+1)}\right) + m\left(U_a^{(1)} - U_b^{(1)}\right)V_a^{(1)}W_a^{(m-1)}\\ &\quad + mU_b^{(1)}\left(V_a^{(1)} - V_b^{(1)}\right)W_a^{(m-1)}\\ &\quad + mU_b^{(1)}V_b^{(1)}\left(W_a^{(m-1)} - W_b^{(m-1)}\right).\end{aligned}$$

The right-hand side of (2.31), denoted by $h(x,t)$, is bounded by

$$\|h(\cdot,t)\|_{L^\infty} \leqq m\,\delta_{m+1}(t) + 2mM_0^m\delta_1(t) + mM_0^2\delta_{m-1}(t) \leqq 4m\,\delta_{m+1}(t).$$

By the assumption of Theorem 2.5, $W_a^{(m)}(x,0) = W_b^{(m)}(x,0)$. Integration of (2.31) along its characteristic line will give

$$\left\|W_a^{(m)}(\cdot,t) - W_b^{(m)}(\cdot,t)\right\|_{L^\infty} \leqq 4m\int_0^t \delta_{m+1}(s)\,ds.$$

Moreover one can show that, for $1 \leqq p \leqq m$,

$$M_0^{m-p}\left\|W_a^{(p)}(\cdot,t) - W_b^{(p)}(\cdot,t)\right\|_{L^\infty} \leqq 4m\int_0^t \delta_{m+1}(s)\,ds.$$

A similar result applies to the U and V components. Thus we conclude that

$$(2.32)\qquad \delta_m(t) \leqq 4m\int_0^t \delta_{m+1}(s)\,ds.$$

Note that $\delta_m(t) \leqq M_0^m$. By induction one can show that

$$(2.33)\qquad \delta_m(t) \leqq 4^k\frac{(m+k-1)!}{(m-1)!}\frac{t^k}{k!}M_0^{m+k} \quad \text{for} \quad k = 0,1,\cdots.$$

For $T < 1/4M_0$, the right-hand side of (2.33) tends to zero as $k \to \infty$. Hence $\delta_m(t) \equiv 0$. This proves the theorem for the case I.

Case II: α is a rational number, $\alpha = m/n$. We assume that $|m/n| < 1$. Let us choose initial data of type a to be

$$U_a(x,y,0) = \sin(2\pi(m+n)y) + 1.0,$$

$$V_a(x,y,0) = \cos(2\pi(m-n)y) + 1.0,$$

$$W_a(x,y,0) = 8.0(1 + 0.5\sin(4\pi ny)),$$

and choose initial data of type b to be

$$U_b(x, y, 0) = \sin(2\pi(m+n)y) + 1.0,$$

$$V_b(x, y, 0) = \cos(2\pi(m-n)y) + 1.0,$$

$$W_b(x, y, 0) = 8.0(1 + 0.5\cos(4\pi n y)).$$

Then equalities (2.28a–c) are satisfied at time $t = 0$. However, direct computation shows that

$$\int_0^1 \left(W_a(x, y, 0) \frac{1}{n} \int_0^n U_a(x, y + (m/n - 1)z, 0) \right.$$

$$\left. \cdot V_a(x, y + (m/n + 1)z, 0)\, dz \right) dy = 9.0,$$

and

$$\int_0^1 \left(W_b(x, y, 0) \frac{1}{n} \int_0^n U_b(x, y + (m/n - 1)z, 0) \right.$$

$$\left. \cdot V_b(x, y + (m/n + 1)z, 0)\, dz \right) dy = 8.0.$$

Thus for the above choices of initial data, we get

$$\left(\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial x} \right) \int_0^1 \left(W_a(x, y, t)^2 - W_b(x, y, t)^2 \right) dy = 2.0 \quad \text{at} \quad t = 0, \tag{2.34}$$

and we conclude from (2.34) that

$$\int_0^1 \left(W_a(x, y, t)^2 - W_b(x, y, t)^2 \right) dy > 0 \quad \text{for} \quad t > 0 \text{ small}.$$

Therefore (2.28c) cannot be true for all $t > 0$. This compietes the proof of Theorem 2.5.

