Naval Research Laboratory

Washington, DC 20375-5000

(2)

NRL Memorandum Report 6046

DTIC FILE COPY

AD-A198 785

Bearings Only Air-to-Air Ranging

ROBERT RUBIN

Sachs/Freeman Associates, Inc.

JOSEPH V. MICHALOWICZ

Advanced Concepts Branch
Optical Sciences Division

DTIC
ELECTE
SEP 23 1988
S D
D

July 25, 1988

Approved for public release; distribution unlimited.

88 9 22 010

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimi. .d.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NRL Memorandum Report 6046	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Research Laboratory 6b. OFFICE SYMBOL (If applicable) 6520	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) Washington, DC 20375-5000	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Office of Naval Technology 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code) Arlington, VA 22217-5000	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO.: 62111N PROJECT NO.: RL1A/01 TASK NO.: RA11W53 WORK UNIT ACCESSION NO.:

11. TITLE (Include Security Classification)

Bearings Only Air-to-Air Ranging

12. PERSONAL AUTHOR(S)

Rubin,* Robert and Michalowicz, Joseph V.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
	FROM June 85 TO May 86	1988 July 25	79

16 SUPPLEMENTARY NOTATION

*Sachs/Freeman Associates, Inc.

17 COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Air-to-Air Ranging; Range Estimation; Minimal Estimators; Passive Ranging; Target Bearings; IRST; Infrared Detection

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Passive infrared search and track (IRST) devices do not provide immediate estimates of a target's location, range and velocity. Rather, information on the target takes the form of collections of bearings of the target relative to the observer, who himself is in motion. Target parameters, such as location and velocity, are then derived as functions of the bearings.

This report presents results of a study of the performance of various air-to-air passive ranging techniques. Attention is concentrated on the cases of a fixed target and a constant velocity target. Range estimation methods are investigated which are based on least squares fits using all observed bearings, least squares fits using azimuths only, and on minimal estimates using only a selected small number of bearings.

Simulations are run to assess the performance of the various methods, with bearings taken at regular intervals and assumed independent Gaussian errors in azimuth and altitude measurements. Keywords:)

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL Joseph V. Michalowicz	22b TELEPHONE (Include Area Code) (202) 767-3045 22c. OFFICE SYMBOL 6520

DD Form 1473, JUN 86 Previous editions are obsolete. SECURITY CLASSIFICATION OF THIS PAGE

S/N 0102-LF-014-6603

CONTENTS

DTIC QUALITY INSPECTED 1

Accesion For
NTIS CRA&I ☑
DTIC TAB ☐
Unannounced ☐
Justification

By
Distribution /

Availability Codes

Dist | Avail and / or Special
A-1

BEARINGS ONLY AIR-TO-AIR RANGING

1. INTRODUCTION

Airborne passive infrared search and track (IRST) devices, once they have detected and started to acquire the track of a point target, do not provide immediate estimates of the target's location, range and velocity. It is crucial, of course, to obtain such estimates. As a result, it is desirable to have effective ranging algorithms as a component of a passive IR detection system.

It is in the nature of passive IR detection from an airborne platform that information regarding the target takes the form of collections of bearings (azimuth and altitude angles) of the target relative to the observer, who himself is in motion. Target parameters, such as location and velocity, are then derived from non-linear functions (typically as ratios of sums of trigonometric functions) of the bearings.

Passive ranging has not yet received the attention given to the problem of target detection. As a result, the literature on the subject is relatively sparse, and the publications that do exist [1-5] deal mainly with the simpler two-dimensional problem of surface-to-surface passive ranging. Furthermore, the focus of each of the published papers has been on the presentation of a particular methodology, with only a few examples of its performance and no real attempt to relate performance to the parameters of observer and target motion and sensor characteristics.

The purpose of this report is to present the results of a study of the performance of a variety of air-to-air passive ranging techniques, particularly with regard to the behavior of the mean and standard deviation of range estimates as target location and motion, observer motion, sensor resolution

Manuscript approved January 21, 1988.

and length of track are varied. Attention has been concentrated on the cases of a fixed target and of a target moving at constant velocity, although the computer programs developed to simulate and analyze these cases can be used with any specified type of observer and target motion. The methods selected involve least squares fits using all observed bearings, least squares fits using observed azimuths only (effectively reducing the problem to two dimensions), and combining "minimal estimates," which are closed-form solutions for a target that produce a selected small number of bearings. Details of each method are presented in the appropriate sections of the sequel.

In this report, the process of obtaining estimates of the range and, where appropriate, velocity of a target by a moving observer is simulated. It is assumed that bearings are obtained at regular intervals (taken to be 1 second in nearly all cases considered) and that errors consist of independent Gaussian errors in azimuth and altitude measurements with zero mean and standard deviation of 0.333 mrad. For fixed targets, accurate estimates of target location (to within a few percent of the true values) can be obtained within about 30 seconds (assuming one set of bearings is obtained each second) for nearly any combination of target location and observer motion. If the target is initially well off-axis, such accuracy can be obtained in considerably less time, while if the target is directly in front of the observer when first detected, more time will be needed for a good estimate. A sound strategy then is for the observer, having detected the target, to accelerate in order to put the target in an off-axis position as quickly as possible. Furthermore, range estimates can be obtained in a computationally simple manner by using the sequence of azimuths to obtain successive estimates of the target's x and y coordinates and a single altitude angle to provide an estimate of the z component. Moving targets commonly require some 60 seconds for good estimates of target location and velocity; this figure is quite sensitive to the specifics of the target's initial position and the relative motion between observer and target. Unlike the fixed target case, where a good strategy for the observer can be determined a priori, highly effective maneuvers for the observer in the case of a moving target can only be determined after a reasonable estimate of the target's velocity has been obtained. Moreover, it appears that least squares estimation of the location of a moving target will, for certain combinations of observer and target motions, consistently produce estimates based on 40 to 60 sets of bearings that are far from the true values. In other cases, estimates cluster, in proportions that depend on target and observer motion parameters, about two or more widely separated values, one of which may be the actual range of the target. Further study is required for a more complete understanding of this phenomenon.

It should be pointed out that the most commonly cited method for performing passive ranging has been the application of an extended Kalman filter to the sequence of obtained bearings. While this method has the advantage of producing estimates recursively, and therefore (perhaps) more rapidly, it has the disadvantages of requiring linearization of the problem, which could entail a loss of accuracy in the estimator, and of requiring initial estimates (actually no more than guesses) of target parameters and estimation error statistics, which makes estimates depend on other quantities than the observed bearings. See [3] for more on this aspect of the Kalman filter. Instead, non-linear least-squares estimation, which in the linear case is equivalent to Kalman filtering, is used to obtain the desired estimates in this report.

2. BACKGROUND

A point target, located at $T = (x, y, z)$, has bearings θ and ϕ relative to an observer at the origin, where

$$\tan\theta = y/x, \quad \sin\phi = z/r, \quad r^2 = x^2 + y^2 + z^2.$$

Bearings-only ranging methodology requires a moving observer, an appropriate assumption for an air-to-air scenario. The coordinate system used will be chosen so that at the outset of the ranging procedure, when the target is first detected (i.e., at time $t = 0$), the observer is considered to have an initial velocity v_0 in the $+x$ direction.

In bearings-only ranging the observer obtains a sequence of sets of measurements of the target's bearings, and from this sequence, together with knowledge of his own trajectory, attempts to determine the relevant target motion parameters, the most important of which are the target's initial location, range and velocity. A single set of bearings is not sufficient to estimate any of these features of the target's motion, but it will provide information about the line of sight between the observer and the target at the time the bearings were obtained. Additional sets of bearings then allow estimation of target motion parameters through what are effectively triangulation procedures; i.e., for a fixed target, intersections of lines of sight are determined.

This report will be concerned with observed (measured) bearings, which are estimates (corrupted values) of the target's true bearings. Estimates are, of course, realizations of random variables referred to as estimators. To be consistent with statistical usage, estimators will be denoted by upper case symbols, whereas their estimates will be written using the corresponding lower case characters. For example estimators of θ, the target's azimuth, and ϕ, its elevation, may be denoted by H and P respectively; corresponding estimates will be designated h and p. In general, true values of quantities derived from θ and ϕ will be denoted by lower case Greek letters, estimators by upper case Roman letters, and measured values by corresponding lower case Roman symbols. Frequently, bearings will be identified by subscripts or by functional dependence ($\theta(t)$, $\phi(t)$). In such cases, H, P, and other estimators and all estimates will be similarly modified. If the true bearings are θ and ϕ, then the measured bearings, h and p, are assumed to be realizations of independent Gaussian random variables H and P with a common variance; i.e. H $\sim$ N (θ,σ^2) and P $\sim$ N (ϕ,σ^2). The purpose of this report is to specify the statistical behavior of various bearings-only ranging techniques in terms of observer motion, target motion and measurement errors.

3. LINE OF SIGHT STATISTICS

Although a single line of sight does not specify anything about the range of the target, it does provide an estimate of the direction in which the target lies. As a result, given that the estimate of the direction is sufficiently accurate, all that is needed to locate the target is an estimate of any of its coordinates. From the point of view of determining statistical behavior, the process of estimating a single coordinate of the target's location is much more amenable to analysis and simple computational procedures than is that of directly estimating its range. Present day infrared sensors provide estimates of lines of sight that are extremely good, so that we may choose to pursue bearings-only ranging as a problem in estimation of a single target location coordinate with negligible loss of accuracy. This is illustrated in Figure 1, where for a fixed "cone" of probability, say p, a single line of sight is seen to yield a small spread compared to the area common to two such cones, which is a region that has probability p^2 of containing the target. Figure 1 also illustrates that the intersection of two lines of sight produces a biased estimate of the target's true position, and that the estimation region becomes more elongated as the target's initial azimuth approachs 0 degrees. Furthermore, while the present topic is referred to as "passive ranging," it is really "passive location," since to know only that a potential target is, say, 35 nautical miles away is to know practically nothing about the position of the target. Fortunately, when line-of-sight errors are small, the issues of range and location estimation are effectively equivalent.

Given the true bearings θ and ϕ of a target relative to an observer at the origin, the true line of sight from the observer to the target is along the unit vector $<\cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi>$, which we will denote by $<\xi_1, \xi_2, \xi_3>$. On the other hand, observed bearings h and p are realizations of Gaussian random variables H and P, with $H \sim N(\theta, \sigma^2)$ and $P \sim N(\theta, \sigma^2)$, and they determine an observed line of sight with unit vector

$$<e_1, e_2, e_3> = <\cos h \cos p, \sin h \cos p, \sin p>,$$

GEOMETRIC ISSUES
in
FIXED TARGET RANGING

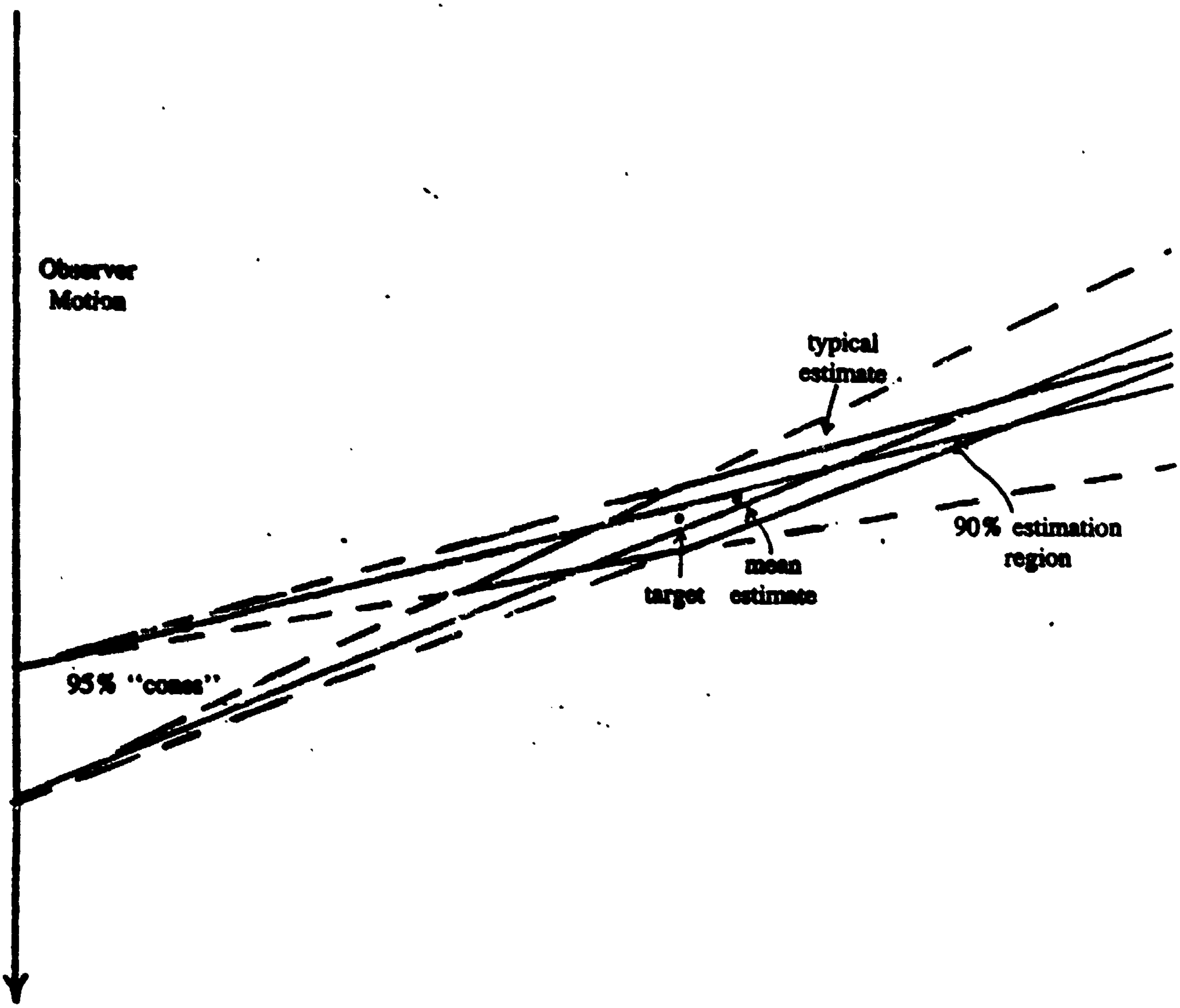


Figure 1

which is in turn a realization of the random vector

$$<E_1, E_2, E_3> = < \cos H \cos P, \sin H \cos P, \sin P >.$$

Furthermore, we may write $H = \theta + U$, and $P = \phi + V$, where both U and V are independent Gaussian random variables with mean 0 and variance σ^2. Thus,

$$E_1 = \cos H \cos P = \cos(\theta + U) \cos(\phi + V)$$

$$= (\cos\theta \cos U - \sin\theta \sin U)(\cos\phi \cos V - \sin\phi \sin V).$$

In practice we expect to have $\sigma << 1$, so that we may use the approximations $\sin U \approx U$ and $\cos U \approx 1 - U^2/2$, and similarly for V. Thus

$$E_1 \approx [(1 - U^2/2)\cos\phi - U \sin\theta]\,[1 - V^2/2)\cos\theta - V\sin\phi].$$

From the independence of U and V the expected value of E_1 is easily obtained:

$$E(E_1) \approx \cos\theta \cos\phi\,(1 - \sigma^2/2)^2 \approx \xi_1\,(1 - \sigma^2).$$

After a bit of algebra we also obtain, ignoring terms higher than σ^2,

$$Var\,(E_1) = E(E_1^2) - (E(E_1))^2 = \sigma^2\,(\cos^2\theta \sin^2\phi + \sin^2\theta \cos^2\phi).$$

Similarly,

$$E_2 = \sin H \cos P \approx [(1 - U^2/2)\sin\theta + U\cos\theta]\,[(1 - V^2/2)\cos\phi - V\sin\phi],$$

from which it follows that

$$E(E_2) \approx \sin\theta\cos\phi\,(1 - \sigma^2/2)^2 \approx \xi_2(1 - \sigma^2), \text{ and}$$

$$Var(E_2) \approx \sigma^2(\cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi).$$

Although, E_3 can always be obtained from the condition that $E_1^2 + E_2^2 + E_3^2 = 1$, we still specify the marginal distribution of $E_3 = \sin P \approx (1 - V^2/2)\sin\phi + V\cos\phi$:

$$E(E_3) \approx \sin\phi(1 - \sigma^2/2) = \xi_3(1 - \sigma^2/2), \text{ and}$$

$$Var(E_3) \approx \sigma^2\cos^2\phi.$$

Note that for certain values, e.g. $\theta = \phi = 0$, or $\theta = \phi = \pi/2$, the above formulas yield variances of 0 in some cases. This is of course due to the omission of high order terms. In reality, for $\theta = \phi = 0$ or $\pi/2$, $Var(E_1) \propto \sigma^4$, for example.

