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# STABILITY ANALYSIS OF FINITE DIFFERENCE APPROXIMATIONS TO HYPERBOLIC SYSTEMS, AND PROBLEMS IN APPLIED AND COMPUTATIONAL MATRIX THEORY

Period: 1 May 1983 - 30 April 1988



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### Marvin Marcus

**STABILITY CRITERIA FOR DIFFERENCE APPROXIMATIONS TO HYPERBOLIC SYSTEMS, AND MULTIPLICATIVITY OF MATRIX AND OPERATOR**

**Principal Investigator: Moshe Goldberg**

## ABSTRACT


Research completed under Grant AFOSR -83-0150 by Moshe Goldberg consists of the following two topics:

1 ~~(a)~~ Convenient stability criteria for difference approximations to hyperbolic initial-boundary value problems. and 2)

2 ~~(b)~~ Multiplicativity and stability of matrix and operator norms.


Keywords: finite differences, eigenvalues, stochastic processes. (KR)

## STABILITY CRITERIA FOR DIFFERENCE APPROXIMATIONS TO HYPERBOLIC SYSTEMS, AND MULTIPLICATIVITY OF MATRIX AND OPERATOR

**Principal Investigator: Moshe Goldberg**

### 1. Convenient Stability Criteria for Difference Approximations to Hyperbolic Initial-Boundary Value Problems

Consider the first order system of hyperbolic partial differential equations

$$\partial u(x,t)/\partial t = A\partial u(x,t)/\partial x + Bu(x,t) + f(x,t), \quad x \geq 0, \quad t \geq 0, \tag{1.1a}$$

where $u(x,t) = (u^{(1)}(x,t), \ldots, u^{(n)}(x,t))'$ is the unknown vector (prime denoting the transpose), $f(x,t) = (f^{(1)}(x,t), \ldots, f^{(n)}(x,t))'$ is a given *n*-vector, and A and B are fixed $n \times n$ matrices such that A is diagonal of the form

$$A = \begin{bmatrix} A^{I} & 0 \\ 0 & A^{II} \end{bmatrix}, \quad A^{I} > 0, \quad A^{II} < 0, \tag{1.2}$$

with $A^{I}$ and $A^{II}$ *of orders* $k \times k$ and $(n - k) \times (n - k)$, *respectively.*

The solution of (1.1a) is uniquely determined if we prescribe initial values

$$u(x,0) = \mathring{u}(x), \quad x \geq 0, \tag{1.1b}$$

and boundary conditions

$$u^{II}(0,t) = Su^{I}(0,t) + g(t), \quad t \geq 0, \tag{1.1c}$$

where S is a fixed $(n - k) \times k$ coupling matrix, $g(t)$ a given $(n - k)$-vector, and

$$u^{I} = (u^{(1)}, \ldots, u^{(k)})', \quad u^{II} = (u^{(k+1)}, \ldots, u^{(n)})' \tag{1.3}$$

a partition of $\mathbf{u}$ into its outflow and inflow components, respectively, corresponding to the partition of A in (1.2).

In the past five years, E. Tadmor and I [22-24] extended our previous results in [19,20] to achieve versatile, easily checkable stability criteria for a wide class of finite difference approximations to the above initial-boundary value problem.

More specifically, introducing a mesh size $\Delta x > 0$, $\Delta t > 0$, such that $\lambda \equiv \Delta t/\Delta x$ = constant, and using the notation $\mathbf{v}_\nu(t) = \mathbf{v}(\nu\Delta x, t)$, we approximated (1.1a) by a general, basic difference scheme -- explicit or implicit, dissipative or not, two-level or multilevel -- of the form

$$\begin{aligned} &Q_{-1}\mathbf{v}_\nu(t+\Delta t) = \sum_{\sigma=0}^{s} Q_\sigma \mathbf{v}_\nu(t-\sigma\Delta t) + \Delta t\mathbf{b}_\nu(t), \quad \nu = r, r+1, \ldots, \\ &Q_\sigma = \sum_{j=-r}^{p} A_{j\sigma}E^j, \quad E\mathbf{v}_\nu = \mathbf{v}_{\nu+1}, \quad \sigma = -1, \ldots, s, \end{aligned} \tag{1.4}$$

where the $n \times n$ coefficient matrices $A_{j\sigma}$ are polynomials in $\lambda A$ and $\Delta tB$, and the n-vectors $\mathbf{b}_\nu(t)$ depend on $\mathbf{f}(x,t)$ and its derivatives.

The difference equations in (1.4) have a unique solution $\mathbf{v}_\nu(t+\Delta t)$ if we provide initial values

$$\mathbf{v}_\nu(\mu\Delta t) = \mathring{\mathbf{v}}_\nu(\mu\Delta t), \quad \mu = 0, \ldots, s, \quad \nu = 0, 1, 2, \ldots, \tag{1.5}$$

and specify, at each time level $t = \mu\Delta t$, $\mu = s, s+1, \ldots,$ boundary values $\mathbf{v}_\nu(t+\Delta t)$, $\nu = 0, \ldots, r-1$. Such boundary values are determined by boundary conditions of the form

$$T_{-1}^{(\nu)} \mathbf{v}_\nu(t+\Delta t) = \sum_{\sigma=0}^{q} T_\sigma^{(\nu)} \mathbf{v}_\nu(t-\sigma\Delta t) + \Delta t \mathbf{d}_\nu(t), \quad \nu = 0, \ldots, r-1,$$

$$T_\sigma^{(\nu)} = \sum_{j=0}^{m} C_{j\sigma}^{(\nu)} E^j, \quad \sigma = -1, \ldots, q, \tag{1.6a}$$

where the $n \times n$ matrices $C_{j\sigma}^{(\nu)}$ depend on A, $\Delta t$B and S; and the n-vectors $\mathbf{d}_\nu(t)$ are *functions of* $\mathbf{f}(x,t)$, $\mathbf{g}(t)$ *and their derivatives.*

Our intention was to interpret the difficult and often stubborn Gustafsson-Kreiss-Sundstrõm (GKS) stability criterion in [26] in order to obtain simple and convenient stability criteria for approximation (1.4) - (1.6a). While we were unable to meet this goal for general boundary conditions of type (1.6a), we managed to achieve rather satisfactory results under the further assumption that, in accordance with the partition of A in (1.2), the $C_{j\sigma}^{(\nu)}$ can be written as

$$C_{j\sigma}^{(\nu)} = \begin{bmatrix} C_{j\sigma}^{I\,I} & C_{j\sigma}^{I\,II(\nu)} \\ C_{j\sigma}^{II\,I(\nu)} & C_{j\sigma}^{II\,II(\nu)} \end{bmatrix}, \tag{1.6b}$$

where

$$\text{the } C_{j\sigma}^{I\,I} \text{ are independent of } \nu, \tag{1.6c}$$

$$\text{the } C_{j\sigma}^{I\,I} \text{ are diagonal when } B = 0, \tag{1.6d}$$

$$\text{the } C_{j\sigma}^{I\,II(\nu)} = 0 \text{ when } B = 0, \tag{1.6e}$$

$$C_{j\sigma}^{II\,II(\nu)} = 0 \text{ for } j > 0 \text{ and } \sigma > -1 \text{ when } B = 0. \tag{1.6f}$$

The essence of (1.6c)-(1.6e) is that for $B = 0$, the outflow boundary conditions are *translatory* (i.e., determined at all boundary points by the same coefficients), *separable* (i.e., split into independent scalar conditions for the different outflow unknowns), and independent of outflow values. Assumption (1.6f) implies that for $B = 0$, the inflow values at the boundary depend essentially on the outflow.

It should be pointed out that our outflow boundary conditions are quite general, despite the apparent restrictions in (1.6c)-(1.6e). Indeed, (1.6c) is not much of a restriction, since in practice the outflow boundary conditions are translatory. In particular, if the numerical boundary consists of a single point, then the boundary conditions are translatory by definition, so (1.6c) holds automatically. The restrictions in (1.6d), (1.6e) pose no great difficulties either, since they are satisfied by all reasonable boundary conditions, where for $B = 0$ the $C_{j\sigma}^{I\,I}$ usually reduce to polynomials in the diagonal block $A^I$, and the $C_{j\sigma}^{I\,II(v)}$ vanish.

We realize that in view of the restriction in (1.6f) our inflow boundary conditions are not quite as general as the outflow ones. They can, however, be constructed to any degree of accuracy (see [20]); and if the boundary consists of a single point, then such conditions can be achieved in a trivial manner, simply by duplicating the analytic condition (1.1c), which gives

$$v_0^{II}(t + \Delta t) = S v_0^{I}(t + \Delta t) + g(t + \Delta t).$$

Throughout our work we assume, of course, that the basic scheme (1.4) is stable for the pure Cauchy problem, and that the other assumptions which guarantee the validity of the GKS theory in [26], hold.

The first step in our analysis was to reduce the above stability question to that of a scalar, homogeneous problem. This is obtained by considering the outflow scalar equation

$$\partial u/\partial t = a\partial u/\partial x, \quad x \geq 0, \quad t \geq 0, \quad a = \text{constant} > 0, \tag{1.7}$$

for which the basic scheme (1.4) reduces to the homogeneous scheme

$$Q_{-1}v_{\nu}(t+\Delta t) = \sum_{\sigma=0}^{s} Q_{\sigma}v_{\nu}(t-\sigma\Delta t)$$

$$Q_{\sigma} = \sum_{j=-r}^{p} a_{j\sigma}E^{j}, \quad \sigma = -1, \ldots, s, \tag{1.8a}$$

and the boundary conditions (1.6) reduce to translatory conditions of the form

$$T_{-1}v_{\nu}(t+\Delta t) = \sum_{\sigma=0}^{q} T_{\sigma}v_{\nu}(t-\sigma\Delta t)$$

$$T_{\sigma} = \sum_{r=0}^{m} c_{j\sigma}E^{j}, \quad \sigma = -1, \ldots, q, \tag{1.8b}$$

where $a_{j\sigma}$ and $c_{j\sigma}$ are scalar coefficients.

Referring to (1.8) as the *basic approximation*, we proved:

**Theorem 1.1** [24]. *Approximation* (1.4)-(1.6) *is stable if and only if the reduced outflow scalar approximation* (1.8) *is stable for every eigenvalue* $a > 0$ *of* $A^{I}$. *That is, approximation* (1.4)-(1.6) *is stable if and only if the scalar outflow components of its principal part are all stable.*

This reduction theorem implies that from now on we may restrict our stability study to the basic approximation (1.8).

In order to introduce our stability criteria for the basic approximation, we use the coefficients of the basic scheme (1.8a) to define the *basic characteristic function*

$$P(z,\kappa) = \sum_{j=-r}^{p}\left[a_{j,-1} - \sum_{\sigma=0}^{s} a_{j\sigma}z^{-\sigma-1}\right]\kappa^{j}.$$

Similarly, using the coefficients of the boundary conditions in (1.8b) we define the boundary characteristic function

$$R(z,\kappa) = \sum_{j=0}^{m} \left[ c_{j,-1} - \sum_{\sigma=0}^{q} c_{j\sigma} z^{-\sigma-1} \right] \kappa^j .$$

Now putting

$$\Omega(z,\kappa) \equiv | P(z,\kappa) | + | R(z,\kappa) |,$$

we proved:

**Theorem 1.2** [20]. *The basic approximation* (1.8) *is stable if* $\Omega(z,\kappa) > 0$ *for all*

$$\{ |z| = |\kappa| = 1,\ (z,\kappa) \neq (1,1) \} \cup \{ |z| \geq 1,\ 0 < |\kappa| < 1 \} . \quad (1.9)$$

In fact, we often found it convenient to divide the $(z,\kappa)$ domain in (1.9) into three disjoint parts, and restate Theorem 1.2 as follows:

**Theorem 1.2'**. *Approximation* (1.8) *is stable if*

$$\Omega(z,\kappa) > 0 \quad \textit{for all} \quad |z| = |\kappa| = 1, \quad \kappa \neq 1, \quad (1.10a)$$

$$\Omega(z,\kappa = 1) > 0 \quad \textit{for all} \quad |z| = 1, \quad z \neq 1, \quad (1.10b)$$

$$\Omega(z,\kappa) > 0 \quad \textit{for all} \quad |z| \geq 1, \quad 0 < |\kappa| < 1. \quad (1.10c)$$

The advantage of this setting over that of Theorem 1.2 is clarified by the following lemma, in which we provide helpful sufficient conditions for each of the three inequalities in (1.10) to hold:

**Lemma 1.3** [24].

(i) *Inequality* (1.10a) *holds if either the basic scheme* (1.8a) *or the boundary conditions* (1.8b) *are dissipative.*

(ii) *Inequality* (1.10b) *holds if any of the following is satisfied:*

(a) *The basic scheme is two-level.*

(b) *The basic scheme is three-level and*

$$\Omega(z = -1, \kappa = 1) > 0. \tag{1.11}$$

(c) *The boundary conditions are two-level and at least zero-order accurate as an approximation to equation* (1.7).

(d) *The boundary conditions are three-level, at least zero-order accurate, and* (1.11) *is satisfied.*

(iii) *Inequality* (1.10c) *holds if the boundary conditions fulfill the von Neumann condition, and are either explicit or satisfy*

$$T_{-1}(\kappa) \equiv \sum_{j=0}^{m} c_{j,-1}\kappa^{j} \neq 0 \quad \forall\ 0 < |\kappa| \leq 1.$$

We note that if both the basic scheme and the boundary conditions are unitary (i.e., strictly nondissipative), then $\Omega(z = -1, \kappa = -1) = 0$; hence Theorem 1.2 is rendered useless. For such cases we proved

**Theorem 1.4** [24]. *Approximation* (1.8)*is stable if*

$$\left.\frac{\partial P(z,\kappa)}{\partial z} \cdot \frac{\partial P(z,\kappa)}{\partial \kappa}\right|_{z = \kappa = -1} < 0,$$

and $\Omega(z,\kappa) > 0$ for all

$$\left\{ |z| = |\kappa| = 1,\ \ (z,\kappa) \neq \pm(1,1) \right\} \cup \left\{ |z| \geq 1,\ \ 0 < |\kappa| < 1 \right\}$$

The above lemma applies to this theorem precisely in the same way it applied to Theorem 1.2.

The stability criteria obtained in Theorems 1.2 and 1.4 depend both on the basic difference scheme and on the boundary conditions, but not on the intricate and often complicated interaction between the two. Consequently, Theorems 1.2 and 1.4, aided by Lemma 1.3, provide in many cases a convenient alternative to the celebrated stability criteria of Kreiss [31] and of Gustafsson, Kreiss and Sundström [26].

Having the new criteria, we easily established stability for a host of examples that incorporate and generalize most of the cases studied in recent literature; e.g., [4, 5, 19, 20, 22-27, 30, 32, 38, 39, 42- 44, 47, 50]. To mention just a few of our examples, we proved stability for:

(a) Arbitrary two-level schemes, with boundary conditions generated by either the explicit or implicit one-sided Euler schemes.

(b) Arbitrary two-level schemes, with boundary conditions generated by either horizontal extrapolation or by the one-sided three-level Euler scheme.

(c) Arbitrary dissipative schemes, with boundary condition generated by oblique extrapolation or by the Box scheme.

(d) The Crank-Nicolson, Backward-Euler, Leap-Frog and Lax-Friedrichs schemes (all nondissipative), with boundary conditions generated by either oblique extrapolative or by the one-sided Weighted Euler scheme.

We drew great satisfaction from the fact that our theory and examples in [19, 20, 22-24] were used already by a number of authors, including Berger [2], LeVeque [34], South, Hafez and Gottlieb [45], Thuné [49], Trefethen [50, 51], and Yee [53]. Thuné [49], in his effort to provide a software package for stability analysis of finite difference approximations to hyperbolic initial-boundary value problems, says: "...Another approach has been to derive new criteria, based on the Gustafsson-Kreiss-Sundström theory but more convenient for practical use... The most far-reaching work along these lines has been made by Goldberg and Tadmor [19, 20, 22] ..."

We were also pleased to learn that part of our theory in [24] was taught already in several institutions including UCLA and the University of Paris VI.

## 2. Multiplicativity and Stability of Matrix and Operator Norms

Let V be a normed vector space over the complex field **C**, and let $\mathcal{B}(V)$ be the algebra of bounded linear operators on V. As usual, a real-valued function

$$N : \mathcal{B}(V) \rightarrow \mathbf{R}$$

is called a *norm* on $\mathcal{B}(V)$ if for all A, B $\in \mathcal{B}(V)$ and $\alpha \in \mathbf{C}$,

$$\begin{aligned} N(A) &> 0, \quad A \neq 0, \\ N(\alpha A) &= |\alpha| \cdot N(A), \\ N(A+B) &\leq N(A) + N(B). \end{aligned}$$

If in addition N is *multiplicative*, i.e.,

$$N(AB) \leq N(A)\, N(B) \quad \forall\, A, B \in \mathcal{B}(V)\,,$$

we say that N is an *operator norm* on $\mathcal{B}(V)$. If $\mathcal{B}(V)$ is an algebra of (finite) matrices and N is multiplicative, then N is called a *matrix norm.*

The first multiplicative example that comes to mind is of course, the ordinary operator norm

$$\|A\| = \sup\left\{ |Ax| : x \in V,\ |x| = 1 \right\}, \tag{2.1}$$

where $|\cdot|$ is the vector norm on V.

If V is a (finite- or infinite-dimensional) Hilbert space, then perhaps the best known example of a nonmultiplicative norm on $\mathcal{B}(V)$ is the numerical radius (e.g., [1, 6, 21, 28, 41])

$$r(A) = \sup\left\{ |(Ax, x)| : x \in V,\ |x| = (x, x)^{1/2} = 1 \right\} \tag{2.2}$$

which plays an important role in stability analysis of finite difference schemes for multi-space-dimensional hyperbolic initial-value problems [21, 33, 35, 52].

Another example of considerable interest is the $\ell_p$ norm, $1 \le p \le \infty$, of an $n \times n$ complex matrix $A = (\alpha_{ij}) \in \mathbf{C}_{n \times n}$:

$$|A|_p = \left( \sum_{i,j=1}^{n} |\alpha_{ij}|^p \right)^{1/p}. \tag{2.3}$$

Ostrowski [40] has shown that this norm is multiplicative (i.e., a matrix norm) if and only if $1 \le p \le 2$.