3. Discrete Boltzmann Equations with Multiple Scale Initial Data

We choose the simple Carleman model to illustrate the results. Generalization to more complicated models follows directly. The Carleman equations are given as follows (see [6]):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u^2 - v^2 = 0, \tag{3.1a}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} + v^2 - u^2 = 0. \tag{3.1b}$$

We assume that the initial values are given by

$$u(x,0) = u_0\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}\right), \qquad v(x,0) = v_0\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}\right), \tag{3.2}$$

where $u_0(x, y_1, y_2)$ and $v_0(x, y_1, y_2)$ are 1-periodic functions in each y_i variable.

Case 1: $\varepsilon_2/\varepsilon_1 \to 0$ as $\varepsilon_2 \to 0$. The homogenized equations for (3.1)–(3.2) are given by

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U^2 - \int_0^1\int_0^1 V(x, y_1, y_2, t)^2\, dy_1\, dy_2 = 0, \tag{3.3a}$$

$$\frac{\partial V}{\partial t} - \frac{\partial V}{\partial x} + V^2 - \int_0^1\int_0^1 U(x, y_1, y_2, t)^2\, dy_1\, dy_2 = 0, \tag{3.3b}$$

with initial data

$$U(x, y_1, y_2, 0) = u_0(x, y_1, y_2), \qquad V(x, y_1, y_2, 0) = v_0(x, y_1, y_2). \tag{3.4}$$

Case 2: $\varepsilon_1/\varepsilon_2 \to \alpha$, an irrational number, and $|\varepsilon_1/\varepsilon_2 - \alpha| \leqq c(\varepsilon_1)^r$ with $r > 1$. Then the homogenized equations for (3.1)–(3.2) are given by (3.3)–(3.4), the same homogenized equations as for the case 1.

Case 3: $\varepsilon_1/\varepsilon_2 \to m/n \neq 0$, and $|\varepsilon_1/\varepsilon_2 - m/n| \leqq c(\varepsilon_1)^r$ with $r > 1$. Then the homogenized equations for (3.1)–(3.2) are different from (3.3)–(3.4). They are given by

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U^2 - \frac{1}{n}\int_0^n V(x, z, (m/n)z, t)^2\, dz = 0, \tag{3.5a}$$

$$\frac{\partial V}{\partial t} - \frac{\partial V}{\partial x} + V^2 - \frac{1}{n}\int_0^n U(x, z, (m/n)z, t)^2\, dz = 0, \tag{3.5b}$$

with initial data

$$U(x, y_1, y_2, 0) = u_0(x, y_1, y_2), \qquad V(x, y_1, y_2, 0) = v_0(x, y_1, y_2). \tag{3.6}$$

THEOREM 3.1. *Suppose that $u_0(x, y_1, y_2)$, $v_0(x, y_1, y_2)$ are bounded, nonnegative and continuously differentiable. Then we have*

$$\left|u(x,t) - U\left(x, \frac{x-t}{\varepsilon_1}, \frac{x-t}{\varepsilon_2}, t\right)\right| \to 0 \quad \text{as} \quad \varepsilon_1, \varepsilon_2 \to 0,$$

$$\left|v(x,t) - V\left(x, \frac{x+t}{\varepsilon_1}, \frac{x+t}{\varepsilon_2}, t\right)\right| \to 0 \quad \text{as} \quad \varepsilon_1, \varepsilon_2 \to 0,$$

strongly in the L^∞-norm.

