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Fresnel Coefficients for a Phase Conjugator
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ABSTRACT

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I. INTRODUCTION

After the first demonstrations\(^1-4\) of the feasibility of generating phase-conjugated electromagnetic waves (with respect to a reference wave), this technique has found important applications in optical engineering, and especially in the design of devices for the production of high-quality laser beams. If a light ray is reflected by phase-conjugating mirror, then its wave front is reversed. This implies that a diverging beam emanates as converging rather than as diverging, which would be the situation for an ordinary mirror. In this fashion, a distorted wave front can be corrected, after reflection by a phase conjugator (PC), by letting it pass through the same device which built up the distortion.\(^5-7\)

Since wave front reversal by PC’s appears to work so well, one presently anticipates more sophisticated applications. Especially lifetime modifications of atoms, which are due to the fact that the atom radiates its fluorescence (in spontaneous decay) in the vicinity of a PC, are expected to be dramatic.\(^8\)

Emitted dipole radiation diverges from its source, and a subsequent reflection by a PC can focus the wave exactly back on the atom. Stimulated absorption of photons can then conceivably lead to (effective) infinite lifetimes of excited atomic states. This in turn leads to a reduction of the natural linewidth, which might have consequences for frequency standards. It was predicted\(^9\) that the linewidth vanishes identically, if the reflectivity of the mirror equals unity, and this property would be independent of the distance between the atom and the PC.

Although wave front reversal has been proven to be possible in general, there is a little more to it if one wants to extend the horizon of its applications. Let us represent the electric field by its Fourier integral
\[ E(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \ e^{-i\omega t} \hat{E}(\mathbf{r}, \omega) \]  \hspace{1cm} (1.1)

From \( E(\mathbf{r}, t)^* = \hat{E}(\mathbf{r}, t) \) we obtain the relation

\[ \hat{E}(\mathbf{r}, \omega)^* = \hat{E}(\mathbf{r}, -\omega) \]  \hspace{1cm} (1.2)

On the other hand, the phase-conjugated replica of \( E(\mathbf{r}, t) \) follows from the substitution \( \hat{E}(\mathbf{r}, \omega) \to \hat{E}(\mathbf{r}, \omega)^* \) in Eq. (1.1), and combination with Eq. (1.2) then yields

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \ e^{-i\omega t} \hat{E}(\mathbf{r}, \omega)^* = E(\mathbf{r}, -t) \]  \hspace{1cm} (1.3)

Hence, perfect phase conjugation is identical to time reversal, and it is easy to argue that it is impossible to construct a device which can accomplish that.

Time reversal (looking into the future) violates causality. For the example of fluorescence, this would imply that at the time the photon is emitted the atom already knows that the phase-conjugated wave will be reflected back. If the distance between atom and mirror, divided by the speed of light, is much larger than the atomic lifetime, then the presence of the PC should not affect the optical properties of the atom anymore, according to the principles of special relativity. Therefore, more profound understanding of phase conjugation requires a time (or frequency) resolution in the description, and in such a way that causality is preserved. Besides that, the literature on reflection by PC's is largely restricted to the case of normal incidence of the probe field. Dipole radiation, for instance, is a spherical wave, and consequently it is
imperative to take a non-zero parallel component of the incident wave vector into consideration.

II. CONSTITUTIVE EQUATION

If radiation is scattered by a vacuum-material interface, then the reflected field often acquires a phase-conjugated component. Since we are interested in the basic possibilities and limitations of producing phase-conjugated radiation, we consider the most simple configuration, which is experimentally realizable and has all the desired features. As active medium we choose a crystal which is transparent (unit dielectric constant) for the frequency range under consideration, but has a significant third-order susceptibility $\chi^{(3)}(\omega)$. Two strong counterpropagating and linearly-polarized laser beams with frequency $\bar{\omega} > 0$ are assumed to excite the nonlinear interaction. A relatively weak incident (probe) wave then induces a polarization $P(r,t)$ in the crystal. Corresponding oscillating charges then emit radiation, which gives rise to a reflected, phase-conjugated wave. From the theory of four-wave mixing we know that the (Fourier transform of the) polarization is then related to the electric field by

$$P(r,\omega) = \epsilon_0 \hat{f}(\bar{\omega} - \omega) \hat{f}(r,\omega - 2\bar{\omega}) , \quad \omega > 0 . \quad (2.1)$$