Given a norm N on $\mathcal{B}(V)$ and a fixed constant $\mu > 0$, then obviously $N_\mu \equiv \mu N$ is a norm too. Clearly, $N_\mu$ may or may not be multiplicative. If it is, then we call $\mu$ a *multiplicativity factor* for N. That is, $\mu$ is a multiplicativity factor for N if and only if

$$N(AB) \le \mu N(A)N(B) \quad \forall \; A, B \in \mathcal{B}(V).$$

Having this definition one can obtain at once:

**Theorem 2.1** [36, 15]. *Let* N *be a norm on* $\mathcal{B}(V)$. *Then*

(i) N *has multiplicativity factors if and only if*

$$\mu_{min} \equiv \sup \left\{ N(AB) : N(A) = N(B) = 1; \; A, B \in \mathcal{B}(V) \right\} < \infty. \tag{2.4}$$

(ii) *If* $\mu_{min} < \infty$, *then* $\mu$ *is a multiplicativity factor for* N *if and only if* $\mu \ge \mu_{min}$.

In the finite-dimensional case, compactness immediately implies that $\mu_{min} < \infty$; hence N always has multiplicativity factors. In the infinite-dimensional case, however, N may fail to have multiplicativity factors, as was demonstrated by Straus and myself in [15].

While Theorem 2.1 seems to settle the question of characterizing multiplicativity factors, the quantity $\mu_{min}$ in (2.4) is often difficult to compute. A more practical approach towards verifying whether a constant $\mu_{min} > 0$ is the best (least) multiplicativity factor for a given norm N is implied by the following obvious observation:

*A constant* $\mu_{min} > 0$ *is the best (least) multiplicativity factor for* N *if*

$$N(AB) \leq \mu_{min} N(A)N(B) \quad \forall\ A, B \in \mathcal{B}(V),$$

*with equality for some nonzero* $A = A_0$, $B = B_0$.

With this observation in mind, it was shown by Holbrook [29] (and independently by Straus and myself in [13]) that if V is a Hilbert space of dimension at least 2, and if r is the numerical radius defined in (2.2), then $\mu r$ is an operator norm on $\mathcal{B}(V)$ if and only if $\mu \geq 4$; i.e., the best multiplicativity factor for r is $\mu_{min} = 4$.

Similarly, Maitre [36], and Straus and I [17] showed that the best multiplicativity factor for the $\ell_p$ norm on $C_{n \times n}$ defined in (2.3) is

$$\mu_{min} = \begin{cases} 1, & 1 \leq p \leq 2, \\ n^{1-2/p}, & 2 \leq p \leq \infty . \end{cases}$$

Often, when $\mu_{min}$ remains unknown, one may obtain multiplicativity factors via the following somewhat stronger version of a result by Gastinel:

**Theorem 2.2** [3, 13]. *Let* N *and* M *be a norm and an operator norm on* $\mathcal{B}(V)$, *respectively; and let* $\eta \geq \xi > 0$ *be constants such that*

$$\xi M(A) \leq N(A) \leq \eta M(A) \quad \forall\ A \in \mathcal{B}(V).$$

*Then any* $\mu$ *with* $\mu \geq \eta/\xi^2$ *is a multiplicativity factor for* N.

This result was utilized by Straus and myself [13-16, 18] to obtain multiplicativity factors for certain generalizations of the numerical radius, called C-numerical radii.

The above concepts of multiplicativity and multiplicativity-factors were extended by me in 1983 as follows:

**Definition 1.** Let **U**, **V**, and **W** be normed vector spaces over **C**; and let $\mathcal{B}_1 = \mathcal{B}(\mathbf{U}, \mathbf{W})$, $\mathcal{B}_2 = \mathcal{B}(\mathbf{V}, \mathbf{W})$, and $\mathcal{B}_3 = \mathcal{B}(\mathbf{U}, \mathbf{V})$ be the spaces of bounded linear operators from **U** into **W**, **V** into **W**, and **U** into **V**, respectively. If $N_1$, $N_2$, and $N_3$ are norms on $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$, respectively, and $\mu > 0$ is a constant such that

$$N_1(AB) \le \mu N_2(A) N_3(B) \quad \forall\, A \in \mathcal{B}_2,\ \ B \in \mathcal{B}_3,$$

then we say that $\mu$ is a *multiplicativity factor for* $N_1$ *with respect to* $N_2$ *and* $N_3$.

In analogy with Theorem 2.1 we have now:

**Theorem 2.3** [11]. *Let* $N_1$, $N_2$, *and* $N_3$ *be norms as in Definition* 1. *Then*:
(i) $N_1$ *has multiplicativity factors with respect to* $N_2$ *and* $N_3$ *if and only if*

$$\mu_{min} \equiv \sup\left\{ N_1(AB) : N_2(A) = N_3(B) = 1;\ \ A \in \mathcal{B}_2,\ \ B \in \mathcal{B}_3 \right\} < \infty.$$

(ii) *If* $\mu_{min} < \infty$, *then* $\mu$ *is a multiplicativity factor for* $N_1$ *with respect to* $N_2$ *and* $N_3$, *if and only if* $\mu \ge \mu_{min}$.

We observe, of course, that *a constant* $\mu_{min} > 0$ *is the best (least) multiplicativity factor for* $N_1$ *with respect to* $N_2$ *and* $N_3$ *if*

$$N_1(AB) \le \mu_{min} N_2(A) N_3(B) \quad \forall\ A \in \mathcal{B}_2,\ B \in \mathcal{B}_3,$$

*with equality for some nonzero* $A = A_0$, $B = B_0$.

For example, if **V** is a Hilbert space, and if $\|\cdot\|$ and r are the operator norm and numerical radius in (2.1) and (2.2), then Holbrook [29] has shown that

$$r(AB) \leq 2r(A)\| B \| \quad \forall\ A, B \in \mathcal{B}(V),$$

with equality for certain $A = A_0$, $B = B_0$. Thus, $\mu_{min} = 2$ is the best multiplicativity factor for r with respect to r and $\|\cdot\|$.

This example employs only a single vector space and two norms. In order to demonstrate the idea of mixed multiplicativity to its full extent, consider, for $1 \leq p \leq \infty$, the $\ell_p$ norm of an $m \times n$ matrix $A = (a_{ij}) \in \mathbf{C}_{m \times n}$:

$$|A|_p = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_{ij}|^p \right)^{1/p}. \tag{2.5}$$

Defining

$$\lambda_{pq}(m) = \begin{cases} 1, & p \geq q \\ m^{1/p - 1/q}, & q \geq p, \end{cases}$$

I proved:

**Theorem 2.4** [9]. *Let* p, q, r *satisfy* $1 \leq p, q, r \leq \infty$, *and let* q' *be the conjugate of* q (i.e., $1/q + 1/q' = 1$). *Then the best multiplicativity factor for the* $\ell_p$ *norm on* $\mathbf{C}_{m \times n}$ *with respect to the* $\ell_q$ *norm on* $\mathbf{C}_{m \times k}$ *and the* $\ell_r$ *norm on* $\mathbf{C}_{k \times n}$ *is*

$$\mu_{min} = \lambda_{pq}(m)\, \lambda_{pr}(n)\, \lambda_{q'r}(k).$$

*That is, for all* $A \in \mathbf{C}_{m \times k}$ *and* $B \in \mathbf{C}_{k \times n}$

$$| AB |_p \le \lambda_{pq}(m)\, \lambda_{pr}(n)\, \lambda_{q'r}(k)\, | A |_q\, | B |_r \ , \tag{2.6}$$

*where this inequality is sharp.*

Theorem 2.4 (which generalizes some of the results in [7,8]) has quite a few applications. For example (see [9, 12]), taking (2.6) with $m = n = 1$, we get an upper bound for the standard inner product $(x, y)$ on $\mathbf{C}^n$ in terms of $| x |_p$ and $| y |_q$; and if we further set $r = q'$ we obtain the classical Hölder inequality.

Another application of Theorem 2.4 concerns the "ordinary" $\ell_p$ operator-norm on $\mathbf{C}_{m \times n}$:

$$\| A \|_p = \sup \left\{ | Ax |_p \ : \ x \in \mathbf{C}^n,\ | x |_p = 1 \right\}, \tag{2.7}$$

for which I proved:

**Theorem 2.5** [11]. *Let* $p, q, r$ *satisfy* $1 \le p, q, r, \le \infty$. *Then for all* $A \in \mathbf{C}_{m \times k}$, $B \in \mathbf{C}_{k \times n}$,

$$\| AB \|_p \le \lambda_{pq}(m)\, \lambda_{qp}(k)\, \lambda_{pr}(k)\, \lambda_{rp}(n)\, \| A \|_q\, \| B \|_r \ ,$$

*where the inequality is sharp if either* $q \le p \le r$ *or* $r \le p \le q$.

Another consequence of (2.6) describes the equivalence relation between the norms in (2.5) and (2.7):

**Theorem 2.6** [10]. *Let* $p, q$ *satisfy* $1 \le p, q \le \infty$, *and let* $q'$ *be the conjugate of* $q$. *Then for all* $A \in \mathbf{C}_{m \times n}$,

$$| A |_p \leq \lambda_{pq}(mn) | A |_q ,$$

$$\| A \|_p \leq \lambda_{pq}(m)\, \lambda_{qp}(n)\, \| A \|_q ,$$

$$\| A \|_p \leq \lambda_{pq}(m)\, \lambda_{q'p}(n)\, | A |_q ,$$

$$| A |_p \leq (mn)^{1/p} \| A \|_q ,$$

*where the first three inequalities are sharp.*

Having the above results, it should be possible to construct a complete table of best (least) multiplicativity factors and equivalence constants for r, $| \cdot |_p$ and $\| \cdot \|_p$, as well as for other useful norms such as the Householder norms described in [47].

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2. M. J. Berger, Stability of Interfaces with Mesh Refinement, Math. Comp. 45 (1985), 301-318.

3. N. Gastinel, *Linear Numerical Analysis*, Academic Press, New York, 1970.

4. M. Goldberg, On a boundary extrapolation theorem by Kreiss, Math. Comp. 31 (1977), 469-477.

5. M. Goldberg, On boundary extrapolation and dissipative schemes for hyperbolic problems, Proceedings of the 1977 Army Numerical Analysis and Computer Conference, ARO Report 77-3, 157-164.

6. M. Goldberg, On certain finite dimensional numerical ranges and numerical radii, Linear and Multilinear Algebra 7 (1979), 329-342.

7. M. Goldberg, Some inequalities for $\ell_p$ norms of matrices, in "General Inequalities 4", edited by W. Walter, Birkhäuser Verlag, Basel, 1984, pp. 185-189.

8. M. Goldberg, Multiplicativity of $\ell_p$ norms for matrices. II, Linear Algebra Appl. 62 (1984), 1-10.

9. M. Goldberg, Mixed multiplicativity and $\ell_p$ norms for matrices, Linear Algebra Appl. 73 (1986), 123-131.

10. M. Goldberg, Equivalence constants for $\ell_p$ norms of matrices, Linear and Multilinear Algebra 21 (1987), 173-179

11. M. Goldberg, Multiplicativity factors and mixed multiplicativity, Linear Algebra Appl. 97 (1987), 45-56.

12. M. Goldberg, Mixed-multiplicativity for $\ell_p$ norms of matrices, in "Linear Algebra in Signals, Systems, and Control", edited by B.N. Datta et al., SIAM, Philadelphia, 1988, 44-47.

13. M. Goldberg and E.G. Straus, Norm properties of C-numerical radii, Linear Algebra and Appl. 24 (1979), 113-131.

14. M. Goldberg and E.G. Straus, Combinatorial inequalities, matrix norms, and generalized numerical radii, in "General Inequalities 2", edited by E.F. Beckenbach, Birkhäuser Verlag, Basel, 1980, 37-46.

15. M. Goldberg and E.G. Straus, Operator norms, multiplicativity factors, and C-numerical radii, Linear Algebra Appl. 43 (1982), 137-159.

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17. M. Goldberg and E.G. Straus, Multiplicativity of $\ell_p$ norms for matrices, Linear Algebra Appl. 52-53 (1983), 351-360.

18. M. Goldberg and E.G. Straus, Multiplicativity factors for C-numerical radii, Linear Algebra Appl. 54 (1983), 1-16.

19. M. Goldberg and E. Tadmor, Scheme-independent stability criteria for difference approximations of hyperbolic initial-boundary value problems. I, Math. Comp. 32 (1978), 1097-1107.

20. M. Goldberg and E. Tadmor, Scheme-independent stability criteria for difference approximations of hyperbolic initial-boundary value problems. II, Math. Comp. 36 (1981), 605-626.

21. M. Goldberg and E. Tadmor, On the numerical radius and its applications, Linear Algebra Appl. 42 (1982), 263-184.

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38. J. Oliger, Fourth order difference methods for the initial-boundary value problem for hyperbolic equations, Math. Comp. 28 (1974), 15-25.

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41. C. Pearcy, An elementary proof of the power inequality for the numerical radius, Mich. Math. J. 13 (1966), 289-291.

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43. G. Skolermo, How the Boundary Conditions Affect the Stability and Accuracy of Some Implicit Methods for Hyperbolic Equations, Report No. 62, 1975, Dept. of Computer Science, Uppsala University, Uppsala, Sweden.

44. G. Skolermo, Error Analysis for the Mixed Initial Boundary Value Problem for Hyperbolic Equations, Report No. 63, 1975, Dept. of Computer Science, Uppsala University, Uppsala, Sweden.

45. J.C. South, Jr., M.M. Hafez and D. Gottlieb, Stability analysis of intermediate boundary conditions in approximate factorization schemes, ICASE Report 86-36, NASA, 1986.

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49. M. Thuné, IBSTAB - A Software System for Automatic Stability Analysis of Difference Methods for Hyperbolic Initial-Boundary Value Problems, Report No. 93, 1984, Department of Computer Science, Uppsala University, Uppsala, Sweden.

50. L.N. Trefethen, Wave Propagation and Stability for Finite Difference Schemes, Ph.D. Thesis, Report No. STAN-CS-82-905, Computer Science Department, Stanford University, Stanford, California, 1982.

51. L.N. Trefethen, Stability of Finite Difference Models Containing Two Boundaries of Interfaces, Math. Comp. 45 (1985), 279-300.

52. E. Turkel, Symmetric hyperbolic difference schemes and matrix problems, Linear Algebra Appl. 16 (1977), 109-129.

53. H.C. Yee, Numerical Approximation of Boundary Conditions with Applications to Inviscid Equations of Gas Dynamics, NASA Technical Memorandum 81265, 1981, NASA Ames Research Center, Moffett Field, California.

## VITA

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**Military Service:** Israel Defense Force, 1965-1968, First Lieutenant

**Academic Degrees:**

1965 B.Sc., Applied Mathematics, Tel Aviv University

1970 M.Sc. (magna cum laude), Applied Mathematics, Tel Aviv University

1973 Ph.D. Applied Mathematics, Tel Aviv University

**Academic Appointments:**

1972- 1974 Instructor, Department of Mathematics, Tel Aviv University, Tel Aviv, Israel

1974- 1979 Assistant Professor, Department of Mathematics, University of California, Los Angeles

1979- 1985 Senior Lecturer, Department of Mathematics, Technion, Haifa, Israel

1985 present Associate Professor, Department of Mathematics, Technion, Haifa, Israel

Moshe Goldberg

**Professional Experience:**

Associate Research Mathematician, Institute for the Interdisciplinary Application of Algebra and Combinatorics, University of California, Santa Barbara Summers of 1980-86

Invited Guest, Department of Mathematics, University of California, Los Angeles, Summers of 1981-88

Visiting Professor, Department of Mathematics, University of California, Los angeles, academic year 1985/86

Visiting Associate Mathematician, Department of Mathematics, California Institute of Technology, Pasadena, California, Fall 1985

Visiting Professor, Centre de Rechereche de Mathematiques de la Decision, University of Paris IX, Paris, Spring 1986

Associate Research Mathematician, Center for Computational Sciences and Engineering, University of California, Santa Barbara, Summers of 1987-88

**Editorial Activities:**

Editor, "Linear Algebra and its Applications", Elsevier Science Publishing Co., New York

Editor,"Linear and Multilinear Algebra", Gordon and Breach Science Publisher, New York

Editor, "Algebras, Groups and Geometries", Hadronic Press Inc., Nonantum, Massachusetts

Referee for several mathematical journals and for "Letters in Physics"

Referee for the Applied Mathematics Section of the U.S. National Science Foundation

**Selected Administrative Activities:**

Officer, Executive Committee, Society for Industrial and Applied Mathematics (SIAM), Southern California Section, 1975-78

Chairman, Applied Mathematics Colloquium University of California, Los Angleles, 1975-77

Organizing Committee, Joint AMS-MAA-SIAM Meeting, Pomona College, Claremont, California, October 19-21, 1978

Organizer, Minisymposium on Finite Difference Approximations to Hyperbolic Initial-Boundary Value Problems, as part of the First International Conference on Industrial and Applied Mathematics, Paris June 29 - July 4, 1987.

Chairman, Mathematics Colloquium, Technion, 1981-83

Organizer, Minisymposium on Finite Difference Approximations to Hyperbolic Initial-Boundary Value Problems, as part of the First International Conference on Industrial and Applied Mathematics, Paris June 29 - July 4, 1987

Organizer, Workshop on Numerical Methods for Solving Partial Differential Equations, as part of the 1988 Annual Meeting of the Israel Mathematical Union, Tel Aviv University, Tel Aviv, March 29, 1988

Treasurer, Israel Mathematical Union, 1988

**Talks at Conferences and Meetings:** See attached list.

**Memberships:**

Israel Mathematical Union

American Mathematical Society

**Grants and Awards:**

| | |
|---|---|
| 1976-80 | Principal Investigator, U.S. Air Force Grant AFOSR-76-3046 |
| 1979-83 | Principal Investigator, U.S. Air Force Grant AFOSR-79-0127 |
| 1983-88 | Principal Investigator, U.S. Air Force Grant AFOSR-83-0150 |
| 1986-87 | Distinguished Lecturer Award, Technion |
| 1988- | Principal Investigator, U.S. Air Force Grant AFOSR-88-0175 |

**Publications**: See attached list.

**Talks at Universities and Research Centers**: See attached list.

**Graduate Students:**

Professor Eitan Tadmor, M.Sc., 1975
Thesis: "The Numerical Radius and Power Boundedness"
(Co-supervisor with Professor G. Zwas)

## TALKS AT CONFERENCES AND MEETINGS

**Moshe Goldberg**

1. Invited speaker, International Conference on Computational Methods in Nonlinear Mechanics, The Texas Institute of Computational Mechanics, The University of Texas at Austin, Austin, Texas, September 1974, title: "Stable approximations for hyperbolic systems with moving boundary conditons".

