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HARMONIZABILITY, V-BOUNDEDNESS, (2P)-BOUNDEDNESS OF STOCHASTIC PROCESSES

by

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HARMONIZABILITY, V–BOUNDEDNESS, (2,P)–BOUNDEDNESS OF STOCHASTIC PROCESSES*

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Abstract Some new classes of discrete time non–stationary processes, related to the harmonizable and V–bounded classes, are introduced. A few characterizations are obtained which, in turn, unify the V–bounded theory. Our main results depend on a special form of Grothendieck inequality.

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1. Introduction

In many applied problems, e.g., signal estimation, time series analysis, econometrics, etc., a wide sense stationary (WSS) assumption is unacceptable. Various non-stationary models have thus been studied in connection with non-stationary phenomena. In system theory, the (finite dimensional linear) state space model has been favored; in time series analysis the ARMA model is preferred, while periodically correlated processes are models for economical data which exhibit some periodicity. Very simple transformations of WSS processes do not preserve the stationary structure, for example, finite or infinite sampling, deterministic or random scaling, linear transformations, etc. To study the effects of those transformations on stationary pocesses, as well as to encompass the various models mentioned above, general non-stationary notions have to be studied and characterized.

The main successes of the theory of WSS processes and its applications rely on harmonic analysis techniques and in particular on two Fourier integral representations. On the one hand, the shift invariant covariance kernel is the Fourier transform of a positive measure. On the other hand, the process itself is the Fourier transform of an orthogonally scattered stochastic measure. Hence, it is natural in extending the WSS concept, to try to preserve a potential use of Fourier analysis techniques. Various generalizations in that direction have been presented, among others, the classes of harmonizable processes introduced by Loève [16] and Rozanov [27], as well as Bochner's [4] V-bounded class. In the present work, this line of investigation is pursued, and some new classes of non-stationary processes are introduced then characterized.

A brief synopsis of the paper is as follows: Section 2 is mainly introductory, various non-stationary concepts are recalled and related to one another. Theorem 2.4 characterizes the orthogonal processes which are harmonizable in the sense of

Loève and Proposition 2.5 clarifies a minor point. The third section is the core of the paper. A new notion, (2,p)–boundedness, is introduced and the stationary processes which are (2,p)–bounded characterized. We then prove a special form of Grothendieck inequality (Theorem 3.6) and this leads to various characterizations of (2,p)–boundedness (Theorem 3.8). Finally, some important practical examples are shown to be (2,2)–bounded.

Notations and Conventions: $\mathbb{R}$ is the real field, $\mathbb{C}$ the complex one, $\mathbb{Z}$ the integers, $\mathbb{N}^* = \{1,2,3,\ldots\}$. $L^2(\Omega,B,P)$ ($L^2(P)$ for short) is the usual Hilbert space of complex valued random variables with finite second moments. A *process* x is always taken to be of discrete time and L^2–bounded, i.e., $x\colon \mathbb{Z} \to L^2(P)$, with $Ex_n\bar{x}_n = E|x_n|^2 = \|x_n\|^2_{L^2(P)} \le K$, $K > 0$, $n \in \mathbb{Z}$ (E denotes expectation and overbars complex conjugates). It is also always assumed that $E|x_n|^2 > 0$ for at least one $n \in \mathbb{Z}$. The covariance kernel of x is the doubly indexed sequence $\{R(n,m)\}_{n,m\in\mathbb{Z}}$ with $R(n,m) = Ex_n\bar{x}_m$, $n,m \in \mathbb{Z}$.

The usual identification is made between 2π–periodic functions on $\mathbb{R}$ and functions on $\Pi = \mathbb{R}/_{2\pi\mathbb{Z}}$ with $]-\pi,\pi]$ a model for Π. For $1 \le p \le +\infty$, $L^p(\Pi)$, $L^p(\Pi^2)$, denote the Lebesgue spaces on Π and $\Pi^2 = \Pi \times \Pi$ associated to the normalized Lebesgue measure $d\theta$, $d\theta d\psi$. The corresponding norms are denoted by $\|\cdot\|_{L^p(\Pi)}$, $\|\cdot\|_{L^p(\Pi^2)}$. For $1 \le p < +\infty$, $\ell^p(\mathbb{Z})$ denote the usual discrete spaces with corresponding norms $\|\cdot\|_{\ell^p(\mathbb{Z})}$. A (complex) *measure* will always be a complex valued (regular) Borel measure on Π. When added, the adjective positive will refer to non–negative valued measures. A *stochastic measure* is a σ–additive set function $\zeta\colon \mathcal{B}(\Pi) \to L^2(P)$ ($\mathcal{B}(\Pi)$ is the Borel σ–algebra of Π). A stochastic measure is said to be *orthogonally scattered* whenever $E\zeta(A)\overline{\zeta(B)} = 0$, $A, B \in \mathcal{B}(\Pi)$, $A \cap B = \phi$. The integration of scalar functions with respect to stochastic measures is taken in the sense of Bartle, Dunford and Schwartz [2], the reader being referred to Dunford and Schwartz [6,IV.10] for further details. Finally, K denotes a generic absolute constant whose value might change from one expression to another.

2. Harmonizability and V–boundedness

The simplest processes admitting a "harmonic decomposition" are the *wide sense stationary* (WSS) processes. As is well–known, their covariance kernel R has a Toeplitz structure. Hence, and this is also well known, a process x is WSS if and only if there exists a (unique) finite positive measure μ on Π such that

$$R(n,m) = \hat{\mu}(n-m) = \int_{-\pi}^{\pi} e^{i(n-m)\theta} d\mu(\theta), \quad n,m \in \mathbb{Z}. \tag{1}$$

Equivalently, there exists a (unique) orthogonally scattered stochastic measure ζ such that

$$x_n = \hat{\zeta}(n) = \int_{-\pi}^{\pi} e^{in\theta} d\zeta(\theta), \quad n \in \mathbb{Z}. \tag{2}$$

On the model of (1), Loève introduced, as follows, a first generalization of the WSS class.

Definition 2.1. A process x is *L–harmonizable* if there exists a (unique) complex measure μ on Π^2 such that

$$R(n,m) = \hat{\mu}(n,m) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\theta} e^{-im\psi} d\mu(\theta,\psi), \quad n,m \in \mathbb{Z}. \tag{3}$$

In Loève's original definition, μ is given via a distribution function and is also unnecessarily assumed, as first noticed by Hurd [14], to be positive definite. The positive definiteness of μ and R are equivalent, in fact, even in a more general framework (see Proposition 2.5). L–harmonizable processes are also known as strongly harmonizable, Loève harmonizable or simply harmonizable. We introduced the terminology L–harmonizable to avoid confusion with another class of harmonizable processes first studied by Rozanov and which are also known as Rozanov harmonizable, weakly harmonizable or simply, harmonizable.

