A COUNTEREXAMPLE CONCERNING THE EXTREMAL INDEX

by

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August 1988
163. M. O'Sullivan and T.R. Fleming, Statistics for the two-sample survival analysis
problem based on product limit estimators of the survival functions. Nov. 86.

164. F. Avrasc, On bilinear forms in Gaussian random variables, Toeplitz matrices and
Parsivel's relation. Nov. 86.


166. R.J Wilson, Model fields in crossing theory-a weak convergence perspective. Nov. 86.

167. D.B.H Cline, Consistency for least squares regression estimators with infinite
variance data. Dec. 86.

86.


170. H. Godaia, Freidlin-Wentzell type estimates for a class of self-similar processes
represented by multiple Wiener Integrals. Dec. 86.


172. R. Menich and R.F. Serfozo. Optimality of shortest queue routing for dependent
service stations. Dec. 86.


174. F. Mertx and R.L. Taylor. Strong laws of large numbers for arrays of orthogonal
random variables. Dec. 86.

175. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations driven by nuclear

176. E. Merzbach, Point processes in the plane. Feb. 87.


178. G. Kallianpur, A.C. Niamte and H. Niemi. On the prediction theory of two parameter


180. R. Bigong, Stochastic filtering solutions for ill-posed linear problems and their
extension to measurable transformations. Mar. 87.


182. H.L. Hurd, Representation of harmonizable periodically correlated processes and their
covariance. Apr. 87.


184. T. Mort and H. Gudair, Freidlin-Wentzell estimates and the law of the iterated
logarithm for a class of stochastic processes related to symmetric statistics. May
87.

Processes. to appear.


187. O. Kallenberg, Decoupling identities and predictable transformations in
exchangeability, June 87.

188. O. Kallenberg, An elementary approach to the Daniell-Kolmogorov theorem and some

to appear.

190. D. Ruinart, L. Sam and N. Zakai, On the relations between increasing functions
associated with two-parameter continuous martingales, June 87.

191. F. Avrasm and N. Taqgu. Weak convergence of sums of moving averages in the a-stable
domain of attraction, June 87.


195. J. Rosinski, Series representations of infinitely divisible random vectors and a


197. I. Kuznetsova-Shilpo and S.T. Rachev, Explicit solutions of moment problems I, July


201. N. Marques and S. Cambanis, Admissible and singular translates of stable processes,
Aug. 88.

202. O. Kallenberg, One-dimensional uniqueness and convergence results for exchangeable
processes. Aug. 87.


204. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations driven by nuclear

The concept of an extremal index, which is a measure of local dependence amongst the exceedances over a high threshold by a stationary sequence, has a natural interpretation as the reciprocal of mean cluster size. We exhibit a counterexample which shows that this interpretation is not necessarily correct.
A COUNTEREXAMPLE CONCERNING THE EXTREMAL INDEX

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Summary
The concept of an extremal index, which is a measure of local dependence amongst the exceedances over a high threshold by a stationary sequence, has a natural interpretation as the reciprocal of mean cluster size. We exhibit a counterexample which shows that this interpretation is not necessarily correct.

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Guildford, GU2 5XH, U.K.
Let \((\xi_n, n=0,1,2,...)\) denote a stationary sequence and define 
\[ M_n = \max(\xi_1, \ldots, \xi_n). \]
Under suitable conditions it is possible to prove results of the form

\[
np(\xi_1 > u_n) \to \tau \iff p(M_n \leq u_n) \to e^{-\Theta \tau} (0 < \tau < \infty) \tag{1}
\]

where \(0 < \Theta < 1\). The parameter \(\Theta\) was termed the extremal index by
Leadbetter (1983), though the concept had occurred earlier in papers of
For a general overview of extremes in stationary sequences, see

It is possible to define an exceedance point process \(N_n\) on \((0,1)\),
such that \(N_n(s,t]\) is the number of exceedances of the level \(u_n\) among \([\xi_r: ns < r < nt]\). Convergence of \((N_n)\) as \(n \to \infty\) is studied by
Hsing, Hüsler and Leadbetter (1988). One of their main results is that, if a limiting
point process exists, then it must be compound Poisson. The atoms of this
limiting process correspond to clusters of exceedances. Somewhat parallel
results have also been obtained by Alpuim (1987).

A natural interpretation of \(\Theta\) is that \(1/\Theta\) is the main cluster size in
the limiting point process. Hsing et al. were not, however, able to prove
this without making additional assumptions. The following example shows
that the result is false without such assumptions.

The example is a regenerative sequence of the form

\[
\xi_n = \zeta_j \quad \text{for} \quad \sum_{i=0}^{j-1} N_i < n < \sum_{i=0}^{j} N_i \quad (j \geq 1) \tag{2}
\]

where

(i) \(\zeta_j, j \geq 1\) are independent with a common distribution function \(F\)
satisfying \( P(1) = 0 \).