2. Principal speaker, The 104th Regular Meeting of the Association for Computer Machinery (ACM), Los Angeles Chapter, Special Interest Group on Numerical Mathematics, Los Angeles, California, April 1975, title: "Stable approximations for hyperbolic systems with moving internal boundaries".

3. Invited speaker, American Mathematical Society 1975 Summer Meeting, Special Session on Numerical Ranges, Western Michigan University, Kalamazoo, Michigan, August 1975, title: "Inclusion relations between certain sets of matrices".

4. Speaker, The 1977 Army Numerical Analysis and Computer Conference, Mathematics Research Center, University of Wisconsin, Madison, Wisconsin, March 1977, title: "On boundary extrapolation and dissipative schemes for hyperbolic problems".

5. Invited speaker, The 746th American Mathematical Society Meeting, Special Session on Matrix Theory, California State University, Hayward, California, April 1977, title: "Some inclusion relations for c-numerical ranges".

6. Speaker, The 1977 Dundee Biennial Conference on Numerical Analysis, University of Dundee, Dundee, Scotland, June 1977, title: "Dissipative schemes for hyperbolic problems and boundary extrapolation".

7. Principal speaker, The National Science Foundation Conference on Linear and Multilinear Algebra, University of California, Santa Barbara, California, December 1977, title: "Numerical ranges and numerical radii".

8. Speaker, The Eighth U.S. National Congress of Applied Mechanics, University of California, Los Angeles, California, June 1978, title: "Spectral analysis of hydroelastic problems" (with R.S. Chadwick).

9. Invited speaker, The Second International Conference on General Inequalities, Mathematical Research Institute, Oberwolfach, West Germany, August 1978, title: "Some combinatorial inequalities and C–numerical radii".

10. Principal speaker, Workshop Series, Five One-Hour Talks, Institute for the Interdisciplinary Applications of Algebra and Combinatorics, University of California, Santa Barbara, California, September 1979, title: "Numerical ranges and numerical radii".

11. Principal speaker, The October 1979 Meeting of the Association for Computing Machinery (ACM), Los Angeles Chapter, Special Interest Group in Numerical Mathematics, Los Angeles, California, October 1979, title: "Stability theory for difference approximations of hyperbolic partial differential equations".

12. Invited speaker, The 1980 Annual Meeting of the Israeli Society for the Applications of Mathematics, Safad, Israel, May 1980, title: "Boundary-dependent stability criteria for difference approximations of hyperbolic initial-boundary value problems".

13. Speaker, The 1981 International Conference on Convexity and Graph Theory, University of Haifa, Haifa, Israel, March 1981, title: "On the convexity of numerical ranges".

14. Invited speaker, The 1981 Annual Meeting of the Israeli Society for Applications of Mathematics, The Weizmann Institute of Science, Rehovot, Israel, April 1981, title: "Scheme-independent stability criteria for difference approximations of hyperbolic initial-boundary value problems".

15. Invited speaker, The Third International Conference on General Inequalities, Mathematics Research Institute, Oberwolfach, West Germany, May 1981, title: "Better stability bounds for Lax-Wendroff schemes in several space dimensions".

16. Invited speaker, The Toeplitz Memorial Conference, Tel Aviv University, Tel Aviv, Israel, May 1982, title: "The numerical radius: from Toeplitz to modern numerical analysis" (with G. Zwas).

17. Invited speaker (two talks), The Fourth International Conference on General Inequalities, Mathematics Research Institute, Oberwolfach, West Germany, May 1983, titles (two talks): "New inequalities for $\ell_p$ norms of matrices", and "In memoriam Edwin Beckenbach".

18. Invited speaker, The AMS-SIAM Summer Seminar on Large-scale Computations in Fluid Mechanics, Scripps Institute of Oceanography, University of California, La Jolla, California, June-July 1983, title: "New stability criteria for difference approximations of hyperbolic initial-boundary value problems".

19. Invited speaker, The 1984 Annual Meeting of the Israel Mathematical Union, Applied Mathematics Session, Tel Aviv University, Tel Aviv, Israel, April 1984, title: "Convenient stability criteria for difference approximations of hyperbolic initial-boundary value problems".

20. Speaker, The Gatlinburg IX Conference on Numerical Algebra, Univesity of Waterloo, Waterloo, Canada, July 1984, titles (two talks): "Generalizations of the Perron-Frobenius Theorem", and "Norms and multiplicativity".

21. Invited attendee, The U.S.-Israel Binational Workshop on the Impact of Supercomputers on the Next Decade of Computational Fluid Dynamics, Jerusalem, Israel, December 1984, Panel Discussion.

22. Invited speaker, The 1984 Haifa Conference on Matrix Theory, Technion - Israel Institute of Technology and the University of Haifa, Haifa, Israel, December 1984, title: "Submultiplicativity of matrix norms and operator norms".

23. Invited speaker, Joint French-Israeli Mathematical Symposium on Linear and Nonlinear Partial Differential Equations, Numerical Analysis, and Geometry of Banach Spaces, The Israel Academy of Sciences and Humanities, Jerusalem, Israel, March 1985, title: "Stability criteria for difference approximations of hyperbolic initial-boundary value problems".

24. Principal speaker, Mathematics Research Conference, California Institute of Technology, Pasadena, California, October 1985, title: "Submultiplicativity of matrix norms and operator norms".

25. Principal speaker, Southern California Functional Analysis Seminar (SCFAS), California State University, Los Angeles, California, October 1985, title: "Submultiplicativity of matrix norms and operator norms".

26. Invited speaker, The 1985 Haifa Conference on Matrix Theory, Technion - Israel Institute of Technology and the University of Haifa, Haifa, Israel, December 1985, title: "Submultiplicativity and mixed submultiplicativity of matrix norms and operator norms".

27. Principal speaker, The 187th Meeting of the Association for Computing Machinery (ACM), Los Angeles Chapter, Special Interest Group in Numerical Analysis, Los Angeles, California, February 1986, title: "Stability criteria for finite difference approximations of hyperbolic initial-boundary value problems".

28. Invited speaker, The Fifth International Conference on General Inequalities, Mathematics Research Institute, Oberwolfach, West Germany, May 1986, title: "Multiplicativity and mixed-multiplicativity of operator norms and matrix norms".

29. Speaker, SIAM Conference on Linear Algebra in Signals, Systems and Control, Boston, Massachusetts, August 1986, title: "Mixed multiplicativity for $\ell_p$ norms of matrices".

30. Invited speaker, The Third Haifa Matrix Theory Conference, Technion - Israel Institute of Technology, Haifa, Israel, January 1987, title: "Equivalence constants for $\ell_p$ norms of matrices".

31. Invited speaker and Session Chairman, First International Conference on Industrial and Applied Mathematics, Minisymposium on Finite Difference Approximations to Hyperbolic Initial-boundary Value Problems, Paris, June-July 1987, title: "Convenient stability criteria for difference approximations of hyperbolic initial-boundary value problems".

32. Invited speaker, Meeting on Numerical Problems for Initial and Initial-boundary Value Problems, Mathematics Research Institute, Oberwolfach, West Germany, August 1987, title: "Stability criteria for finite difference approximations to hyperbolic initial-boundary value problems".

33. Invited speaker, The Fourth Haifa Matrix Theory Conference, Technion - Israel Institute of Technology, Haifa, Israel, January 1988, title: "On monotone and semi-monotone matrix functions".

34. Speaker and Session Chairman, Second International Conference on Hyperbolic Problems, RWTH Aachen, Aachen, West Germany, March 1988, title: "Convenient stability criteria for difference approximations of hyperbolic initial-boundary value problems".

35. Invited speaker and Session Chairman, The 1988 Annual Meeting of the Israel Mathematical Union, Tel Aviv University, Tel Aviv, Israel, March 1988, title: "Simple stability criteria for difference approximations of hyperbolic initial-boundary value problems".

## TALKS AT UNIVERSITIES AND RESEARCH CENTERS

**Moshe Goldberg**

1. T.J. Watson Research Center, IBM, Yorktown Heights, New York, Mathematics Seminar, September 1970.

2. NASA Ames Research Center, Moffet Field, California, Thermo and Gas Dynamics Divison, Computation Seminar, November 1974.

3. University of California, Berkeley, California, Department of Mathematics, Numerical Analysis and Applied Mathematics Colloquium, May 1975.

4. NASA Langley Research Center, Hampton, Virginia, Institute for Computer Applications in Science and Engineering (ICASE), ICASE Seminar, August 1975.

5. University of California, Santa Barbara, California, Institute for the Interdisciplinary Applications of Algebra and Combinatorics, June 1976.

6. Tel Aviv University, Tel Aviv, Israel, Department of Mathematical Sciences, Colloquium, December 1976.

7. Tel Aviv University, Tel Aviv, Israel, Department of Mathematical Sciences, Numerical Analysis Seminar, December 1976.

8. The Weizmann Institute of Science, Rehovot, Israel, Department of Mathematics, Colloquium, December 1976.

9. The Hebrew University, Jerusalem, Israel, Institute of Mathematics, Colloquium, December 1976.

10. University of California, Santa Barbara, California, Department of Mathematics, Colloquium, January 1977.

11. University of California, Los Angeles, California, Department of Mathematics, Applied Mathematics Colloquium, February 1977.

12. Stanford University, Stanford, California, Computer Science Department, Numerical Analysis Seminar, March 1977.

13. Case Western Reserve University, Cleveland, Ohio, Department of Mathematics and Statistics, Colloquium, March 1977.

14. University of Southern California, Los Angeles, California, Department of Mathematics, Colloquium, November 1977.

15. University of California, Los Angeles, California, Department of Mathematics, Applied Mathematics Colloquium, March 1978.

16. Polytechnic Institute of New York, Brooklyn, New York, Department of Mathematics, Colloquium, March 1978.

17. Georgia Inistitute of Technology, Atlanta, Georgia, Department of Mathematics, Colloquium, May 1978.

18. University of Georgia, Athens, Georgia, Department of Mathematics, Colloquium, May 1978.

19. Technion - Israel Institute of Technology, Haifa, Israel, Department of Mathematics, Colloquium, October 1978.

20. Ben Gurion University, Beer-Sheva, Israel, Department of Mathematics, Colloquium, November 1978.

21. Tel Aviv University, Tel Aviv, Israel, Department of Mathematics, Colloquium, November 1978.

22. University of California, Santa Barbara, California, Institute for the Interdisciplinary Applications of Algebra and Combinatorics, Colloquium, May 1979.

23. Technion - Israel Institute of Technology, Haifa, Israel, Department of Mathematics, Analysis Seminar, three one-hour talks, May 1980.

24. University of California, Santa Barbara, California, Department of Mathematics, Linear and Multilinear Algebra Seminar, six 75-minute talks, August 1980.

25. Tel Aviv University, Tel Aviv, Israel, School of Mathematical Sciences, Numerical Analysis Seminar, May 1981.

26. Technion - Israel Institute of Technology, Haifa, Israel, Department of Mathematics, Analysis Seminar, June 1981.

27. University of California, Santa Barbara, California, Institute for the Interdisciplinary Applications of Algebra and Combinatorics, Colloquium, August 1981.

28. New York University, New York, New York, Courant Institute of Mathematical Sciences, Numerical Analysis Seminar, September 1981.

29. The Hebrew University, Jerusalem, Israel, Institute of Mathematics, Colloquium, February 1982.

30. The Hebrew University, Jerusalem, Israel, Institute of Mathematics, Partial Differential Equations Seminar, March 1982.

31. Tel Aviv University, Tel Aviv, Israel, Department of Mathematical Sciences, Operator Theory Seminar, April 1982.

32. University of California, Santa Barbara, California, Institute for the Interdisciplinary Applications of Algebra and Combinatorics, Colloquium, September 1982.

33. New York University, New York, New York, Courant Institute of Mathematical Sciences, Numerical Analysis Seminar, September 1983.

34. Tel Aviv University, Tel Aviv, Israel, Department of Mathematical Sciences, Numerical Analysis Seminar, January 1984.

35. Polytechnic Institute of New York, Brooklyn, New York, Department of Mathematics, Colloquium, September 1984.

36. University of California, Los Angeles, California, Department of Mathematics, Applied Mathematics Colloquium, October 1985.

37. California Institute of Technology, Pasadena, California, Department of Applied Mathematics, Colloquium, November 1985.

38. California Institute of Technology, Pasadena, California, Department of Mathematics, Combinatorics Seminar, November 1985.

39. University of Southern California, Los Angeles, California, Department of Mathematics, Colloquium, March 1986.

40. Centre National de la Recherche Scientifique et Université Pierre et Marie Curie (Paris VI), Paris, France, Numerical Analysis Seminar, April 1986.

41. École Polytechnique, Centre de Mathématiques Apliquées, Paris, France, Applied Mathematics Seminar, April 1986.

42. École Normale Supérieure, Paris, France, Centre de Mathématiques, Applied Mathematics Seminar, April 1986.

43. Institute National de Recherche en Informatique et en Automatique (INRIA), Rocquencourt, France, Seminar, May 1986.

44. University of Paris IX, Paris, France, Centre de Recherche de Mathématiques de la Décision, Colloquium, May 1986.

45. New York University, New York, New York, Courant Institute of Mathematical Sciences, Numerical Analysis Seminar, September 1986.

46. Tel Aviv University, Tel Aviv, Israel, School of Mathematical Sciences, Colloquium, January 1987.

## PUBLICATIONS

### Moshe Goldberg

Theses:

1. M.Sc. Thesis, "Quasi-conservative hyperbolic systems", Tel Aviv University, Tel Aviv, Israel, 1970.

2. Ph.D. Thesis, "Stable approximations for hyperbolic systems with moving internal boundary conditions", Tel Aviv University, Tel Aviv, Israel, 1973.

Published Papers:


1. Numerical solution of quasi-conservative hyperbolic systems - The cylindrical shock problem (with S. Abarbanel), Journal of Computational Physics 10 (1972), 1-21.

2. A note comparing the root condition and the resolvent condition (with W.L. Miranker), Information Sciences 4 (1972), 285-288.

3. A note on the stability of an iterative finite-difference method for hyperbolic systems, Mathematics of Computation 27 (1973), 41-44.

4. Stable approximations for hyperbolic systems with moving internal boundary conditions (with S. Abarbanel), Mathematics of Computation 28 (1974), 413-447.

5. Stable schemes for hyperbolic systems with moving internal boundaries (with S. Abarbanel), in "Computational Mechanics", edited by J.T. Oden, Texas Institute for Computational Mechanics (TICOM), 1974, 469-478.

6. On matrices having equal spectral radius and spectral norm (with G. Zwas), Linear Algebra and Its Applications 8 (1974), 427-434.

7. The numerical radius and spectral matrices (with E. Tadmor and G. Zwas), Linear and Multilinear Algebra 2 (1975), 317-326.

8. Numerical radius of positive matrices (with E. Tadmor and G. Zwas), Linear Algebra and Its Applications 12 (1975), 209-214.

9. On inscribed circumscribed conics (with G. Zwas), Elemente der Mathematik 31 (1976), 36-38.

10. Inclusion relations between certain sets of matrices (with G. Zwas), Linear and Multilinear Algebra 4 (1976), 55-60.

11. A test problem for numerical schemes for nonlinear hyperbolic equations (with S. Abarbanel), Computer Methods in Applied Mechanics and Engineering 8 (1976), 331-334.

12. Inclusion relations involving k-numerical ranges (with E.G. Straus), Linear Algebra and Its Applications 15 (1976), 261-270.

13. On characterizations and integrals of generalized numerical ranges (with E.G. Straus), Pacific Journal of Mathematics 69 (1977), 45-54.

14. On a boundary extrapolation theorem by Kreiss, Mathematics of Computation 31 (1977), 469-477.

15. Elementary inclusion relations for generalized numerical ranges (with E.G. Straus), Linear Algebra and Its Applications 18 (1977), 1-24.

16. On a theorem by Mirman (with E.G. Straus), Linear and Multilinear Algebra 5 (1977), 77-78.

17. On boundary extrapolation and dissipative schemes for hyperbolic problems, Proceedings of the 1977 U.S. Army Numerical Analysis and Computer Conference, ARO Report 77-3 (1977), 157-164.

18. Scheme-independent stability criteria for difference approximations of hyperbolic initial-boundary value problems. I, (with E. Tadmor), Mathematics of Computation 32 (1978), 1097-1107.

19. Norm properties of C-numerical radii (with E.G. Straus) Linear Algebra and Its Applications 24 (1979), 113-132.

20. On certain finite dimensional numerical ranges and numerical radii, Linear and Multilinear Algebra 7 (1979), 329-342.

21. Combinatorial inequalities, matrix norms, and generalized numerical radii (with E.G. Straus), in "General Inequalities 2", edited by E.F. Beckenbach, Birkhäuser Verlag, Basel, 1980, 37-46.

22. Scheme-independent stability criteria for difference approximations of hyperbolic initial-boundary value problems. II, (with E. Tadmor), Mathematics of Computation 36 (1981), 603-626.

23. On the numerical radius and its applications (with E. Tadmor), Linear Algebra and Its Applications 42 (1982), 263-284.

24. Operator norms, multiplicativity factors, and C-numerical radii (with E.G. Straus), Linear Algebra and its Applications 43 (1982), 137-159.

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28. Multiplicativity factors for C-numerical radii (with E.G. Straus), Linear Algebra and Its Applications 54 (1983), 1-16.

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30. In Memoriam Edwin F. Beckenbach, in "General Inequalities 4", edited by W. Walter, Birkhäuser Verlag, Basel, 1984, 3-11.

31. Some inequalities for $\ell_p$ norms of matrices, in "General Inequalities 4", edited by W. Walter, Birkhäuser-Verlag, Basel, 1984, 185-189.

32. Multiplicativity of $\ell_p$ norms for matrices. II, Linear Algebra and Its Applications 62 (1984), 1-10.

33. Ernst G. Straus (1922-1983), Linear Algebra and Its Applications 63 (1985), 1-19.

34. New stability criteria for difference approximations of hyperbolic initial-boundary value problems (with E. Tadmor), in "Large-Scale Computations in Fluid Mechanics", edited by B.E. Engquist, S.J. Osher, and R.C.J. Somerville, American Mathematical Society, Providence, Rhode Island, 1985, 177-192.