Definition 2.2. A process x is *R–harmonizable* if there exists a (unique) complex bimeasure β such that

$$R(n,m) = \hat{\beta}(n,m) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\psi}d\beta(\theta,\psi), \quad n,m \in \mathbb{Z}. \tag{4}$$

Remark 2.3. The basic difference between (3) and (4) lies in the fact that β is a bimeasure, i.e., $\beta(\cdot,B)$ and $\beta(A,\cdot)$ are complex measures for all $A,B \in \mathcal{B}(\Pi)$. In other words $\beta(\cdot,\cdot)$ is a separately σ–additive function on $\mathcal{B}(\Pi) \times \mathcal{B}(\Pi)$ which does not necessarily extend to a measure on $\mathcal{B}(\Pi) \otimes \mathcal{B}(\Pi)$. Hence, in order to define the integral in (4), a non–absolute integration technique has to be used. This integral has to be understood in a restricted Morse–Transue sense as defined in Houdré [11]. The exponentials being continuous, the Morse–Transue integral or any of its restricted versions can also be used (see [11] and the references cited there for more details). When β is of bounded variation, it uniquely extends to a measure on Π^2 and (4) reduces to (3). For β concentrated on the diagonal of $\mathcal{B}(\Pi) \times \mathcal{B}(\Pi)$, i.e., $\beta(A,B) = 0$ whenever $A \cap B = \phi$, $A,B \in \mathcal{B}(\Pi)$, (4) becomes (1) and the WSS case is recovered. In analogy with the stationary case, β is called the *bispectrum* of the corresponding R–harmonizable process.

The distinction between L–harmonizable and R–harmonizable processes is non vacuous and in fact quite important. To be of interest, the harmonizable classes have to include the simplest cases of non–stationary processes. The Loève class does not do so.

Let x be a (non–stationary) *white noise*, i.e., $R(n,m) = \sigma_n^2\,\delta_{n,m}$, $n,m \in \mathbb{Z}$, with

$\sigma_n^2 \le K$, $n \in \mathbb{Z}$, where $\delta_{n,m}$ is the Kronecker symbol. Then x is R–harmonizable (see Definition 2.6 and Theorem 2.7) but not necessarily L–harmonizable. White noises which are in Loève's class can be characterized.

<u>Theorem 2.4</u>. A white noise is L–harmonizable if and only if there exists a complex measure ν on Π such that $\sigma_n^2 = \hat{\nu}(n)$, for all $n \in \mathbb{Z}$.

<u>Proof</u>. For the necessity, it is enough to show (see Zygmund [29,p.314]) that

$$\int_{-\pi}^{\pi} \Big| \sum_{-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) \sigma_n^2 e^{in\theta} \Big| d\theta \le K, \tag{5}$$

(K independent of N).

Since $\sigma_n^2 = \hat{\mu}(n,n)$ and by Fubini's theorem, the left hand side of (5) is majorized by

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Big| \sum_{-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) e^{-in\theta} e^{in\theta_1} e^{-in\psi_1} \Big| d\theta d|\mu|(\theta_1, \psi_1)$$

$$= |\mu|\ (\Pi^2), \text{ since the above integrand is non-negative.}$$

For the sufficiency, it is enough to show that the two dimensional version of (5) holds when $\sigma_n^2 = \hat{\nu}(n)$. But,

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Big| \sum_{-N}^{N} \sum_{-M}^{M} \Big(1 - \frac{|n|}{N+1}\Big) \Big(1 - \frac{|m|}{M+1}\Big) R(n,m) e^{-in\theta_1} e^{im\psi_1} \Big| d\theta_1 d\psi_1$$

$$= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Big| \int_{-\pi}^{\pi} \sum_{-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big)^2 e^{in\theta} d\nu(\theta)\, e^{-in\theta_1} e^{in\psi_1} \Big| d\theta_1 d\psi_1$$

$$\le |\nu|\,(\Pi)$$

by the same arguments as above. ∎

This elementary proof was set to illustrate a use of one of the various criteria for a sequence to be a Fourier–Stieltjes transform. A disintegration of μ over the map $(\theta,\psi) \longrightarrow \theta + \psi$ trivially gives the result.

Using Theorem 2.4, R–harmonizable processes which are not L–harmonizable are now easy to find. Let x be a white noise such that $\lim_{n\to+\infty} \sigma_n^2 = a$ and $\lim_{n\to-\infty} \sigma_n^2 = b$ with $a \neq b$, then x is not L–harmonizable. This is a direct consequence of Theorem 2.4 and of the following classical result: let μ be a measure on Π such that $\hat{\mu}(n)$ has a limit as $n \longrightarrow +\infty$, then $\hat{\mu}(n)$ has the same limit as $n \longrightarrow -\infty$. In particular, let x be a *unilateral white noise*, e.g., $\sigma_n^2 = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$, then x is not L–harmonizable. We thus recover a classical counterexample which first appeared in Helson and Lowdenslager [9] and was subsequently used for similar purposes by various authors.

An extra assumption, as for example in [25,p.305], is sometimes imposed on the bimeasure β in (4), namely, β is assumed to be positive definite (p.d.), i.e., $\Sigma_{i=1}^N \Sigma_{j=1}^N z_i\beta(A_i,A_j)\,\bar{z}_j \geq 0$, for all $N \in \mathbb{N}^*$, $z_1, \ldots z_N \in \mathbb{C}$, $A_1, \ldots, A_N \in \mathcal{B}(\Pi)$. This is unnecessary; the positive definiteness of the bimeasure β and of the sequence $\{R(n,m)\}_{n,m\in\mathbb{Z}}$ are equivalent.

<u>Proposition 2.5</u>. Let $\{a_{n,m}\}_{n,m\in\mathbb{Z}}$ be a doubly indexed sequence such that $a_{n,m} = \hat{\beta}(n,m)$ for some bimeasure β. Then, β is positive definite if and only if $\{a_{n,m}\}$ is positive definite, namely, $\Sigma_{i=1}^N \Sigma_{j=1}^N z_i\, a_{n_i,n_j} \bar{z}_j \geq 0$, for all $N \in \mathbb{N}^*$, $n_1,\ldots,n_N \in \mathbb{Z}$, $z_1,\ldots,z_N \in \mathbb{C}$.