To obtain a more complete description of the behavior of lines of sight, the pairwise behavior of E_1, E_2 and E_3 needs to be examined. Recall that the covariance of two random variables X and Y, denoted Cov (X, Y), is obtained by Cov (X, Y) = E (XY) − E(X)E(Y). Calculations yield:

$$Cov(E_1,E_2) \approx \sigma^2 \cos\theta \sin\theta (1 - 2\cos^2\phi),$$

$$Cov(E_1,E_3) \approx -\sigma^2 \cos\theta \cos\phi \sin\phi, \text{ and}$$

$$Cov(E_2,E_3) \approx -\sigma^2 \sin\theta \cos\phi \sin\phi.$$

The point of the above analysis is that errors in range estimates due to deviations of estimated lines of sight from their true values are on the order of $\sigma^2 R$, where R is the true range of the target. For example, using $\sigma = 0.333$ mrad, then even for $h = \theta + 3\sigma$ and $p = \phi + 3\sigma$, the range and individual coordinate estimates are within 0.1% of their true values. Thus with negligible additional loss of accuracy, range estimates may often be reduced to estimates of a single target location coordinate. Furthermore, with knowledge of the individual and joint statistical behavior of the line of sight components, one could obtain slight improvements on estimates based on lines of sight. (There appears to be little practical advantage in doing so, however.)

4. FIXED TARGET RANGING

In a number of situations it is necessary to estimate the distance to a target which is known (or assumed) to be stationary. Examples are distant small clouds, nearby cloud edges, or objects fixed on the ground.

Three distinct methods will be presented for estimating the range of a fixed target. The reason for using a multiplicity of approaches is that bearings only range estimation is a non-linear estimation problem, and so the investigator is faced with certain trade-offs when selecting the estimator to be used in a particular application. These trade-offs involve bias and efficiency of the estimator, computational requirements and ease of use in recursive updating schemes, and analytic tractability. (In linear estimation problems, such considerations do not arise since least squares estimators are "the" choice according to all of the preceding criteria.)

All of the methods operate on a common set of assumptions. Specifically, N sets of bearings are assumed, resulting in 2N measurements (N azimuths and N altitudes) with which to estimate the three target coordinates. The kth set of bearings in the collection is denoted by $h(k)$ and $p(k)$ and the location (assumed known without error) from which the observer obtained the kth set of bearings by $(x(k), y(k), z(k))$. When it is relevant to the discussion, we will make use of the fact that the bearings are obtained in a sequential manner.

In assessing the performance of an estimator, many different combinations of target location, observer motion and other parameters of the observation process must be measured. This has been done for each of the selected estimators; results are given in the following sections.

4.1 Least Squares Range Estimation

When $N > 1$, the most commonly used technique is that of least squares estimation of the unknown parameters. For the bearings only ranging problem, two different functions suggest themselves as criterion functions whose sum of squares is to be minimized. The first is simply the difference between observed and fitted bearings; i.e. we minimize with respect to x, y and z:

$$S_1 = \sum_k \left[h(k) - \text{Arctan}\left(\frac{y - y(k)}{x - x(k)} \right) \right]^2 + \sum_k \left[p(k) - \text{Arcsin}\left(\frac{z - z(k)}{r(k)} \right) \right]^2 . \quad (1)$$

where $r^2(k) = (x - x(k))^2 + (y - y(k))^2 + (z - z(k))^2$. The second function transforms this difference between angles into a difference between functions of the angles. Specifically, we minimize with respect to x, y and z:

$$S_2 = \sum_k \left[\tan h(k) - \frac{y - y(k)}{x - x(k)}\right]^2 + \sum_k \left[\sin p(k) - \frac{z - z(k)}{r(k)}\right]^2 \tag{2}$$

Simulations have shown that in many cases these functions yield very similar estimates, but in those cases where there is a practical difference, minimizing S_1 more often produces estimates that are closer to the true target coordinates. Consequently, studies of simultaneous least squares estimation of all three target coordinates have been based upon minimizing S_1. A closed form solution of the normal equations for S_1 (i.e. the system of three equations resulting from setting the partial derivatives of S_1 with respect to each of x, y and z equal to zero) does not appear to be readily obtainable, so non-linear least squares estimation software (the SNLSE non-linear least squares estimation package of the MATHLIB subrouting library) was used to produce estimates from simulated sample data.

In order to assess the performance of this least squares estimator, a variety of initial target locations and observer motions of either constant velocity or constant acceleration were selected. For each such target/observer combination the number of bearings taken and the time between successive bearings were varied methodically. With all relevant parameters chosen, simulations were performed by determining the correct bearings, corrupting them by addition of independent samples from a Gaussian N $(0, \sigma^2)$ distribution, and then using the non-linear least squares software to obtain an estimate of the target location. Typically this was repeated 100 times, so that for a selected scenario the statistics (mean, median, standard deviation, third moment, and selected percentiles for each target coordinate as well as range) for a sample of size 100 were determined. Tables A1 and A2 (see Appendix) show the results of two such analyses, while Fig. 2 is a plot of the sample standard deviations as a function of the number of points on the track, for a selected set of values of time between bearings. Examination of the tables and the plot supports the intuitively appealing notion that for a

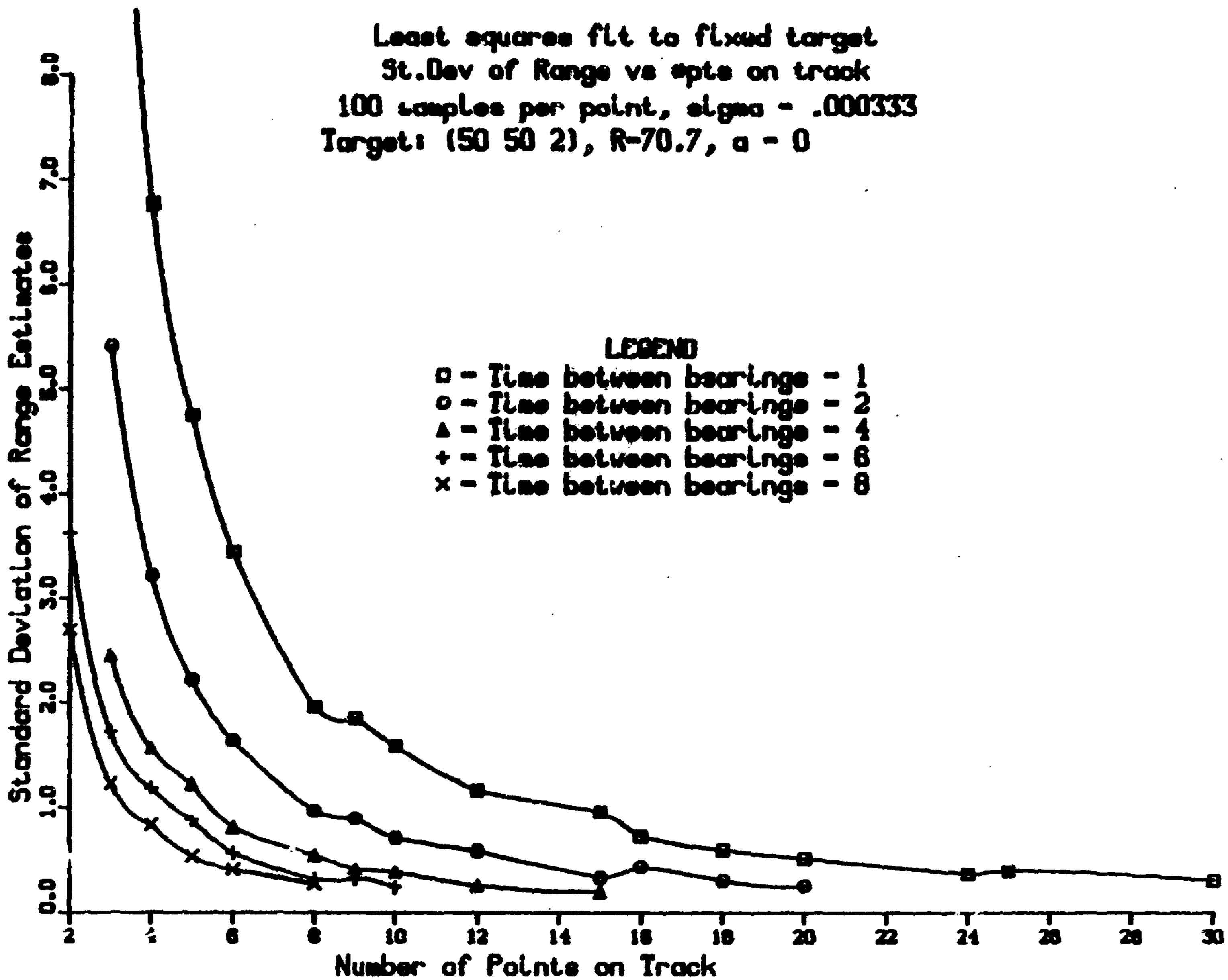

Least squares fit to fixed target
St.Dev of Range vs #pts on track
100 samples per point, sigma = .000333
Target: (50 50 2), R=70.7, a = 0
LEGEND
□ = Time between bearings = 1
○ = Time between bearings = 2
△ = Time between bearings = 4
+ = Time between bearings = 6
× = Time between bearings = 8
Standard Deviation of Range Estimates
Number of Points on Track
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30

Figure 2

given amount of time, the estimator that involves the largest number of bearings provides the best estimate of target location. Consequently, all further analyses were carried out with bearings being obtained at the rate of one set per time unit (assumed to be 1 second). Further examination of Tables A1 and A2 also supports the argument based on the geometry of Figure 1 that range estimates are biased towards overestimating the true range.

Even after the interval between successive bearings has been fixed, a large number of combinations of sensor, observer and target parameters still remain. In order to reduce the number of cases to a manageable one, while preserving the geometric flavor of the relation between observer motion and target position, two particular quantities were kept fixed for most of the analyses. These quantities are the standard deviation, σ, of the measurement errors in the target's azimuth and altitude, which was held at 0.333 mrad, and the initial observer velocity, which was fixed at 0.16 nm/s in the x direction. These values were chosen because 0.333 mrad is a realistic value for errors in modern passive infrared sensors, while 0.16 nm/s represents a typical speed for a patrol aircraft. Two examples of the effects of varying the sensor's resolution are given in Figures 3 and 4. These plots and their accompanying tables (Tables A3 and A4 in the Appendix) imply that, with all other parameters held fixed, the standard deviation of the least squares range estimator varies approximately linearly with the standard deviation of sensor error.

The plots that follow will be of standard deviations of range estimates (or their logarithms) versus number of points on the track, with the latter being equivalent to the length of time the target is being tracked. Standard deviations have been chosen because they are the most common measure by which to express expected variability. Recall that for a Gaussian random variable approximately 68%, 95% and 99.5% of sample values will fall within 1, 2, and 3 standard deviations, respectively, of the mean of the random variable. Although least squares range estimators do not appear to be Gaussian, particularly for short tracks, for sufficiently long tracks (of length at least 20) the above percentages appear to be reasonable estimates. For example, based on Table A2 the statement can be

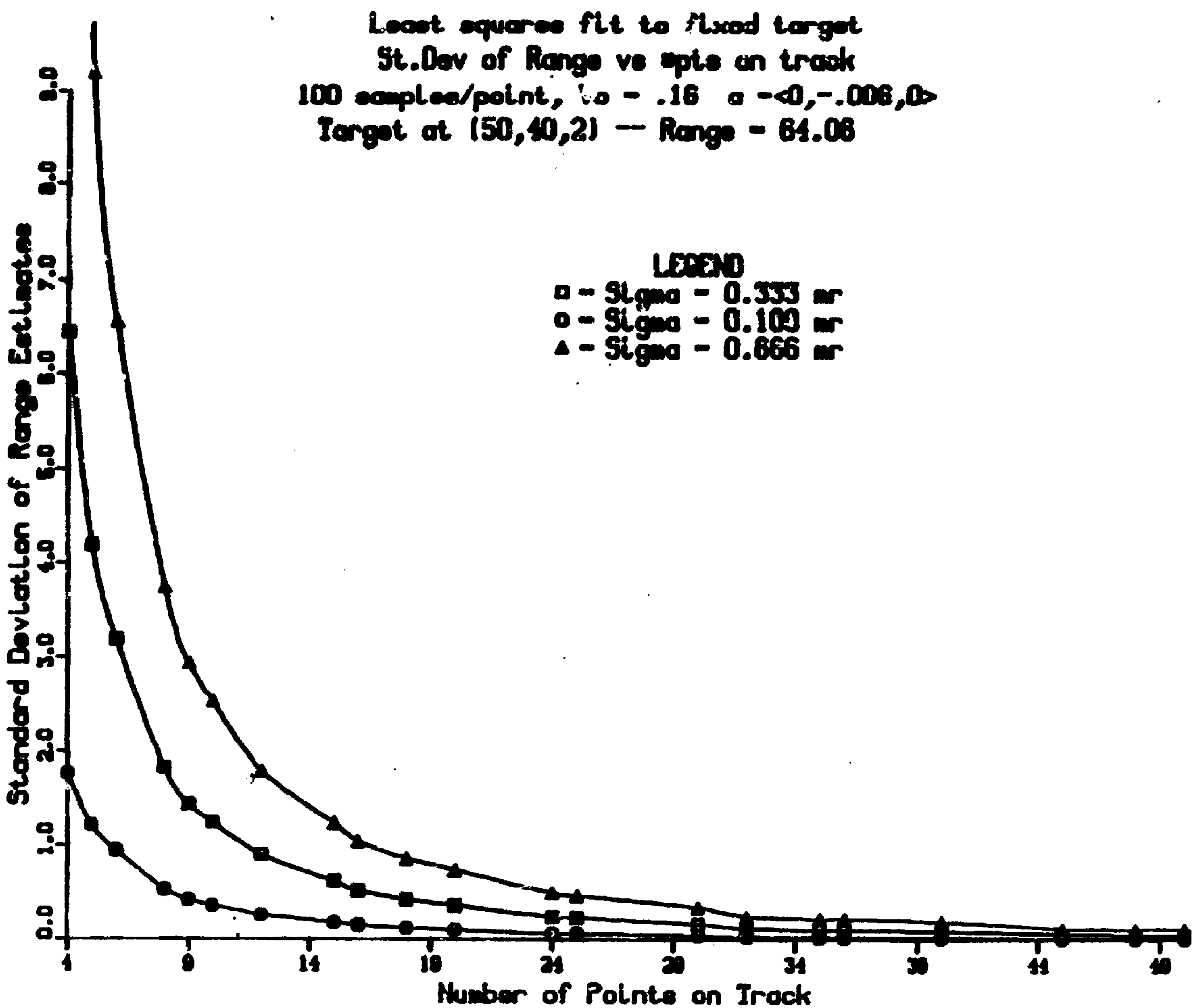

Least squares fit to fixed target
St.Dev of Range vs #pts on track
100 samples/point, Vo = .16 a = <0,-.006,0>
Target at (50,40,2) -- Range = 64.06
LEGEND
□ = Sigma = 0.333 mr
○ = Sigma = 0.100 mr
△ = Sigma = 0.666 mr
Standard Deviation of Range Estimates
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
4
9
14
19
24
29
34
39
44
49
Number of Points on Track

Figure 3

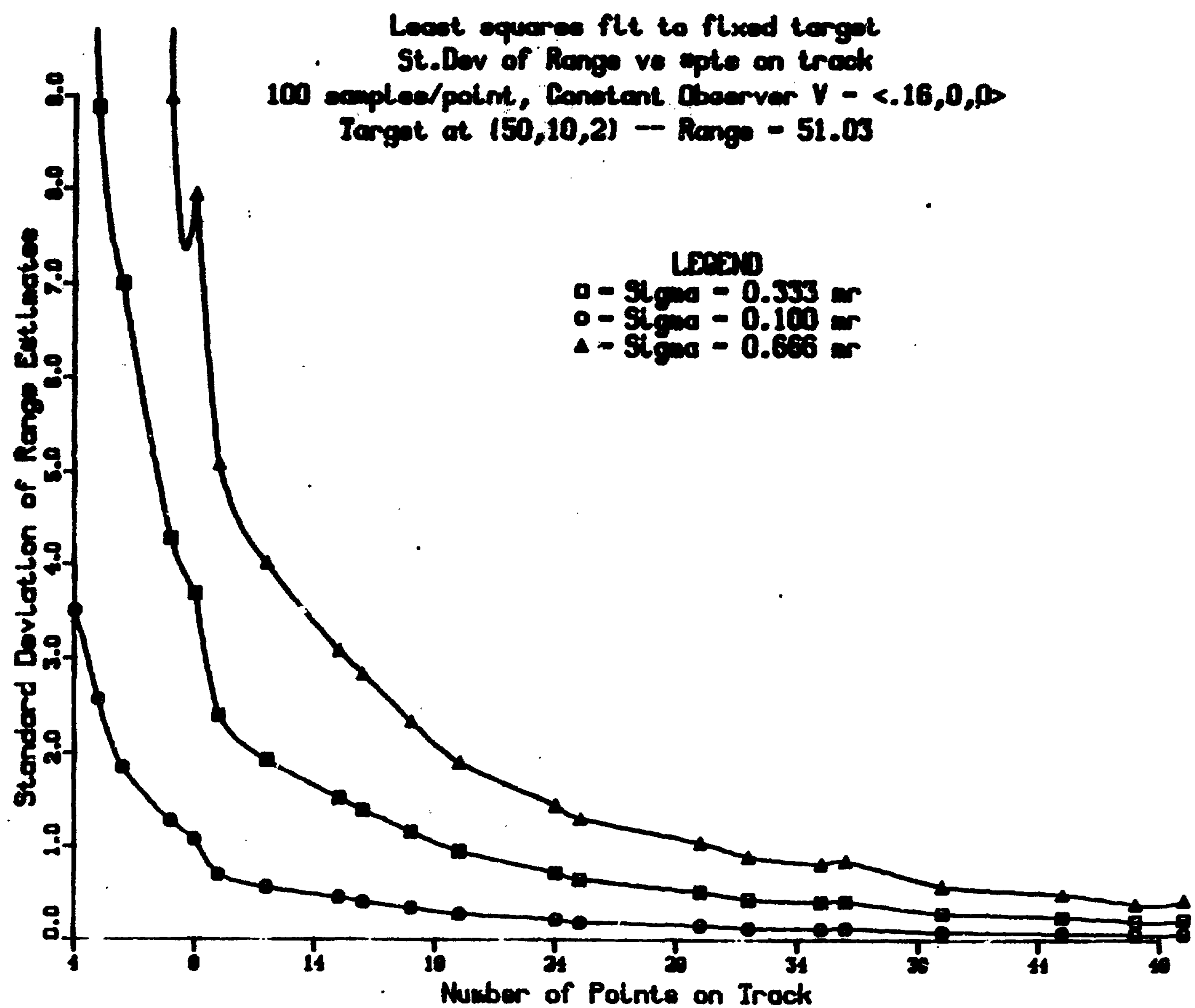

Least squares fit to fixed target
St.Dev of Range vs #pts on track
100 samples/point, Constant Observer V = <.16,0,0>
Target at (50,10,2) -- Range = 51.03
LEGEND
□ = Sigma = 0.333 mr
○ = Sigma = 0.100 mr
△ = Sigma = 0.666 mr
Standard Deviation of Range Estimates
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
Number of Points on Track

Figure 4

made with a high degree of assurance that, at least 90% of the range estimates of a target initially located at the point (50, 40, 2) nm would be within 0.5 nm of the true range of 64.06 nm for an observer traveling with initial velocity <.16, 0, 0> nm/s and constant acceleration <0, -.006, 0> nm/s^2, obtaining bearings every second for 25 seconds. Also, although least squares estimators are biased, the bias has been found to be small compared to expected sampling error, so that nearly all of the information about the reliability of least squares estimators lies in the sample standard deviation.