Here, $\hat{f}$ represent the electric field at $r$ in the crystal, but it does not include the two pump fields. The function $\hat{f}(\omega)$ represents the nonlinear material and will further remain unspecified. This quantity has the same status as the frequency-dependent first-order susceptibility $\chi^{(1)}(\omega)$ for dielectrics. In the present situation, the $\omega$-dependence of $\hat{f}$ equals the $\omega$-dependence of $\chi^{(3)}$, shifted
over the 'setting frequency' \( \bar{\omega} \) of the phase conjugator. Furthermore, \( \bar{f} \) is proportional to the products of the amplitudes of the two pump beams.

From \( \bar{P}(r,t)^* = \bar{P}(r,t) \) we find \( \bar{f}(r,\omega)^* = \bar{f}(r,-\omega) \), and with Eq. (1.2) we then obtain

\[
\bar{f}(r,\omega) = \epsilon_0 \bar{f}(\bar{\omega} + \omega) \bar{E}(r,2\bar{\omega}+\omega) , \quad \omega < 0 ,
\]

(2.2)

the polarization for negative frequencies. Equations (2.1) and (2.2) give \( \bar{f}(r,\omega) \) for every \( \omega \), and together they will be considered as the constitutive relation for a phase conjugator.

Because the frequency dependence of \( \bar{f}(\omega) \) represents the frequency dependence of \( \chi^{(3)}(\omega) \) around the setting \( \bar{\omega} \), the function \( \bar{f}(\omega) \) should be strongly peaked around \( \omega = 0 \). From Eq. (2.2) we then see that \( \bar{P}(r,\omega) \) can only be nonzero for frequencies \( \omega = -\bar{\omega} \). Its value is proportional to \( \bar{P} \), evaluated at the frequency \( 2\bar{\omega} + \omega - \bar{\omega} \). In other words, Eq. (2.2) expresses that \( \bar{P} \) around \( -\bar{\omega} \) is determined by \( \bar{P} \) around \( \bar{\omega} \), and is zero if \( \omega \) is sufficiently far away from \( -\bar{\omega} \).

With this in mind, the Fourier inverse of Eq. (2.2) is readily found to be

\[
\bar{P}(r,t) = \epsilon_0 e^{i\bar{\omega}t} \int_0^\infty d\tau \ f(\tau) \ e^{i\bar{\omega}(t-\tau)} \bar{E}(r,t-\tau) ,
\]

(2.3)

where \((-)\) and \((+)\) indicate the negative and positive frequency parts, respectively. The function \( f(\tau) \) is related to \( \bar{f}(\omega) \) by

\[
f(\omega) = \int_0^\infty d\tau \ e^{i\omega\tau} f(\tau) ,
\]

(2.4)
which is a Fourier integral if we set \( f(\tau < 0) = 0 \). Equation (2.3) expresses that the polarization at time \( t \) is determined by the electric field in the past only. Therefore, the mechanism of phase conjugation is causal.

III. COUPLED WAVE EQUATIONS

In terms of a polarization, the Fourier-transformed Maxwell equations read

\[
\nabla \cdot (\varepsilon \mathbf{E}(r, \omega) + \mathbf{P}(r, \omega)) = 0 ,
\]

(3.1)

\[
\nabla \cdot \mathbf{H}(r, \omega) = 0 ,
\]

(3.2)

\[
\nabla \times \mathbf{E}(r, \omega) - i\omega \mathbf{H}(r, \omega) = 0 ,
\]

(3.3)

\[
\mu_0^{-1} \nabla \times \mathbf{H}(r, \omega) + i\omega (\varepsilon \mathbf{E}(r, \omega) + \mathbf{P}(r, \omega)) = 0 ,
\]

(3.4)

which should be obeyed for every \( \omega \) separately. Outside the phase conjugator we have \( \mathbf{P} = 0 \), inside we set \( \mathbf{P} \) equal to expression (2.1) or (2.2), depending on the sign of \( \omega \), and furthermore, Eqs. (3.1)-(3.4) imply the boundary conditions at the interface in the usual way.