<u>Proof</u>. Since the Borel functions are the pointwise limits of continuous functions and by the dominated convergence theorem for vector measures ([6,p.328]) it is equivalent to show that

$$\sum_{i=1}^N \sum_{j=1}^N z_i a_{n_i,n_j} \bar{z}_j \geq 0 \iff \int_\Pi \int_\Pi f d\beta \bar{f} \geq 0, \tag{6}$$

for any f continuous on Π. Let the left hand of (6) be satisfied and let $a_{n,m} = \hat{\beta}(n,m)$. Then, since continuous functions are uniform limits of trigonometric polynomials, another application of the dominated convergence theorem for vector measures gives the direct implication. For $a_{n,m} = \hat{\beta}(n,m)$, the reversed implication is immediate. ■

L– as well as R–harmonizable processes are modelled after (1). Another class of non–stationary processes modelled after (2) has been introduced and studied by Bochner [4]. It is as follows:

<u>Definition 2.6</u>. A process x is *V-bounded* if there exists a constant $K > 0$ such that

$$\|\sum_{j=1}^{N} P_j x_{n_j}\|_{L^2(P)} \le K\|P\|_{L^\infty(\Pi)} \tag{7}$$

for all trigonometric polynomials P of the form $\Sigma_{j=1}^{N} P_j e^{-in_j\theta}$.

As already noticed by Bochner, it immediately follows from (3) and (7) that L–harmonizable processes are V–bounded. However, this inclusion is strict since for a white noise, (7) is always satisfied. The condition (7) just says that T: $P(\cdot) = \Sigma_{j=1}^{N} P_j e^{-in_j\cdot} \to \Sigma_{j=1}^{N} \hat{P}(n_j)x_{n_j}$ extends to a bounded linear operator from $C(\Pi)$ to $L^2(P)$. Hence, as in the scalar case, (see Phillips [22], Bartle, Dunford and Schwartz [2,p.301], or Kluvánek [15]), T has an integral representation, and (7) characterizes the Fourier tranforms of stochastic measures on Π. In other words, the V–bounded processes are exactly the Fourier transforms of stochastic measures.

The recent studies on V–bounded processes have been initiated by Niemi; in his thesis and a sequence of papers [18–20], he essentially obtained the equivalence of the conditions (ii), (iii) and (iv) below.

<u>Theorem 2.7</u>. The following are equivalent:

(i) x is V–bounded,

(ii) x is the Fourier transform of a stochastic measure,

(iii) x is R–harmonizable,

(iv) there exist $L^2(\tilde{P}) \supset L^2(P)$ and a WSS process y on $L^2(\tilde{P})$ such that $x = \mathbb{Q}y$, i.e., $x_n = \mathbb{Q}y_n$, $n \in \mathbb{Z}$, where $\mathbb{Q}$ is the orthogonal projection from $L^2(\tilde{P})$ onto $L^2(P)$.

The condition (iv) is not only a purely theoretical result and is in fact of great practical importance. It allows, by just interchanging $\mathbb{Q}$ and lim, V–bounded generalizations of the asymptotic mean squared results, such as a law of large numbers, valid for WSS processes. In particular, the bispectrum can be recovered from its transform, i.e., an inversion formula holds. Since typical examples of projections are conditional expectation operators, (iv) also identifies conditional expectations of WSS processes. Combined with (7), Theorem 2.7 also easily shows that white noises are R–harmonizable with bispectrum given by $\beta(\theta,\psi) \sim \Sigma_{n\in\mathbb{Z}} \sigma_n^2 e^{-in(\theta-\psi)}$.

Results on bimeasures usually rely on *Grothendieck inequality* [7], such is the theory of V–bounded processes. For example, the proof of (iii)=>(iv) as given in Miamee and Salehi [17] relies heavily on the following form of Grothendieck inequality: Let ζ be a stochastic measure, then there exists a finite positive measure μ on Π such that

$$\|\int_\Pi f d\zeta\|^2_{L^2(P)} \le \int_\Pi |f|^2 d\mu, \tag{8}$$

for all continuous functions f on Π. The (non–unique) μ in (8) is usually called a *Grothendieck measure*, a *dominating measure* or a *2–majorant.*

3. <u>(2,p)–boundedness</u>

The V–bounded class is of interest, since it is potentially subject to harmonic analysis studies. However, it also has the disadvantage of being too broad since typically one is more interested in some specific subclass of non-stationary processes. In the WSS case, for example, the processes with absolutely continuous or discrete spectrum play a particular rôle, while the ones with singular continuous spectrum are pathological. To initiate such studies, in the case of V–bounded processes, a natural step is to replace the L^∞ norm in (7) by a smaller one, an L^p norm for example. This is done now.

Definition 3.1. A process x is *(2,p)–bounded*, $1 \le p < +\infty$ if there exists a constant $K > 0$ such that

$$\left\| \sum_{j=1}^{N} P_j x_{n_j} \right\|_{L^2(P)} \le K \|P\|_{L^p(\Pi)} \tag{9}$$

for all trigonometric polynomials P of the form $\Sigma_{j=1}^{N} P_j e^{-in_j\theta}$.

Bochner [3], [4] also introduced and studied a (2,p)–boundedness notion for stochastic measures. With the help of Theorem 2.7, it is immediate to verify that his definition of (2,p)–boundedness and the one above are dual of one another, i.e., a process is (2,p)–bounded if and only if it is the Fourier transform of a (2,p)–bounded stochastic measure.

Let $\mathcal{M}^p$, $1 \le p < +\infty$ and $\mathcal{V}$ denote respectively the classes of (2,p)–bounded and of V–bounded processes. Then since Π has finite Lebesgue measure, we have $\mathcal{M}^p \subset \mathcal{M}^q \subset \mathcal{V}$, $1 \le p \le q < +\infty$. While WSS processes are always V–bounded, they are not necessarily (2,p)–bounded. It is readily seen that for $1 \le p < 2$, a stationary white noise does not satisfy the condition (9) while for $2 \le p < +\infty$ a WSS process with discrete spectrum also violates (9). These two types of counterexamples reflect a more general situation. We say that a WSS process has *L^p–spectrum*, $1 \le p \le +\infty$ if its spectral measure is absolutely continuous with Radon–Nikodym derivative in $L^p(\Pi)$.

Theorem 3.2. A WSS process is (2,p)–bounded, $2 \le p < +\infty$ if and only if it has $L^{p/(p-2)}$ spectrum (L^∞–spectrum when p=2). For $1 \le p < 2$, the only (2,p)–bounded WSS process is the zero process.

Proof. If x is a WSS process with $L^{p/(p-2)}$–spectrum, (9) follows directly from Hölder's inequality and x is (2,p)–bounded. Let x be a WSS process of type (2,p), and let μ be its spectral measure. Then,

$$\{\int_\Pi |P|^2 d\mu\}^{\frac{1}{2}} \le K\{\int_\Pi |P|^p d\theta\}^{1/p}, \tag{10}$$

for all trigonometric polynomials P. By the density of the trigonometric polynomials, (10) can be extended to $L^p(\Pi)$ and becomes

$$\int_\Pi |g|^2 d\mu \le K\{\int_\Pi |g|^p d\theta\}^{2/p}, \quad g \in L^p(\Pi). \tag{11}$$

In particular, let $g = \chi_A$ with $|A| = 0$ ($|A|$ denotes the Lebesgue measure of A); then $\mu(A) = 0$; hence $d\mu = f d\theta$, $f \ge 0$, $f \in L^1(\Pi)$. Now let $2 < p < +\infty$, and let $g = f^{1/(p-2)}$, then $g \in L^p(\Pi)$ and by (11), $\int_\Pi f^{p/(p-2)} d\theta \le K\{\int_\Pi f^{p/(p-2)} d\theta\}^{2/p}$, i.e., $\|f\|_{L^{p/(p-2)}(\Pi)} \le K$. For p = 2, let $g = \chi_E$, $|E| > 0$, then arguments similar to the ones above give $\frac{1}{|E|}\int_E f d\theta \le K$, i.e., $f \le K$ a.s. (Leb.).