With the resolution of the sensor and the initial velocity of the observer fixed, the remaining relevant parameters are the target's range, its initial direction relative to the observer, and the motion of the observer. As far as initial target direction and observer motion are concerned, two characteristic cases will be considered for each: (a) targets that are initially "in front of" the observer; i.e. targets which have initial azimuth of approximately 10 degrees or less, and (2) targets that are "well off-axis" — targets with initial azimuth approximately 40 degrees or more. It is tacitly assumed that the sensor is limited to azimuths -45 and 45 degrees, although on occasion this limitation is ignored. The observer was taken to be moving with either constant velocity or constant acceleration. In the latter case accelerations of approximately 1g (.006 nm/sec^2) were used in the simulations. For targets initially in front of the observer, two different constant accelerations were considered, while for off-axis targets only a constant velocity observer and one with constant acceleration in the positive x direction are treated. The reason for the latter restriction is that once the target is well off axis, any appreciable acceleration in any but the x direction will either remove the target from the sensor's field of view or bring the target into a less favorable head-on position.

The effect of range alone is seen in Figures 5 through 9, where for fixed observer motion and target direction we see the effects of ever increasing range. Figures 5 and 6 deal with a target initially well off-axis (initial azimuth = 38.7 degrees) for a constant velocity and accelerating observer respectively, while Figs. 7, 8 and 9 repeat the situation for a head-on target (initial azimuth = 11.3 degrees). Note that these are plots of the logarithms of sample standard deviations against track

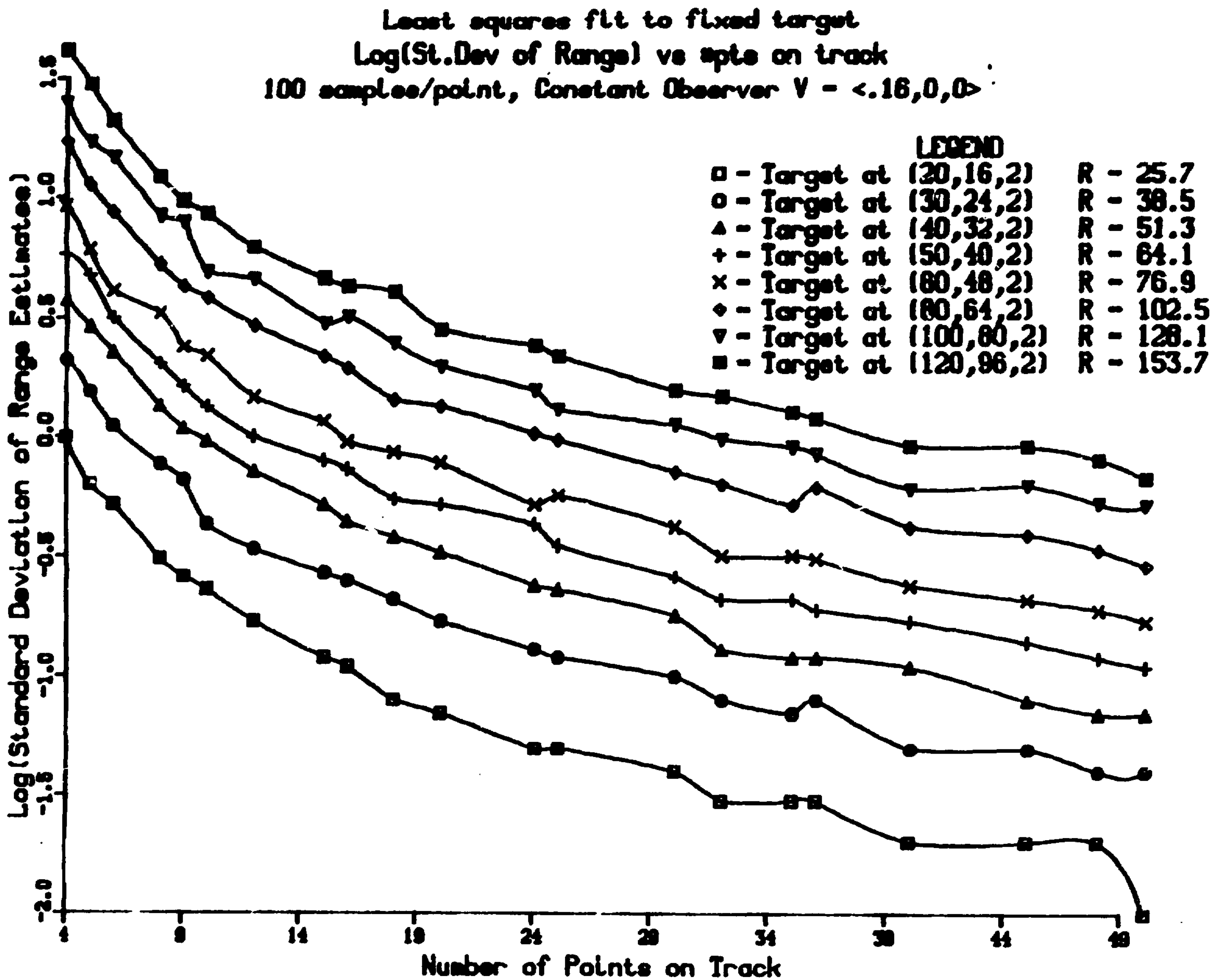
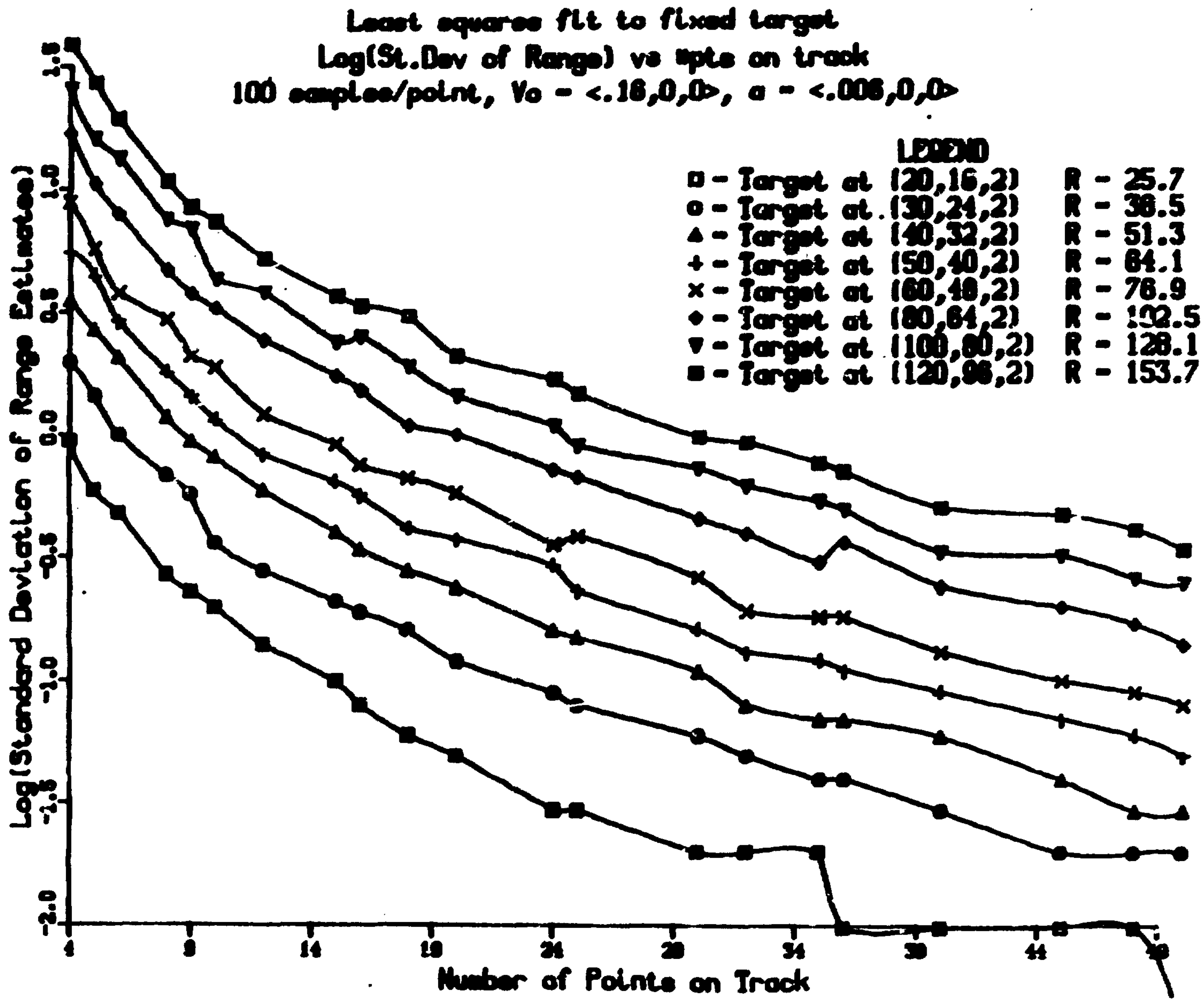

Least squares fit to fixed target
Log(St.Dev of Range) vs #pts on track
100 samples/point, Constant Observer V = <.16,0,0>
LEGEND
□ - Target at (20,16,2) R = 25.7
o - Target at (30,24,2) R = 38.5
▲ - Target at (40,32,2) R = 51.3
+ - Target at (50,40,2) R = 64.1
× - Target at (60,48,2) R = 76.9
◆ - Target at (80,64,2) R = 102.5
▼ - Target at (100,80,2) R = 128.1
■ - Target at (120,96,2) R = 153.7
Log(Standard Deviation of Range Estimates)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
4
9
14
19
24
29
34
39
44
49
Number of Points on Track

Figure 5



Least squares fit to fixed target
Log(St.Dev of Range) vs #pts on track
100 samples/point, Vo - <.16,0,0>, a - <.006,0,0>
LEGEND
□ - Target at (20,16,2) R - 25.7
○ - Target at (30,24,2) R - 38.5
▲ - Target at (40,32,2) R - 51.3
+ - Target at (50,40,2) R - 64.1
× - Target at (60,48,2) R - 76.9
◆ - Target at (80,64,2) R - 102.5
▼ - Target at (100,80,2) R - 128.1
■ - Target at (120,96,2) R - 153.7
Log(Standard Deviation of Range Estimates)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
Number of Points on Track
4
9
14
19
24
29
34
39
44
49

Figure 6

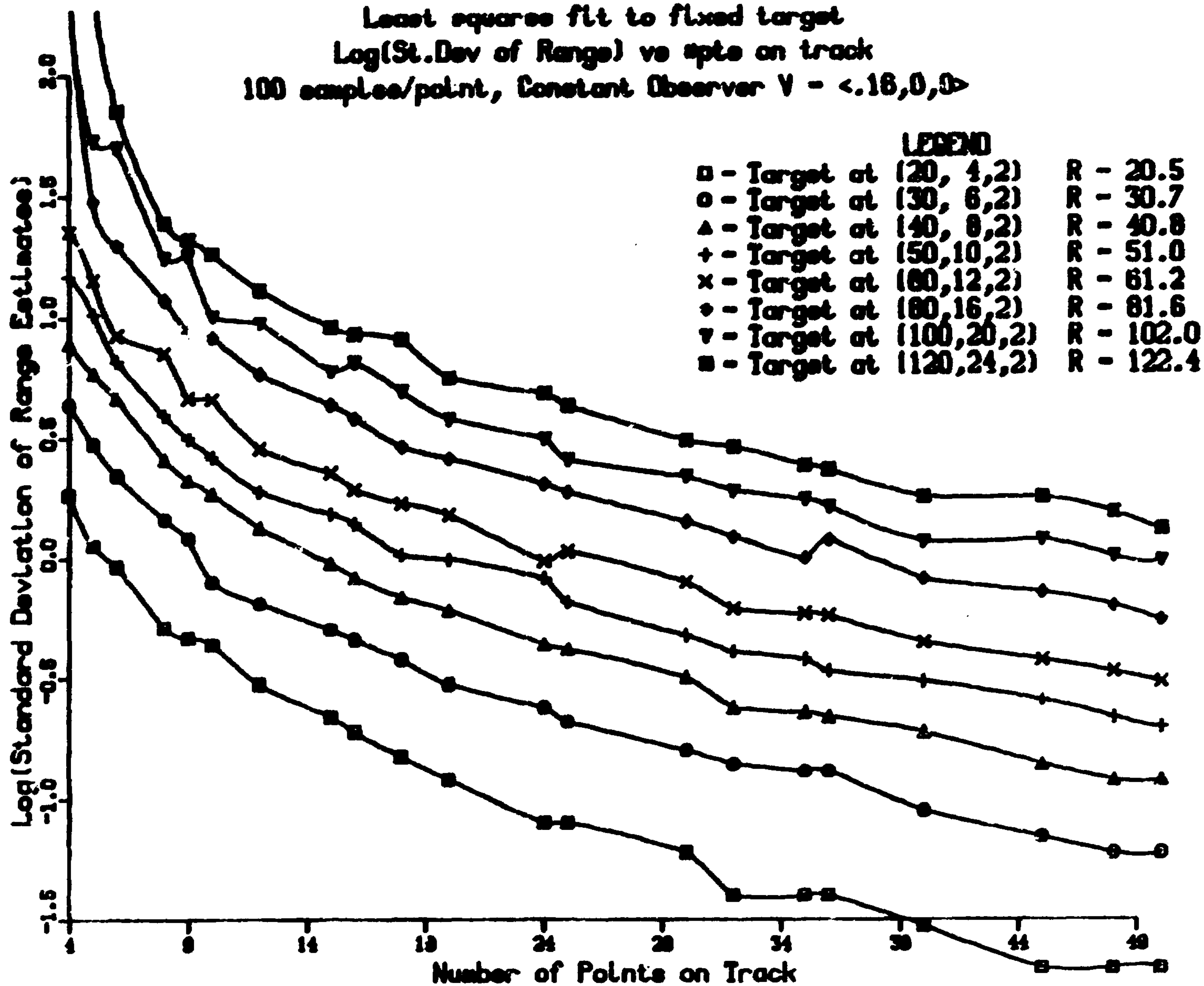

Least squares fit to fixed target
Log(St.Dev of Range) vs #pts on track
100 samples/point, Constant Observer V = <.16,0,0>
LEGEND
□ - Target at (20, 4,2) R - 20.5
o - Target at (30, 6,2) R - 30.7
▲ - Target at (40, 8,2) R - 40.8
+ - Target at (50,10,2) R - 51.0
× - Target at (60,12,2) R - 61.2
◆ - Target at (80,16,2) R - 81.6
▼ - Target at (100,20,2) R - 102.0
■ - Target at (120,24,2) R - 122.4
Log(Standard Deviation of Range Estimates)
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
Number of Points on Track
4
14
24
34
44
49