The polarization \( \mathbf{P} \) at frequency \( \omega \) is expressed in \( \mathbf{E} \) at a different frequency, and therefore Eqs. (3.1) and (3.4) couple sets of Maxwell equations for different frequencies. If we take a fixed frequency \( \omega_1 - \bar{\omega} \), then the polarization \( \mathbf{P}(r, \omega_1) \) is brought about by the electric field at \( \omega_1 - 2\bar{\omega} < 0 \), according to Eq. (2.1). On the other hand, the polarization at \( \omega_1 - 2\bar{\omega} \) is induced by \( \mathbf{E}(r, \omega_1) \), as follows from Eq. (2.2). Therefore, Maxwell's equations
couple positive and negative frequencies two by two. If we take \( \omega_1 - \tilde{\omega} > 0 \), then this frequency couples with \( \omega_2 - - \tilde{\omega} < 0 \), and their relation is

\[
\omega_1 - \tilde{\omega} = \omega_2 + \omega .
\] (3.5)

For the fields inside the PC we eliminate \( \mathbf{E} \) with Eq. (3.3),

\[
\mathbf{B}(r, \omega) = -i\omega^{-1} \nabla \times \mathbf{E}(r, \omega) , \text{ for } \omega = \omega_1, \omega_2 .
\] (3.6)

Then Eq. (3.2) is automatically satisfied, and Maxwell's equations are equivalent to

\[
\nabla \cdot \mathbf{E}(r, \omega) = 0 , \text{ for } \omega = \omega_1, \omega_2 ,
\] (3.7)

\[
\nabla^2 \mathbf{E}(r, \omega_1) + (\omega_1/c)^2 (\mathbf{E}(r, \omega_1) + f(\tilde{\omega} - \omega_1) \mathbf{E}(r, \omega_2)) = 0 .
\] (3.8)

\[
\nabla^2 \mathbf{E}(r, \omega_2) + (\omega_2/c)^2 (\mathbf{E}(r, \omega_2) + f(\tilde{\omega} + \omega_2) \mathbf{E}(r, \omega_1)) = 0 .
\] (3.9)

Equation (3.7) expresses that the electric field is transverse, and Eqs. (3.8) and (3.9) show that the fields at \( \omega_1 \) and \( \omega_2 \) obey a set of coupled wave equations.

IV. PLANE WAVES

Solutions of the set (3.7)-(3.9) are easily found. If we try

\[
\mathbf{E}_a(r, \omega_1) = E_a e^{i k_a \cdot r} ,
\] (4.1)
\[ \hat{\Phi}_a(r, \omega_2) = \eta_a \exp^{-ik_a \cdot r}, \quad (4.2) \]

then Eqs. (3.7)-(3.9) become

\[ k_a \cdot \hat{\Phi}_a = 0, \quad (4.3) \]

\[ k_a^2 = (\omega_1/c)^2(1+i\eta_a \hat{f}(\omega-\omega_1)^*), \quad (4.4) \]

\[ k_a^2 = (\omega_2/c)^2(1+i\eta_a^{-1} \hat{f}(\omega+\omega_2)). \quad (4.5) \]

Equations (4.4) and (4.5) both express the (complex-valued) wave number in the ratio of amplitudes \( \eta_a \) of the \( \omega_1 \)- and \( \omega_2 \)-waves. Equating the right-hand sides gives a quadratic equation for \( \eta_a \), which admits two acceptable solutions. For reasons which will become clear in due course, we choose

\[ \eta_a = \frac{(1-(\omega_1/\omega_2)^2)(1-D)}{2(\omega_1/\omega_2)^2 \hat{f}(\omega-\omega_1)^*}, \quad (4.6) \]

in terms of the parameter

\[ D = \sqrt{1 + \frac{4(\omega_1/\omega_2)^2 \hat{f}(\omega-\omega_1)^* \hat{f}(\omega+\omega_2)}{(1-(\omega_1/\omega_2)^2)^2}}. \quad (4.7) \]
Equations (4.6) and (4.7) express $n_a$ in the given function $f$, and with Eq. (4.4) we find $k_a^2$. For a given frequency dependence of $f(\omega)$, Eq. (4.4) is then the dispersion relation for the phase conjugator.