35. Convenient stability criteria for difference approximations of hyperbolic initial-boundary value problems (with E. Tadmor), Mathematics of Computation 44 (1985), 361-377.

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40. Multiplicativity factors and mixed multiplicativity, Linear Algebra and its Applications 97 (1987), 45-56.

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42. On monotone and semi-monotone matrix functions, Linear and Multilinear Algebra, accepted.

43. Simple stability of criteria for difference approximations of hyperbolic initial-boundary value problems (with E. Tadmor), to appear.

44. Multiplicativity factors for seminorms (with R. Arens), in preparation.

## Research of Marvin Marcus
## 1983-1988

## I. Introduction

This report covers the research of Marvin Marcus for the period May 1, 1983 - April 30, 1988 sponsored by the Air Force Office of Scientific Research, grant number AFOSR-83-0150.

The sequel is separated into the following sections:

### II. General Area of Research

This section contains: an exposition of the basic mathematical theory of the finite dimensional numerical range; a description of two algorithms that permit effective visualization of the structure of the numerical range; an example of the implementation of algorithms for visualizing the numerical range and how these can be used to refute or substantiate important conjectures; a list of continuing problems currently under investigation.

### III. Research of M. Marcus, 1983 - 1988

This section contains a list of the publications completed by M. Marcus in the period 1983-1988 with short summaries of their contents. At the end of the section is the result of a computer search of the Science Citation Index which contains the total number of references to the work of M. Marcus since 1983. Self references have been excluded in the search criteria. This data provides some quantitative information of the extent to which the research of M. Marcus has been used by other investigators working in the general area of applied and numerical linear algebra.

### IV. Numerical Range Bibliography

This is a preliminary version of a bibliography of 779 citations covering the numerical range. It has been sorted alphabetically by first author. All references in this report refer to the Bibliography. We are currently in the process of identifying the Mathematical Reviews numbers and preparing brief summaries of each of the papers. In view of the very large number of citations, this latter project will probably take several months to complete, and will be incorporated as part of the report on the current grant, AFOSR-88-0175.

### V. Appendix

The appendix contains the vita and publication list of Marvin Marcus.

## II. General Area of Research

Let V be an n-dimensional unitary space and let A be a linear transformation, $A : V \to V$. The numerical range, or field of values, of A is the set of complex numbers

$$W(A) = \{ (Ax, x) \mid \|x\| = 1 \}. \tag{1}$$

The numerical radius of A, w(A), is the maximum distance of any point in W(A) from the origin. By choosing an orthonormal (o.n.) basis of V, and replacing the inner product in V by the standard inner product in the space of complex column n-tuples, the computation of W(A) is reduced to an equivalent matrix problem. Thus we assume that A is an *n*-square complex matrix and that the inner product of two column *n*-tuples (*n*-vectors) is

$$(x, y) = \sum_{k=1}^{n} x_k \bar{y}_k. \tag{2}$$

Elementary results concerning W(A) were known in the last century [165], [398] and in the first decade of this century [69], [317]. These early results were usually formulated in terms of bounding rectangles for the spectrum of A, $\sigma(A)$, which were, in fact, containment rectangles for W(A). However, it was not until 1918 and 1919 that the first important results concerning W(A) were proved by Hausdorff and Toeplitz.

It is a classical result due to Hausdorff [309] and Toeplitz [712] that the numerical range, W(A), is a convex set. Many proofs of this interesting result have appeared in the intervening years since the original Hausdorff-Toeplitz theorem was published. Most of these (e.g., see [281]) depend on reducing the problem to the computation of the numerical range of a 2-square matrix.

There have been a number of interesting papers on geometric properties of the numerical range and their relation to the similarity invariants of A (e.g. [729], [41], [175], [213], [351], [376], [728], [561], [144], [154], [184], [497], [583]). From a numerical standpoint, the numerical range arises in many contexts: the constrained eigenvalue problem [388]; the theory of small vibrations [47], [48]; Tchebychev iteration for linear systems [446]. Much of the interest in the numerical range of a matrix A is motivated by the fact that it is a containment region for the spectrum of A. In fact, for normal A, W(A) is the convex hull of the spectrum of A. It might be conjectured that this geometric property of W(A) is equivalent to A being normal. In fact, M. Marcus and B.N. Moyls [527] showed that for $n \leq 4$ this is indeed the case, but for $n > 4$ it is not. This result led to a sequence of related papers [19], [216], [29], [30], [463], [474], [486] and the introduction of a class of operators called <u>convexoid</u>.

The numerical range of any linear operator is the union of the numerical ranges all 2-dimensional real compressions of A. This fact is the basis for the first algorithm described below. If $1 \leq k \leq n$ and P is a k-dimensional orthogonal projection, then the restriction of PAP to the range of P is called a k-dimensional compression of A. For $k = 2$ and A an n-square complex matrix, a 2-dimensional real orthogonal compression of A is the 2-square matrix

$$A_{xv} = \begin{bmatrix} (Ax, x) & (Av, x) \\ (Ax, v) & (Av, v) \end{bmatrix}. \tag{3}$$

where x and v are real o.n. column n-tuples.

The following are well known properties of the numerical range. The set W(A) is unitarily invariant and is identical with the set of all diagonal elements appearing in all unitary transforms of A (i.e., in all matrices unitarily similar to A). The numerical range of every principal submatrix of A is a subset of the numerical range of A. If $A = B \oplus C$ then $W(A) = H(W(B) \cup W(C))$. (H denotes the convex hull.) The set W(A) is a closed bounded convex region of the plane containing $\sigma(A)$, the spectrum of A, i.e., containing $\lambda_1, \ldots, \lambda_n$, the eigenvalues of A. Since W(A) is convex, it also contains

$$P(A) \equiv H(\lambda_1, \ldots, \lambda_n). \tag{4}$$

If A is normal then $W(A) = P(A)$. This last result implies that if A is normal then the extreme points of W(A) are eigenvalues. If $W(A) = \{ \lambda \}$ then $A = \lambda I$ and if $W(A) \subseteq \mathbb{R}$ then $A = A^*$, i.e., A is hermitian.

If $n = 2$, then W(A) is an ellipse with foci the eigenvalues of A; if A has the form

$$\begin{bmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{bmatrix},$$

then the length of the semi-minor axis of the ellipse is $|\alpha|/2$. The precise equation for the boundary of the numerical range of a non-normal matrix for $n \geq 2$ has been given by Murnaghan [528] and, in more explicit form, by Kippenhahn [376]. Kippenhahn also gives bounds for the diameter of W(A). M. Feidler obtained [213] an equation for the boundary of W(A).

Since the numerical range contains the spectrum of A it is of considerable importance from the standpoint of eigenvalue localization. In fact, this was the starting point for a number of papers on classical eigenvalue localization theory including work by W.V. Parker [554], [556] and A.B. Farnell [206], [208].

P. Henrici [312] related the distance between the boundary of W(A) and P(A) with a measure of the departure of A from normality.

In a paper written in 1952 [254] W. Givens defined for $A \in M_n(\mathbb{C})$ the following set:

$$F_H(A) = \left\{ \frac{(HAx, x)}{(Hx, x)} : x \in \mathbb{C}^n \right\}, \quad H \text{ p.d.}$$

Givens proved that if H is p.d. (positive define hermitian), $H = T^*T$, then $F_H(A) = W(TAT^{-1})$. He also showed that if A has an elementary divisor of degree at least 2 associated with the root $\lambda$, then $\lambda$ is an interior point of $F_H(A)$ for every p.d. H. An immediate consequence of this last result is that if $\lambda$ is an eigenvalue on the boundary of W(A) then $\lambda$ occurs only in linear elementary divisors. Givens' main result was

$$P(A) = \bigcap_{H \text{ p.d.}} F_H(A).$$

He also showed that a necessary and sufficient condition that $F_H(A) = P(A)$ for some p.d. H is that the elementary divisors corresponding to roots on the boundary of P(A) are all linear. Givens' results suggest that for an appropriate choice of H one might obtain information about the eigenvalues of A, particularly those on the boundary of P(A), from properties of the numerical range.

In [175] W.F. Donoghue proved that every non-differentiable boundary point of W(A) is an eigenvalue of A.

O. Taussky [690] has shown that if $A \neq 0$ and $tr(A) = 0$ then 0 is in the interior of W(A).

A result of C.R. Putnam [580] states that if $C = AB - BA$ then 0 is in the interior of W(C).

Ballantine [41] has presented a series of algorithms to determine for a given complex number z and a given $A \in M_n(\mathbb{C})$ whether or not z is in W(A), z is a boundary point of W(A), or z is an extreme point of W(A).

In a paper written in 1963 [457] M. Marcus and R.C. Thompson examined the numerical range of the Hadamard product $A * B$ of two matrices. They showed that if A and B are normal and $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are the eigenvalues of A and B respectively then $W(A * B)$ is a subset of the convex polygon spanned by $(\alpha_i\beta_j + \alpha_j\beta_i)/2$, $1 \leq i \leq j \leq n$. This result was used to yield localization theorems for permanents and determinants.

T. Saitô [600] considered the question: When is the relation $W(A \otimes B) = H(W(A)W(B))$ valid? In a particular answer to this question he proved that if $W(A \otimes B) = H(\sigma(A \otimes B))$ then the above equality holds. He also showed that in general $H(W(A)W(B)) \subseteq W(A \otimes B)$ but that there exist A and B for which the inclusion is strict.

In a series of papers [348], [349], [351A], [356] Johnson examined various inclusion relations involving W(A). In [351], for $n = 2$, he determined the major and minor axes of the ellipse W(A) in terms of the entries of A when A is real. He then utilized this result to determine

$$S(A) = H(\bigcup_{i,j} W(A[\, i, j \mid i, j\, ])$$

for A *n*-square real and showed that $S(A) \subseteq W(A)$.

In [354], [353] Johnson studied the Hadamard product $A * B$ of A and B ($A * B = [a_{ij}b_{ij}]$). He proved that if $A \in M_n(\mathbb{C})$ and for some $0 \leq \theta \leq 2\pi$, $e^{i\theta}H$ is p.d. then $W(H \otimes A) = W(H)W(A)$. Furthermore, if $A \in M_n(\mathbb{C})$ and $N \in M_m(\mathbb{C})$ is normal then $W(N \otimes A) \subset H(W(N)W(A))$. A corollary: if N and A are in $M_n(\mathbb{C})$ and N is normal, then $W(N * A) \subseteq H(W(N)W(A))$; if further N is p.d. then $W(N \cdot A) \subseteq W(N)W(A)$.

Since the effective visualization of the set W(A) has been an important part of this research it is important to be aware that several algorithms exist for graphing the convex hull of a set of points. Sedgewick describes the implementation of the package wrapping algorithm in [Algorithms, 2nd ed., Robert Sedgewick, Addison Wesley, 1988] which is not unrelated to one of the algorithms developed below to visualize the numerical range:

1. Find the point with the least y coordinate.
2. Imagine a horizontal line through this point.
3. Sweep that horizontal line through a positive angle $\theta$ until it intersects with another point.
4. Add that point to the boundary of the convex hull.
5. If the new point is not the starting point, goto step 2.

Obviously this algorithm is suitable for finite sets of points only.

To visualize the numerical range it is required to graph its boundary. In the second algorithm described below, an effective means of computating the boundary of W(A) is described.

The mathematical results at the basis for the visualization algorithms will be discussed next. Let $A \in M_n(\mathbb{C})$, let B be any principal submatrix of A and let U $\in M_n(\mathbb{C})$ be any unitary matrix. Also let $\sigma(A)$ denote the spectrum of A, i.e., $\sigma(A)$ is the set of all eigenvalues of A. Then

$$W(cA) = cW(A), \tag{5}$$
$$W(cI_n + A) = c + W(A), \tag{6}$$
$$W(A^*) = \overline{W(A)}, \tag{7}$$
$$W(B) \subset W(A), \tag{8}$$
$$W(U^*AU) = W(A), \tag{9}$$
$$W(A) \subset \mathbb{R} \text{ iff } A \text{ is hermitian}, \tag{10}$$
$$W(A) \subset i\mathbb{R} \text{ iff } A \text{ is skew-hermitian}, \tag{11}$$
$$W(A) = \{0\} \text{ iff } A = 0, \tag{12}$$
$$\sigma(A) \subset W(A), \tag{13}$$
$$W(A) = \{c\} \text{ iff } A = cI_n. \tag{14}$$

The following theorem, known as the elliptical range theorem, completely describes the structure of W(A) for $A \in M_2(\mathbb{C})$. It is the basis for proving the fact that W(A) is always convex, the most important theorem about the numerical range. It is also the basis of an effective visualization algorithm.

Theorem 1. Let $A \in M_2(\mathbb{C})$ with eigenvalues $\lambda$ and $\mu$. Define the following numbers associated with A:

$$\nu = \left( \sum_{i,j=1}^{2} |a_{ij}|^2 \right)^{1/2} \tag{15}$$
$$\alpha = (\nu^2 - |\lambda|^2 - |\mu|^2)^{1/2}. \tag{16}$$

Then W(A) is an elliptical region bounded by an ellipse (possibly degenerate) whose description is as follows:

$$\text{foci: } \lambda\ \mu; \tag{17}$$
$$\text{semi-major axis: } \frac{(\nu^2 - 2\mathrm{Re}\,\lambda\bar{\mu})^{1/2}}{2} \tag{18}$$
$$\text{semi-minor axis: } \frac{\alpha}{2} \tag{19}$$

An important result proved in [489] is contained in

Theorem 2. Let A be an n-square complex matrix. Then W(A) is the union of all the sets

$$W(A_{xv}) \tag{20}$$

as x and v run over all pairs of o.n. vectors in $M_{n,1}(\mathbb{R})$.

From a computational standpoint it is important to note that although W(A) consists of complex numbers of the form $y^*Ay$, $y \in M_{n,1}(\mathbb{C})$, it is nevertheless the case that only real x and v are required in (20).

We can state the following useful result for computing the numerical radius of a $2 \times 2$ matrix.

Theorem 3. Let A be unitarily similar to

$$\begin{bmatrix} \lambda & \alpha \\ 0 & \mu \end{bmatrix}, \quad \alpha \geq 0. \tag{21}$$

For $s \in [0, 1]$ define the function

$$d(s) = | s\lambda + (1 - s)\mu | + \alpha\sqrt{s(1 - s)} \tag{22}$$

then

$$w(A) = \max d(s) \tag{23}$$

where the max in (23) is computed for $s \in [0, 1]$.

For matrices $A \in M_2(\mathbb{C})$ which are unitarily similar to a real matrix it is possible to give an explicit formula for w(A) in terms of the entries of A. In the following theorem, w(A) is explicitly exhibited in terms of the entries in the upper triangular form of A for a 2-square matrix. This result is useful for determining an approximation from below for the numerical radius of an arbitrary A.

Theorem 4. If $A \in M_2(\mathbb{C})$, upper triangular,

$$A = \begin{bmatrix} \lambda & \alpha \\ 0 & \mu \end{bmatrix} \qquad \alpha \geq 0,$$

and if A is unitarily similar to a real matrix then the numerical radius w(A) can be determined as follows:

I. $\lambda$ and $\mu$ are real. Then w(A) is the larger of the two numbers

$$\frac{|\lambda + \mu \pm \sqrt{(\lambda - \mu)^2 + \alpha^2}|}{2}.$$

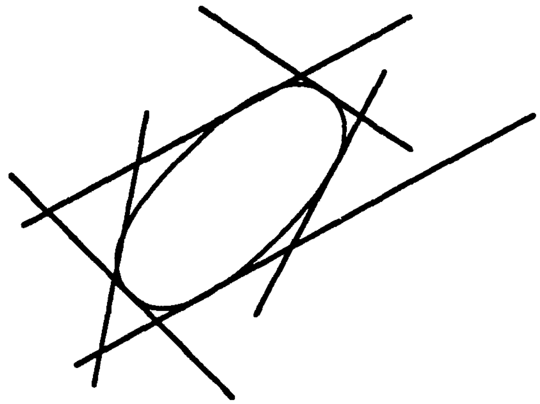
II. $\lambda$ and $\mu$ are complex conjugates: $\lambda = h + ik$, $\mu = h - ik$, $k \neq 0$. If $2k^2 \geq \alpha|h|$ then

$$w(A) = \frac{|\lambda|}{|k|}\frac{\sqrt{\alpha^2 + 4k^2}}{2}.$$

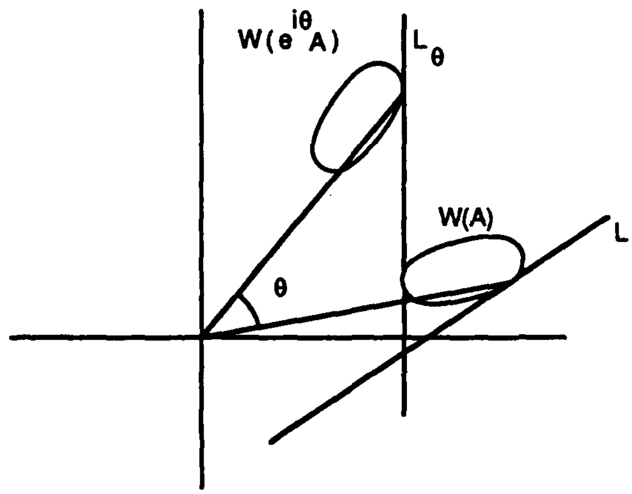
III. $\lambda$ and $\mu$ are complex conjugates: $\lambda = h + ik$, $\mu = h - ik$, $k \neq 0$. If $2k^2 < \alpha|h|$ then

$$w(A) = |h| + \frac{\alpha}{2}.$$

If C is a convex set in $\mathbb{C}$ which is closed, i.e., contains all its limit points, and which is not all of $\mathbb{C}$, then C is the intersection of all its supporting half-planes.