Figure 7

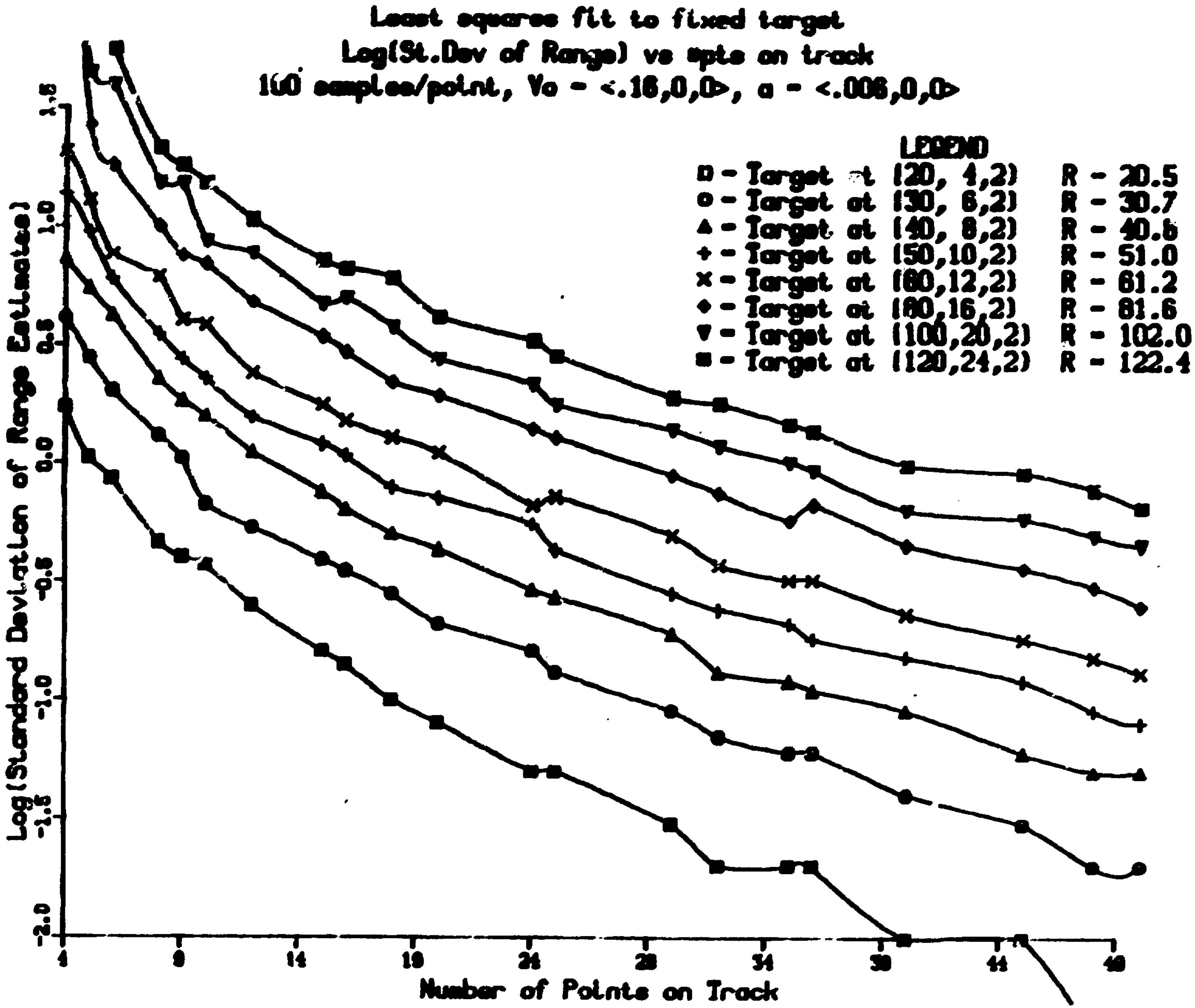
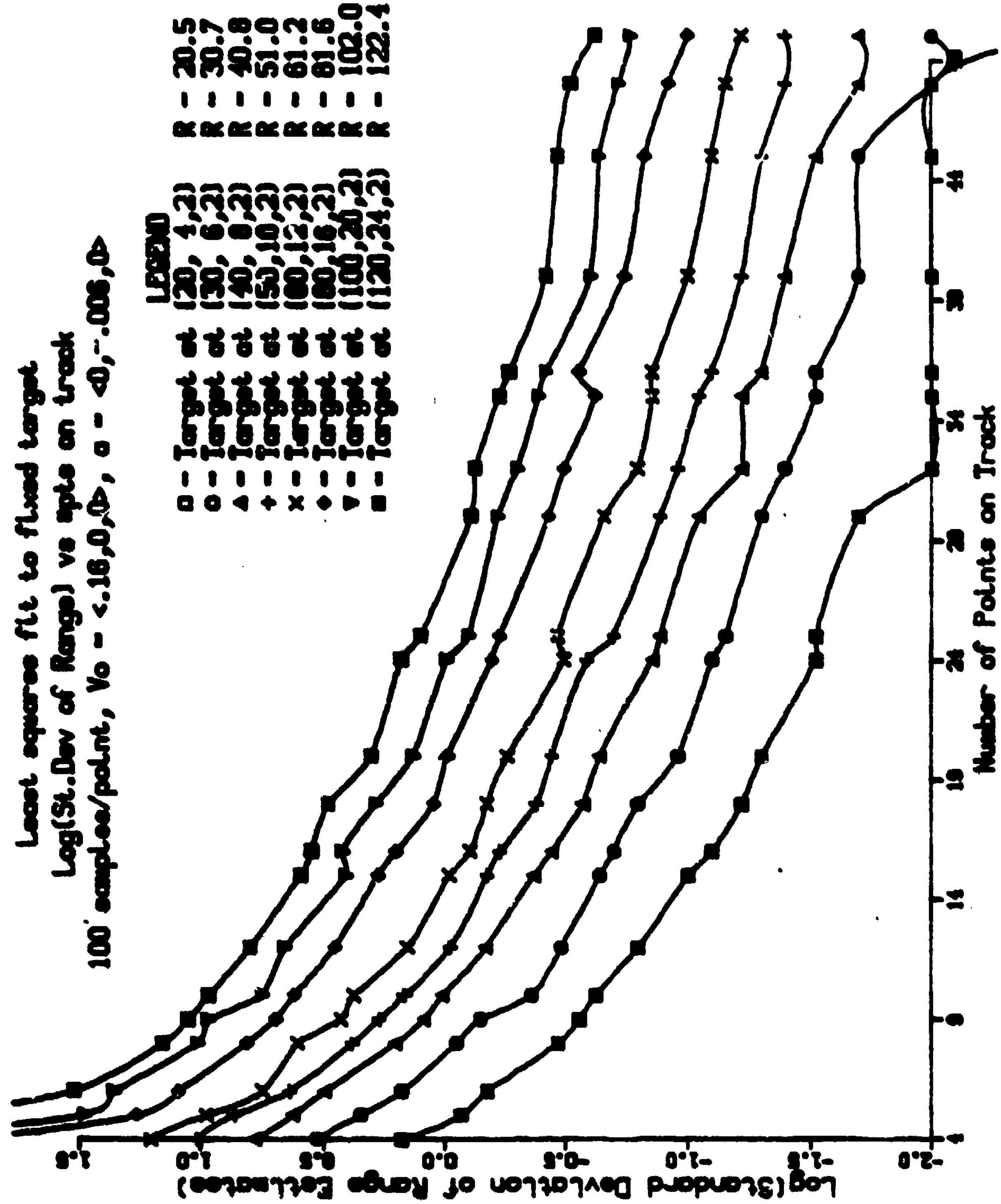

Least squares fit to fixed target
Log(St.Dev of Range) vs npts on track
100 samples/point, Vo = <.16,0,0>, a = <.006,0,0>
LEGEND
□ = Target at (20, 4,2) R = 20.5
○ = Target at (30, 6,2) R = 30.7
△ = Target at (40, 8,2) R = 40.6
+ = Target at (50,10,2) R = 51.0
× = Target at (60,12,2) R = 61.2
◆ = Target at (80,16,2) R = 81.6
▼ = Target at (100,20,2) R = 102.0
■ = Target at (120,24,2) R = 122.4
Log(Standard Deviation of Range Estimates)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
4
14
24
34
44
Number of Points on Track

Figure 8



Least squares fit to fixed target
Log(St.Dev of Range) vs npts on track
100 samples/point, Vo = <.16,0,0>, a = <0,-.006,0>
LEGEND
□ = Target at (20, 4,2) R = 20.5
○ = Target at (30, 6,2) R = 30.7
△ = Target at (40, 8,2) R = 40.8
+ = Target at (50,10,2) R = 51.0
× = Target at (60,12,2) R = 61.2
◆ = Target at (80,16,2) R = 81.6
▽ = Target at (100,20,2) R = 102.0
■ = Target at (120,24,2) R = 122.4
Log(Standard Deviation of Range Estimates)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
4
9
14
19
24
29
34
39
44
Number of Points on Track

Figure 9

length. Logarithm plots were chosen in order to fit a wide range of values on a single plot, but incidentally the plots show that from about 20 seconds on there is a roughly log-linear relation between track length and sample standard deviation. This suggests that after approximately 20 seconds, estimation error falls off approximately exponentially with length of track. Examination of the tables for these graphs (Tables A7 through A11) indicates that, for any of the plots, doubling the target range roughly multiplies the sample standard deviation by four. This suggests that the standard deviation of the least squares range estimator varies as the square of the range, independent of track length, target direction and observer motion.

The preceding five graphs were intended to show the effects of range on accuracy of the estimator. The next two graphs indicate the effect of initial target direction on the estimate. Figures 10 and 11 (also Tables A12, A13) present the errors in estimating the range of a target initially 40 nm away from the observer for initial azimuths of 45°, 40°, 30°, 20°, 15°, 10° and 5° for two different observer motions. Note that for an accelerating observer, the effect of initial target direction essentially disappears after 25 seconds. Figure 12 (also Tables A14, A15) shows the results of varying the direction of a target whose initial x coordinate is fixed, in this instance at 50 nm. Note that for short tracks, target direction effects dominate; i.e. for up to 15 seconds estimates of the target at (50, 10, 2) (Range = 51 nm) are less reliable than those of the target at (50, 40, 2) (Range = 64 nm), despite the fact that in the latter case the target is 13 nm further away. For sufficiently long tracks, however, range plays the dominant role. Figures 13 and 14 are close-up looks at the curves in Figure 12.

The final parameter is observer motion. Figures 15 through 17 (where Fig. 16 is a closeup of the right end of Fig. 15) show the effects of varying observer acceleration for on- and off-axis targets. Note that in the short run very unreliable estimates can arise from an observer accelerating in the "wrong" direction, while over a longer period of time acceleration in any direction produces more accurate range estimates than does remaining at the same velocity. This of course is the result

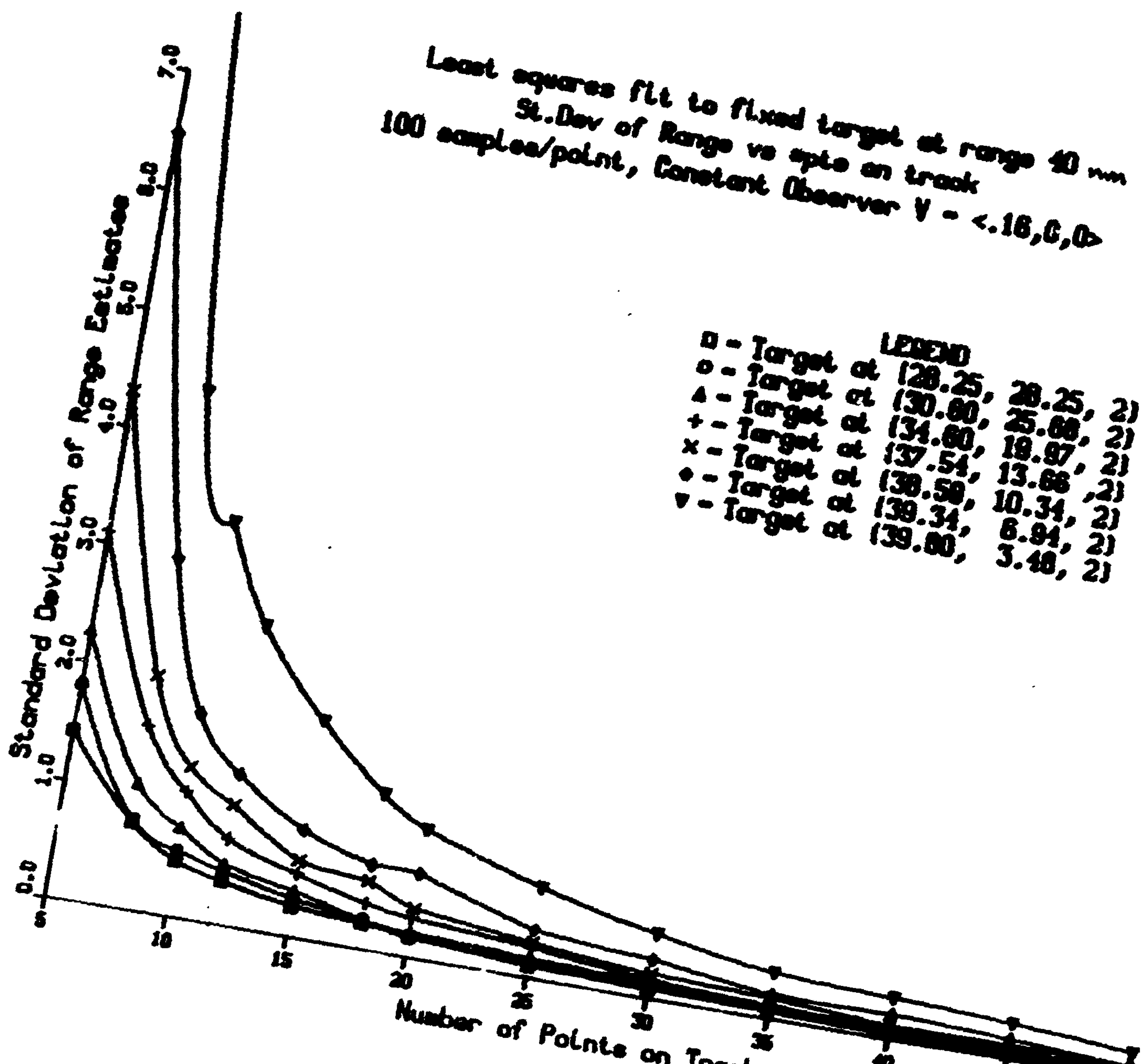

Least squares fit to fixed target at range 40 nm
St.Dev of Range vs spts on track
100 samples/point, Constant Observer V = <.16,0,0>
LEGEND
□ - Target at (28.25, 28.25, 2)
o - Target at (30.80, 25.68, 2)
Δ - Target at (34.80, 19.97, 2)
+ - Target at (37.54, 13.66 ,2)
x - Target at (38.58, 10.34, 2)
◆ - Target at (39.34, 6.94, 2)
▽ - Target at (39.80, 3.48, 2)
Standard Deviation of Range Estimates
Number of Points on Track