Instead of retaining two possible values for $n_a$, we seek a second solution of the form

$$\beta_b(r, \omega_1) = n_b E_b e^{ik_b \cdot r},$$

(4.8)

$$\beta_b(r, \omega_2) = E_b e^{ik_b \cdot r},$$

(4.9)

and along the same lines we find

$$k_b \cdot E_b = 0,$$

(4.10)

$$k_b^2 = (\omega_2/c)^2(1+n_b f(\bar{\omega}+\omega_2)),$$

(4.11)

$$k_b^2 = (\omega_1/c)^2(1+n_b^{-1} f(\bar{\omega}-\omega_1^*)).$$

(4.12)

Again, there are two solutions for $n_b$, but now we take

$$n_b = \frac{((\omega_1/\omega_2)^2-1)(1-D)}{2f(\bar{\omega}+\omega_2)}.$$ 

(4.13)

Substitution of $n_b$ into Eq. (4.11) then gives a second branch of the dispersion relation. It is easy to check that the b-solution is precisely the discarded a-solution.
In the limit \( f \to 0 \), corresponding to a switch-off of the interaction, we find

\[
\eta_a \to 0 \ , \ k_a^2 = (\omega_1/c)^2 \ , \quad (4.14)
\]

and hence the \( \omega_2 \)-wave disappears. The dispersion relation for the \( \omega_1 \)-wave is the same as for a wave in a vacuum. This implies that the \( a \)-solution is essentially a \( \omega_1 \)-wave, but due to the nonlinear interaction \( \dot{f} \neq 0 \) there is a mixing with the \( \omega_2 \)-wave, which is excited with a relative amplitude \( \eta_a \). The second, suppressed solution for \( \eta_a \) would give \( \eta_a \to \infty \) for \( \dot{f} \to 0 \). Since the amplitude of the \( \omega_2 \)-wave must remain finite, this would imply \( E_{a*} + 0 \), \( \eta_a E_a \) finite, indicating that the \( \omega_1 \)-wave disappears in comparison with the \( \omega_2 \)-wave. The property \( \eta_a \to \infty \) for \( \dot{f} \to 0 \) is inconvenient. Therefore we have introduced the second solution with the normalization as given by Eqs. (4.8) and (4.9). For \( \dot{f} \to 0 \) we now obtain

\[
\eta_b \to 0 \ , \ k_b^2 = (\omega_2/c)^2 \ , \quad (4.15)
\]

showing that the \( b \)-solution is essentially an \( \omega_2 \)-wave, modified by the nonlinear interaction.

From Eqs. (4.1) and (4.2) we notice that the \( \omega_1 \)- and \( \omega_2 \)-waves seem to have the same wave vector and polarization, and that their ratio of amplitudes equals \( \eta_a \). However, due to the fact that \( \omega_1 > 0 \) and \( \omega_2 < 0 \), the waves are counterpropagating. In the time domain, the \( \omega_2 \)-wave has a factor

\[\exp(ik_a \cdot r - i\omega_2 t) + \text{c.c.},\]

which represents a wave with wave vector \(-k_a\). Finally, we mention two interesting relations between the \( a \)- and \( b \)-solutions. For the amplitude factors we have
and the wave numbers are related according to

\[ k_a^2 + k_b^2 = (\omega_1/c)^2 + (\omega_2/c)^2, \tag{4.17} \]

independent of the interaction strength \( f \).