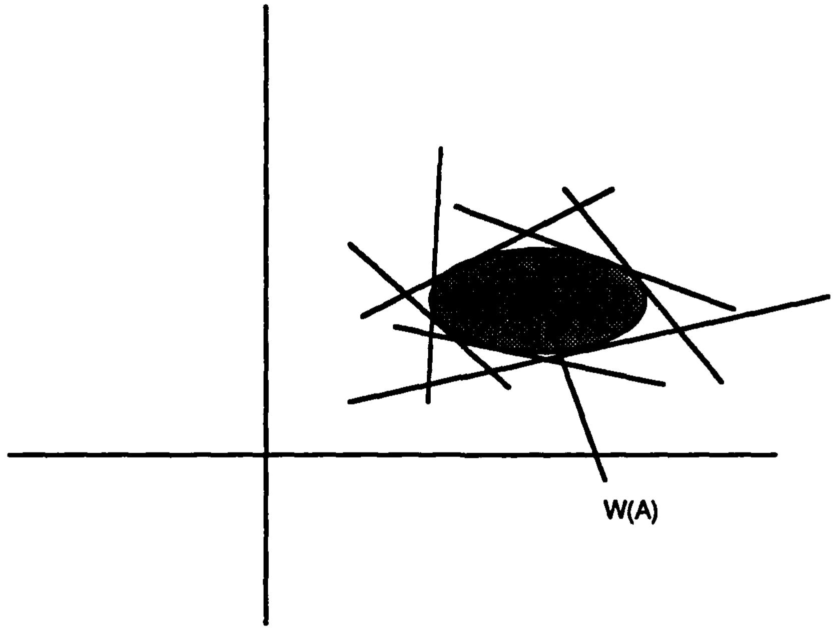


This fact is geometrically evident and it is not difficult to prove. The geometry of this situation is very useful in developing an algorithm for constructing W(A). Theorem 2 provides us with one effective method for constructing W(A). The present discussion will lead us to another such method. Thus, let W(A) be the numerical range of an arbitrary $A \in M_n(\mathbb{C})$.




(24)

The idea is simple: we want to construct a relatively dense set of support lines for W(A). Then W(A) will be accurately depicted as the intersection of the corresponding support half-planes. In fact, simply drawing a sufficiently dense set of such support lines will define W(A) with great accuracy.




(25)

Of course, the problem is to devise a computationally reasonable method of determining the support lines L. We will make our method depend on computing the dominant eigenvalue of a sequence of appropriate hermitian matrices. Thus let L be a fixed but arbitrary support line for W(A). Then perform a counterclockwise rotation in the plane through an angle $\theta$ chosen so that the rotated image of L, call it $L_\theta$, is perpendicular to the x-axis (see (24)). Clearly each such support line L determines a unique $L_\theta$. For a given $\theta$, if we can determine the equation for $L_\theta$, then the equation for L is obtained by elementary geometry. Now, $L_\theta$ is a support line for $W(e^{i\theta}A) = e^{i\theta}W(A)$. Write $A(\theta) = e^{i\theta}A$ and let $H(\theta)$ and $K(\theta)$ be the hermitian parts of $A(\theta)$:

$$A(\theta) = H(\theta) + iK(\theta).$$

Then if $u^*u = 1$,

$$u^*A(\theta)u = u^*H(\theta)u + iu^*K(\theta)u$$

and

$$\operatorname{Re} u^*A(\theta)u = u^*H(\theta)u.$$

Hence

$$\max_{u^*u=1} \operatorname{Re} u^*A(\theta)u = \max_{u^*u=1} u^*H(\theta)u.$$

But $H(\theta)$ is hermitian and thus

$$\max_{u^*u=1} u^*H(\theta)u = \lambda(\theta)$$

where $\lambda(\theta)$ is the largest eigenvalue of $H(\theta)$. In fact, the maximizing u is an eigenvector of $H(\theta)$ corresponding to $\lambda(\theta)$. Assume that it is feasible to compute $\lambda(\theta)$. Then the equation of $L_\theta$ is obviously

$$x = \lambda(\theta),$$

or in complex number notation

$$\operatorname{Re} z = \lambda(\theta).$$

Now a point $z = |z|e^{i\varphi} = x + iy$ lies on the line L iff $e^{i\theta}z$ lies on the line $L_\theta$, i.e, iff

$$\begin{aligned}
\operatorname{Re} e^{i\theta}z &= \lambda(\theta), \\
\operatorname{Re} e^{i\theta}|z|e^{i\varphi} &= \lambda(\theta), \\
\operatorname{Re} |z|e^{i(\theta+\varphi)} &= \lambda(\theta), \\
|z|\cos(\theta+\varphi) &= \lambda(\theta), \\
|z|\cos\varphi\cos\theta - |z|\sin\varphi\sin\theta &= \lambda(\theta), \\
x\cos\theta - y\sin\theta &= \lambda(\theta). \qquad (26)
\end{aligned}$$

Thus (26) is the equation for L in rectangular coordinates. The line (26) is known once $\theta$ is specified and $\lambda(\theta)$ is computed. A sensible scheme might be to specify a sequence of N values of $\theta$ of the form

$$\theta_k = k\frac{2\pi}{N}, \quad k = 0, 1, \ldots, N-1$$

and construct the lines

$$L_k : \; x\cos\theta_k - y\sin\theta_k = \lambda(\theta_k), \qquad k = 0, \ldots, N-1.$$

The method for depicting W(A) just described can also be used to determine the numerical radius w(A). In fact, we can easily verify that

$$w(A) = \max_{\theta \in [0, 2\pi]} \lambda(\theta).$$

**Algorithm 1**

The first algorithm is based on Theorem 1, Theorem 2, and Theorem 4. Theorem 1 is the elliptical range theorem. Theorem 2 states that W(A) is the union of the numerical ranges of all the real 2-square orthogonal compressions of A. Theorem 4 is the explicit formula for w(A) of a 2-square matrix unitarily similar to a real matrix. The algorithm approximates W(A) from the inside.

1 Generate a random o.n. pair of vectors, x and v
2. Compute $A_{xv}$
3. Apply the elliptical range theorem to $A_{xv}$ to obtain $W(A_{xv})$
4. Graph $W(A_{xv})$
5. Update w(A) with maximal value of $w(A_{xv})$
6. Goto 1.

The Macintosh has many built in ROM routines for drawing objects on the screen. The routines that were used in this program were paintoval and lineto. The paintoval command draws an oval inside a specified rectangle. This rectangle is situated in the plane with its sides parallel to the axes. Thus, it was not possible to draw an inclined ellipse. The foci of the ellipse had to lie along the real or the imaginary axis. This means that the eigenvalues of the 2-square compressions of A had to be real or complex conjugates of one another. This fact restricted our ability to quickly depict W(A) for an arbitrary complex matrix using this algorithm.

Step 1 consists of generating a pair of o.n. vectors. This is done by randomly generating two n-vectors with components $\in [-1, 1]$. These two vectors are checked to make sure their lengths are greater than 0 and then orthonormalized using the Gram-Schmidt process.

Step 2 consists of computing $A_{xv}$. This is done by computing

$$A_{xv} = \begin{bmatrix} (Ax,x) & (Av,x) \\ (Av,x) & (Av,v) \end{bmatrix}.$$

When it was necessary to compute a vector-matrix-vector product as above, the operations were applied as follows:

$$(Ax,x) = (x^*(Ax)).$$

Step 3 consists of applying the elliptical range theorem to $A_{xv}$. This entailed computing the eigenvalues of $A_{xv}$, r1 and r2. The eigenvalues are computed by solving the characteristic polynomial of $A_{xv}$. These eigenvalues are the foci of the ellipse which is the numerical range of $A_{xv}$. Next, the value alpha is computed. This is the length of the minor axis. Depending on the values of alpha, r1, and r2, $W(A_{xv})$ has different features and is graphed accordingly.

Step 4 graphing $W(A_{xv})$. If alpha = 0 then $W(A_{xv})$ collapses to a line segment joining r1 and r2. If this is the case $W(A_{xv})$ is drawn using a straight line.

If r1 and r2 are complex conjugates, alpha > 0, then $W(A_{xv})$ is situated in the plane with its major axis parallel to the imaginary axis.

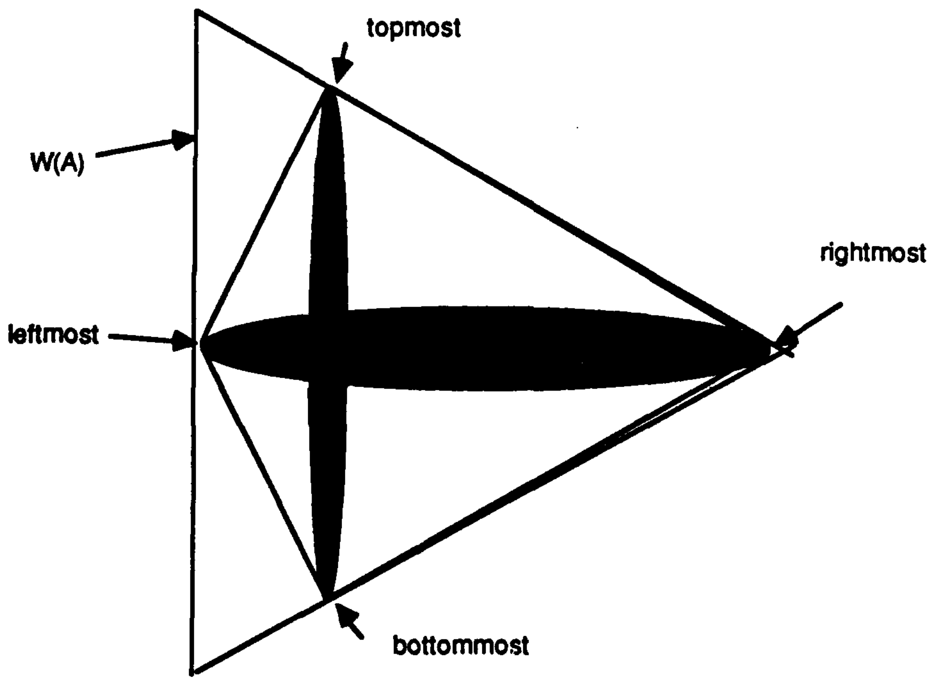
If r1 an r2 are real, alpha > 0, then $W(A_{xv})$ is situated in the plane with its major axis along the real axis.

After each $W(A_{xv})$ is drawn, it is checked to see whether any of its points are either the topmost, bottommost, leftmost, or rightmost points in W(A) exhibited so far. After each iteration, the convex polygon is drawn connecting these four extreme points. This speeds the approximation of the convex hull of A from inside.

To illustrate this, suppose we are trying to approximate the numerical range of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

We know that W(A) is a triangle (A is normal and its numerical range is the convex hull of its eigenvalues). After two iterations we may have two ellipses situated as depicted below, that approximate the triangle from the inside. If we connect the extreme points on the ellipses we get a closer approximation to W(A). Joining the extreme values is performed at every iteration past the first one.




Step 5. update w(A). Theorem 4 provides us with a closed form formula for evaluating w(A) for a 2-square matrix unitarily similar to a real matrix. The

theorem has three alternatives: I) the eigenvalues are real; II) the eigenvalues are complex conjugates of one another, $h \pm ik$ and $2k^2 \geq \alpha \mid h \mid$; and III) the eigenvalues are complex conjugates, $h \pm ik$, and $2k^2 < \alpha \mid h \mid$. The conditions of the theorem are checked against the eigenvalues, r1 and r2, and alpha to see which case holds. Then the value of $w(A_{xv})$ is computed. This is compared to the maximum value to date, and the maximum value is updated if necessary.

Step 6 - Goto Step 1. This program has no set stopping criteria. It is programmed to run indefinitely. Theorem 2 states that W(A) is the union of **all** $W(A_{xv})$ and x and v are being generated randomly. When the image of W(A) appears to have stabilized into a convex shape then it is interrupted.

**Algorithm 2**

The second algorithm is based the fact that the numerical range is a convex set. It implements the algorithm that visualizes W(A) as the intersection of half-spaces of support lines. The algorithm approximates W(A) from the outside.

The algorithm goes as follows:

1. Determine an angle $\gamma$
2. $n \leftarrow \text{trunc}(2\pi/\gamma + 0.5)$
3. for $j := 0$ to n do
    1. $\theta \leftarrow j * \gamma$
    2. $H(\theta) \leftarrow (e^{i\theta} A + e^{-i\theta} (A^*))/2$
    3. $w \leftarrow \lambda_{max}(H(\theta))$
    4. maxw $\leftarrow$ max(w, maxw)
    5. graph the support line corresponding to $\lambda_{max}(H(\theta))$

Unlike the implementation of Algorithm 1, this algorithm is able to exhibit W(A) for an arbitrary complex matrix. It is not restricted to matrices that are unitarily similar to a real matrix.

Step 1. The user is able to enter any choice for $\gamma \in [0, 2\pi]$.

Step 2. Here we compute the number of iterations for the program. This is unlike the first algorithm. This program will terminate after a predetermined number of iterations with an outer approximation to W(A) and an upper bound for w(A). The smaller the angle $\gamma$ specified, the more iterations and the better the approximation to W(A).

Step 3.1. Each successive angle $\theta$ is computed.

Step 3.2. $H(\theta)$, the hermitian part of $e^{i\theta}A$, is calculated.

Step 3.3. The power method is run on $H(\theta)$ to determine its largest eigenvalue. The version of the power method implemented here is the Rayleigh Quotient method. This method will find the largest eigenvalue in modulus of the given matrix. For this algorithm the rightmost eigenvalue is required. To get around this problem the matrix T was computed, $T = H(\theta) + \| H(\theta) \|_1 * I_n$. (Here $\| A \|_1$ is the 1-norm, $\max_i \sum_{j=1}^{n} | a_{ij} |$ $i = 1, \ldots, n$.) This ensures that T is a positive semi-definite-matrix ($\lambda_i \geq 0$, $i = 1, \ldots, n$). Thus the rightmost eigenvalue of T is the eigenvalue of maximum modulus. The rightmost eigenvalue of $H(\theta)$ was computed by $\lambda_{max}(H(\theta)) = \lambda_{max}(T) - \| H(\theta) \|_1$.

Step 3.4. The maximum of the values $\lambda_{max}(H(\theta))$, $\theta \in [0, 2\pi]$, is an approximation to the numerical radius of A. This value is maximized at every iteration.

Step 3.5. The equation of the support line at the point $e^{-i\theta}\lambda_{max}(H(\theta))$, rotated counterclockwise through $\theta$, is $x = \lambda_{max}(H(\theta))$. Thus the equation of the support line itself is

$$x \cos\theta - y \sin\theta = \lambda(H(\theta)).$$

A problem was encountered with the implementation of this algorithm. In Step 3.3 when the power method is applied, an initial estimation, $x_0$, of an eigenvector is required. Originally, the code was written so that $x_0 = [1, \ldots, 1]^T$. This presented no problem with the majority of examples for which the algorithm was tested. But the program consistently failed for any doubly stochastic matrix.

This failure can be explained however, the simplest solution computationally is to generate a random starting vector.

**Nilpotent Matrices**

Using the visualization algorithms it is easy to construct examples of nilpotent matrices whose numerical ranges are disks centered at the origin. The question arises: what are necessary and sufficient conditions on a nilpotent A so that W(A) is a disk centered at the origin?

Let A to be an arbitrary n-square matrix. There is no loss in generality in assuming that $0 \in W(A)$.

Theorem 5. Let $A = H + i K$ be the hermitian decomposition of A. Let $\lambda(\theta)$ be the maximum eigenvalue of

$$\cos\theta\, H - \sin\theta\, K \tag{27}$$

If $0 \in W(A)$ then W(A) is a disk centered at the origin iff the maximum eigenvalue $\lambda(\theta)$ of (27) is independent of $\theta$, $0 \leq \theta \leq 2\pi$.

Theorem 6. Let A be an n-square real nilpotent matrix. For $n = 3$, W(A) is a disk centered at the origin iff

$$\mathrm{tr}((A^2)^T A) = 0.$$

For $n = 4$, W(A) is a disk centered at the origin iff

$$\mathrm{tr}((A^2)^T A) = 0$$

and

$$\mathrm{tr}((A^3)^T A) = 0.$$

Research is currently underway to extend Theorem 6 to general n-square matrices.

## Normal Matrices and Symmetry

Let A be a linear operator on a finite dimensional unitary space V of dimension n. The $k^{th}$ higher numerical range of A, denoted by $W_k(A)$, is the totality of complex numbers tr(PAP) where P runs over all k-dimensional orthogonal projections on V. Very recently [490] we were able to prove that $W_k(A)$ is polygon with the real axis as a line of symmetry, $k = 1, \ldots, n$, if and only if A is normal with a real characteristic polynomial. We also constructed several nonnormal examples to investigate the extent to which the symmetry of all the $W_k(A)$ is *required*.

## Visualization of the Numerical Range

As an example of the use of the visualization algorithms described above, consider the following problem. Is it the case that

$$W_k(A) = W_k(B), \quad k = 1, \ldots, n \tag{28}$$

suffice to conclude that the two n-square matrices A and B are unitarily similar? *Recall that*

$$W_k(A) = \left\{ z \mid z = \sum_{j=1}^{k} (Ax_j, x_j),\ x_1, \ldots, x_k \text{ o.n.} \right\},$$

so that $W_1(A)$ is simply $W(A)$. To investigate this conjecture we take $n = 3$ and then directly confirm that the conditions (28) are equivalent to

$$\begin{aligned} W(A) &= W(B), \\ \mathrm{tr}(A) &= \mathrm{tr}(B). \end{aligned} \tag{29}$$

Consider the matrix

$$A = \begin{bmatrix} 3 & i & 0 \\ i & 2 & i \\ 0 & i & 1 \end{bmatrix}.$$

The visualization algorithm produces the following image for W(A).