Figure 10

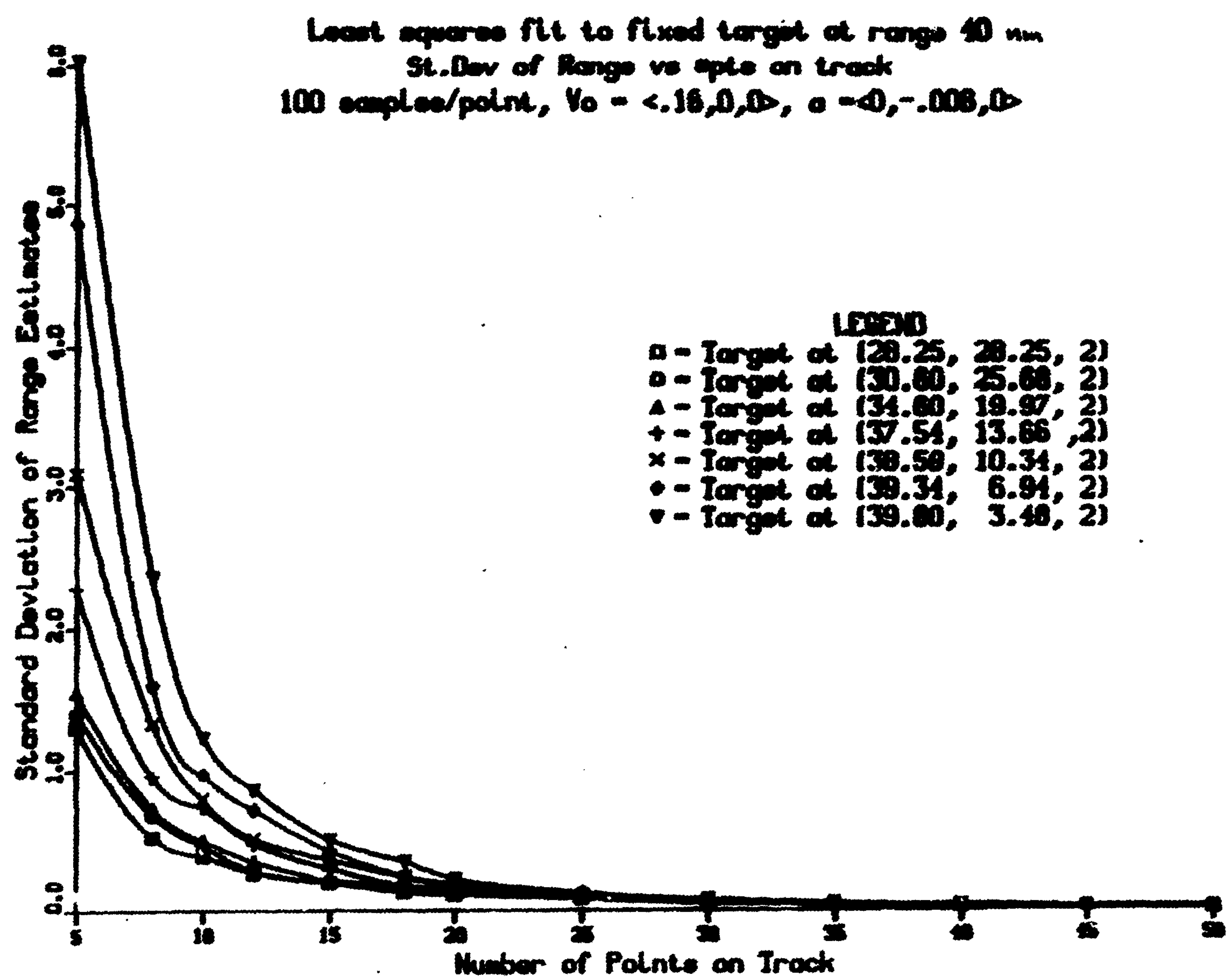
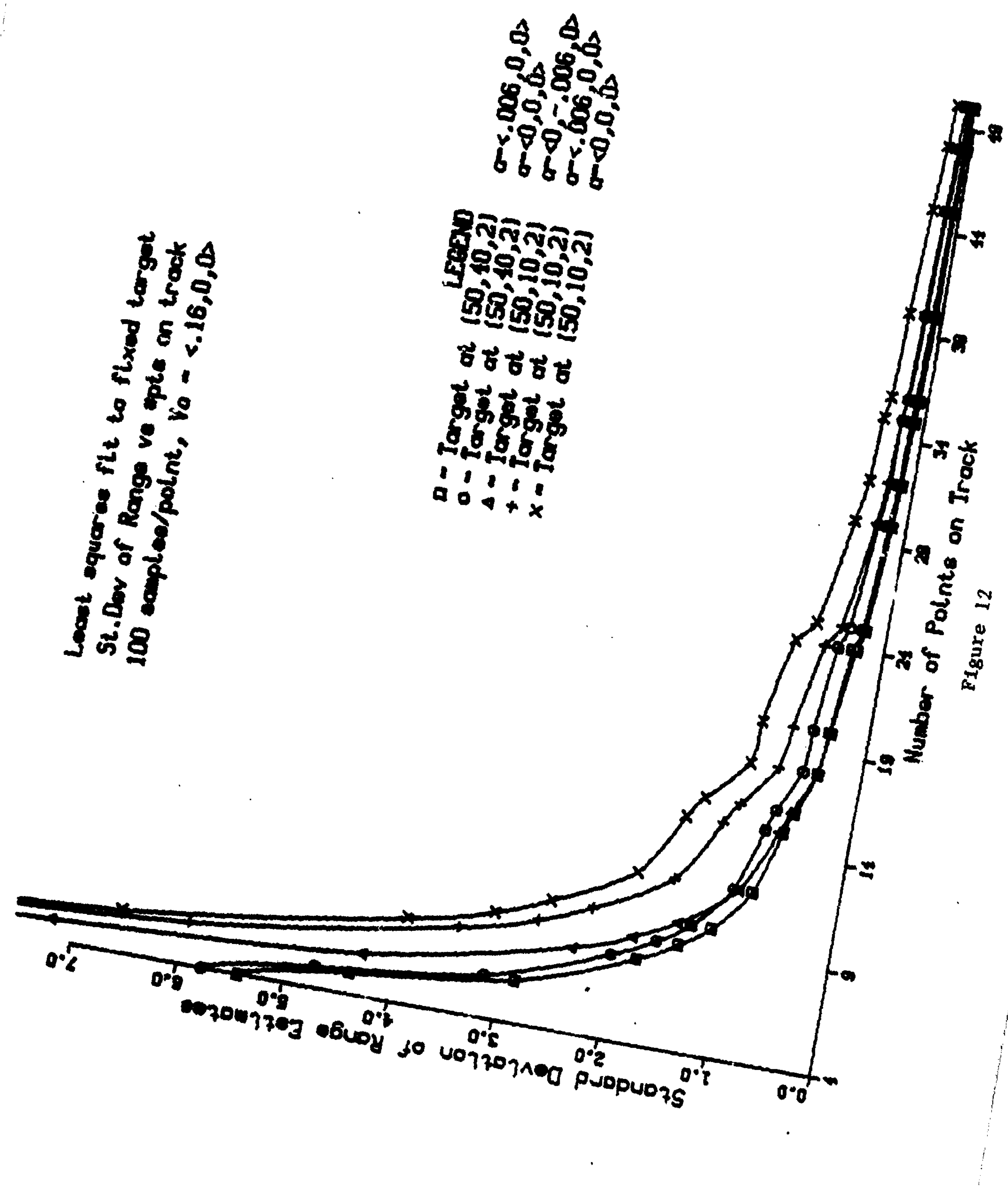

Least squares fit to fixed target at range 40 nm
St.Dev of Range vs #pts on track
100 samples/point, Vo = <.16,0,0>, a =<0,-.008,0>
LEGEND
□ - Target at (28.25, 28.25, 2)
○ - Target at (30.60, 25.68, 2)
▲ - Target at (34.60, 19.97, 2)
+ - Target at (37.54, 13.66 ,2)
× - Target at (38.58, 10.34, 2)
◆ - Target at (39.34, 6.94, 2)
▼ - Target at (39.80, 3.48, 2)
Standard Deviation of Range Estimates
0.0
1.0
2.0
3.0
4.0
5.0
6.0
5
10
15
20
25
30
35
40
45
50
Number of Points on Track

Figure 11



Least squares fit to fixed target
St.Dev of Range vs #pts on track
100 samples/point, Vo = <.16,0,0>
LEGEND
□ = Target at (50,40,2) σ=<.006,0,0>
○ = Target at (50,40,2) σ=<0,0,0>
△ = Target at (50,10,2) σ=<0,-.006,0>
+ = Target at (50,10,2) σ=<.006,0,0>
X = Target at (50,10,2) σ=<0,0,0>
Standard Deviation of Range Estimates
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
Number of Points on Track

Figure 12

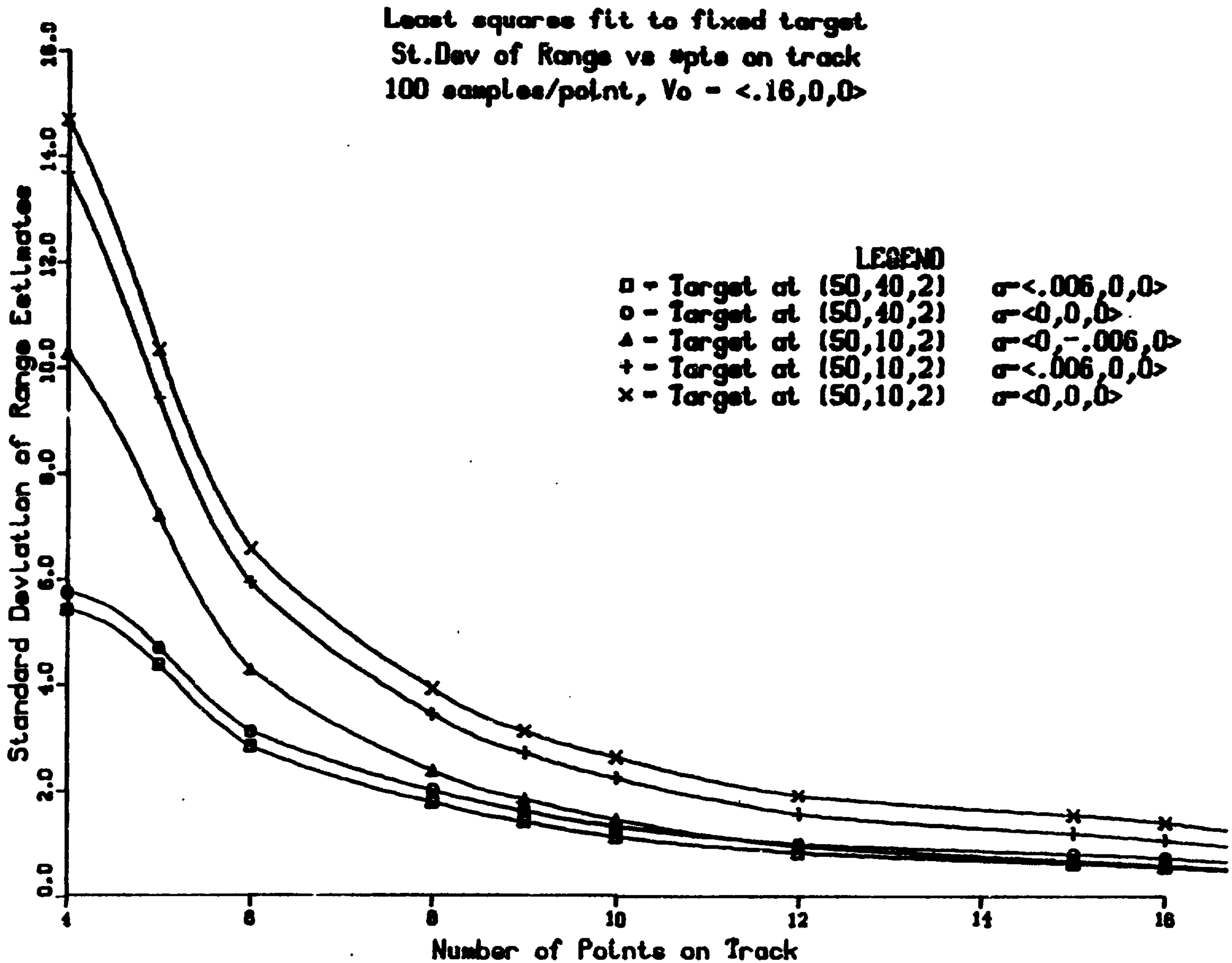

Least squares fit to fixed target
St.Dev of Range vs #pts on track
100 samples/point, Vo = <.16,0,0>
LEGEND
□ - Target at (50,40,2) σ=<.006,0,0>
○ - Target at (50,40,2) σ=<0,0,0>
▲ - Target at (50,10,2) σ=<0,-.006,0>
+ - Target at (50,10,2) σ=<.006,0,0>
× - Target at (50,10,2) σ=<0,0,0>
Standard Deviation of Range Estimates
Number of Points on Track
0.0
2.0
4.0
6.0
8.0
10.0
12.0
14.0
16.0
4
6
8
10
12
14
16

Figure 13

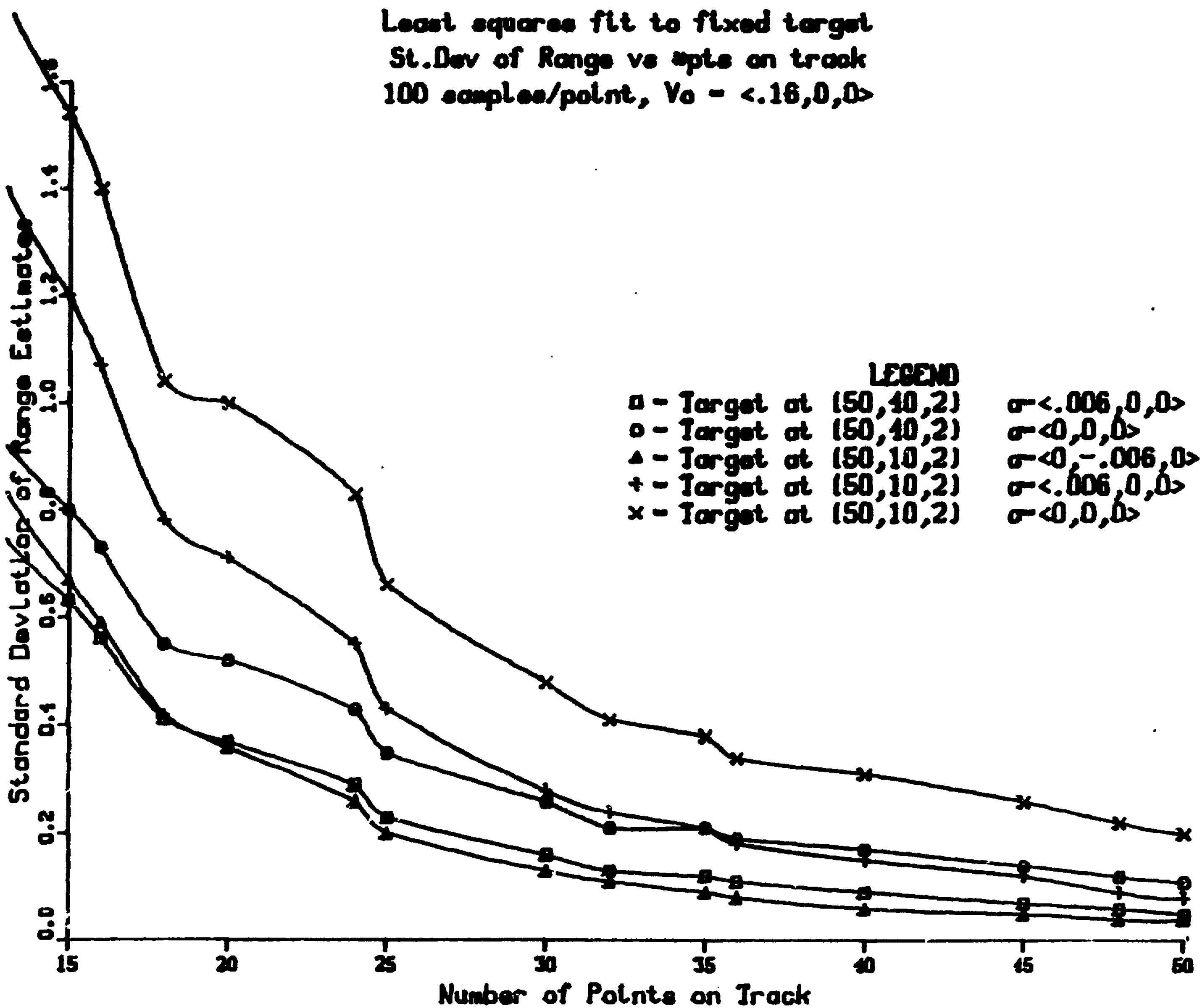

Least squares fit to fixed target
St.Dev of Range vs #pts on track
100 samples/point, Vo = <.16,0,0>
LEGEND
□ = Target at (50,40,2) σ=<.006,0,0>
○ = Target at (50,40,2) σ=<0,0,0>
▲ = Target at (50,10,2) σ=<0,-.006,0>
+ = Target at (50,10,2) σ=<.006,0,0>
× = Target at (50,10,2) σ=<0,0,0>
Standard Deviation of Range Estimates
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
15
20
25
30
35
40
45
50
Number of Points on Track