V. WAVES AT AN INTERFACE

Consider position space to be divided in vacuum \((z>0)\) and a PC \((z<0)\), separated by a plane boundary \((z=0)\). Incident upon this interface, and from the vacuum side, is a plane monochromatic wave with frequency \( \omega_1 > 0 \),

\[ \vec{E}_1(\vec{r},\omega_1) = \vec{E}_0 e^{-ik_1 \cdot \vec{r}}, \tag{5.1} \]

with amplitude and polarization \( \vec{E}_0 \), and wave vector \( \vec{k} \) supposed to be given. From Maxwell's equations \((3.1)-(3.4)\) we find the constraints

\[ k^2 = (\omega_1/c)^2, \quad \vec{k} \cdot \vec{E} = 0. \tag{5.2} \]

If we write \( \vec{k} = k_\parallel \vec{1}_\parallel + k_\perp \), where \( \parallel \) and \( \perp \) refer to the plane \( z = 0 \), then the \( z \)-component of \( \vec{k} \) is

\[ k_z = -\sqrt{k^2 - k_\parallel^2}. \tag{5.3} \]
The question is what the reflected and transmitted waves are. In this section we shall establish which waves occur, and shall determine their wave vectors. In the next section we shall then evaluate their amplitudes (including phase and polarization), e.g., the Fresnel coefficients.

Waves in \( z > 0 \) and in \( z < 0 \) must match at \( z = 0 \) according to the boundary conditions. Every plane wave contains a factor \( \exp(\mathbf{ik} \cdot \mathbf{r}) \), which equals \( \exp(\mathbf{ik}_a, \mathbf{z} \cdot \mathbf{r}) \) at \( z = 0 \). Boundary conditions can only be satisfied if these exponentials cancel, implying that the parallel component of every wave vector must be identical, e.g.,

\[
\mathbf{k}_a, \mathbf{z} = \mathbf{k}_\parallel \quad (5.4)
\]

for any wave vector \( \mathbf{k}_a \) in \( z > 0 \) and \( z < 0 \). Then, both in \( z > 0 \) and \( z < 0 \), the value of \( k_a^2 \) follows from the dispersion relation, given the frequency, which in turn gives for the perpendicular component the two possibilities \( k_{a, z} = \pm \sqrt{(k_a^2 - k_\parallel^2)} \). The sign in front of the square root determines whether the wave travels in the \(+z\)- or \(-z\)-direction. Since the incident wave has frequency \( \omega_1 \), which only couples to \( \omega_2 \), these considerations limit the number of possible plane waves to four in \( z > 0 \) and eight in \( z < 0 \).

Let us first look at the region \( z > 0 \). Besides the incident wave at frequency \( \omega_1 \), the only other possibility for waves at \( \omega_1 \) is the common reflected wave with

\[
k_{r, z} = k_z > 0 \quad . \quad (5.5)
\]
Therefore, the most general expression for the electric field at frequency $\omega_1$ in $z > 0$ reads

$$E(r, \omega_1) = E_0 e^{-ik_r \cdot r} + E_0 e^{-ik_r \cdot r},$$

with only $E_0$ yet to be determined. For the field at $\omega_2$, we again have two possible waves, which only differ in their sign of the $z$-component of the wave vector. Of course, it is tacitly assumed that the incident wave is the only external field, which implies that an $\omega_2$-wave in $z > 0$ can only travel in the $+z$-direction. We shall refer to this wave as the phase-conjugated wave, and it is represented by

$$E(r, \omega_2) = E_{PC} e^{ik_{PC} \cdot r}.$$  

(5.7)

The fact that a wave with $\omega_2 < 0$ propagates in a direction opposite to its wave vector then gives

$$k_{PC,z} = -\sqrt{((\omega_2/c)^2 - k_\parallel^2)}.$$  

(5.8)