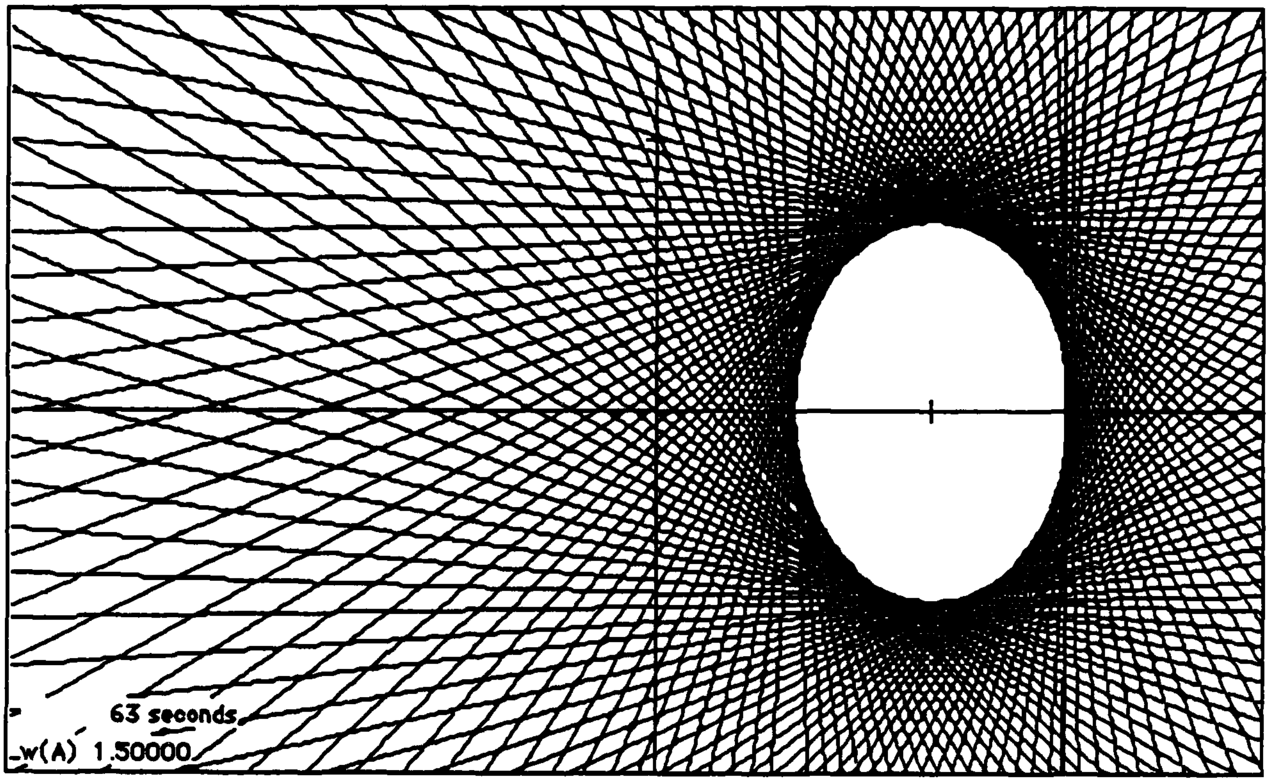



**Figure 1**

The image of W(A) in Figure 1 is scaled by 2, which means that every entry in A is divided by 2 before W(A) is computed.

Next consider the matrix

$$B = \begin{bmatrix} 3 & 0 & \sqrt{2}\,i \\ 0 & 2 & 0 \\ \sqrt{2}\,i & 0 & 1 \end{bmatrix}.$$

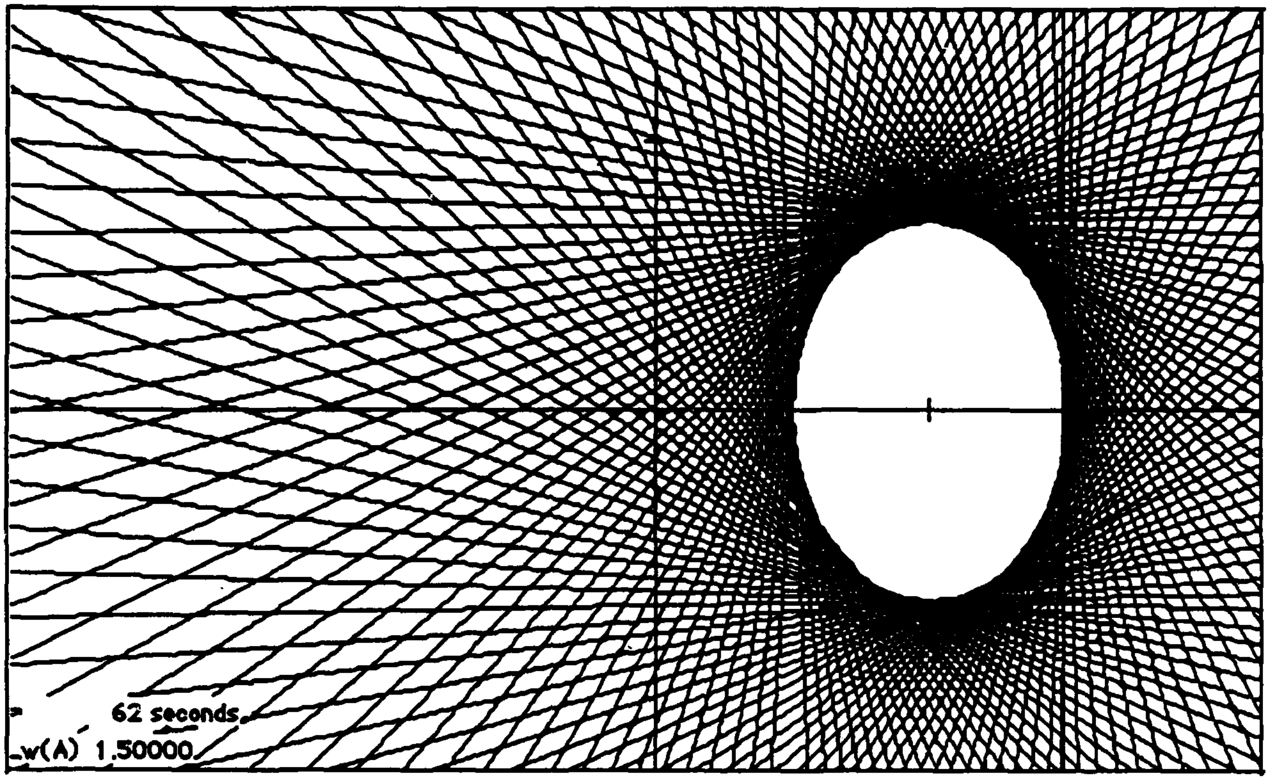



**Figure 2**

The image of W(B) in Figure 2 is scaled by 2.

The numerical ranges of A and B are identical. We know that W(A) = W(B) is a necessary but not sufficient condition for two matrices to be unitarily similar. The matrices A and B are not unitarily similar. To explain how these matrices can have the same numerical range and not be unitarily similar we present the following discussion.

First consider matrix A. Define the matrix

$$C = A - 2I_3 = \begin{bmatrix} 3 & i & 0 \\ i & 2 & i \\ 0 & i & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & i & 0 \\ i & 0 & i \\ 0 & i & -1 \end{bmatrix}.$$

The matrix C has a unique decomposition into real and imaginary parts, $H_C$ and

$K_C$, both symmetric:

$$C = H_C + i\,K_C$$
$$= \frac{C + C^*}{2} + i\,\frac{C - C^*}{2i}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + i \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

For a general matrix, H + iK, Algorithm 2 graphs the support lines

$$x \cos\theta - y \sin\theta = \lambda(\theta)$$

where $\lambda(\theta)$ is the largest eigenvalue of $H(\theta) = \cos\theta\, H - \sin\theta\, K$. We compute from above that for C,

$$\cos\theta\, H_C - \sin\theta\, K_C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & 0 & -\sin\theta \\ 0 & -\sin\theta & -\cos\theta \end{bmatrix}.$$

The characteristic polynomial of the preceding matrix $H_C(\theta)$ is

$$\lambda^3 + [-2\sin^2\theta - \cos^2\theta]\lambda - [-\sin^2\theta\cos\theta + \sin^2\theta\cos\theta] = 0,$$

or

$$\lambda^3 - [1 + \sin^2\theta]\lambda = 0.$$

Solving for $\lambda$ we obtain

$$\lambda = \pm\sqrt{1 + \sin^2\theta}\ .$$

Thus

$$\lambda(\theta) = \sqrt{1 + \sin^2\theta}\,.$$

Now consider the matrix B. Note that B is unitarily (permutation) similar to the matrix

$$B' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & \sqrt{2}\,i \\ 0 & \sqrt{2}\,i & 1 \end{bmatrix}.$$

Define the matrix

$$D = B' - 2I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{2}\,i \\ 0 & \sqrt{2}\,i & -1 \end{bmatrix}.$$

As above, $D = H_D + iK_D$, for some unique hermitian $H_D$ and $K_D$:

$$H_D = \frac{D + D^*}{2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$K_D = \frac{D - D^*}{2i} = \sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

To graph W(B) using the second visualization algorithm, the lines $x\cos\theta - y\sin\theta = \lambda(\theta)$ are graphed where $\lambda(\theta)$ is the largest eigenvalue of

$H(\theta) = \cos\theta\, H - \sin\theta\, K$. For B',

$$H_D(\theta) = \cos\theta\, H_D - \sin\theta\, K_D$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta & -\sqrt{2}\sin\theta \\ 0 & -\sqrt{2}\sin\theta & -\cos\theta \end{bmatrix}.$$

The characteristic polynomial of $H_D(\theta)$ is

$$\lambda^2 + [-\cos^2\theta - 2\sin^2\theta] = 0,$$

or

$$\lambda^2 + [-\cos^2\theta - \sin^2\theta - \sin^2\theta] = 0,$$

or

$$\lambda^2 = 1 + \sin^2\theta.$$

Solving for $\lambda$ we have

$$\lambda = \pm\sqrt{1 + \sin^2\theta}.$$

Thus

$$\lambda(\theta) = \sqrt{1 + \sin^2\theta}.$$

Hence the support lines are the same for W(D) and W(C). But W(A) = W(C) + 2 and W(B) = W(D) + 2, and thus W(A) = W(B). The important thing to note here is that the matrices A and B have the property that

$$\lambda(\theta) = \lambda_{max}(\cos\theta\, H - \sin\theta\, K)$$

is the same for all $\theta$ for both A and B. If we denote the maximum eigenvalue of the hermitian part of $e^{i\theta}A$ by $\lambda_A(\theta)$ (and similarly for $\lambda_B(\theta)$) then the geometric condition

$$\lambda_A(\theta) = \lambda_B(\theta) \text{ for all } \theta \in [0, 2\pi]$$

is not equivalent to the algebraic condition that A is unitarily similar to B.

If A and B were unitarily similar then $U^*AU = B$ would imply that

$$U^*H_AU = H_B,$$

and

$$U^*K_AU = K_B.$$

Hence if A were unitarily similar to B then $C = A - 2I_3$ would be unitarily similar to $D = B' - 2I_3$. Thus we can work with the matrices C and D in our discussion. Consider

$$U^*H_DU = H_C \tag{30}$$

and

$$U^*K_DU = K_C. \tag{31}$$

From (30) we can solve the system for the matrix U:

$$H_DU = UH_C,$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} U = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ u_{21} & u_{22} & u_{23} \\ -u_{31} & -u_{32} & -u_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & -u_{13} \\ u_{21} & 0 & -u_{23} \\ u_{31} & 0 & -u_{33} \end{bmatrix}.$$

The above equalities lead to $u_{11} = u_{13} = u_{22} = u_{23} = u_{32} = u_{31} = 0$. So, if a unitary U exists satisfying (30) and (31) then it must have the form

$$U = \begin{bmatrix} 0 & u_{12} & 0 \\ u_{21} & 0 & 0 \\ 0 & 0 & u_{33} \end{bmatrix}. \tag{32}$$

From (31) we have the equalities

$$K_D U = U K_C,$$

$$\sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & u_{12} & 0 \\ u_{21} & 0 & 0 \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 0 & u_{12} & 0 \\ u_{21} & 0 & 0 \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & u_{33} \\ u_{21} & 0 & 0 \end{bmatrix} = \begin{bmatrix} u_{12} & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

After we have computed the first column of the righthand side of the last equality, we need go no further. The equality shows that $u_{12} = 0$. This fact combined with (32) contradicts the unitary property of U.

## Open Questions

There is a nearly unlimited number of open questions in this field. However, much current research is along the following general lines.

Let $A:V \to V$ be an operator on a unitary space V. Let X be a subset of V and $f:\mathbb{C} \to \mathbb{C}$ be a complex function. Let M be a subset of $\mathbb{C}$. Describe the set

$$W(A, X, M, f) = \left\{ z \mid f((Ax, x)) \in M \text{ for all } x \in X \right\}.$$

There are many variations on this question. For example, suppose $W(A, X, M, f)$ has certain geometric properties, e.g., symmetry with respect to a line or a point, then what can be concluded about A?

Here are some simple instances of this type of question:

1. If A is nilpotent and W(A) is a disk, must it be centered at the origin?

2. If W(A) is a disk centered at the origin, is A nilpotent? The answer is "no":

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

3. If A and B are $3 \times 3$ and $W(A) = W(B)$, $W(A^{-1}) = W(B^{-1})$ and $tr(A) = tr(B)$, is it true that A and B are unitarily similar? ($W(A) = W(B)$ is not enough to conclude that A and B are unitarily similar.)

A brief perusal of the bibliography in Section IV indicates the broad scope of this research.

## III. Research of M. Marcus, 1983 - 1988

1. On the equality of decomposable symmetrized tensors (with J. Chollet), Linear and Multilinear Algebra, 13 (1983), 253-266.

This paper continues earlier work by the authors on finding necessary and sufficient conditions for two decomposable symmetrized tensors to be equal. In the previous paper [Linear and Multilinear Algebra 6 (1978), 317-326] the linear independence of the vectors forming such tensors was assumed. In the present paper, this assumption is dropped and much simpler requirements for equality are obtained. The paper also includes conditions for a decomposable symmetrized tensor to be 0. This research is related to recent work of J.A. Dias da Silva, R. Merris, S. Pierce, G.N. de Oliveira, and S.G. Williamson.

2. Solution to problem 6366, American Mathematical Monthly, 90 (1983, 409-410.

3. Products of doubly stochastic matrices (with K. Kidman and M. Sandy), Linear and Multilinear Algebra, 15 (1984), 331-340.

In studying Westwick's theorem on higher numerical ranges, the theory of elementary doubly stochastic matrices arises. This concept is related to the work of M. Goldberg and E.G. Straus [Linear Algebra Appl, 18 (1977), 1-24] on the representation of a doubly stochastic matrix as a product of elementary doubly stochastic matrices. This paper studies the class of doubly stochastic matrices that can be written as products of elementary doubly stochastic matrices. The same questions for orthostochastic matrices are also investigated.

4. Unitarily invariant generalized matrix norms and Hadamard product (with K. Kidman and M. Sandy), Linear and Multilinear Algebra, 16 (1984), 197-213.

Let $\|\cdot\|$ be a unitarily invariant generalized matrix norm on $M_n(C)$, the space of n-square complex matrices. Theorems are developed relating the Hadamard product (entrywise product) of two matrices $A, B \in M_n(C)$ to the singular values of A and B. For $p \geq 1$, $1 \leq k \leq n$, let

$$\|A\|_p^k = \left(\sum_{i=1}^{k} \alpha_i(A)^p\right)^{1/p},$$

where $\alpha_1(A) \geq \cdots \geq \alpha_n(A)$ are the singular values of A. In this paper the following inequality is proved: $\|A \cdot B\|_p^k \le \|A\|_p^k \ \|B\|_p^k$. If $1 < k \le n$ it is also proved that $\|A \cdot B\|_p^k = \|A\|_p^k \|B\|_p^k$ if and only if $A = a_{ij}E_{ij}$, and $B = b_{ij}E_{ij}$, where $E_{ij}$ is the matrix with 1 in position (i,j) and zeros elsewhere. The case $k = 1$ is also discussed.

5. An exponential group, Linear and Multilinear Algebra, 14 (1984), 293-296.

   Let M(r) be the n-square matrix whose (i,j) entry is

$$\binom{i-1}{j-1} r^{i-j}.$$

   It is proved that the mapping $r \to M(r)$ establishes an isomorphism from the additive group of the real numbers into the multiplicative structure of the $n \times n$ matrices. This paper investigates M(r) as an exponential matrix.

6. Solution to problem 6430 (with J. Bruno), American Mathematical Monthly, 92 (1985), 148-149.

7. *Conditions for the generalized numerical range to be real* (with M. Sandy), Linear Algebra and Appl., 71 (1985), 219-239.

   If A and C are n-square complex matrices then the C-numerical range of A is the totality of numbers $tr(CU^*AU)$ as U varies over all unitary matrices. This paper obtains necessary and *sufficient* conditions for the C-numerical range of A to be a subset of the real axis. The principal condition is that both A and C must be translates of Hermitian matrices.

8. Ryser's permanent identity in symmetric algebra (with M. Sandy), Linear and Multilinear Algebra, 18 (1985), 183-196.

   The polynomial algebra over a field is canonically isomorphic to the symmetric algebra over a vector space. Several identities expressing homogeneous polynomials in terms of sums of powers of linear polynomials are exploited to obtain Ryser's permanent identity [Combinatorial Mathematics, MAA Carus Monograph No. 14, Wiley, New York, 1963] as well as extensions of identities due to Bebiano [Pacific J. Math., 101 No. 1, (1982), 1-9]

9. Singular values and numerical radii (with M. Sandy), Linear and Multilinear Algebra, 18, No. 3, (1985), 337-353.

   The purpose of this paper is to prove the following result relating the singular values and the numerical radius of a matrix: For any n-square, complex matrix A with singular values $\alpha_1 \ge \cdots \ge \alpha_n \ge 0$ *and numerical radius r(A)*

$$\frac{\alpha_1 + \cdots + \alpha_n}{n} \le r(A),$$

with equality if and only if $A/r(A)$ is unitarily similar to the direct sum of a diagonal unitary matrix and unit multiples of $2 \times 2$ matrices of the form

$$\begin{bmatrix} 1 & d \\ -\bar{d} & -1 \end{bmatrix},$$

where $0 < |d| \le 1$.

10. Construction of orthonormal bases in higher symmetry classes of tensors (with J. Chollet), Linear and Multilinear Algebra, 19 (1986), 133-140.
A method is presented for constructing an orthonormal basis for a symmetry class of tensors from an orthonormal basis of the underlying vector spaces. The basis so obtained is not composed of decomposable symmetrized tensors. Indeed, we show that, for symmetry classes of tensors whose associated character has degree higher than 1, it is impossible to construct an orthogonal basis of decomposable symmetrized tensors from any basis of the underlying vector space. The paper poses an open problem on the possibility of a symmetry class having an orthonormal basis of decomposable symmetrized tensors.