Remark 3.1. The above homogenization results can be generalized to the case where the initial data are of finitely many scales. For instances, if the initial values for (3.1) are given by

$$u(x,0) = u_0\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \frac{x}{\varepsilon_3}\right),$$

$$v(x,0) = v_0\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \frac{x}{\varepsilon_3}\right),$$

and if $\varepsilon_1/\varepsilon_2 \to m/n \neq 0$, $\varepsilon_3/\varepsilon_2 \to 0$, then the homogenized equations will be

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + U^2 - \frac{1}{n}\int_0^n\int_0^1 V(x, z_1, (m/n)z_1, z_2, t)^2\, dz_2\, dz_1 = 0,$$

$$\frac{\partial V}{\partial t} - \frac{\partial V}{\partial x} + V^2 - \frac{1}{n}\int_0^n\int_0^1 U(x, z_1, (m/n)z_1, z_2, t)^2\, dz_2\, dz_1 = 0,$$

with initial data

$$U(x, y_1, y_2, y_3, 0) = u_0(x, y_1, y_2, y_3),$$

$$V(x, y_1, y_2, y_3, 0) = v_0(x, y_1, y_2, y_3).$$

To prove Theorem 3.1, we need a few technical lemmas.

LEMMA 3.1. *Suppose that $f(x, y, z)$ is continuously differentiable and is 1-periodic in y and z. Assume that $\varepsilon_2/\varepsilon_1 \to 0$ as $\varepsilon_1 \to 0$; then, for any constants, a, b and x_0,*

(3.7)

$$\lim_{\varepsilon_1 \to 0} \frac{1}{(b-a)}\int_a^b\left(f\left(x, \frac{x-x_0}{\varepsilon_1}, \frac{x-x_0}{\varepsilon_2}\right) - \int_0^1\int_0^1 f(x, y, z)\, dy\, dz\right) dx = 0.$$

Proof: Arguing as in Lemma 2.2, we need only to prove the lemma in the case when f is independent of the first variable. We assume for simplicity that f has been normalized so that

$$\int_0^1\int_0^1 f(y, z)\, dy\, dz = 0. \tag{3.8}$$

For simplicity, we let $x_0 = 0$. By change of variable, the left-hand side of (3.7) becomes

$$\frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1} f\left(x, \frac{\varepsilon_1 x}{\varepsilon_2}\right) dx = \frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1}\left(\int_0^1 f(x,z)\,dz\right) dx \tag{3.9}$$
$$+ \frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1}\left(f\left(x, \frac{\varepsilon_1 x}{\varepsilon_2}\right) - \int_0^1 f(x,z)\,dz\right) dx.$$

Since $f(y,z)$ is assumed to be 1-periodic in y and z, the first term on the right-hand side of (3.9) is bounded by $c\varepsilon_1$ in light of (3.8). Define

$$g(y,z) \equiv f(y,z) - \int_0^1 f(y,z)\,dz.$$

Then $g(y,z)$ is 1-periodic in z and satisfies $\int_0^1 g(y,z)\,dz = 0$. The last term in (3.9) is then bounded by

$$\left|\frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1} g\left(x, \frac{\varepsilon_1 x}{\varepsilon_2}\right) dx\right| \leqq \sup_n \left|\int_n^{n+1} g\left(x, \frac{x}{\varepsilon_2/\varepsilon_1}\right) dx\right| + O(\varepsilon_1). \tag{3.10}$$

Applying Lemma 2.1 to $\int_n^{n+1} g(x, x/(\varepsilon_2/\varepsilon_1))\,dx$, we conclude that the left-hand side of (3.9) converges to zero as $\varepsilon_1 \to 0$. This completes the proof of Lemma 3.1.

LEMMA 3.2. *Suppose that $f(x,y,z)$ is continuously differentiable and is 1-periodic in y and z. Assume that $\varepsilon_1/\varepsilon_2 = \alpha + O((\varepsilon_1)^r)$ with $\alpha \neq 0$ and $r > 1$; then, for any constants a, b and x_0,*

$$\lim_{\varepsilon_1 \to 0} \frac{1}{b-a}\int_a^b f\left(x, \frac{x-x_0}{\varepsilon_1}, \frac{x-x_0}{\varepsilon_2}\right) dx$$
$$= \begin{cases} \dfrac{1}{b-a}\displaystyle\int_a^b\int_0^1\int_0^1 f(x,y,z)\,dy\,dz\,dx & \text{if } \alpha \text{ is irrational}, \\ \dfrac{1}{b-a}\displaystyle\int_a^b \frac{1}{n}\int_0^n f(x,z,(m/n)z)\,dz\,dx & \text{if } \alpha = m/n. \end{cases} \tag{3.11}$$