If the incident field is exactly on resonance with the PC, we then have $\omega_1 = \tilde{\omega}$, $\omega_2 = -\tilde{\omega}$ and $k_{PC,z} = k_z$. Combining this with Eq. (5.4) shows that in this case $k_{PC} = k$, and hence the PC-wave and the incident wave counterpropagate exactly. Therefore, the field at $\omega_2$ in $z > 0$ is the phase-conjugated replica of the incident field (possibly apart from polarization and amplitude), if $\omega_1 = \tilde{\omega}$. Furthermore, we notice that for $-\omega_2 = \omega_1$, $k_{PC,z}$ can be imaginary, corresponding to an evanescent wave.
In the PC we have the independent a- and b-solutions from the previous section. Both solutions can occur with a + or - sign in the definition of their z-component of the wave vector. This amounts to four different solutions, and every one of them consists of two waves (at $\omega_1$ and $\omega_2$). For the corresponding problem with a dielectric, we know that we can discard the wave which propagates in the +z-direction, but for a PC it is not obvious which waves are the causal waves. This results from the fact that every solution consists of two counterpropagating waves. The only clue at this stage is that in the limit $\xi \to 0$, the incident field must propagate undisturbed through the crystal, and every other component must vanish. This corresponds to the a-solution with

$$k_{a,z} = -\sqrt{(k^2_a - k^2_\parallel)} \quad (5.9)$$

Whether the other three solutions are excited by the incident field or not seems to be impossible to find out a priori. We have to consider the full solution for $z < 0$, calculate all amplitudes by matching boundary values, and then require that all fields disappear in the limit $\xi \to 0$. In carrying out this procedure, we have found that only the b-solution with

$$k_{b,z} = \sqrt{(k^2_b - k^2_\parallel)} \quad (5.10)$$

is excited by the incident wave. This field corresponds to an $\omega_2$-like wave, which propagates in the -z-direction. For the field in $z < 0$ we can now write

$$\mathbf{E}(x,\omega) = E_a e^{-ik_a \cdot x} + \eta_b E_b e^{ik_b \cdot x} \quad (5.11)$$
and only $E_a$ and $E_b$ remain to be determined.

VI. FRESNEL COEFFICIENTS

The fields from the previous section must be matched across $z = 0$, with the conditions that $(\mathbf{e}_\alpha \mathbf{E} + \mathbf{p})_\perp$, $\mathbf{E}_\parallel$ and $\mathbf{E}$ are continuous, both for $\omega_1$ and $\omega_2$. Furthermore, we have restrictions $k \cdot E = 0$ for every wave. This procedure fixes the amplitudes $E_r$, $E_{PC}$, $E_a$ and $E_b$ in terms of $E$. Notice that two components of the field for $z < 0$ acquire an additional factor of $\eta_b$ or $\eta_a$ in their amplitudes, according to Eqs. (5.11) and (5.12).

As usual, it is advantageous to distinguish between an s (≈ surface) and a p (≈ plane) polarized incident wave. Then all other waves are s- or p-polarized. In Fig. 1 we have summarized the polarization convention which will be adopted here. All unit vectors are normalized as $\mathbf{e} \cdot \mathbf{e} = 1$, and their directions follow from $\mathbf{e}_\alpha \cdot k = 0$ in combination with the convention from Fig. 1. Since the $k_\alpha$'s are not necessarily real, the unit vectors for p-polarization will be complex, in general. Then the Fresnel coefficients for s-waves are defined in the notation

$$E = E_s$$

$$E_r = R_s E_{a,s}, \quad E_{PC} = P_s E_{PC,s}$$

$$E_a = T_a E_{a,s}, \quad E_b = T_b E_{b,s}$$
and for an incident p-wave we replace the subscripts s by p. We obtain the result

\[ R_s = \frac{1}{Z_s} \{ (k_z - k_{a,z})(k_{PC,z} - k_{b,z}) + \eta_a \eta_b (k_{a,z} - k_{PC,z})(k_z - k_{b,z}) \} \]  \hspace{1cm} (6.4)

\[ P_s = \frac{1}{Z_s} 2 \eta_a k_z (k_{a,z} - k_{b,z}) \] \hspace{1cm} (6.5)

\[ T_{a,s} = \frac{1}{Z_s} 2 k_z (k_{PC,z} - k_{b,z}) \] \hspace{1cm} (6.6)

\[ T_{b,s} = \frac{1}{Z_s} 2 \eta_a k_z (k_{a,z} - k_{PC,z}) \] \hspace{1cm} (6.7)