Figure 14

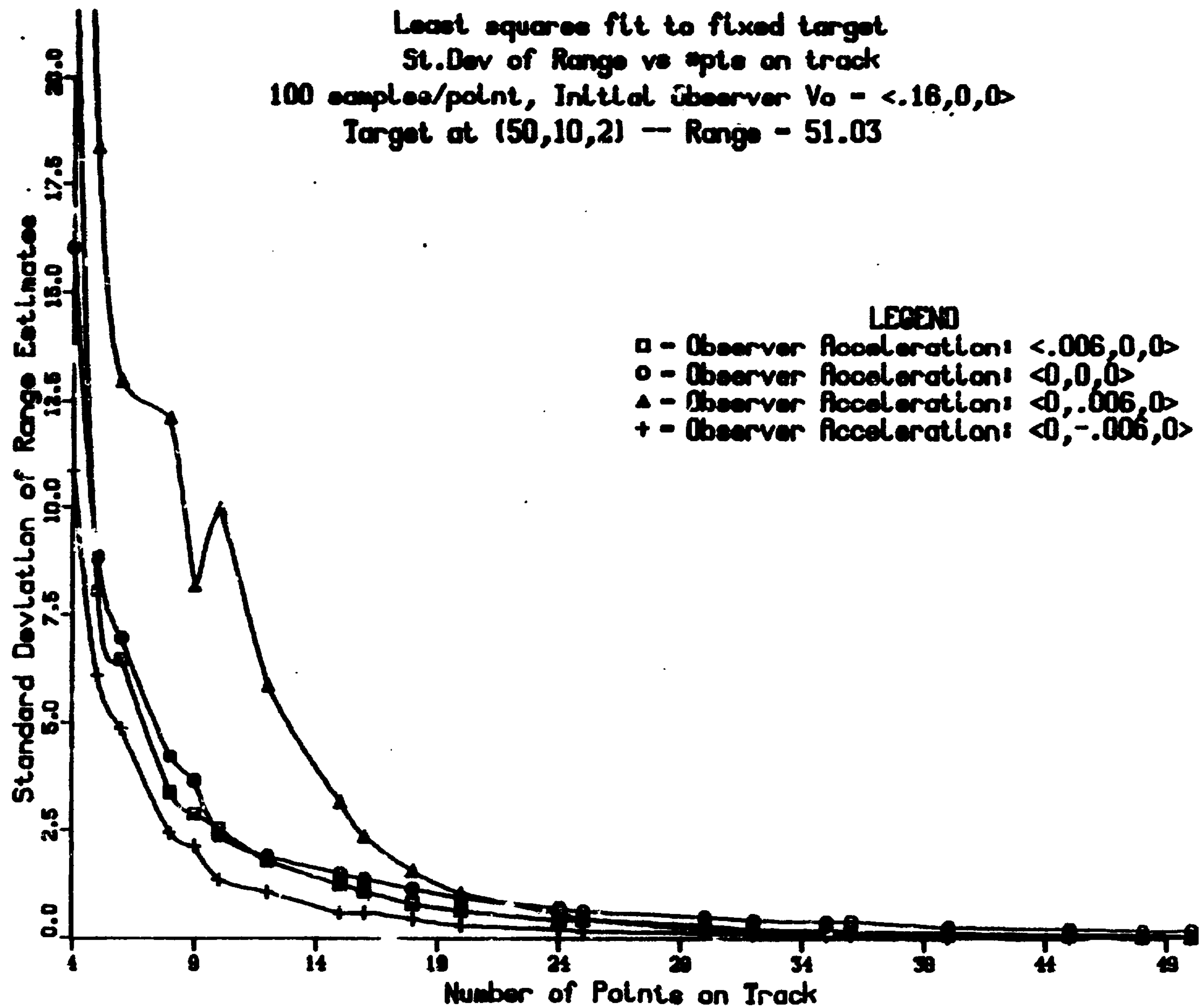

Least squares fit to fixed target
St.Dev of Range vs #pts on track
100 samples/point, Initial Observer Vo = <.16,0,0>
Target at (50,10,2) — Range = 51.03
LEGEND
□ = Observer Acceleration: <.006,0,0>
○ = Observer Acceleration: <0,0,0>
▲ = Observer Acceleration: <0,.006,0>
+ = Observer Acceleration: <0,-.006,0>
Standard Deviation of Range Estimates
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
4
9
14
19
24
29
34
39
44
49
Number of Points on Track

Figure 15

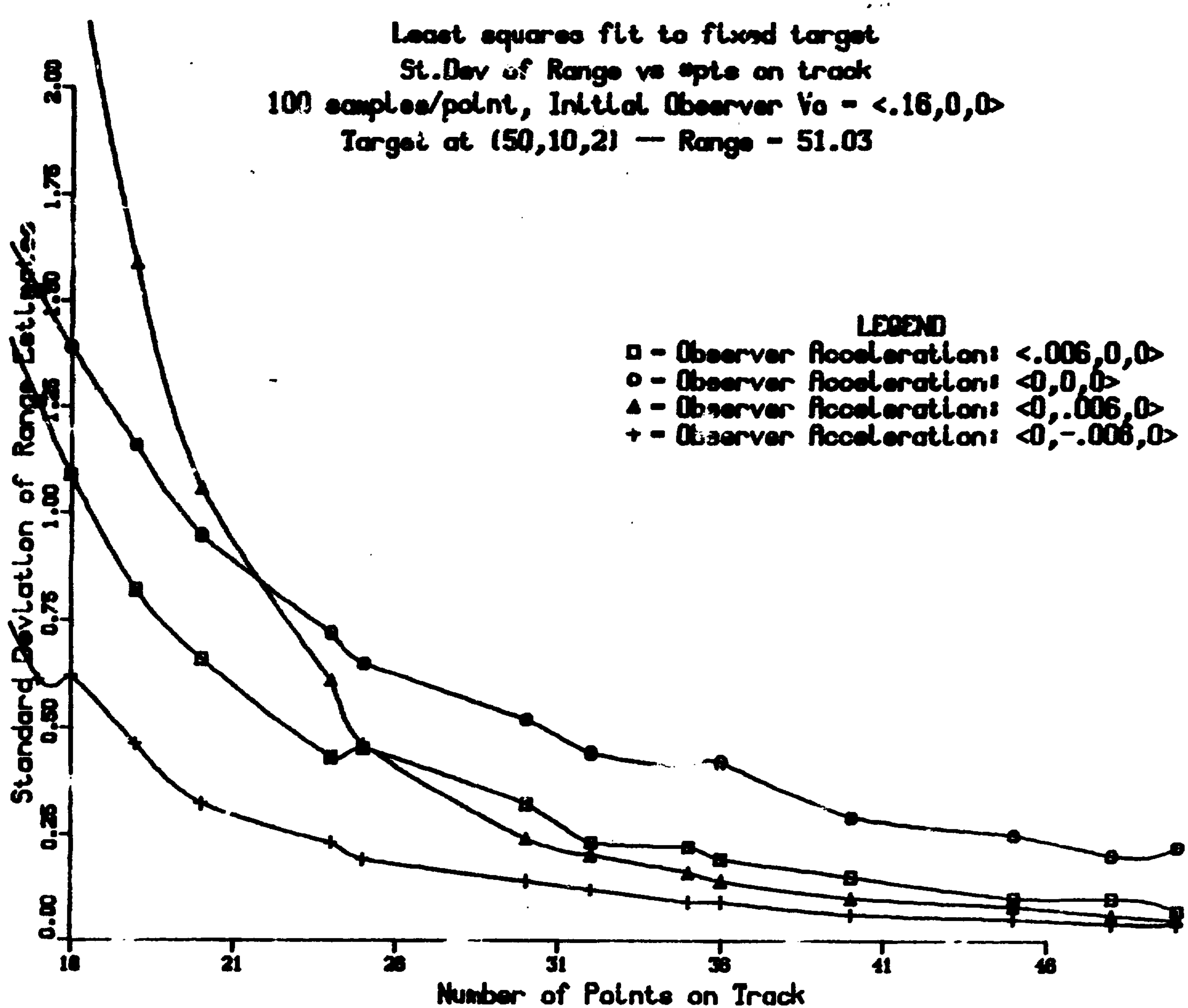
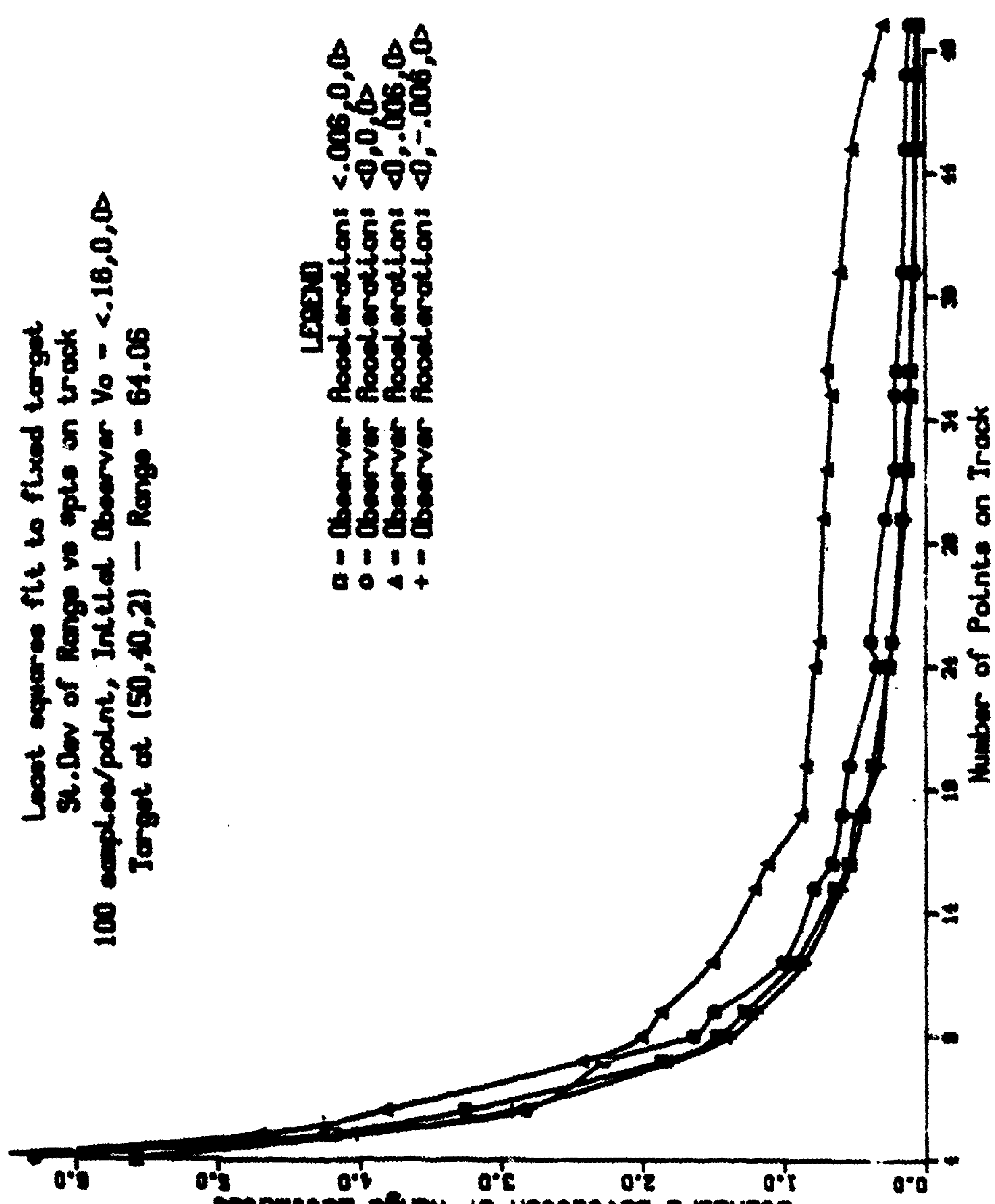

Least squares fit to fixed target
St.Dev of Range vs #pts on track
100 samples/point, Initial Observer Vo - <.16,0,0>
Target at (50,10,2) — Range - 51.03
LEGEND
□ - Observer Acceleration: <.006,0,0>
o - Observer Acceleration: <0,0,0>
▲ - Observer Acceleration: <0,.006,0>
+ - Observer Acceleration: <0,-.006,0>
Standard Deviation of Range Estimates
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
16
21
26
31
36
41
46
Number of Points on Track

Figure 16



Least squares fit to fixed target
St.Dev of Range vs npts on track
100 samples/point, Initial Observer Vo = <.16,0,0>
Target at (50,40,2) — Range = 64.06
LEGEND
□ = Observer Acceleration: <.006,0,0>
○ = Observer Acceleration: <0,0,0>
△ = Observer Acceleration: <0,.006,0>
+ = Observer Acceleration: <0,-.006,0>
Standard Deviation of Range Estimates
0.0
1.0
2.0
3.0
4.0
5.0
6.0
Number of Points on Track

Figure 17

of the observer moving so that the target passes directly in front of him somewhere along his trajectory. Not included here are plots of other possible observer motions, such as acceleration out of the $x - y$ plane. Such accelerations were in fact considered, but all cases examined strongly suggest that the only important features of observer acceleration are the magnitude of the acceleration in the x direction and the magnitude orthogonal to the x direction.

In summary, the salient features of least squares estimation of position and range of a stationary target are: (1) the most reliable estimates come from targets that are well off axis, (2) for short tracks reliability is determined more by the direction of the target than by its range, while the situation reverses itself for longer tracking times, (3) regardless of target direction and range, after some 30 seconds the standard deviation of the range estimate is no more than a percent or two of the true range of the target.

While the least squares estimator presented here offers reliable estimates within a reasonably short period of time, the method has some distinct drawbacks. The first disadvantage is that of requiring computationally complex non-linear least squares software. While such software is readily available, its use consumes considerable amounts of computer time. For example, it required an average of one and a half hours of CPU time on a VAX 11/780 to yield the 168 sample standard deviations that were plotted in each of Figures 5 through 9. Another disadvantage of this approach is that it does not take advantage of the sequential manner in which bearings are obtained by the observer. An ideal scheme would use each new pair of bearings to update the current estimate of target location in a computationally simple manner. (A truly ideal method would be simple enough to allow updating to be done by hand, particularly during test and debugging phases.) For linear estimation problems the desired adaptive estimation procedure is available in the form of the Kalman filter, which is a recursive method for determining the standard linear least squares fit of the model to the data. Kalman filters have been applied to ship-to-ship passive ranging problems [2,3], where the problem is viewed as two-dimensional, bearings consist only of azimuths, and the model has been linearized in order to apply the filter. One of these references [3] dwells on the pitfalls of this method, particularly regarding the sensitivity of short track estimates to the values of required initial estimates that must be supplied before any measurements have actually been taken. Fundamental to the use of the Kalman filter is an initial estimate of the error covariance matrix for the quantities being estimated, in addition to an initial estimate of their values. Unfortunately, the covariance depends on the target's range, so initial estimates are apt to be very unreliable, and it may take quite a long time for the data to correct for a poor guess. On the other hand, non-recursive least squares fitting requires only an initial position estimate, to which the procedure is not particularly sensitive. A final drawback of both is that they are purely numerical schemes, providing no analytic insight into the roles played by target range and direction or by observer motion in the statistical behavior of the estimator.

4.2 Azimuth Only Least Squares Range Estimation

Since it has already been established that, given small errors in bearings measurements, accurate estimation of a single target coordinate is sufficient to yield good estimates of the other coordinates, and since much of the difficulty in dealing analytically with the S_1 and S_2 criterion functions arises from terms involving altitude angles, azimuth-only estimates of x and y were considered. These involved minimizing, with respect to x and y,

$$S_3 = \sum_k \left[h(k) - \mathrm{Arctan}\left(\frac{y - y(k)}{x - x(k)} \right) \right]^2 \tag{3}$$

$$S_4 = \sum_k \left[\tan(h(k)) - \frac{y - y(k)}{x - x(k)} \right]^2 \tag{4}$$

However, S_3 does not lend itself to a simple closed form solution of its normal equations. Neither does S_4, but a modified form of it does provide the desired simplicity. Setting $g(k) = \tan(h(k))$, define

$$S_5 = \sum_k [(y - y(k)) - g(k)(x - x(k))]^2. \tag{5}$$

Equating to zero the partial derivatives of S_5 with respect to x and y, we obtain the normal equations for S_5:

$$y\sum g(k) - \sum g(k)y(k) - x\sum g^2(k) + \sum g^2(k)x(k) = 0$$

and

$$Ny - \sum y(k) - x\sum g(k) + \sum g(k)x(k) = 0.$$

Solving these equations, we obtain solutions $\hat{x}$ and $\hat{y}$ that satisfy

$$\hat{x} = \frac{\bar{y}\sum g(k) - \bar{g}\sum g(k)x(k) - \sum g(k)y(k) + \sum g^2(k)x(k)}{\sum g^2(k) - \bar{g}\sum g(k)} \tag{6}$$

and

$$\hat{y} = \bar{y} + \hat{x}\bar{g} - \frac{1}{N}\sum g(k)x(k), \tag{7}$$

where $\bar{y} = \frac{1}{N}\sum y(k)$ and $\bar{g} = \frac{1}{N}\sum g(k)$.

These estimates, which can be very easily updated as new azimuths are obtained, appear deceptively similar to estimates of slope and intercept in linear regression problems. Here however, unlike the linear regression case, $\hat{x}$ and $\hat{y}$ are non-linear functions of the observations, which we may take to be the $g(k)$'s, and so their statistical behavior cannot be readily determined in an analytic manner. Instead, as was done in the previous section, simulations are performed. Once again sensor resolution is set at 0.333 mrad, initial observer velocity at 0.16 nm/s in the x direction, and the same combinations of target range and direction and observer motion are used. In addition, the same sequence of random numbers (using IMSL's GGNML generator of values from the standard normal distribution) is employed in performing both sets of simulations. This azimuth-only procedure yields estimates of x and y only. Estimation of target range requires an estimate of the target's z-coordinate as well. Based on the argument that a single line of sight is a very good estimator of the true target direction, estimates of z were obtained via the formula

$$\hat{z} = (\tan p(0))\sqrt{(\hat{x}^2 + \hat{y}^2)}, \tag{8}$$

where $p(0)$ is the elevation observed at time 0. The results of these simulations are presented in Table A16. For comparison, the corresponding results using the standard least squares approach of the previous section can be found in Table A7. From these tables it is clear that azimuth-only least squares estimation matches the accuracy of estimation using all obtained bearings. Also of interest is the fact that for short tracks azimuth-only estimates tend to underestimate the target's true range. An examination of the procedure with σ set to zero shows that, for a typical ranging scenario, the equations for $\hat{x}$ and $\hat{y}$ do indeed yield x and y as solutions, but they are not especially well conditioned. This suggests that the underestimation of range might be a computational artifact, so the procedure was repeated using double precision arithmetic. This produced mean ranges that, surprisingly, were even farther from the true range, albeit by rather small amounts and with the anticipated smaller standard deviations. The unavoidable implication is that the observed underestimation is the result of bias in the estimator. Further study of the bias in azimuth-only estimators is proceeding. Azimuth-only estimation has been briefly mentioned in the literature [5], where it is compared with the Kalman filter in the ship-to-ship ranging problem (where there are only azimuths) and where it appears in a rather different formulation. It is of significant interest that the average amount of CPU time required to generate the data for an azimuth-only estimation equivalent to that in Figures 5 through 9 was about seven minutes, compared with the hour-and-a-half needed for the least squares method using all bearings.