Let $1 \le p < 2$ and let x be a non zero (2,p)–bounded WSS process. Then, since $\mathcal{M}^p \subset \mathcal{M}^2$ the inequality (11) holds with $d\mu = f d\theta$, $f \ge 0$, $f \in L^\infty(\Pi)$. First, if f is bounded below the result is immediate. If f is not bounded below but is continuous, and for n large enough, $|E_n| = |\{\frac{1}{(n+1)^{2-p/p}} \le f < \frac{1}{n^{2-p/p}}\}| > 0$. Hence, using (11) with $g = \chi_{E_n}$ gives $1/(n+1) \le K|E_n|$ and summing up over n leads to a contradiction. To finish the job, just notice that for f in $L^\infty(\Pi)$, there exists a sequence $\{f_n\}$ of continuous functions such that $f = \lim f_n$ a.s. (Leb.). ∎

The connections between the L–harmonizable and (2,p)–bounded classes are less natural and no inclusion type relation has been obtained. In fact, x such that $R(n,m) = 1$, $n, m \in \mathbb{Z}$ is L–harmonizable with bispectrum the unit mass at (0,0) but not (2,p)–bounded. More generally, as a direct consequence of (9), if a process is L–harmonizable and in $\mathcal{M}^p$, its bispectrum must be jump free. In particular, a L–harmonizable process with spectral measure $d\mu(\theta,\psi) = f(\theta)\bar{f}(\psi)d\theta d\psi$, $f \in L^{p/(p-1)}(\Pi)$, is (2,p)–bounded. The above examples do not provide necessary conditions for L–harmonizable processes to be in $\mathcal{M}^p$. Again, the white noise processes play a particular role, since it is trivially verified that for $p \geq 2$ they are (2,p)–bounded. Moreover, for $1 \leq p < 2$, the only (2,p)–bounded white noise is the zero process. It also readily follows from this, that any bounded sequence $\{a_n\}_{n\in\mathbb{Z}}$ of non–negative elements is the Fourier transform of a (2,2)–bounded p.d. bimeasure. Furthermore, in contrast to Theorem 2.4, the diagonal sequence of a R–harmonizable covariance is not, in general, a one dimensional Fourier–Stieltjes transform.

Processes in the boundary class $\mathcal{M}^2$ possess another particular covariance structure: Let T be a bounded linear operator on $\ell^2(\mathbb{Z})$, i.e., from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$. It is well–known that the continuity of the inner product and the existence of the canonical basis on $\ell^2(\mathbb{Z})$, ensure for T a representation as a (doubly) infinite matrix $\{T_{n,m}\}_{n,m\in\mathbb{Z}}$. Reciprocally, any infinite matrix $\{T_{n,m}\}_{n,m\in\mathbb{Z}}$ such that

$$|\sum_n \sum_m x_n T_{n,m} \bar{y}_m| \leq K\|x\|_{\ell^2(\mathbb{Z})} \|y\|_{\ell^2(\mathbb{Z})} \tag{12}$$

represents a bounded linear operator on $\ell^2(\mathbb{Z})$. In our framework, this simple fact can be restated as

<u>Proposition 3.3</u>. A process is (2,2)–bounded if and only if its covariance kernel, in infinite matrix form, is a (positive) bounded linear operator on $\ell^2(\mathbb{Z})$.

The Hilbert space isometry between $\ell^2(\mathbb{Z})$ and $L^2(\Pi)$, makes the analysis of the (2,2)–bounded case easy to handle. For $p \neq 2$, Proposition 3.3 admits only partial generalizations. For $1 \leq p < 2$, since $\|\cdot\|_{L^p(\Pi)} \leq \|\cdot\|_{L^2(\Pi)} = \|\cdot\|_{\ell^2(\mathbb{Z})} \leq \|\cdot\|_{\ell^p(\mathbb{Z})}$, the covariance (in infinite matrix form) of a (2,p)–bounded process is a bounded linear operator from $\ell^p(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$. In general, the converse does not hold: a diagonal matrix $\{R(n,n)\}_{n\in\mathbb{Z}}$ with $0 \leq R(n,n) \leq K$, $n \in \mathbb{Z}$ maps $\ell^p(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$ boundedly but a white noise is not (2,p)–bounded, $1 \leq p < 2$. The case $p > 2$ is also recalcitrant. If a covariance R is a bounded linear operator from $\ell^p(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$, then again since $L^p(\Pi) \subset L^2(\Pi)$ and $\ell^2(\mathbb{Z}) \subset \ell^p(\mathbb{Z})$, R is the covariance of a (2,p)–bounded process. Conversely, a white noise is (2,p)–bounded. However, the associated diagonal matrix $\{R(n,n)\}$ maps $\ell^p(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$ boundedly when and only when $\{R(n,n)\}_{n\in\mathbb{Z}} \in \ell^{p/(p-2)}(\mathbb{Z})$. Finally, similar arguments show that the covariance of processes in $\mathcal{M}^p$, $1 \leq p < 2$ are bounded linear operators from $\ell^2(\mathbb{Z})$ to $\ell^q(\mathbb{Z})$, $q = p/p-1$, while if R maps $\ell^2(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$ boundedly then the associated process is in $\mathcal{M}^q$. Again, these conditions are not characterizations.

The main objective in the rest of this section is to give a few characterizations of the classes $\mathcal{M}^p$. These results, which are the (2,p)–bounded versions of the characterization stated in Theorem 2.7 rely on a specialization of (8). Our goal is to obtain (8) with special types of dominating measures. To do so, we "generalize" (since $\mathcal{M}^\infty = \mathcal{V}$ and since the elements of the dual of $C(\Pi)$ are the complex measures) Pietsch's [23] proof of the classical inequality (see also Miamee and Salehi [17] and Remark 3.7). Towards this, we first state a standard result obtained by Rogge [26] in the real case and generalized to the complex case in [17].