\[ R_p = \frac{1}{Z_p} \{ (k_z - k_{a,z})(k_{PC,z} - k_{b,z}) \} \] \hspace{1cm} (6.8)

\[ P_p = \frac{1}{Z_p} 2 \eta_a k_z (k_{a,z} - k_{b,z}) \] \hspace{1cm} (6.9)

\[ T_{a,p} = \frac{1}{Z_p} 2 k_z (k_{PC,z} - k_{b,z}) \] \hspace{1cm} (6.10)

\[ T_{b,p} = \frac{1}{Z_p} 2 \eta_a k_z (k_{a,z} - k_{PC,z}) \] \hspace{1cm} (6.11)
where we have introduced the abbreviations

\[
Z_s = (k_z + k_{a,z})(k_{PC,z} - k_{b,z})
\]

\[
+ \eta_a \eta_b (k_{a,z} - k_{PC,z})(k_z + k_{b,z})
\]

(6.12)

\[
Z_p = (k_z \frac{k_a^2}{k^2} + k_{a,z})(k_{PC,z} \frac{k_b^2}{k_{PC}} - k_{b,z})
\]

\[
+ \eta_a \eta_b (k_{a,z} - k_{PC,z})(k_z \frac{k_a^2}{k_{PC}} + k_{b,z})
\]

(6.13)

With the explicit expressions for the z-components of the various wave vectors from the preceding section, Eqs. (6.4)-(6.13) determine the Fresnel coefficients in terms of \( f(\omega) \) and \( k_\parallel^2 \) (or \( f(\omega) \) and the angle of incidence).

VII. PHASE CONJUGATION

The incident wave gives rise to an (ordinary) reflected wave, a reflected phase-conjugated wave and four transmitted waves. It is easy to check from the formulas above that in the limit \( f \to 0 \) both \( T_{a,s} \) and \( T_{a,p} \) approach unity, and that the other Fresnel coefficients vanish. In this limit the PC is transparent, as it should be. For \( f \neq 0 \) the PC-wave appears, but also the r-wave (not to be confused with reflection at a dielectric; this r-wave is merely a result of the four-wave mixing). The presence of the r-wave already indicates that this device cannot be a perfect phase conjugator, even if the PC-wave would be the phase conjugated signal with respect to the incident wave.
Perfect phase conjugation would be achieved if $R_s = R_p = 0$, $P_s = P_p = 1$, the transmission coefficients arbitrary, and $\omega_2 = -\omega_1$. With our expressions for the wave vectors and the Fresnel coefficients it is easy to investigate the quality of phase conjugation in a particular situation. Let us first assume that the incident field is on resonance with the PC ($\omega_1 = \bar{\omega}$), which can always be managed by tuning the pump fields. Then we write

$$f(0) = \gamma e^{i\phi}, \quad \gamma > 0, \ \phi \ \text{real}, \quad (7.1)$$

so that $\gamma$ measures the strength of the nonlinear interaction. From Eqs. (4.6), (4.7) and (4.13) we then find

$$\eta_a = -\delta e^{i\phi}, \quad \eta_b = \delta e^{-i\phi}, \quad (7.2)$$

where

$$\delta = \begin{cases} 
1, & \text{if } \omega < \bar{\omega}, \\
-1, & \text{if } \omega > \bar{\omega}.
\end{cases} \quad (7.3)$$

First, we observe that $|\eta_a| = |\eta_b| = 1$, and in view of Eqs. (4.1) and (4.2) or Eqs. (4.8) and (4.9), this means that the $\omega_1$- and $\omega_2$-waves have the same amplitude, independent of the interaction strength $\gamma$. Second, we notice that $\eta_a$ and $\eta_b$ are discontinuous if we pass $\omega$ over the resonance $\bar{\omega}$, and third, Eq. (7.2) might seem to be in conflict with Eqs. (4.14) and (4.15). From Eq. (4.7), however, we see that close resonance in fact means
\[
\frac{\omega_1 - |\omega_2|}{\omega_1} \ll \gamma , \quad (7.4)
\]

or, the relative detuning must be much smaller than the coupling strength.