Proof: Arguing as in Lemma 2.2, we need only to prove the lemma in the case when f is independent of the first variable. For simplicity, we assume $x_0 = 0$. By change of variable, we obtain

$$\frac{1}{b-a}\int_a^b f\left(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_1}\right) dx = \frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1} f(y, \alpha y)\,dy \tag{3.12}$$
$$+ \frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1}\left(f\left(y, \left(\frac{\varepsilon_1}{\varepsilon_2}\right) y\right) - f(y, \alpha y)\right) dy.$$

The second term on the right-hand side of (3.12) is bounded by

$$\left|\frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1}\left(f\left(y,\left(\frac{\varepsilon_1}{\varepsilon_2}\right)y\right)-f(y,\alpha y)\right)dy\right| \leq \frac{C\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1}|y|\left|\frac{\varepsilon_1}{\varepsilon_2}-\alpha\right|dy \leq C\varepsilon_1^{r-1}. \tag{3.13}$$

For the first term on the right-hand side of (3.12), if α is irrational, Lemma 2.2 implies

$$\lim_{\varepsilon_1\to 0}\frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1}f(y,\alpha y)\,dy = \int_0^1\int_0^1 f(y,z)\,dy\,dz. \tag{3.14}$$

If $\alpha = m/n$, then $f(y,(m/n)y)$ is an n-periodic function in y. Thus we have

$$\lim_{\varepsilon_1\to 0}\frac{\varepsilon_1}{b-a}\int_{a/\varepsilon_1}^{b/\varepsilon_1}f\left(y,\left(\frac{m}{n}\right)y\right) = \frac{1}{n}\int_0^n f\left(y,\left(\frac{m}{n}\right)y\right)dy. \tag{3.15}$$

Lemma 3.2 then follows from (3.10)–(3.15).

Proof of Theorem 3.1: The solutions of the Carleman equations are known to be bounded for all time for bounded non-negative initial data (see [10], [13]). Thus Theorem 3.1 can be proved by using the similar techniques we use in the proof of Theorem 2.1 and the two lemmas above. We omit the proof.

Remark 3.1. If $\varepsilon_1/\varepsilon_2 \to \alpha \neq 0$ and the rate of convergence is of order $O((\varepsilon_1)^r)$ with $r \leq 1$, then Lemma 3.2 may not hold in general.

For examples, suppose that $\varepsilon_1/\varepsilon_2 = 1 + \varepsilon_1$ $(r = 1)$. We take $f(y,z) = \cos(2\pi y)\sin(2\pi z)$, $x_0 = 0$, $a = 0$, $b = \frac{1}{2}$. Then a direct calculation shows that

$$2\int_0^{1/2} f\left(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2}\right)dx - \int_0^1 f(z,z)\,dz = 1/\pi + O(\varepsilon_1) \neq 0.$$

4. General Model for a Gas with Discrete Velocity Distribution

Consider a gas composed of identical particles of mass m. The velocities of these particles are restricted to a given finite set of p vectors: $\mathbf{u}_1,\cdots,\mathbf{u}_p$. $N_i = N_i(x,t)$ denotes the number density of particles with velocity $\mathbf{u}_i$ at the point x and at the time t.