Lemma 3.4. Let $\mathbb{R}^n$, $n \geq 2$ be the real Euclidean space with inner product $<\cdot,\cdot>$ and norm $\|\cdot\|$ and let m be the normalized Haar measure on the unit sphere S of $\mathbb{R}^n$. Let the kernel $L(\cdot,\cdot)$ be defined on $\mathbb{R}^n \times \mathbb{R}^n$ by $L(r,s) = \int_S \operatorname{sign}(r,t)\operatorname{sign}(s,t)dm(t)$. Then,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \langle t_i,t_j\rangle \lambda_i\overline{\lambda}_j \le \frac{\pi}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} L(t_i,t_j) \|t_i\| \|t_j\|\lambda_i\overline{\lambda}_j, \tag{13}$$

for all $N \in \mathbb{N}^*$, $t_1,\dots,t_N \in \mathbb{R}^n$, $\lambda_1,\dots,\lambda_N \in \mathbb{C}$.

Before presenting our version of Grothendieck inequality, another preliminary result is needed. With the uniform norm replacing the p/2 norm, and for real valued f_i, this result is also due to Pietsch [23]. The extension to complex valued f_i as given in [17] does not hold (F there is implicitly assumed to be real). However, our arguments can be used to obtain the corresponding version for complex valued f_i and V–bounded processes.

<u>Lemma 3.5</u>. *Let x be a (2,p)–bounded process, $p \ge 2$, with spectral stochastic measure ζ. Then there exists a constant $K > 0$ such that*

$$\sum_{i=1}^{N} \| \int_\Pi f_i d\zeta\|^2_{L^2(P)} \le K \| \sum_{i=1}^{N} |f_i|^2\|_{L^{p/2}(\Pi)} \tag{14}$$

for all $N \in \mathbb{N}^*$, $f_1,\dots,f_N$, continuous functions on Π.

<u>Proof</u>. Since continuous functions on Π are uniform limits of simple functions, it is enough to show that (14) holds for simple functions. Let f_i be real valued. By eventually taking common partitions of Π, let $f_i = \sum_{k=1}^{M} f_i(\theta_k)\chi_{A_k}$, then

$$\sum_{i=1}^{N} \| \int_\Pi f_i d\zeta\|^2_{L^2(P)} = \sum_{i=1}^{N} \sum_{k=1}^{M} \sum_{\ell=1}^{M} f_i(\theta_k)f_i(\theta_\ell)\, E\zeta(A_k)\overline{\zeta(A_\ell)}$$

$$= \sum_{k=1}^{M} \sum_{\ell=1}^{M} \langle t_k,t_\ell\rangle\, E\zeta(A_k)\overline{\zeta(A_\ell)}$$

where $t_k = (f_1(\theta_k),\dots,f_N(\theta_k)) \in \mathbb{R}^N$. Let $\{e_\alpha\}_\alpha$ be an orthonormal basis of $L^2(P)$, then by Parseval identity $E\zeta(A_k)\overline{\zeta(A_\ell)} = \sum_\alpha E\zeta(A_k)\bar{e}_\alpha \overline{E\zeta(A_\ell)\bar{e}_\alpha}$, hence from Lemma 3.4, we get

$$\sum_{i=1}^N \|\int_\Pi f_i d\zeta\|^2_{L^2(P)} = \sum_\alpha \sum_{k=1}^M \sum_{\ell=1}^M \langle t_k,t_\ell\rangle E\zeta(A_k)\bar{e}_\alpha \overline{E\zeta(A_\ell)\bar{e}_\alpha}$$

$$\le \frac{\pi}{2}\sum_\alpha \sum_{k=1}^M \sum_{\ell=1}^M L(t_k,t_\ell)\|t_k\|\,\|t_\ell\| E\zeta(A_k)\bar{e}_\alpha \overline{E\zeta(A_\ell)\bar{e}_\ell}$$

$$= \frac{\pi}{2}\sum_{k=1}^M \sum_{\ell=1}^M L(t_k,t_\ell)\|t_k\|\,\|t_\ell\| E\zeta(A_k)\overline{\zeta(A_\ell)}$$

$$= \frac{\pi}{2}\int_S \|\int_\Pi \sum_{k=1}^M \operatorname{sign}(t_k,t)\|t_k\|\,\chi_{A_k}(\theta)d\zeta(\theta)\|^2_{L^2(P)}\,dm(t).$$

Now, x is (2,p)–bounded hence by applying (9) (extended to Borel bounded functions by the density of trigonometric polynomials) to the above expression, we obtain

$$\sum_{i=1}^N \|\int_\Pi f_i d\zeta\|^2_{L^2(P)} \le K\frac{\pi}{2}\int_S \Big(\int_\Pi \Big|\sum_{k=1}^M \operatorname{sign}(t_k,t)\|t_k\|\,\chi_{A_k}(\theta)\Big|^p d\theta\Big)^{2/p} dm(t)$$

$$= K\frac{\pi}{2}\int_S \Big\{\int_\Pi \sum_{k=1}^M \|t_k\|^p\,\chi_{A_k}(\theta)d\theta\Big\}^{2/p} dm(t)$$

$$= K\frac{\pi}{2}\Big\{\int_\Pi \Big(\sum_{k=1}^M \|t_k\|^2\,\chi_{A_k}(\theta)\Big)^{p/2} d\theta\Big\}^{2/p}$$

$$= K\frac{\pi}{2}\Big\{\int_\Pi \Big(\sum_{k=1}^M \sum_{i=1}^N (f_i(\theta_k))^2\,\chi_{A_k}(\theta)\Big)^{p/2} d\theta\Big\}^{2/p}$$

$$= K\frac{\pi}{2}\Big\{\int_\Pi \Big(\sum_{i=1}^N f_i^2(\theta)\Big)^{p/2} d\theta\Big\}^{2/p},$$

and the result follows for any set $\{f_1,\dots,f_N\}$ of real valued continuous functions. For the general case, decomposing the f_k's as well as ζ in real and imaginary parts, f_k^r, f_k^i, ζ^r, ζ^i, we get

$$0 \le \| \int_{\Pi} f_k^r d\zeta \|^2_{L^2(P)} = \| \int_{\Pi} f_k^r d\zeta^r \|^2_{L^2(P)} + \| \int_{\Pi} f_k^r d\zeta^i \|^2_{L^2(P)},$$

and similarily for the f_k^i 's. Hence,

$$0 \le \sum_{k=1}^{N} \| \int_{\Pi} f_k d\zeta \|^2_{L^2(P)}$$

$$= \sum_{k=1}^{N} (\| \int_{\Pi} f_k^r d\zeta^r - \int_{\Pi} f_k^i d\zeta^i \|^2_{L^2(P)} + \| \int_{\Pi} f_k^r d\zeta^i + \int_{\Pi} f_k^i d\zeta^r \|^2_{L^2(P)})$$

$$\le 2 \sum_{k=1}^{N} (\| \int_{\Pi} f_k^r d\zeta^r \|^2_{L^2(P)} + \| \int_{\Pi} f_k^i d\zeta^i \|^2_{L^2(P)} + \| \int_{\Pi} f_k^r d\zeta^i \|^2_{L^2(P)} + \| \int_{\Pi} f_k^i d\zeta^r \|^2_{L^2(P)})$$

$$= 2 \sum_{k=1}^{N} (\| \int_{\Pi} f_k^r d\zeta \|^2_{L^2(P)} + \| \int_{\Pi} f_k^i d\zeta \|^2_{L^2(P)})$$

Since f_k^r and f_k^i are real valued, the previous result applies and

$$\sum_{k=1}^{N} \| \int_{\Pi} f_k d\zeta \|^2_{L^2(P)} \le K\pi(\| \sum_{k=1}^{N} (f_k^r)^2 \|_{L^{p/2}(\Pi)} + \| \sum_{k=1}^{N} (f_k^i)^2 \|_{L^{p/2}(\Pi)})$$

$$\le 2K\pi(\| \sum_{k=1}^{N} |f_k|^2 \|_{L^{p/2}(\Pi)}),$$

since $(f_k^r)^2 \le |f_k|^2$ and $(f_k^i)^2 \le |f_k|^2$. ■

We are now ready to prove the following Grothendieck type inequality.