Conversely, the limit \( f \to 0 \) should read as \( \gamma >> 1 - |\omega_2/\omega_1| \).

As a second condition we take

\[ \gamma << 1 , \quad (7.5) \]

and subsequently we choose the angle of incidence to be zero \( (k_\parallel = 0) \). Then the Fresnel coefficients simplify considerably, and we obtain

\[ R_s = \frac{1}{2} \delta \gamma << 1 , \quad (7.6) \]
\[ P_s = - \delta e^{i\phi} , \quad (7.7) \]
\[ T_{a,s} = 1 , \quad (7.8) \]
\[ T_{b,s} = \frac{1}{4} \gamma e^{i\phi} << 1 . \quad (7.9) \]

Equation (7.8) expresses that the amplitude of the incident beam is not affected by the crystal (up to order \( \gamma \)), as could be expected from the fact that for \( \gamma \to 0 \) the PC becomes transparent. Furthermore, the ordinary reflected wave and the \( b \)-waves disappear for \( \gamma \) small. Most remarkable is that \( |P_s| = 1 \) in this limit, which implies perfect phase conjugation for a monochromatic wave on resonance and of perpendicular incidence. This feature, which is present for an arbitrarily
small nonlinear interaction strength $\gamma$, justifies the name PC for this device. Figure 2 illustrates the dependence of $|P_s|$ and $|P_p|$ on the angle of incidence.

VIII. CONCLUSIONS

In order to investigate quantitatively the possibilities and limitations of optical phase conjugation, we have worked out the realistic case where the conjugated wave is generated by four-wave mixing in a nonlinear medium. Since every electromagnetic wave, which is incident on the PC, can be decomposed into plane waves, it is sufficient to evaluate the response of the PC to an external plane wave. We found that a PC reflects an ordinary and a phase-conjugated wave back into the vacuum, and that the transmitted wave has four components for a half-infinite medium. The wave vectors and Fresnel coefficients were obtained in terms of the incident wave vector and frequency, and the function $f(\omega)$, which represents the PC. Reflection and transmission angles for the rays follow from the wave vectors, and the intensities and polarizations are determined by the Fresnel coefficients.

It was shown that the device indeed operates as a PC if the angle of incidence is zero, the wave is on resonance with the PC, and the nonlinear interaction is weak. Conversely, for every other situation the PC conjugates the wave only partially, and in addition emits an ordinary reflected wave. In spectroscopic applications, where spherical waves irradiate the PC, many plane-wave components have a nonzero angle of incidence for which the PC is nonideal. Because it is the interference of all reflected and incident waves, which determines, for instance, the lifetime of an atom near the surface, it is inevitable that these imperfections must be taken into account. Moreover, it was pointed out in the Introduction that a perfect phase-conjugating medium should not exist, due to causality requirements.
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FIGURE CAPTIONS

Fig. 1. Polarization convention for the various waves. All s-polarization vectors, perpendicular to the plane of incidence, point in the same direction. For p-polarized waves we choose the unit vectors such that their z-components are negative in the limit $\bar{f} \to 0$ (for $\bar{f} \neq 0$, the z-components might be complex). The arrows with the letters next to them indicate the wave vectors. Their parallel components are all equal to $k_\parallel$, and the figure shows the sign of their z-component in the limit $\bar{f} \to 0$.

Fig. 2. Reflectivity coefficients $|P_s|$ (curve a) and $|P_p|$ (curve b) for the phase-conjugated wave as a function of the cosine of the angle of incidence. For angles between 0° and 60° we have $|P_s| \approx |P_p| \approx 1$, which corresponds to perfect phase conjugation. For larger angles, however, the operation of the device is far from perfect, and for parallel incidence the phase-conjugated wave disappears completely. The peaks in the two curves reflect an amplification of the intensity of the phase-conjugated wave with respect to the intensity of the incident wave. The parameters are $\omega_1 = \bar{\omega}$, $\gamma = 0.1$ and $\phi = 0$. 
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