In summary, azimuth-only range estimation is computationally very efficient and practically matches the accuracy of the more complicated standard least squares method. Its main drawbacks are that it appears to entail a bias of presently unknown nature and once again is not sufficiently amenable to analysis to make it explicit how the mean and variance of the range estimator depend on target location, observer motion and sensor resolution. In addition, azimuth-based methods do not apply in the general moving target case.

4.3 Range Estimation Based on Minimal Estimators

The third method for estimating range presented in this report aims at uncovering the manner in which target and observer parameters affect the statistics of the estimator, at the expense of increased uncertainty in the estimator. It is intended not so much as a practical estimation scheme (at least in the fixed target case, where the azimuth-only least squares approach is preferable), but as a means of throwing light on the nature of the biases and variability inherent in bearings only estimation. The basic idea is to use only as many of the bearings as are required in order to obtain a closed form solution for the target location that yielded those bearings. Because the resulting range estimator uses the smallest possible number of bearings, it is referred to as a "minimal" estimator. Since minimal range estimators turn out to have simple formulations in terms of the bearings that define them, it is possible to obtain formulas for their means and variances that hold to very good approximations. Being based on very few bearings, minimal estimators are of course not very accurate. Accuracy can be increased, and use made of all obtained sets of bearings, by combining collections of minimal estimates into a single better estimate.

In the stationary target case the three target coordinates are to be estimated, so that, at the minimum, two sets of bearings must be obtained. Two sets of bearings yield four measurements, and since they are corrupted values of the true bearings, they produce an overdetermined system. There is a choice of methods for dealing with this situation. Having four measurements available to estimate three location coordinates allows, by selecting all possible sets of three measurements, four distinct minimal estimates of the target location. These estimates may be considered singly, or as a composite estimate such as a mean or median.

While the two sets of bearings to be taken could be obtained from any two points in space, intended applications of this study as well as avoidance of computational complexity suggest that the first bearings, (h_0, p_0), be taken from the origin, and the second set, (h_1, p_1), from the point $(d, 0, 0)$.

This choice of second location is also justified by the fact that minimal estimates are often quick estimates, and as such are based on bearings taken in rapid succession. As a result, since we are assuming the x-axis of our coordinate system to be oriented with the observer's initial velocity, we expect that in a short time interval only a net change in the observer's x coordinate can realistically be achieved.

With the observer motion specified in the previous paragraph, and with realistic target locations relative to an assumed airborne observer, various estimation procedures were examined for the purpose of selecting one for use in the ensuing analysis. While the performances of the various methods were expected to be comparable for a target well off-axis, it is desired that the chosen estimator perform notably better, in terms of both bias and variance, when the off-axis azimuth θ_0 is near zero. Conceptually, the chosen estimation procedure uses h_0 and h_1 to obtain an azimuth-only estimate of the target's x and y coordinates, and then uses p_0 to obtain a further estimate of the z coordinate, whence the range can be estimated.

As mentioned at the beginning of this report, the target is initially located at (x, y, z) relative to the observer. Bearings h_0, p_0, and h_1, with h_1 being the target's azimuth relative to $(d, 0, 0)$, yield a unique estimate of the target location, $\hat{r} = (\hat{x}, \hat{y}, \hat{z})$, obtained as follows. Letting $a_0 = \tan h_0$ and $a_1 = \tan h_1$ we may write

$$a_0 = \tan h_0 = \hat{y}/\hat{x} \tag{9}$$

and

$$a_1 = \tan h_1 = \hat{y}/(\hat{x} - d). \tag{10}$$

Solving for $\hat{x}$ and $\hat{y}$ yields

$$\hat{x} = da_1/(a_1 - a_0) \tag{11}$$

and

$$\hat{y} = a_0\hat{x} = da_0a_1/(a_1 - a_0). \tag{12}$$

Using these estimates and the line of sight determined by h_0 and p_0 we can determine $\hat{z}$. Letting

$$b_0 = \sec h_0 \tan p_0 \tag{13}$$

it follows from projecting $(\hat{x}, \hat{y}, \hat{z})$ into the x-y plane that

$$\hat{z} = b_0 \hat{x} = d b_0 a_1 / (a_1 - a_0). \tag{14}$$

Thus the estimated target position is $T = (\hat{x}, \hat{y}, \hat{z}) = \hat{x} < 1, a_0, b_0 >$. Now, as discussed in the section on the statistics of lines of sight, for the small values of σ under consideration, $< 1, a_0, b_0 >$ is a very good estimate of $< 1, \tan\theta_0, \sec\theta_0 \tan\phi_0 >$, the true vector relating the coordinates of the target relative to the origin. Errors in estimating T, then, arise mainly from errors in $\hat{X}$. Consequently, attention will be concentrated on the analysis of the random variable $\hat{x}$, particularly with regard to its dependence on both T and d.

Note: The preceding discussion does not apply directly when $x \approx 0$, for then both a_0 and b_0 might be arbitrarily large. In this case, we would simply solve the same equations explicitly for $\hat{y}$, and obtain $T = \hat{y} < a_0', 1, b_0' >$, for appropriate a_0' and b_0', and perform a similar analysis for $\hat{y}$. This, however, will not be a problem here, since we are assuming that $|\theta_0^*|$ does not exceed 45 degrees.

$\hat{X}$ is given by $\hat{X} = dA_1/(A_1 - A_0)$, where $A_0 = \tan H_0$, and $A_1 = \tan H_1$. Now $H_0 = \theta_0 + U$, and $H_1 = \theta_1 + V$, where θ_0 is the true azimuth of the target from the origin, θ_1 is the true azimuth from $(d, 0, 0)$, and U and V are independent Gaussian variables with mean 0 and a common variance $\sigma^2 (\sigma^2 << 1)$. Now $\tan H_0 = \tan(\theta_0 + U)$ $(\tan\theta_0 + \tan U)/(1 - \tan\theta_0 \tan U)$. Letting

$$\alpha_0 = \tan\theta_0 = y/x\,, \tag{15}$$

and recalling that for small ψ, $\tan\psi \approx \psi$, we have

$$A_0 \approx (\alpha_0 + U)/(1 - \alpha_0 U). \tag{16}$$

Similarly, defining

$$\alpha_1 = \tan\theta_1 = y/(x - d), \tag{17}$$

we have

$$A_1 = (\alpha_1 + V)/(1 - \alpha_1 V). \tag{18}$$

Then

$$\hat{X} = \frac{dA_1}{(A_1 - A_0)} = \frac{\dfrac{d(\alpha_1 + V)}{1 - \alpha_1 V}}{\dfrac{\alpha_1 + V}{1 - \alpha_1 V} - \dfrac{\alpha_0 + U}{1 - \alpha_0 U}}. \tag{19}$$

This becomes

$$\hat{X} = \frac{d(\alpha_1 + V)(1 - d_0 U)}{(\alpha_1 - \alpha_0) + (1 + \alpha_0\alpha_1)(V - U) + (\alpha_1 - \alpha_0)UV} \tag{20}$$

Factoring $(\alpha_1 - \alpha_0)$ out of the denominator, and setting

$$K = \frac{d}{\alpha_1 - \alpha_0}, \quad C = \frac{1 + \alpha_0\alpha_1}{\alpha_1 - \alpha_0} \tag{21}$$

we obtain

$$\hat{X} = \frac{K(\alpha_1 + V)(1 - \alpha_0 U)}{1 + C(V - U) + UV}. \tag{22}$$

The following relations among C, K, α_0, α_1, x, y and d will be useful in determining the behavior of $\hat{X}$, and are immediate from the definitions:

$$\alpha_1 - \alpha_0 = \frac{yd}{x(x - d)},$$

$$K = \frac{x(x - d)}{y},$$

$$K\alpha_1 = x, \quad K\alpha_0 = x - d,$$

$$C = \frac{x(x - d) + y^2}{yd} = \frac{K + y}{d}. \tag{23}$$

To obtain estimates of $E(\hat{X})$ and $Var(\hat{X})$ that are accurate up to the σ^2 term, note that since U and V are independent and each has mean zero, the lowest order term to which UV can make a contribution is the σ^4 term. Thus we may ignore UV terms, and write

$$\hat{X} \approx \frac{K(\alpha_1 - \alpha_0\alpha_1 U + V)}{1 + C(V - U)} \tag{24}$$

Under the approximation $1/(1 + w) \approx 1 - w + w^2$, this becomes

$$\hat{X} \approx K(\alpha_1 - \alpha_0\alpha_1 U + U)(1 - C(V - U) + C^2(V - V)^2).$$

Performing the specified multiplication, and ignoring terms whose expected values are of degree three and higher in σ, we obtain

$$\hat{X} \approx K(\alpha_1 - \alpha_1 C(V - U) + \alpha_1 C^2(V - U)^2 - \alpha_0\alpha_1 U + \alpha_0\alpha_1 CU(V - U) + V - CV(V - U))$$

Taking expected values and recalling that $E((V - U)^2) = 2\sigma^2$,

$$\begin{aligned} E(\hat{X}) &\approx K(\alpha_1 + \sigma^2 C(2\alpha_1 C - \alpha_0\alpha_1 - 1)) \\ &= K\alpha_1 + \sigma^2 C(2CK\alpha_1 - K\alpha_0\alpha_1 - K). \end{aligned} \tag{25}$$

Since $K\alpha_1 = x$ and $y = \alpha_0 x$, this becomes

$$E(\hat{X} \approx x + \sigma^2 C(2xC - y - K).$$

Using the relation $K + y = Cd$, we have

$$E(\hat{X}) \approx x + \sigma^2 C^2(2x - d). \tag{26}$$

Squaring $\hat{X}$, dropping higher order and mixed terms, taking expected values and using the relations (23) once again we obtain

$$E(\hat{X}^2) \approx x^2 + \sigma^2(6C^2x^2 + y^2 + K^2 - 4\alpha_0 Cx^2 - 4KCx).$$

Then

$$Var(\hat{X}) = E(\hat{X}^2) - E(\hat{X})^2 \approx$$

$$\sigma^2(2C^2x^2 + 2C^2xd + y^2 + K^2 - 4\alpha_0 Cx^2 - 4KCx). \tag{27}$$

Using the relations $K = Cd - y$ and $y^2 = Cdy - x(x - d)$, this simplifies to

$$Var(\hat{X}) \approx \sigma^2[C^2(2x^2 - 2xd + d^2) - 2x(x - d)]. \tag{28}$$

Remark: In most applications $x >> d$, so with very little additional loss in accuracy the moments of $\hat{X}$ take on much simpler forms; viz.

$$E(\hat{X}) \approx x(1 + 2\sigma^2 C^2) \tag{29}$$

$$Var(\hat{X}) \approx \sigma^2[2x^2(C^2 - 1)]. \tag{30}$$

Furthermore, in this case, $C^2 >> 1$, so that we may use

$$Var(\hat{X}) \approx 2x^2C^2\sigma^2. \tag{31}$$

Note that for a fixed range and displacement d, C increases as y decreases, i.e. as the target becomes more "head-on." As a result both the bias and the variance of $\hat{X}$ increase. Not only that, but the second order approximation used for $1/(1 + C(V - U))$ becomes less accurate as C increases. Thus even the large values of $E(\hat{X})$ and $Var(\hat{X})$ obtained above are actually underestimates of the true values when C is large. That C becomes infinite when the target is directly in front of the observer is, of course, a consequence of the assumption that the second set of bearings is also taken from a point on the x-axis. In practice, one need not take the second bearings on the x-axis, but in realistic situations the point from which they are taken will be quite near the x-axis. As a result, there will be increases in both the bias and variance of $\hat{X}$ as an estimator of x when range increases, when d decreases, and for fixed range and d, when y decreases.

To improve estimates of $E(\hat{X})$ and $Var(\hat{X})$ when C is large, fourth order approximations of $\hat{X}$ were obtained. Using $1/(1 + w) \approx 1 - w + w^2 - w^3 + w^4$, including UV terms, performing the required algebra, noting that those terms in which C has the same exponent as σ dominate the others, taking expected values and again assuming that $x >> d$ and $C^2 >> 1$, the results are

$$E(\hat{X}) \approx x(1 + 2\sigma^2C^2 + 12\sigma^4C^4) \tag{32}$$

$$Var(\hat{X}) \approx 2x^2C^2\sigma^2 + 32x^2C^4\sigma^4. \tag{33}$$

The preceding estimates are the results of a great many truncations, and it is reasonable at this point to inquire as to how good they are in practice. Table 1 shows the estimated means and standard deviations using both the second-order and fourth-order approximations to $\hat{X}$ for a few cases. Also included in this table are the results of simulations in which the means and standard deviations of samples of size 1000 were obtained. Note that for large values of C even the fourth order approximation may grossly underestimate the sample standard deviation. This arises from the fact that in reality $\hat{X}$ has infinite mean, and that in any finite sample soı e extremely large or even negative values are likely to occur. Fortunately these extreme values do ıוסt correspond to realistic situations, so in practice such an estimate can be ignored if it arises. With this in mind, extreme scores were effectively eliminated by excluding the upper and lower 1% of the scores in the samples with large C. The statistics of these middle 98% of the samples are shown in parentheses in the table, where appropriate. With this proviso we see that for targets well off axis or with sufficient separation between bearings, the second order approximations are good ones, while in other cases including the fourth order term produces better estimates.

Table 1 — Minimal (Two-point) Estimation of Fixed Target Position
Bearings taken from (0, 0, 0) and (d, 0, 0)
Sample values are from samples with N = 1000 and σ = 0.333 mrad
Sample values in parentheses refer to middle 98% of scores

x	y	d	C	σC	2nd order approx. $E(\hat{X})$	2nd order approx. $Var(\hat{X})$	4th order approx. $E(\hat{X})$	4th order approx. $Var(\hat{X})$	$\bar{x}$	s
25	5	0.16	807.5	0.27	28.6	9.5	30.2	14.0	29.4 (29.3)	43.4 (17.9)
25	5	0.64	198.1	0.07	25.2	2.3	25.2	2.4	25.2	2.4
50	5	0.64	779.1	0.26	56.7	18.3	59.4	26.4	64.1 (57.2)	183.4 (28.5)
50	5	1.28	384.5	0.13	51.6	9.1	51.8	10.2	52.1 (51.5)	16.3 (9.4)
50	10	0.64	401.3	0.13	51.8	9.4	52.0	10.7	52.4	11.3
50	40	0.64	158.9	0.05	50.3	3.7	50.3	3.8	50.2	3.8

In practice, of course, one does not expect to be restricted to two sets of bearings in estimating the location of the target. Instead, a collection of sets of bearings would be available, the collection being obtained in a sequential manner, with new pairs of bearings being added at regular intervals. A method was sought to select pairs of sets of bearings and combine the resulting minimal estimates into a single effective range estimate, preferably a scheme that can be easily updated as new sets of bearings are obtained. Examination of sampling distributions of minimal estimators indicate that the sample median is a much better estimator of the true target location parameters than is the sample mean, especially for small separations between the points from which the pair of bearings is taken. (That the median is a better estimator than the mean also follows from the fact that the form of the estimator of x is, effectively, $x/(1 + W)$ where W is normally distributed with mean 0. Since, on the average, half of the W's in a sample are positive, and the other half negative, half of the estimates will be less than x, and half will be larger. Thus the sample median is an unbiased estimator of x, while the mean of $x/(1 + W)$ is approximately $x(1 + Var(W))$.) The procedure chosen takes a sequence of sets of bearings, forms minimal estimates of the target's x coordinate from each pair of azimuths using formula (6), and then takes the median of this collection of minimal estimates to be the estimate of x. Estimates of y, z, and the range are then obtained using this x estimate and the initial line of sight. This procedure is still under investigation, particularly regarding the best choice of pairs of points to be used, so results are still preliminary. Table A17 gives the results of using this method in two familiar cases.