Theorem 3.6. Let x be a (2,p)–bounded process, $p \ge 2$, with spectral stochastic measure ζ. Then, there exits a non–negative function g in $L^{p/(p-2)}(\Pi)$ such that

$$\| \int_{\Pi} f d\zeta \|^2_{L^2(\Pi)} \le \int_{\Pi} |f|^2 g d\theta, \tag{15}$$

for all continuous functions f on Π.

Proof. For f continuous real valued on Π and for K any constant in (14) let

$$\mathbb{Q}(f) = \operatorname{Inf}\{(\int_{\Pi} |f+K\sum_{i=1}^{N} |f_i|^2|^{p/2}d\theta)^{2/p} - \sum_{i=}^{N} \|\int_{\Pi} f_i d\zeta\|^2\}$$

where the infimum is taken over all finite sets $\{f_1,\ldots,f_N\}$ of complex valued continuous functions on Π. Then $\mathbb{Q}$ is an homogeneous subadditive functional on $C^{\mathbb{R}}(\Pi)$ (the space of real valued continuous functions on Π) such that $-\|f\|_{L^{p/2}(\Pi)} \leq \mathbb{Q}(f) \leq \|f\|_{L^{p/2}(\Pi)}$. To prove these assertions, let us define

$$S(f,f_1,\ldots,f_N) = (\int_{\Pi} |f+K\sum_{i=1}^{N} |f_i|^2|^{p/2})^{2/p} - \sum_{i=1}^{N} \|\int_{\Pi} f_i d\zeta\|^2.$$

Then for $\alpha = 0$, $S(\alpha f,f_1,\ldots,f_N) = K(\int_{\Pi} (\sum_{i=1}^{N} |f_i|^2)^{p/2})^{2/p} - \sum_{i=1}^{N} \|\int_{\Pi} f_i d\zeta\|^2) \geq 0$, where we also used (14). Now $f_1 = f_2 = \ldots = f_N = 0$, gives the infimum hence $\mathbb{Q}(0) = 0$. Let $\alpha > 0$, then $S(\alpha f,f_1,\ldots,f_N) = \alpha S(f,\alpha^{-\frac{1}{2}}f_1,\ldots,\alpha^{-\frac{1}{2}}f_N) \geq \alpha\mathbb{Q}(f)$, hence $\mathbb{Q}(\alpha f) \geq \alpha\mathbb{Q}(f)$. On the other hand, $\alpha S(f,f_1,\ldots,f_N) = S(\alpha f, \alpha^{\frac{1}{2}}f_1,\ldots,\alpha^{\frac{1}{2}}f_N) \geq \mathbb{Q}(\alpha f)$, hence $\alpha\mathbb{Q}(f) \geq \mathbb{Q}(\alpha f)$ and $\mathbb{Q}$ is homogeneous. For the sublinearity, let $\{f_1,\ldots,f_N\}$ and $\{g_1,\ldots,g_M\}$ be two arbitary sets of continuous functions. It follows from Minkowski's inequality that $\mathbb{Q}(f+g) \leq S(f+g,f_1,\ldots,f_N,g_1,\ldots,g_M) \leq S(f,f_1,\ldots,f_N) + S(g,g_1,\ldots,g_M)$, and $\mathbb{Q}(f+g) \leq \mathbb{Q}(f) + \mathbb{Q}(g)$. For the last assertion, again by Minkowski inequality, $S(f,f_1,\ldots,f_N) \leq \|f\|_{L^{p/2}(\Pi)} + S(0,f_1,\ldots,f_N)$. Hence, $\mathbb{Q}(f) \leq \|f\|_{L^{p/2}(\Pi)} + \mathbb{Q}(0) = \|f\|_{L^{p/2}(\Pi)}$. Similarly, $S(f,f_1,\ldots,f_N) \geq -\|f\|_{L^{p/2}(\Pi)} + S(0,f_1,\ldots,f_N)$, hence $\mathbb{Q}(f) \geq -\|f\|_{L^{p/2}(\Pi)} + \mathbb{Q}(0) = -\|f\|_{L^{p/2}(\Pi)}$ and the three assertions are proved. Since $\mathbb{Q}$ is real homogeneous and subadditive, by the Hahn—Banach theorem, there exists a real linear functional L on $C^{\mathbb{R}}(\Pi)$ such that $-\mathbb{Q}(-f) \leq L(f) \leq \mathbb{Q}(f)$, hence such that $-\|f\|_{L^{p/2}(\Pi)} \leq -\mathbb{Q}(-f) \leq L(f) \leq \mathbb{Q}(f) \leq \|f\|_{L^{p/2}(\Pi)}$. Now, L can be extended to $C(\Pi)$ via $L(f_1+jf_2)$† $= L(f_1) + jL(f_2)$ (Hahn—Banach again).

†Mathematicians beware: j is not the intensity of an electrical current.

Furthemore, $|L(f_1+jf_2)| = (|L(f_1)|^2+|L(f_2)|^2)^{1/2} \leq (\|f_1\|_{L^{p/2}(\Pi)}+\|f_2\|_{L^{p/2}(\Pi)})^{1/2}$ $\leq \sqrt{2}\ \|f_1+jf_2\|_{L^{p/2}(\Pi)}$ hence, L can also be extended to $L^{p/2}(\Pi)$. Now, by the Riesz representation theorem there exists $g_0 \in L^{p/(p-2)}(\Pi)$ such that $Lf = \int_\Pi fg_0 d\theta$, for all $f \in L^{p/2}(\Pi)$. For f continuous ≥ 0, $S(f) \geq S(0)$ hence $Q(f) \geq Q(0) = 0$, while for $f \leq 0$, i.e, $f = -h$, $h \geq 0$, we have $Q(f) \leq S(f,K^{-\frac{1}{2}}h^{\frac{1}{2}},0,\dots,0) = -\|\int_\Pi K^{-\frac{1}{2}}h^{\frac{1}{2}}d\zeta\|^2 \leq 0$. Hence, L is a positive linear functional and $g_0 \geq 0$. To finish the proof, let $f \in C(\Pi)$, then $Q(-K|f|^2) \leq S(-K|f|^2,f,0,\dots,0) = -\|\int_\Pi fd\zeta\|^2_{L^2(P)}$. Finally, $-KL(|f|)^2 = L(-K|f|^2) \leq Q(-K|f|^2) \leq -\|\int_\Pi fd\zeta\|^2_{L^2(P)}$, i.e., $\|\int_\Pi fd\zeta\|^2_{L^2(P)} \leq K\int_\Pi |f|^2 g_0 d\theta$, and the result follows by taking $g = Kg_0$. ∎