We consider the binary collisions only. Denote by $\mathbf{u}_i$ and $\mathbf{u}_j$ the velocities of two molecules before an encounter, after the encounter these molecules have velocities $\mathbf{u}_k$ and $\mathbf{u}_l$. They must satisfy the following two relations expressing the conservation of momentum and the conservation of energy:

$$\mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_k + \mathbf{u}_l,$$

$$|\mathbf{u}_i|^2 + |\mathbf{u}_j|^2 = |\mathbf{u}_k|^2 + |\mathbf{u}_l|^2.$$

A transition probability A_{ij}^{kl} is associated with the collision "$\mathbf{u}_i$, $\mathbf{u}_j$ to $\mathbf{u}_k$, $\mathbf{u}_l$". $A_{ij}^{kl}N_iN_j$ is the number of collisions $\mathbf{u}_i$, $\mathbf{u}_j$ to $\mathbf{u}_k$, $\mathbf{u}_l$ per unit time and unit volume, satisfying the particle indistinguishability:

$$A_{ij}^{kl} = A_{ij}^{lk} = A_{ji}^{lk}.$$

Then the Boltzmann equation is replaced by a system of p nonlinear partial differential equations (see [10]):

$$\frac{\partial N_i}{\partial t} + \mathbf{u}_i \cdot \nabla N_i = \frac{1}{2}\sum_{j,k,l}\left(A_{kl}^{ij}N_kN_l - A_{ij}^{kl}N_iN_j\right), \qquad i = 1, 2, \cdots, p. \tag{4.1}$$

We are especially interested in the case when the initial values are of the form:

$$N_{i,0}(x) = \phi_i\left(x, \frac{x}{\varepsilon}\right), \qquad i = 1, 2, \cdots, p, \tag{4.2}$$

where $\phi_i(x, y)$ are 1-periodic functions in each component of y.

ASSUMPTION. Let $\{\gamma_i,\ 1 \leqq i \leqq n\}$ be a sequence of non-zero numbers. Suppose that $\{\gamma_1, \cdots, \gamma_k\}$ is the largest linearly independent set of $\{\gamma_1, \cdots, \gamma_n\}$ among the integers in the following sense:

We say that $\{\gamma_1, \cdots, \gamma_k\}$ is linearly independent among integers provided $\mathbf{m} \cdot (\gamma_1, \cdots, \gamma_k) = 0$ for some integer-valued vector $\mathbf{m}$ implies that $\mathbf{m} = 0$.

Since $\{\gamma_1, \cdots, \gamma_k\}$ is the largest linearly independent set of $\{\gamma_1, \cdots, \gamma_n\}$, there exist integers l and m such that

$$\gamma_i = \frac{1}{m_i}\sum_{j=1}^{k} l_{i,j}\gamma_j \quad \text{for} \quad i = k+1, \cdots, n. \tag{4.3}$$

We denote by N_k the smallest common integral multiple of $m_{k+1}, \cdots, m_n$.

LEMMA 4.1. *Suppose $f(x_1, \cdots, x_n)$ is a continuous function, and is 1-periodic in each component. Let $\{\gamma_i,\ 1 \leq i \leq n\}$ be defined as in the above assumption.*

Then we have

Case 1: $k = 1$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x_1 + \gamma_1 t, \cdots, x_n + \gamma_n t)\, dt \tag{4.4}$$
$$= \frac{\gamma_1}{N_1}\int_0^{N_1/\gamma_1} f(x_1 + \gamma_1 t, \cdots, x_n + \gamma_n t)\, dt;$$

Case 2: $k = n$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x_1 + \gamma_1 t, \cdots, x_n + \gamma_n t)\, dt \tag{4.5}$$
$$= \int_0^1 \cdots \int_0^1 f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n;$$

Case 3: $1 < k < n$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x_1 + \gamma_1 t, \cdots, x_n + \gamma_n t)\, dt$$
$$= \frac{1}{(N_k)^k}\int_0^{N_k} \cdots \int_0^{N_k} f\Bigg(x_1 + z_1, \cdots, x_k + z_k, \tag{4.6}$$
$$x_{k+1} + \sum_{j=1}^{k} \frac{l_{k+1,j}}{m_{k+1}} z_j, \cdots, x_n + \sum_{j=1}^{k} \frac{l_{n,j}}{m_n} z_j\Bigg)\, dz_1 \cdots dz_k.$$