11. Computer generated numerical ranges and some resulting theorems (with C. Pesce), Linear and Multilinear Algebra, 21 (1987), 121-157.
The numerical range, W(A), of an arbitrary n-square matrix A is the union of the numerical ranges of all 2-square real compressions of A. As a result, a simple graphics program is written that accurately exhibits W(A) for real A, and suggests several conjectures relating the geometry of W(A) to algebraic properties of A. Some of these conjectures are analyzed in the final sections of the paper.

12. Solution to problem 1231, Mathematics Magazine, 60 No. 1 (1987), 42.

13. Vertex points in the numerical range of a derivation (with M. Sandy), Linear and Multilinear Algebra, 21 (1987), 385-394.
This paper contains a number of results on the distribution of values of subdeterminants of normal matrices. It is a continuation of earlier work of M. Marcus [Indiana University Math. J., 22 (1973), 1137-1149].

14. Solution to problem E3179, submitted to American Mathematical Monthly.

15. Solution to problem 1248, A Curious property of 1/7, (with C. Pesce), Mathematics Magazine, 60 (1987), 42.

16. Two Determinant Condensation Formulas, Linear and Multilinear Algebra, 22 (1987), 95-102.

    This paper corrects and extends several classical results that express the determinant of a block matrix in terms of determinants of the constituent blocks.

17. Symmetry properties of higher numerical ranges (with M. Sandy), Linear Algebra and Appl., 104 (1988), 141-164.

    Let A be a linear operator on a finite dimensional unitary space V of dimension n. The $k^{th}$ higher numerical range of A, denoted by $W_k(A)$, is the totality of complex numbers tr(PAP) where P runs over all k-dimensional orthogonal projections on V. It is proved that $W_k(A)$ is polygon with the real axis as a line of symmetry, $k = 1, \ldots, n$, if and only if A is normal with a real characteristic polynomial. Several non-normal examples are exhibited that reveal the extent to which the symmetry of all the $W_k(A)$ is required.

18. Advanced problem, Triangular Kronecker Products, (with C. Pesce), accepted for publication, American Math. Monthly.

19. Bessel's inequality in tensor space (with M. Sandy), Linear and Multilinear Algebra, in press.

    Let A be an n-square complex matrix and define $A_A$ to be the n!-square matrix whose entries are

$$\prod_{i=1}^{n} a_{\sigma(i),\tau(i)},$$

where $\sigma$ and $\tau$ run lexicographically over $S_n$. If A is positive definite Hermitian and $\chi$ is a unit n!-tuple then

$$(A_A\chi, \chi) \geq \det(A) + |\sum \chi(\sigma)|^2 c(A)$$

where c(A) is the largest of the numbers

$$\prod_{i=1}^{n} | a_{ij} |^2 / a_{jj}^n, \quad j = 1, \ldots, n,$$

and the summation is over $\sigma \in S_n$. For $n = 3$, if A is not permutation similar to a direct sum and $\chi$ is a unit n!-tuple then $(A_A\chi, \chi) = \det(A)$ iff $\chi$ is a multiple of the alternating character. The relationships among recent results of Bapat and Sunder, Chollet, and Gregorac and Henzel are also discussed.

20. A unified approach to some classical matrix theorems, submitted.
An elementary inequality is proved that obtains the lower bound of the product of forms $(Ax,x)\,(A^{-1}x,x)$, where x is a unit vector and A is a positive definite Hermitian matrix. Using this inequality it is possible to provide a unified treatment of the following theorems: the Hadamard determinant theorem; the Fischer inequality; the Kantorovich inequality; Weyl's inequalities.

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4. Computing Without Mathematics: BASIC, Pascal, Applications (with J. Marcus), Computer Science Press, Rockville, Maryland, 1986.

5. Introduction to Linear Algebra (with H. Minc), Dover Publications Inc., New York, 1988.

6. Foundations of Numerical Linear Algebra I, Center for Computational Sciences and Engineering, University of California, Santa Barbara, Monographs in Scientific Computation, 1988.

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| 680 | Tam | Tin Yau | | Linear operators on matrices: the invariance of the decomposable numerical range | Linear Algebra Appl. | 85 | 1-7 | 1987 |
| 681 | Tam | Tin Yau | | Linear operators on matrices: the invariance of the decomposable numerical radius | Linear Algebra Appl. | 87 | 147-153 | 1987 |
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| 684 | Taussky | O. | | A recurring theorem on determinants | Amer. Math. Monthly | 56 | 672-676 | 1949 |
| 685 | Taussky | O. | | Commutativity in finite matrices | Amer. Math. Monthly | 64 | 227-235 | 1957 |
| 686 | Taussky | O. | | Research Problems 11 | Bull. Amer. Math. Soc. 64 | 3 | 124 | 1958 |
| 687 | Taussky | O. | | A weak property L for pairs of matrices | Math. Z. | 71 | 463-465 | 1959 |
| 688 | Taussky | O. | | Commutators of unitary matrices that commute with one factor | J. Math. and Mech. | 10 | 175-178 | 1961 |
| 689 | Taussky | O. | | A Generalization of a Theorem of Lyapunov | J. Math. Anal. Appl., also J. SIAM 9, 640-643 (1961) | 2 | 105-107 | 1961 |
| 690 | Taussky | O. | | Matrices with trace zero | Amer. Math. Monthly | 69 | 40-42 | 1962 |
| 691 | Taussky | O. | | Eigenvalues of finite matrices: Some topics concerning bounds for eigenvalues of finite matric | Survey of Numerical Analysis, McGraw Hill, New York | | 279-297 | 1962 |
| 692 | Taussky | O. | | A remark concerning the similarity of a finite matrix A and A* | Math. Z. | 117 | 189-190 | 1970 |
| 693 | Taussky | O. | | The role of symmetric matrices in the study of general matrices | Linear Algebra Appl. | 5 | 147-154 | 1972 |
| 694 | Taussky | O. | | Sets of complex matrices which can be transformed to triangular form | Colloq. Math. Soc. Janos Bolyai | 22 | 579-290 | 1977 |
| 695 | Taussky | O. | | Eigenvalues, circles, ovals, and periods of complex functions | Linear and Multilinear Algebra | 18 | 87-88 | 1985 |
| 696 | Taussky-Todd | O. | | Positive definite matrices in: Inequalities | Academic Press, New York | | 309-319 | 1967 |
| 697 | Thompson | Robert C. | | Principal submatrices X: Orthostochastic matrices and the independence of the eigenvalues of | to appear | | | |
| 698 | Thompson | R.C. | | Simultaneous conjunctive reduction of a pair of indefinite hermitian matrices | Institute for the Interdisciplinary Applications of Algebra and | | | |
| 699 | Thompson | R.C. | | Principal submatrices VIII: Principal sections of a pair of forms | Rocky Mountain J. Math. | | | |
| 700 | Thompson | R.C. | | Prinicpal submatrices and the independence of the eigenvalues of different principal submatrice | | | | |
| 701 | Thompson | R.C. | | A note on normal matrices | Canad. J. Math. | 15 | 220-226 | 1963 |
| 702 | Thompson | Robert C. | | Principal submatrices of normal and Hermitain matrices | Illinois J. Math. | 10 | 296-308 | 1966 |
| 703 | Thompson | R.C. | | Principal submatrices IV: On the independence of the eigenvalus of different principal submatri | Linear Algebra Appl. | 2 | 355-374 | 1969 |
| 704 | Thompson | Robert C. | | Principal submatrices IX: Interlacing inequalities for singular values of submatrices | Linear Algebra Appl. | 5 | 1-12 | 1972 |
| 705 | Thompson | R. C. | | The bilinear field of values | Monatsh. Math. | 81 | 153-167 | 1976 |
| 706 | Thompson | R.C. | | Singular values, diagonal elements and convexity | SIAM J. Appl. Math. | 32 | 39-63 | 1977 |
| 707 | Thompson | R.C. | | Singular values and diagonal elements of complex symmetric matrices | Linear Algebra Appl. | 26 | 65-106 | 1979 |
| 708 | Thompson | Robert C. | | The Congruence Numerical Range. | Linear and Multilinear Aglebra | 8 | 197-206 | 1980 |
| 709 | Thompson | Robert C. | | Research problem: The matrix numerical range | Linear and Multilinear Algebra | 21 | 321-323 | 1987 |
| 710 | Thorp | E.O. | | The relation between a compact linear operator and its conjugate | Amer. Math. Monthly | 66 | 764-769 | 1959 |
| 711 | Todd | J. | | Survey of numerical analysis, editor | McGraw-Hill, New York | | | 1962 |
| 712 | Toeplitz | O. | | Das algebraische Analogen zu einem Satze von Fejer | Math. Z. | 2 | 187-197 | 1918 |
| 713 | Tong | Wen-Ting | | Disk theory for tensor product of matrices | to appear | | | |
| 714 | Tong | Wen-Ting | | On the spectrum of tensor product of matrix | Acta Math. Sinca | 23 | 128-134 | 1980 |
| 715 | Tong | Wen-Ting | | On the Spectral Radius of Matrices. | Linear and Multilinear Aglebra | 20 | 175-182 | 1987 |
| 716 | Tsing | N.K. | | On the shape of the generalized numerical ranges | Linear and Multilinear Algebra | 10 | 173-182 | 1981 |
| 717 | Tsing | Nam-Kiu | | Diameter and Minimal Width of the Numerical Range. | Linear and Multilinear Aglebra | 14 | 179-185 | 1983 |
| 718 | Tsing | Nam-Kiu | | The Constrained Bilinear Form and the C-Numerical Range | Linear Algebra Appl. | 56 | 195-206 | 1984 |
| 719 | Turkel | E. | | Symmetric hyperbolic difference schemes and matrix problems | Linear Algebra Appl. | 16 | 109-129 | 1977 |
| 720 | Uhlig | Frank | | A canonical form for a pair of real symmetric matrices that generate a nonsingular pencil. | Linear Algebra Appl., in press | | | |
| 721 | Uhlig | Frank | | On the maximal number of linearly independent real vectors annihilated simultaneously by two | Pacific J. Math. | 49 | 543-560 | 1973 |
| 722 | Uhlig | Frank | | The number of vectors jointly annihilated by two real quadratic forms determines the inertia of | Pacific J. Math. | 49 | 537-542 | 1973 |
| 723 | Uhlig | Frank | | Definite and semidefinite matrices in a real symmetric matrix pencil. | Pacific J. Math. | 2 | 561-568 | 1973 |
| 724 | Uhlig | Frank | | Simultaneous block diagonalization of two real symmetric matrices | Linear Algebra Appl. | 7 | 281-289 | 1973 |

Numerical Range References

| | Auth. Last | Auth. First | Auth2 Last | Title | Journal | Vol | Pge | Yr |
|---|---|---|---|---|---|---|---|---|
| 727 | Uhlig | F. | | Explicit polar decomposition and a near-characteristic polynomial: The 2 x 2 case | Linear Algebra Appl. | 38 | 239-249 | 1981 |
| 728 | Uhlig | Frank | | Relations between the fields of values of a matrix and of its polar factors: the 2x2 real and cor | Linear Algebra Appl. | 52 | 701-715 | 1983 |
| 729 | Uhlig | Frank | | The field of values of a complex matrix, an explicit desciption in the 2 x 2 case | SIAM J. Algebraic Discrete Methods | 6 | 541-545 | 1985 |
| 730 | Vagabov | N.G. | | Numerical range and multiple completeness. I. Numerical range and specturm of operator-functi | "Elm", Baku | | 102-141 | 1980 |
| 731 | Viswanth | K. | | Normal operators on quaternionic Hilbert spaces | Trans. Amer. Math. Soc. | 162 | 337-351 | 1971 |
| 732 | Vogg | H. | | Ordnungstheoretische behandlung von wertevorraten | Dissertation, Technische Universitat München | | | 1972 |
| 733 | Wadhoa | B.L. | | Spectral M hyponormal decomposable operators | Ph.D. Thesis, Indiana Univ. | | | 1971 |
| 734 | Walker | A.G. | Weston | Inclusion theorems for the eigenvalues of a normal matrix | J. London Math. Soc. | 24 | 28-31 | 1949 |
| 735 | Wang | J. | Fan | Structured singular value and geometry of m-Form numerical range | University of Maryland, College Park, SRC, TR 87-186 | | | 1987 |
| 736 | Waterhouse | W. | | Pairs of quadratic forms | Inventiones Math. | 37 | 157-164 | 1976 |
| 737 | Waterhouse | William C. | | A conjectured property of Hermitian pencils | Linear Algebra Appl. | 51 | 173-177 | 1983 |
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| 739 | Wedderburn | J.H.M. | | Lectures on matrices | Amer. Math. Soc. Colloq. Pub. | 17 | | 1934 |
| 740 | Westwick | R. | | A theorem on numerical range | Linear and Multilinear Algebra | 2 | 311-315 | 1975 |
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| 744 | Wielandt | H. | | Ueber die Unbescgränkheit der Operatoren der Quantenmechanik | Math. Annalen | 121 | 21 | 1949 |
| 745 | Wielandt | H. | | Die Einschliessung von Eigenwerten normaler Matrizen | Math. Ann. | 121 | 234-241 | 1949 |
| 746 | Wielandt | H. | | Inclusion theorems for eigenvalues | Nat. Bur. of Stand. Appl. Math. Series 29 | | 75-78 | 1953 |
| 747 | Wielandt | H. | | An extremum property of sums of eigenvectors | Proc. Amer. Math. Soc. | 6 | 106-110 | 1955 |
| 748 | Wielandt | H. | | On eigenvalues of sums of normal matrices | Pacific J. Math. | 5 | 622-638 | 1955 |
| 749 | Wielandt | H. | | review of Commutators of unitary matrices that commute with one factor by O. Taussky | in Zentralblatt für Mathematik und Ihre Grenzgebiete | 96 | 9 | 1962 |
| 750 | Wielandt | H. | | On the eigenvalues of A + B and AB | J. Res. Nat. Bur. Standards, Sect. B | 77 | | 1973 |
| 751 | Williams | J. | | Spectral sets and finite dimensional operators | Thesis, University of Michigan | | | 1965 |
| 752 | Williams | J.P. | | Spectra of products and numerical ranges | J. Math. Anal. Appl. | 17 | 214-220 | 1967 |
| 753 | Williams | J. P. | Crimmins | On the numerical radius of a linear operator | Amer. Math. Monthly | 74 | 832-833 | 1967 |
| 754 | Williams | J.P. | Radjavi | Products of self adjoint operators | Michigan Math. J. | 16 | 177-185 | 1969 |
| 755 | Williams | J. P. | | Operators similar to their adjoints | Proc. Amer. Math. Soc. | 20 | 121-123 | 1969 |
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| 757 | Williams | J. | | Finite operators | Proc. Amer. Math. Soc. | 26 | 129-136 | 1970 |
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| 762 | Williamson | J. | | The conjunctive equivalence of pencils of Hermitian and anti-Hermitian matrices | Amer. J. MAth. | 59 | 399-413 | 1937 |
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| 772 | Zafiriou | E. | Morari | Robust H2-Type IMC Controller Design Via the Structured Singular Value | Proc. 10th IFAC World Congress 2, July | | 275-280 | 1987 |
| 773 | Zarantonello | E.G. | | The closure of the numerical range contains the spectrum | Bull. Amer. Math. Soc. | 70 | 781-787 | 1964 |
| 774 | Zassenhaus | H. | | A remark on a paper of O. Taussky | J. Math. and Mech. | 10 | 179-180 | 1961 |
| 775 | Zenger | Chr. | | On convexity properties of the Bauer field of values of a matrix | Num. Math. | 12 | 96-105 | 1968 |
| 776 | Zenger | Chr. | Deutsch | Inclusion domains for the eigenvalues of stochastic matrices | Num. Math. | 18 | 182-192 | 1971 |
| 777 | Zenger | C. | | A comparison of some bounds for the nontrivial eigenvalues of stochastic matrices | Numer. Math. | 19 | 209-211 | 1972 |
| 778 | Zenger | Chr. | | Minimal subadditive inclusion domains for the eigenvalues of matrices | Linear Algebra Appl. | 17 | 233-268 | 1977 |
| 779 | Zhang | Fu-Zhang | | Another Proof of a Singular Value Inequality Concerning Hadamard Products of Matrices. | Linear and Multilinear Aglebra | 22 | 307-311 | 1987 |

## V. Appendix

### VITA

### PERSONAL BACKGROUND

| | |
|---|---|
| **Born:** | Albuquerque, New Mexico, July 31, 1927 |
| **Education:** | Attended public schools in California |
| **Military Service:** | United States Navy, 1944-1946, honorable discharge |
| **Married:** | Rebecca Elizabeth Marcus |
| **Children:** | Jeffrey (employed, Micropoint, Los Angeles)<br>Karen (Ph.D. student, Stanford University) |

**Academic Degrees:**

| | |
|---|---|
| 1950 | B.A. (highest honors in Mathematics) University of California, Berkeley |
| 1953 | Ph.D. University of California, Berkeley |

### PROFESSIONAL EXPERIENCE

| | |
|---|---|
| 1987 - present | Professor of Computer Science, UCSB |
| 1983 - 1987 | Professor of Mathematics and Computer Science, University of California, Santa Barbara |
| 1979 - present | Founder, Microcomputer Laboratory,University of California, Santa Barbara |
| 1978 - 1986 | Associate Vice Chancellor and Dean, Research and Academic Development, University of California, Santa Barbara |

| | |
|---|---|
| 1973 - 1979 | Director, Institute for the Interdisciplinary Applications of Algebra and Combinatorics,University of California, Santa Barbara |
| 1963 - 1968 | Chairman, Department of Mathematics, University of California, Santa Barbara |
| 1962 - 1983 | Professor of Mathematics, University of California, Santa Barbara |
| 1960 - 1961 | Research Mathematician, U.S. National Bureau of Standards, Washington, D.C. |
| 1954 - 1961 | Instructor, Assistant and Associate Professor of Mathematics, University of British Columbia |