<u>Remark 3.7</u>. A few comments on the above results are in order. The version of Theorem 3.6 where $C(\Pi)$ replaces $L^{p/2}(\Pi)$ is due to Pietsch, who also introduced a "scanning" sublinear functional as above. The function g is trivially non unique but, by simple modifications of the arguments of [23], it can easily be seen that there exists a unique g such that $\|g\|_{L^{p/(p-2)}(\Pi)} = \text{Inf } K$ appearing in (14). Construction of such a minimal g can also be obtained by adaptation of the techniques and results of Niemi [21]. If x is L–harmonizable with spectral measure ν, Abreu [1] showed that $\mu(A) = \frac{1}{2}(|\nu|(A\times\Pi) + |\nu|(\Pi\times A))$, $A \in \mathcal{B}(\Pi)$, $|\nu|$ the total variation of ν, defines a Grothendieck measure. For ζ of bounded variation, then $\mu(\cdot) = |\zeta|(\cdot)[|\zeta|(\Pi)]^{-1}$ also defines a dominating measure (Chatterji [5]). Both these results hold in special form in the (2,p)–bounded case. For (2,2)–bounded processes, Lemma 3.5 and Theorem 3.6 are immediate since in this case the right hand side of (9) is just $(\int_{-\pi}^{\pi}|P(\theta)|^2d\theta)^{1/2}$. Hence, the Lebesgue measure is always a dominating measure for processes in $\mathcal{H}^2$, and Theorem 3.6 can be viewed as an interpolation result.

We now present a few characterizations of (2,p)–boundedness which are essentially based on the previous result.

Theorem 3.8. Let x be a $L^2(P)$–valued process and let $2 \leq p < +\infty$. Then, the following are equivalent.

(i) x is (2,p)–bounded,

(ii) x is V–bounded with a Grothendieck measure in $L^{p/(p-2)}(\Pi)$,

(iii) there exists $L^2(\tilde{P}) \supset L^2(P)$ and a (2,p)–bounded WSS process y on $L^2(\tilde{P})$ such that $x = \mathbf{Q}y$, where $\mathbf{Q}$ is the orthogonal projection from $L^2(\tilde{P})$ onto $L^2(P)$.

Proof. Our proof is cyclical.

(i) => (ii) (7), (9) and Theorem 3.6.

(ii) => (iii) We exhibit the projection by a method due to Abreu in the L–harmonizable case and which has been subsequently used by various authors in the R–harmonizable one. Let g be as in Theorem 3.6, then there exists a measure ν on $\mathcal{B}(\Pi^2)$ (the Borel σ–algebra of Π^2) which is concentrated on its diagonal and such that $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\theta,\psi)d\nu(\theta,\psi) = \int_{-\pi}^{\pi} f(\theta,\theta)g(\theta)d\theta$, for any f continuous on Π^2. Let $\beta(\cdot,\cdot) = E\zeta(\cdot)\overline{\zeta(\cdot)}$, then (see [11]), $E(\int_\Pi h_1 d\zeta)\overline{(\int_\Pi h_2 d\zeta)} = \int_\Pi\int_\Pi h_1 d\beta \bar{h}_2$, h_1, $h_2 \in C(\Pi)$. Hence from (15), $\langle h_1,h_2\rangle = \int_\Pi\int_\Pi h_1\bar{h}_2 d\nu - \int_\Pi\int_\Pi h_1\bar{h}_2 d\beta$ defines a semi–inner product on $C(\Pi)$. After having identified the functions h such that $\langle h,h\rangle = 0$ (i.e., taking the quotient) and after completion under $\langle\cdot,\cdot\rangle$, the resulting space is a separable Hilbert space H. Hence there exists a probability triple $(\Omega_1,\mathcal{B}_1,P_1)$ such that $H = L^2(\Omega_1,\mathcal{B}_1,P_1)$. Now, let I be the canonical projection $C(\Pi) \to L^2(\Omega_1,\mathcal{B}_1,P_1)$, let $w_n = I(e^{in\cdot})$, $n \in \mathbb{Z}$, and let $(\tilde{\Omega},\tilde{\mathcal{B}},\tilde{P}) = (\Omega,\mathcal{B},P) \otimes (\Omega_1,\mathcal{B}_1,P_1)$. Then, $L^2(\Omega,\mathcal{B},P) \oplus L^2(\Omega_1,\mathcal{B}_1,P_1)$ can be naturally identified with a subspace of $L^2(\tilde{\Omega},\tilde{\mathcal{B}},\tilde{P})$, and so is $L^2(P) = L^2(P) \oplus \{0\}$. Finally, let $y_n = x_n + w_n$. Since x and w are mutually orthogonal, we have $x = Qy$, where Q is the orthogonal projection from $L^2(\tilde{P})$ onto $L^2(P)$. So it just remains to show that y is a WSS process with $L^{p/(p-2)}$-spectrum. But,

$$Ey_n\bar{y}_m = Ex_n\bar{x}_m + \langle w_n, w_m\rangle$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\psi}d\beta(\theta,\psi) + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\psi}d\nu(\theta,\psi) - \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\psi}d\beta(\theta,\psi)$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\psi}d\nu(\theta,\psi)$$

$$= \int_{-\pi}^{\pi} e^{i(n-m)\theta}g(\theta)d\theta$$

which proves the result. We note too, that in the extreme situation where $\langle w_n, w_m\rangle = 0$, we have

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\psi}d\beta(\theta,\psi) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\psi}d\nu(\theta,\psi) = \int_{-\pi}^{\pi} e^{i(n-m)\theta}g(\theta)d\theta.$$

(iii) => (i)

$$\|\sum_{j=1}^{N} P_j x_{n_j}\|^2_{L^2(P)} = \|Q(\sum_{j=1}^{N} P_j y_{n_j})\|^2_{L^2(P)}$$

$$\le \|Q\|^2\|\sum_{j=1}^{N} P_j y_{n_j}\|^2_{L^2(\tilde{P})}$$

$$= \int_{-\pi}^{\pi} |\sum_{j=1}^{N} P_j e^{in_j\theta}|^2 g(\theta)d\theta$$

$$\le \left(\int_{-\pi}^{\pi} |\sum_{j=1}^{N} P_j e^{in_j\theta}|^p d\theta\right)^{2/p} \left(\int_{-\pi}^{\pi} g(\theta)^{p/(p-2)}d\theta\right)^{(p-2)/p}. \blacksquare$$

<u>Remark 3.9</u>. For $1 \le p < 2$, we do not know if a result similar to Theorem 3.8 holds.