(ii) For \( j > 1 \), given \( N_1, \ldots, N_{j-1}, \xi_1, \ldots, \xi_j \) with \( m < \xi_j < m+1 \), the probability of the event \( N_j = i \) is \( q_{mi} \). Here \( (q_{mi}, m \geq 1, i \geq 1) \) is a sequence of probabilities with \( q_{mi} \geq 0 \), \( \sum_i q_{mi} = 1 \) for each \( m \).

In words, the process remains in state \( \xi_i \) for a random number of time epochs determined by the probability distribution \( q_{mi}(i \geq 1) \) with \( m = [\xi_i] \), and then moves to a new state which is independently chosen from \( P \).

Let \( P_m = P(m \leq \xi_j < m-1) \), \( \mu_m = \sum_i iq_{mi} \) and suppose \( \mu = \sum_i P_m \mu_m = \infty \).

Then \( \mu \) is the mean recurrence time of the process. The process may be made stationary by a suitable choice of distribution of \( N_1 \). It may also be regarded as a function of a Harris chain, and may therefore be treated by extreme value arguments of O'Brien (1987) and Rootzén (1987).

Now let us specify \( (q_{mi}) \) to be

\[
q_{mi} = \begin{cases} 
\frac{(m-1)}{m}, & i=1, \\
\frac{1}{m}, & i=m+1, \\
0, & \text{otherwise}.
\end{cases}
\]

Then \( \mu_m = 2 \) for all \( m \), and so \( \mu = 2 \) also. Let \( (u_n, n \geq 1) \) be a sequence of thresholds such that \( n P(\xi_1 > u_n) \rightarrow \tau, 0 < \tau < \infty \).

**Proposition 1** \( \theta = \frac{1}{2} \).

**Proof.** By Theorem 3.1 of Rootzén (1987), for \( \delta > 0 \) and \( \delta' = \delta' + 1/n \) we have

\[
|P(M_n < x) - P(\xi_1 < x)^{n/\mu}| \leq 2\delta' + P(|\nu_n/n - 1/2| > \delta) \tag{4}
\]

where \( \nu_n \) is the number of regenerations up to time \( n \). By choosing \( \delta = \delta_n > 0 \),
appropriately and setting \( x = u_n \), we deduce

\[
P(M_n < x) \to e^{-\tau/2}
\]
as required.

**Proposition 2.** The exceedance point process \( N_n \) converges to a simple Poisson process of intensity \( \frac{1}{2} \), as \( n \to \infty \).

**Proof.** This follows from Theorem 4.2 of Hsing, Hüsler and Leadbetter (1988). Positive recurrent Harris chains are strong mixing (cf. O'Brien 1987), and hence satisfy the mixing condition \( \Delta(u_n) \) used by Hsing et al.

For a suitable sequence \( (r_n) \) satisfying \( r_n \to \infty, r_n/n \to 0 \), let

\[
\pi_n(j) = P(\bigcap_{i=1}^{x_n} X_{n,i} = j \mid \bigcap_{i=1}^{x_n} X_{n,i} > 0), \ j=1,2,\ldots
\]

where \( X_{n,i} \) is 1 if \( \xi_i > u_n \), 0 otherwise. Let \( \pi(j) = \lim_{n \to \infty} \pi_n(j) \) for \( j=1,2,\ldots \)

The theorem of Hsing, Hüsler and Leadbetter asserts that the point process \( N_n \) converges to a compound Poisson process with compounding distribution \( \pi(\cdot) \).

However, under (3) it is easy to see that \( \pi(j) = 1 \) for \( j=1 \), 0 for \( j>1 \). Hence the limiting process in this case is simple Poisson, with a mean cluster size of 1. This completes our description of the example.

From a statistical point of view, the most natural way to estimate the extremal index is via the point process of high-level exceedances. Such a procedure was in fact proposed by Smith (1984). The example here reveals a possible fallacy in this procedure, though it may not be possible to do much about it in practice.
1. What is going wrong is the lack of tightness of the sequence $(q_{mi}, i>1)$ as $m \to \infty$. In a similar way it is possible to construct an example of the same phenomenon for any $\theta<1$. Hsing et al. show that the extremal index is in general given by $\theta = \lim_{n} \left( \sum_{j} \pi_{n}(j) \right)^{-1}$, so what is at issue is whether

$$\sum_{j} \pi_{n}(j) \to \sum_{j} \pi(j) \text{ as } n \to \infty. \quad (6)$$

This is false for the example considered here.

2. It is possible to exhibit other kinds of extremal behaviour by taking other sequences $(q_{mi})$. For instance, if $\mu_{m} \to \infty$ we very easily obtain an example of a process with extremal index 0. Taking this one step further, if $q_{mi} \to 0$ as $m \to \infty$ for each $i$ but the distribution $(q_{mi}, i>1)$ converges under some renormalisation to the distribution of a continuous random variable as $m \to \infty$ (example: take $q_{mi} = 1/m$ for $i=1,2,\ldots,m$), then the point process $N_{n}$ does not converge but a suitably renormalised sequence converges to a compound Poisson point process with continuous compounding distribution. Such behaviour is admitted in the general theory of Hsing, Hüsler and Leadbetter, but they do not give any examples.

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