Proof: We only give the proof in the case $1 < k < n$. The other cases can be proved similarly. Let $M = N_k/\gamma_1$. We have

$$H \equiv \frac{1}{T}\int_0^T f(x_1 + \gamma_1 t, \cdots, x_n + \gamma_n t)\, dt \tag{4.7}$$
$$= \frac{1}{T}\sum_{j=0}^{[T/M]-1} \int_{jM}^{(j+1)M} f(x_1 + \gamma_1 t, \cdots, x_n + \gamma_n t)\, dt + O(1/T).$$

By change of variable $t = z_1 + jM$, we have

$$H = \frac{1}{T}\sum_{j=0}^{[T/M]-1} \int_0^M f\Bigg(x_1 + \gamma_1 z_1, x_2 + \gamma_2 z_1 + j\gamma_2 M, \cdots, x_k + \gamma_k z_1 + j\gamma_k M,$$
$$x_{k+1} + \gamma_{k+1} z_1 + \sum_{m=2}^{k} \frac{l_{k+1,m}}{m_{k+1}}(j\gamma_m M), \cdots,$$
$$x_n + \gamma_n z_1 + \sum_{m=2}^{k} \frac{l_{n,m}}{m_n}(j\gamma_m M)\Bigg)\, dz_1 + O(1/T).$$

Define function $g(z_2, \cdots, z_k)$ by

$$g(z_2, \cdots, z_k) \equiv \int_0^M f\Bigg(x_1 + \gamma_1 z_1, x_2 + \gamma_2 z_1 + z_2, \cdots, x_k + \gamma_k z_1 + z_k,$$

$$x_{k+1} + \gamma_{k+1} z_1 + \sum_{m=2}^{k} \frac{l_{k+1,m}}{m_{k+1}} z_m,$$

$$\cdots, x_n + \gamma_n z_1 + \sum_{m=2}^{k} \frac{l_{n,m}}{m_n} z_m \Bigg) \, dz_1.$$

Then $g(z_2, \cdots, z_k)$ is a continuous N_k-periodic function in each z_m for $2 \leqq m \leqq k$, and

$$H = \frac{1}{T} \sum_{j=0}^{[T/M]-1} g(j\gamma_2 M, \cdots, j\gamma_k M) + O(1/T).$$

Suppose that there exists an integer-valued vector $\mathbf{m} \in Z^k$ such that

$$m_2(\gamma_2 M) + \cdots + m_k(\gamma_k M) = -m_1 \in Z.$$

Recall that $M = N_k/\gamma_1$. So we have

$$m_1\gamma_1 + m_2 N_k \gamma_2 + \cdots + m_k N_k \gamma_k = 0.$$

The assumption on $\{\gamma_i;\ 1 \leqq i \leqq k\}$ implies that $m_i = 0$ for $1 \leqq i \leqq k$. Thus we conclude that $\{\gamma_i M;\ 2 \leqq i \leqq k\}$ are linearly independent among integers in the sense of [1]. Applying the theorem of "Ergodic Translation of Tori" in [1] to $g(z_2, \cdots, z_k)$ with $\omega = (\gamma_2 M, \cdots, \gamma_k M)$, we obtain

$$\lim_{T \to \infty} H = \frac{\gamma_1}{(N_k)^k} \int_0^{N_k} \cdots \int_0^{N_k} g(z_2, \cdots, z_k) \, dz_2 \cdots dz_k. \tag{4.8}$$

Thus we prove (4.6) by expressing the right-hand side of (4.8) in terms of f. This completes the proof of the lemma.

Now we can define the homogenized equations for the general discrete Boltzmann equations.

Suppose $f(y_1, \cdots, y_n)$ is a continuous 1-periodic function in y_i, $1 \leqq i \leqq n$. Let $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n)$.

DEFINITION. Define functional $F[f; \boldsymbol{\gamma}]$ as the limiting function on the right-hand side of (4.4), (4.5) and (4.6) corresponding to the cases $k = 1$, $k = n$ or $1 < k < n$, respectively.