Since (2,p)–bounded processes ($1 \le p < 2$) are (2,2)–bounded, they also are projections of WSS processes with L^∞–spectrum but, taking $g = 1$ above shows that the inclusion is proper. However, we do not know which additional condition their spectra has to satisfy in order to obtain a characterization. In view of Theorem 3.2, such a characterization seems to be very unlikely since the spectral stochastic measure of a (2,p)–bounded process always has dependent increments. Furthermore, as indicated by white noise processes, this dependency has to be quite strong. Finally, we note that altough we do not recover L^1–spectrum (this is classical: $L^\infty(\Pi)^* \neq L^1(\Pi)$), the bimeasures associated to (2,p)–bounded processes play a rôle similar to the absolutely continuous measures for WSS processes.

To finish this section, we apply some of the methods developed to this point to show that certain classes of processes are (2,p)–bounded.

A non–stationary model of great practical importance is the ARMA model, e.g., $x_{k+1} = \alpha x_k + v_k$, $k \in \mathbb{Z}$, $\alpha \in \mathbb{C}$, $|\alpha| < 1$, where say $\{v_k\}_{k\in\mathbb{Z}}$ is a white noise. When the recursion is on $\mathbb{Z}$, the analysis of such models is not difficult: since $\{v_k\}_{k\in\mathbb{Z}}$ is a white noise, it is (2,2)–bounded and $v_k = \int_{-\pi}^{\pi} e^{ik\theta}\, d\zeta_v(\theta)$, with β_v dominated by the Lebesgue measure. It is then straightforward to verify that $\int_{-\pi}^{\pi} \frac{e^{ik\theta}}{e^{i\theta}-\alpha}\, d\zeta_v(\theta)$ satisfies the recursion and is in fact the unique V–bounded solution to this recursion. Furthermore, $\frac{d\zeta_v}{e^{i\theta}-\alpha}$ is dominated by $\frac{d\theta}{(1-|\alpha|)^2}$ and x is not only V–bounded but also (2,2)–bounded. Its bispectrum is given by $d\beta_x(\theta,\psi) = (e^{i\theta}-\alpha)^{-1} d\beta_v(\theta,\psi)\overline{(e^{i\psi}-\alpha)}^{-1}$. In particular, if v is WSS, we recover the classical formula, $d\beta_x(\theta) = |e^{i\theta}-\alpha|^{-2} d\theta$. When the recursion is not given on $\mathbb{Z}$ but say for $k \ge 0$, an initial condition x_0 is given with $E|x_0|^2 = 1$. For definiteness, we also assume that $x_k = 0$, $k < 0$ and to facilitate the computation that $Ex_k\bar{v}_j = 0$, $j \ge k$ and

$Ev_k\bar{v}_j = \delta_{k,j}$. Then, a simple computation shows that

$$Ex_k\bar{x}_j = \begin{cases} \frac{1}{|\alpha|^2-1}(\alpha^{k+1}\bar{\alpha}^{j+1} - \alpha^{k-(k\wedge j)}\bar{\alpha}^{j-(k\wedge j)}), & k,j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $k\wedge j = \min(k,j)$.

Since $\lim_{k\to+\infty} E|x_k|^2 = \frac{1}{1-|\alpha|^2} \neq \lim_{k\to-\infty} E|x_k|^2 = 0$, the process x is not L–harmonizable. Nevertheless, it is (2,2)–bounded. Stated differently, the covariance of x is a bounded linear operator on $\ell^2(\mathbb{Z})$. Directly verifying that (12) is satisfied is quite impractical and we will not do so. However, a nice sufficent condition for (12) to hold is that $\Sigma_{k\in\mathbb{Z}}|Ex_k\bar{x}_j| \leq K$, $j \in \mathbb{Z}$ (this follows from the Cauchy–Schwarz inequality). In the above time invariant ARMA example , we get the following inequalities, $\Sigma_{k\in\mathbb{Z}}|Ex_k\bar{x}_j| \leq$

$$\frac{1}{1-|\alpha|^2}(\Sigma_{k=0}^{j}|\alpha^{k+1}\bar{\alpha}^{j+1}-\bar{\alpha}^{j-k}| + \Sigma_{k=j+1}^{\infty}|\alpha^{k+1}\bar{\alpha}^{j+1}-\alpha^{k-j}|) \leq \frac{1}{1-|\alpha|^2}(\frac{4}{1-|\alpha|} + \frac{2}{1-|\alpha|}).$$

Hence x is (2,2)–bounded. Its spectral bimeasure which *cannot* determine a measure is given by $\beta(\theta,\psi) \sim \frac{1}{(1-e^{-i\theta}e^{i\psi})(1-\alpha e^{-i\theta})(1-\bar{\alpha}e^{i\psi})}$. These examples are just samples of large classes of non–stationary processes, including the *time varying ARMA* models, which are (2,2)–bounded. The reader is referred to Houdré [10] for a more detailled analysis of the (2,2)–bounded case.

4. Conclusion

This work represents an attempt at presenting a unified theory for some classes of (discrete time) non–stationary processes subject to harmonic analysis studies. Very often, one is not interested in the full generality of the V–bounded class but more likely in some special classes. This is particularly true in applications in which case the class $\mathcal{M}^2$ seems to be the most promising for further studies. In fact, the linear least squares prediction problem for V–bounded processes has a particularly nice solution in the (2,2)–bounded case (see Houdré [10], [12]). The results presented here can also be rephrased or extended in various ways. Theorem 3.6 and 3.8 can be translated in equivalent statements in terms of dominating Toeplitz kernels, projection of orthogonally scattered stochastic measures or p–summing operators. A first type of extension is the multidimensional case, and this is presented in [12]. The order structure of $\mathbb{Z}$ has not been used in our main results, only the compactness of Π and commutativity have some importance. Hence, comparable techniques will give similar results for processes $x: W \longrightarrow L^2(P)$ where W is a compact abelian Hausdorff space and, in particular, for discrete random fields. Non–commutative analogs of our results can also be obtained following the works of Pisier [24], Haagerup [8] and Ylinen [28]; while the approach developed by Chatterji [5] will give finitely additive versions. We mention finally that, for continuous time processes, new difficulties arise due to the non–inclusion of the various spaces $L^p(\mathbb{R})$, i.e., a (2,p)–bounded process is no longer automatically V–bounded. Different techniques to obtain Fourier integral representations have to be developed. This is presented in Houdré [13] where processes of order $1 \leq \alpha < 2$ are also studied.

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