**UNDERGRADUATE ACADEMIC ACTIVITIES**

| | |
|---|---|
| 1964 - 1972 | Lecturer, National Science Foundation Linear Algebra Conference for College Teachers, University of California, Santa Barbara |
| 1965 - 1966 | Lecturer, National Science Foundation In-service Institute for Secondary School Teachers, University of California, Santa Barbara |
| 1965 - 1974 | Visiting lecturer for the Mathematical Association of America, touring four-year undergraduate institutions in the far western area giving lectures on undergraduate mathematics |
| 1965 - present | Author and co-author of 20 undergraduate textbooks (see publication list) |
| 1975 | Principal Investigator, Summer Projects Grant and Regents' Undergraduate Instructional Improvement Grant for training Scientific Information Specialists, University of California, Santa Barbara |

| | |
|---|---|
| 1979 - 1986 | Established Microcomputer Laboratory at the University of California at Santa Barbara, under grants from: the Fund for the Improvement of Postsecondary Education; California Postsecondary Education Commission; Instructional Scientific Equipment Program, NSF. |
| 1979 - 1984 | Principal Investigator, The Comprehensive Program, Fund for the Improvement of Postsecondary Education, Curriculum Development Project in Applied Algebra |
| 1979 - 1984 | Program Director, Intensive Short Course in Basic College Level Mathematics for Adult Reentry Women under grants from the California Postsecondary Education Commission and The Development in Science Education Project of the National Science Foundation |
| 1987 - 1988 | Principal Investigator, National Science Foundation Grant, Computing and Algorithmic Mathematics for Secondary School Teachers |

## GRADUATE ACADEMIC ACTIVITIES

| | |
|---|---|
| 1964 - present | Author and co-author of four graduate textbooks |
| 1970 | Ford Foundation Visiting Distinguished Professor, University of Islamabad, Islamabad, Pakistan; Consultant on curriculum design at the new Pakistan National University |
| 1971 | Visiting Lecturer, University of Victoria, Victoria, British Columbia; Assist in the graduate program |
| 1973, 1977 | Director, Conferences on Matrix Theory, sponsored by the National Science Foundation, University of California, Santa Barbara |
| 1974 | Visiting Distinguished Professor, Laval University, Quebec, Canada; assist in the graduate program |

## GRADUATE STUDENTS

The following mathematicians have completed their Ph.D. work under the direction of M. Marcus:

Dr. Roy Westwick, Professor
University of British Columbia
Vancouver, B.C., Canada
Thesis: Linear transformations of Grassmann algebras
1960

Dr. Nisar A. Khan, Professor
Muslim University, Aligarh, India
Thesis: Matrix commutators
1961

Dr. Peter Botta, Assoc. Professor
University of Toronto
Toronto, Ontario, Canada
Thesis: Linear transformations on algebras
1965

Dr. Stanley G. Williamson, Professor
University of California
San Diego, California
Thesis: Tensor Algebras
1965

Dr. William R. Gordon, Professor
Department of Mathematics
University of Victoria
Victoria, B.C., Canada
Thesis: Inequalities for generalized matrix functions
1965

Dr. George Soules
Institute for Defense Analysis
Princeton, New Jersey
Thesis: Combinatorial functions
1966

Dr. Paul J. Nikolai, Mathematician
Wright-Patterson Air Force Base
Thesis: Mean value properties of generalized matrix functions
(This thesis was supervised jointly with Professor H. J. Ryser (deceased), California Institute of Technology)
1966

Dr. Stephen J. Pierce, Professor
California State University
San Diego
Thesis: Generalized isometries
1968

Dr. William Watkins, Professor
California State University, Northridge
Northridge, California
Thesis: Inequalities for derivation operators on a tensor space
1969

Dr. Russell Merris, Professor
California State University at Hayward
Hayward, California
Thesis: A generalization of the associated transformation
1969

Dr. Mohammad Shafqat Ali, Assoc. Professor
California State University at Long Beach
Long Beach, California
Thesis: Additive commutators, Jordan products and bilinear functions
1970

Dr. Elizabeth Wilson, Mathematician
Naval Labs. Pt. Mugu, California
Thesis: Partial derivations on symmetry classes of tensors
1971

Dr. James Holmes, Assistant Professor
Westmont College
Santa Barbara, California
Thesis: Application of derivations to invariance problems
1971

Dr. Herbert Robinson, Professor
Department of Mathematics
Texas A & M University
College Station, Texas
Thesis: Quadratic & bilinear forms on symmetry classes of tensors
1975

Dr. Patricia Andresen
University of Alaska
Fairbanks, Alaska
Thesis: The finite dimensional numerical range
1976

Dr. Robert Grone
University of Auburn
Auburn, Alabama
Thesis: Isometries of Matrix Algebras
1976

Dr. Ivan Filippenko, Research Mathematician
Lockheed Aircraft
Los Angeles, California
Thesis: Higher and Decomposable Numerical Ranges
1977

Dr. John Chollet, Assistant Professor
University of British Columbia
Vancouver, British Columbia, Canada
Thesis: Equalities of decomposable symmetrized tensors
1979

Dr. Kenneth Moore
Radar Systems Group
Hughes Aircraft Co.
El Segundo, California
Thesis: Determinantal Inequalities
1980

Dr. Kent Kidman
Hughes Aircraft Co.
El Segundo, California
Thesis: Stochastic Matrices and unitarily Invariant Norms
1983

Claire Pesce
Naval Weapons Center, China Lake
China Lake, California
Thesis: Visualization of the Numerical Range
1988

**ACADEMIC AWARDS AND DISTINCTIONS**

| | |
|---|---|
| 1950 | Graduated highest honors in mathematics, University of California, Berkeley |
| 1954 | Fulbright Award |
| 1956-57 | National Research Council, National Science Foundation, Post-doctoral Research Fellowship |
| 1956, 1958-60, 1975-84 | National Science Foundation Research Grants |
| 1962 | Certificate of Award for Distinguished Service, U.S. Department of Commerce, National Bureau of Standards |
| 1962 - present | Principal Investigator on Air Force Office of Scientific Research grants |

1965 Mathematical Association of America Editorial Prize for the article entitled: "Linear Transformations on matrices"

1966 L.R. Ford Memorial Prize awarded by the Mathematical Association of America for the article, "Permanents"

**EDITORIAL ACTIVITIES**

1. Mathematics Editor, Computer Science Press

2. Editor, Linear and Multilinear Algebra, published by Gordon and Breach, Science Publishers Inc.

3. Associate Editor, Linear Algebra and Its Applications, Elsevier Science Publishing Co., Inc.

4. Member of the Editorial Board, Pure and Applied Mathematics Series, Marcel Dekker, Inc.

5. Editor, Linear Algebra Volumes of Encyclopedia of Applicable Mathematics, Addison-Wesley Publishing Co.

6. Member of the Editorial Board, Linear Algebra and Its Applications

7. Associate Editor, Advanced Problem Section, American Mathematical Monthly

8. Referee and Reviewer for the following journals:

   Linear and Multilinear Algebra
   Linear Algebra and Its Applications
   Duke Journal
   Proceedings of the AMS
   Transactions of the AMS
   Bulletin of the AMS
   Mathematical Reviews
   Memoirs of the MAA
   American Mathematical Monthly

Canadian Journal of Mathematics
Pacific Journal of Mathematics
Proceedings of the Cambridge Philosophical Society
Zentralblatt

9. Technical reviewer for the Air Force Office of Scientific Research

10. Technical reviewer for the Mathematics Division of the National Science Foundation

11. Technical reviewer for the National Research Council of Canada

12. Technical reviewer for United States-Israel Binational Science Foundation

13. Editorial advisor for the following publishers:
    Houghton-Mifflin Company
    W.A. Benjamin, Inc.
    Harcourt, Brace and World

14. Advisory Editor, Letters in Linear Algebra

15. Editorial Board, Algebras, Groups, and Geometries

**SELECTED INVITED PAPERS**

1963 International Conference, "Recent Advances in Matrix Theory", U.S. Army Research Center, Madison, Wisconsin

1965 Far-Western meeting of the Mathematical Association of America

1965 Invited speaker, Annual meeting of the American Mathematical Society

1965, 1967, 1969
Symposium on Inequalities, sposored by Aerospace Resarch Laboratories, U.S. Air Force

1967 International Symposium on Combinatorial Analysis, sponsored by the Society for Industrial and Applied Mathematics

1972 Conference on Numerical Algebra, Los Alamos Scientific Laboratory

1974 University of California, Los Angeles

1975 University of Chicago, Chicago, Illinois

1975 University of California, San Diego

1975 California State University of Hayward

1975 AMS meeting, Kalamazoo, Michigan, Special Session on Matrix Theory

1975 California Mathematical Council Conference, Asilomar, California

1976 Northern California Section of the MAA annual meeting, University of California, Davis

1977 Gatlinburg VII, Conference on Numerical Algebra

1978 New York Academy of Science, Second International Conference on Combinatorial Mathematics

1980 California Institute of Technology Colloquium Series

1980 Oberwolfach Conference on General Inequalities

1981 Conference on Numerical Algebra, Oxford University

1982 Mid Atlantic Conference on Educational Computing, Bennett College, Greensboro, N.C.

1984 Invited contribution Special Issue of Linear Algebra and Its Applications honoring Helmut Wielandt

1986 University of California, Riverside

1986 University of California, San Diego

1986 Conference on Computers and Mathematics, Stanford University

1986 Western Educational Computing Consortium, Irvine

1988 SIAM Conference on Applied Linear Algebra, Madison Wisconsin

## MEMBERSHIP IN LEARNED SOCIETIES

American Mathematical Society

Mathematical Association of America

American Association of University Professors

Sigma Psi; Pi Mu Epsilon

American Association for the Advancement of Science

Washington Academy of Science

Society for Industrial and Applied Mathematics

Society for Technical Communication

Association for Computing Machinery

## UCSB UNIVERSITY SERVICE

Associate Vice Chancellor, Research and Academic Development. (1979 - 1987)
The following units reported to this office:

ACTER
Intructional Development/Learning Resources
Microcomputer Laboratory

Off Campus Studies
University Center at Ventura
University Extension
Algebra Institute
Center for Black Studies
Center for Chicano Studies
Community and Organization Research Institute
Computer Systems Laboratory
Intercampus Institute for Research at Particle Accelerators
Institute for Polymers and Organic Solids
Institute of Environmental Stress
Marine Science Institute
Quantum Institute
Social Process Research Institute

Numerous ad hoc personnel review committees

| | |
|---|---|
| 1962 - 1963 | Chairman, Statistics Committee |
| 1962 - 1963 | Academic Senate Educational Policy Committee |
| 1962 - 1963<br>1964 - 1965 | Chairman, Computer Committee |
| 1963 - 1964 | Digital Computer Committee |
| 1969 - 1972<br>1975 - 1977 | Academic Senate Research Committee |
| 1970 - 1972 | Academic Senate Education Abroad Committee |
| 1973 - 1974 | Chairman, Undergraduate Committee |
| 1973 - 1974 | Computer Science Laboratory Director Search Committee |
| 1974 | Chancellor's task force on career development |
| 1974 - 1975 | Academic Senate Athletic Policy Committee |

| | |
|---|---|
| 1975 | Ad Hoc Committee for Scientific Communication |
| 1979 - 1986 | Coordinator, Chinese Exchange Program |
| 1988 - Present | Computing Task Force |

**PROFESSIONAL REFERENCES**

Professor Stephen P. Diliberto
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University of California
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(415) 642-6550

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Professor Stanley G. Williamson
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La Jolla, California 92037
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**EXTRAMURAL SUPPORT**

| | |
|---|---|
| Air Force<br>Multilinear Algebra<br>M. Marcus, H. Minc<br>10/01/66 - 09/30/67 | $51,436 |
| Air Force<br>Multilinear Algebra<br>M. Marcus, H. Minc<br>10/01/67 - 09/30/68 | $61,610 |
| Air Force<br>Multilinear Algebra<br>M. Marcus, H. Minc<br>10/01/68 - 09/30/69 | $60,297 |
| Air Force<br>Multilinear Methods<br>M. Marcus, H. Minc<br>10/01/69 - 09/30/70 | $62,270 |
| Air Force<br>Multilinear Methods<br>M. Marcus, H. Minc<br>10/01/70 - 09/30/71 | $60,416 |
| Air Force<br>Eigenvalue Investigators and Stability<br>R.C. Thompson, M. Marcus, H. Minc<br>10/01/71 - 09/30/72 | $44,606 |
| Air Force<br>Inequalities, Combinatorics and Applications<br>M.Marcus, H. Minc, R.C. Thompson<br>10/01/72 - 09/30/73 | $42,961 |

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|---|---|
| Air Force<br>The Algebraic Eigenvalue Problem with Applications<br>M. Marcus, H. Minc, R.C. Thompson<br>10/01/73 - 09/30/74 | $36,079 |
| National Science Foundation<br>Theoretical Matrix Theory<br>M. Marcus<br>10/15/73 - 10/14/74 | $12,100 |
| Air Force<br>Supplementary Request<br>M. Marcus, H. Minc, R.C. Thompson<br>06/30/74 - 09/30/74 | $ 7,435 |
| Air Force<br>Algebraic Stability: A Linear Algebra Bibliography<br>M. Marcus, H. Minc, R.C. Thompson<br>10/01/74 - 09/30/75 | $55,183 |
| National Science Foundation<br>Undergraduate Research Participation<br>M. Marcus<br>02/15/75 - 05/31/76 | $26,740 |
| Air Force<br>Foundations of Stability, Linear Algebra Bibliography<br>M. Marcus, H. Minc, R.C. Thompson<br>10/01/75 - 09/30/76 | $51,702 |
| National Science Foundation<br>Computer Searchable Information Files<br>M. Marcus<br>07/01/76 - 12/31/78 | $68,348 |
| Air Force<br>Eigenvalue Problems in Stability Theory<br>M. Marcus, R.C. Thompson, H. Minc<br>10/01/76 - 09/30/77 | $58,793 |

National Science Foundation
Dissemination of Scientific Information
M. Marcus
09/07/77 - 01/31/79 $10,000

Air Force
Supplement to: The Localization of Eigenvalues
M. Marcus
10/01/77 - 09/30/78 $102,881

National Science Foundation
Research Conference on Linear Algebra
M. Marcus
11/01/77 - 10/31/78 $ 2,800

Air Force
Stability, Control and Numerical Linear Algebra
M. Marcus
10/01/78 - 09/30/79 $80,949

International Business Machines Corp.
Intensive Short Course in Basic College Mathematics
M. Marcus
08/01/79 - 09/30/80 $ 5,000
Air Force
Foundations of Eigenvalue Distribution Theory
M. Marcus et al
09/30/79 - 09/29/80 $84,143

Cal Post Secondary Education Commission
Intensive Short Course in Basic College Mathematics
M. Marcus
10/01/79 - 06/30/80 $33,000

Cal Post Secondary Education Commission
Intensive Short Course in Basic College Level Math
M. Marcus
07/01/80 - 09/30/81 $40,000

Air Force
Eigenvalue Localization Techniques in Numerical Algebra
M. Marcus et al
09/30/80 - 09/30/81 $101,993

*National Science Foundation*
Microcomputer Equipment for Undergraduate Applied Mathematics
M. Marcus
10/15/80 - 09/30/83 $ 19,319

National Science Foundation
Intensive Computer Based Mathematics Training
M. Marcus
03/01/81 - 10/31/83 $192,012

National Science Foundation
Research Conference on Multilinear Algebra
R. Merris, M. Marcus
03/15/81 - 08/31/81 $ 7,550

Department of Education
A Program In Quantitative Decision Making
M. Marcus
09/15/81 - 09/14/84 $113,961

Air Force
Eigenvalues, Numerical Ranges, Stability Analysis
M. Marcus et al
09/30/81 - 04/30/83 $114,545

Air Force
Questions in Numerical Analysis / Associated Problems
M. Marcus, M. Goldberg
05/01/83 - 04/30/84 $57,515

Department of Education
A Program In Quantitative Decision Making
M. Marcus
05/01/83 - 09/14/84 $ 6,035

Air Force
Stability Analysis of Finite Difference Schemes
M. Marcus, M. Goldberg
05/01/84 - 04/30'85 $56,953

Air Force
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M. Marcus, M. Goldberg
05/01/85 - 04/30/86 $64,444

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Stability Analysis of Finite Difference Schemes
M. Marcus, M. Goldberg
05/01/86 - 04/30/87 $75,030

Air Force
Stability Analysis of Finite Difference Approximations to Hyperbolic Systems, and Problems in Applied and Computational Linear Algebra
M. Marcus, M. Goldberg
5/1/8787 - 4/30/88 $73,693

National Science Foundation
Computing and Algorithmic Mathematics for Secondary School Teachers
M. Marcus, J. Bruno
3/11/87 - 8/31/89 $516,999

National Science Foundation
A National Institute for Secondary School Teachers for the Dissemination of Computer Science and Algorithmic Mathematics
M. Marcus, J. Bruno, R. Mayer
2/1/88 - 8/31/89 $509,560

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Stability Analysis of Finite Difference Approximations to Hyperbolic Systems, and Problems in Applied and Compuational Matrix Theory
M. Marcus, M. Goldberg
5/1/88 - 10/31/88 $78,114

**TOTAL** $3,046,472

**Publication List**

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2. A remark on a norm inequality for square matrices, Proc. Amer. Math. Soc. 6 (1955), 117-119.

3. Some results on the asymptotic behavior of linear systems, Canad. J. Math. 7 (1955), 531-538.

4. Boundedness of a continuous function, Amer. Math. Monthly 62 (1955).

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5. An invariant surface theorem for a non-degenerate system. Contributions to non-linear oscillations, Annals of Math. Study 36 (1956), 243-256.

6. A note on the existence of periodic solutions of differential equations (with S.P. Diliberto), Annals of Math. Study 36 (1956), 237-241.

7. Repeating solutions for a degenerate system, Annals of Math. Study 36 (1956), 261-268.

8. On the optimum gradient method for systems of linear equations, Proc. Amer. Math. Soc. 1 (1956), 77-81.

9. Extramural properties of Hermitian matrices (with J. McGregor), Canad. J. Math. 8 (1956), 524-531.

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11. On the maximum principle of Ky Fan (with B. Moyls), Canad. J. Math. 9 (1957), 313-320.

12. Inequalities for symmetric functions and Hermitian matrices (with L. Lopes), Canad. J. Math. 9 (1957), 304-312.

13. A note on symmetric functions of eigenvalues (with R.C. Thompson), Duke Math. J. 24 (1957), 43-46.

14. A note on the values of a quadratic form, J. Wash. Acad. Sci. 47 (1957), 97-99.

15. Some extreme value results for indefinite Hermitian matrices I. (with B. Moyls and R. Westwick), Illinois J. Math. 1 (1957), 449-457.

16. On subdeterminants of doubly stochastic matrices, Illinois J. Math. 1 (1957), 583-590.

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