For $\mathbf{x}$ and t fixed, we regard $M_j(\mathbf{x},\mathbf{y},t)$ as a function of $\mathbf{y}$ alone. Define the functional $G_i(M_j, M_k)$ as follows:

$$G_i\big(M_j(\mathbf{x},\mathbf{y},t),\, M_k(\mathbf{x},\mathbf{y},t)\big) \equiv \begin{cases} M_j M_k & \text{if } j = k = i, \\ M_j F\big[M_k;\, (\mathbf{u}_i - \mathbf{u}_k)\big] & \text{if } j = i,\ k \neq i, \\ M_k F\big[M_j;\, (\mathbf{u}_i - \mathbf{u}_j)\big] & \text{if } j \neq i,\ k = i, \\ F\big[M_j M_k;\, (\mathbf{u}_i - \mathbf{u}_j, \mathbf{u}_i - \mathbf{u}_k)\big] & \text{if } j \neq i,\ k \neq i, \end{cases}$$

where $F[M_j M_k;\ (\mathbf{u}_i - \mathbf{u}_j, \mathbf{u}_i - \mathbf{u}_k)]$ corresponds to the limiting function in Lemma 4.1 with f given by

$$f(\mathbf{y} + \alpha t, \mathbf{z} + \beta t) = M_j\big(\cdot, \mathbf{y} + (\mathbf{u}_i - \mathbf{u}_j)t, \cdot\big) M_k\big(\cdot, \mathbf{z} + (\mathbf{u}_i - \mathbf{u}_k)t, \cdot\big).$$

Then the homogenized equations are given by

$$\frac{\partial M_i}{\partial t} + \mathbf{u}_i \cdot \nabla_{\mathbf{x}} M_i = \frac{1}{2} \sum_{j,k,l} \big(A_{k,l}^{i,j} G_i(M_k, M_l) - A_{i,j}^{k,l} G_i(M_i, M_j) \big), \tag{4.9}$$

with initial values

$$M_i(\mathbf{x},\mathbf{y},0) = \phi_i(\mathbf{x},\mathbf{y}), \qquad i = 1, \cdots, p. \tag{4.10}$$

THEOREM 4.1. *Let M_i be the solutions of the homogenized equations* (4.9)–(4.10). *Then we have*

$$\sum_{i=1}^{p} \left\| N_i(\mathbf{x}, t) - M_i\left(\mathbf{x}, \frac{\mathbf{x} - t\mathbf{u}_i}{\varepsilon}, t\right) \right\|_{L^\infty(\mathbf{R}^3;\,[0,T])} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

provided that equations (4.1)–(4.2) *and* (4.9)–(4.10) *have bounded solutions for* $0 \leqq t \leqq T$.

Proof of Theorem 4.1: Using Lemma 4.1, we can prove Theorem 4.1 in a similar way to Theorem 2.1. We omit the proof.

Remark 4.1. The global existence theory of equations (4.1)–(4.2) has been investigated by several authors (see e.g. [12], [10], [5], [13] and [2]). The general results in [10] indicate that the bounded global solutions exist for long time if the initial data are small in some sense. The local existence of the homogenized equations (4.9) can be obtained by classical analysis. By combining the known global existence results for (4.1) with Theorem 4.1, we can show that bounded global solutions of (4.9) exist as long as the global solutions of (4.1) exist.

Remark 4.2. The techniques we use here in deriving the homogenized equations can be used for more complicated models. For examples, our techniques will apply to the semi-linear hyperbolic systems of the form

$$\frac{\partial N_i}{\partial t} + \mathbf{u}_i \cdot \nabla N_i = f_i(N_1, \cdots, N_k), \qquad i = 1, \cdots, k,$$

with oscillatory initial data

$$N_i(x, 0) = N_{i,0}\left(x, \frac{x}{\varepsilon}\right), \qquad i = 1, 2, \cdots, k,$$

if the $\mathbf{u}_i$ are constant vectors and the f_i are smooth and separable functions in N_j, $j = 1, \cdots k$.

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