5. CONSTANT VELOCITY TARGETS

Conceptually, the methods for estimating the target motion parameters of a moving target are the same as those for a fixed target. The practical difference is that there are more parameters to be estimated, just how many more depending on the target motion model selected. For instance, if a constant velocity target is assumed, there are three velocity components to be estimated in addition to three location coordinates, while if a target with constant acceleration is assumed, there are yet another three acceleration components to be estimated. Specifying that the target move along a particular type of parametric path, e.g. along the arc of a circle, requires estimating as many quantities as are needed to completely specify the motion. The effects of having more parameters to estimate are that more complicated relations exist between observer and target motions and the reliability of estimators, that longer tracks must be obtained in order to estimate range and other quantities to specified accuracies, and that the vagaries of non-linear estimation procedures with their reliance on numerical methods must be accepted.

In this section attention will be restricted to constant velocity targets. Only one estimation method will be examined in detail—least squares fitting of the constant velocity model to the observed bearings. In this case, there are six quantities to estimate, the three target coordinates, x, y and z, as well as its velocity components denoted by Vx, Vy and Vz.

Studies of alternative ranging methods are ongoing. Unfortunately, moving targets do not necessarily stay at the same altitude, so that the azimuth-only least squares method that works so well for fixed targets is not applicable to moving targets in general. However, in many cases a distant target will not undergo significant changes in its altitude during the period of its observation, so that azimuth-based estimation procedures are currently being considered. Also, as in the fixed target case, minimal closed-form estimates of target motion parameters are available, in this case requiring three sets of bearings to provide sufficient data to estimate the six target values. Collections of such estimates may be combined to form a single improved value. Concurrent with present efforts in the fixed target case, the emphasis of ongoing work in minimal estimation of moving target parameters is on the optimal choice of sets of bearings to be combined for a single minimal estimate.

5.1 Least Squares Range Estimation

A similar procedure is used as in the fixed target case, based on the differences between observed and fitted bearings; the expression

$$S_6 = \sum_k \left[h(k) - \text{Arctan} \left[\frac{y + (k-1)Vy - y(k)}{x + (k-1)Vx - x(k)} \right] \right]^2$$

$$+ \sum_k \left[p(k) - \text{Arcsin} \left[\frac{z + (k-1)Vz - z(k)}{r(k)} \right] \right]^2, \qquad (34)$$

is minimized with respect to x, y, z, Vx, Vy, Vz,

where $r^2(k) = (x + (k-1)Vx - x(k))^2 + (y + (k-1)Vy - y(k))^2 + (z + (k-1)Vz - z(k))^2$.

Once again the properties of the estimator were studied by examining its performance under a wide variety of simulated scenarios, and again the standard deviation of the 100 range estimates obtained for each chosen combination of observer and target parameters was used as the primary measure of reliability. In ranging a constant velocity target, the observer cannot himself travel at constant velocity, for then, even in the absence of any measurement errors, the obtained sequence of bearings does not yield a unique solution for the target parameters. This can be very easily seen by imagining two targets, each flying parallel to the observer, the first target at half the range and speed of the second. As long as the observer maintains the same velocity, these targets will yield the same sets of bearings. Consequently, in all cases examined we have assumed an accelerating observer, and, further, that the acceleration is held constant.

Figures 18 through 21 show the standard deviation of range estimates as a function of track length for a variety of combinations of observer and target motions (also see Tables A18 and A19). In these figures track lengths begin at 15 (still assuming one set of bearings per second) since for shorter tracks the estimates are so poor that they provide no useful information. Figure 18 deals with a target that is initially in front of the observer, while Figs. 19, 20 and 21 examine an off-axis target. Even a cursory examination of these graphs reveals a great deal more complexity when the target's velocity must be estimated in addition to its range. The most apparent difference is that the standard deviation of the range estimates does not monotonically decrease with the length of the track. Indeed there are often dramatic rises in the sample standard deviation. In an effort to understand this phenomenon, percentiles were obtained for range estimates from samples of size 1000 simulating a target originally at (50, 40, 1) nm with velocity $<-.20, -.20, 0)$ nm/s being tracked by an observer

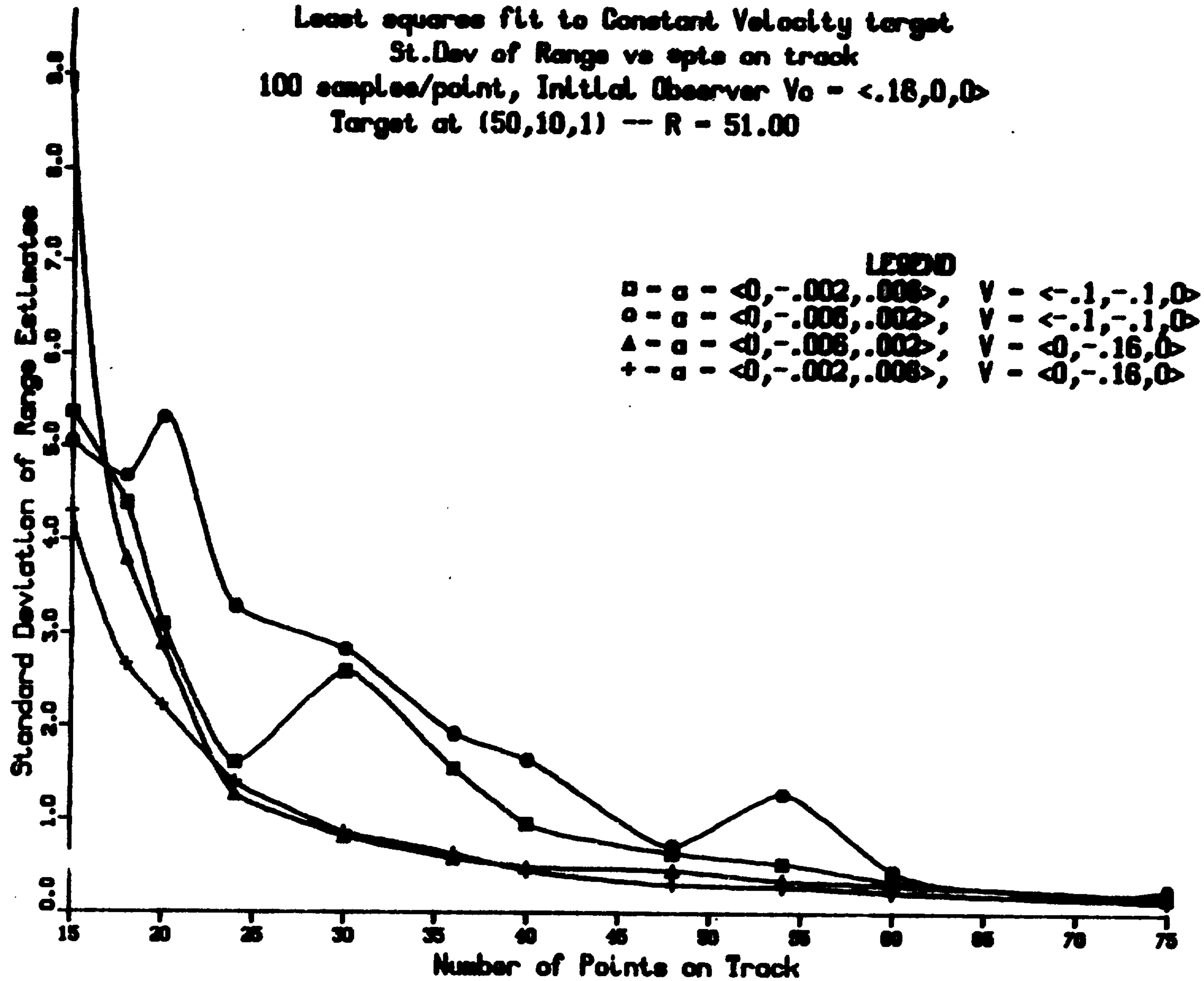

Least squares fit to Constant Velocity target
St.Dev of Range vs #pts on track
100 samples/point, Initial Observer Vo = <.18,0,0>
Target at (50,10,1) -- R = 51.00
LEGEND
□ = a = <0,-.002,.008>, V = <-.1,-.1,0>
○ = a = <0,-.006,.002>, V = <-.1,-.1,0>
▲ = a = <0,-.006,.002>, V = <0,-.16,0>
+ = a = <0,-.002,.008>, V = <0,-.16,0>
Standard Deviation of Range Estimates
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
15
20
25
30
35
40
45
50
55
60
65
70
75
Number of Points on Track

Figure 18

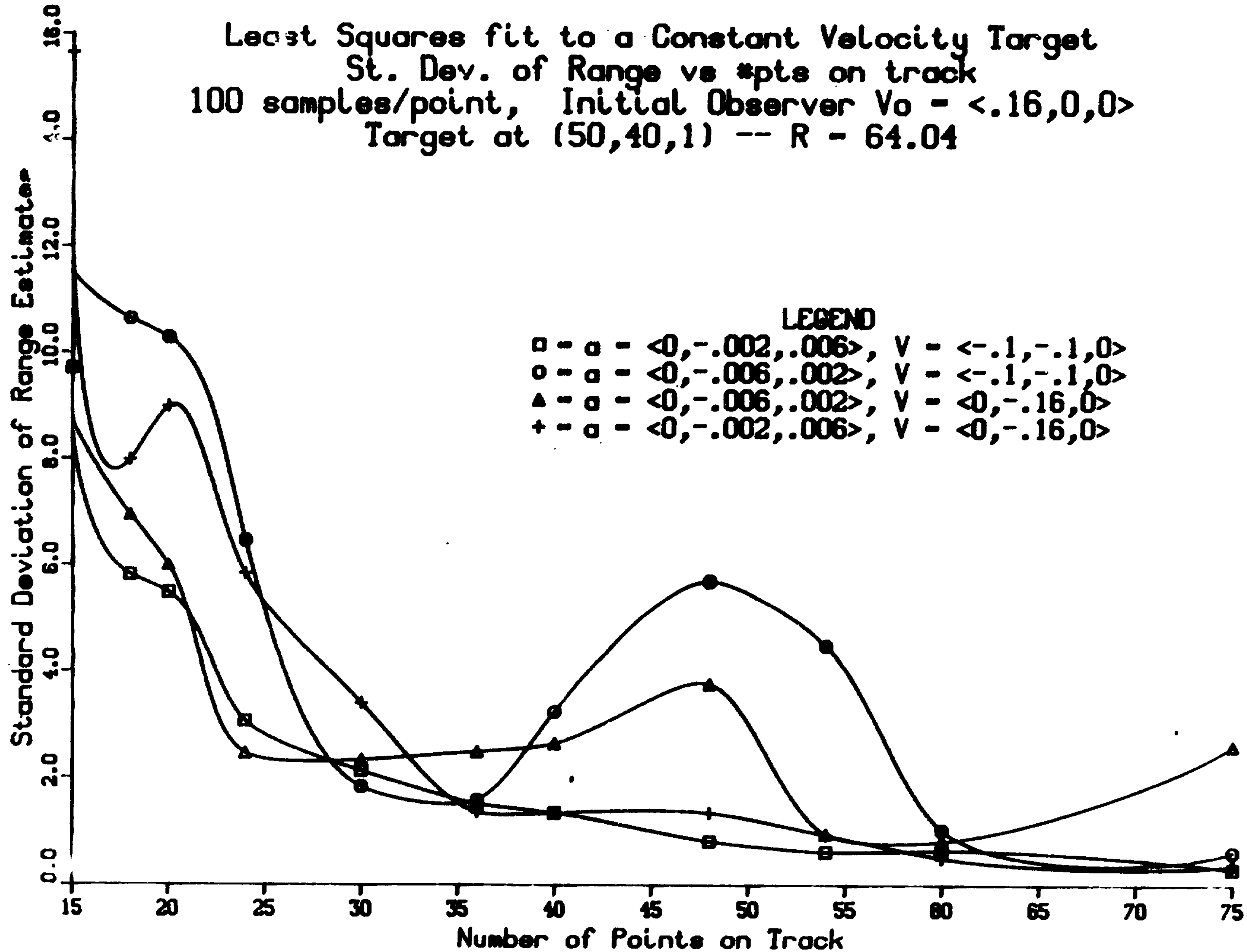
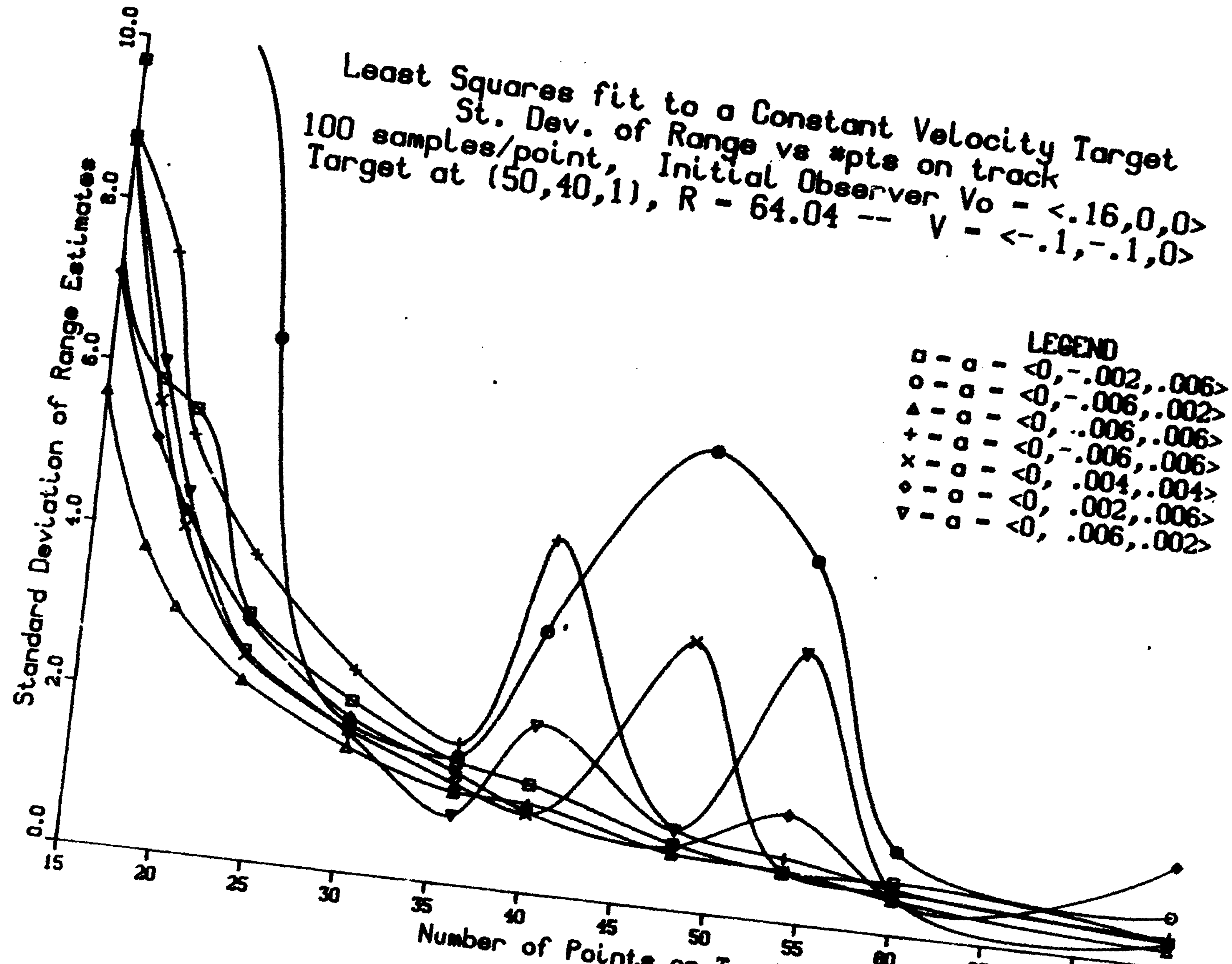

Least Squares fit to a Constant Velocity Target
St. Dev. of Range vs #pts on track
100 samples/point, Initial Observer Vo = <.16,0,0>
Target at (50,40,1) -- R = 64.04
LEGEND
□ = a = <0,-.002,.006>, V = <-.1,-.1,0>
o = a = <0,-.006,.002>, V = <-.1,-.1,0>
▲ = a = <0,-.006,.002>, V = <0,-.16,0>
+ = a = <0,-.002,.006>, V = <0,-.16,0>
Standard Deviation of Range Estimates
0.0
2.0
4.0
6.0
8.0
10.0
12.0
16.0
15
20
25
30
35
40
45
50
55
60
65
70
75
Number of Points on Track

Figure 19


Least Squares fit to a Constant Velocity Target
St. Dev. of Range vs #pts on track
100 samples/point, Initial Observer Vo = <.16,0,0>
Target at (50,40,1), R = 64.04 -- V = <-.1,-.1,0>
LEGEND
□ = a = <0,-.002,.006>
o = a = <0,-.006,.002>
▲ = a = <0,.006,.006>
+ = a = <0,-.006,.006>
x = a = <0,.004,.004>
◆ = a = <0,.002,.006>
▽ = a = <0,.006,.002>
Standard Deviation of Range Estimates
0.0
2.0
4.0
6.0
8.0
10.0
15
20
25
30
35
40
45
50
55
60
65
70
75
Number of Points on Track
Figure 20

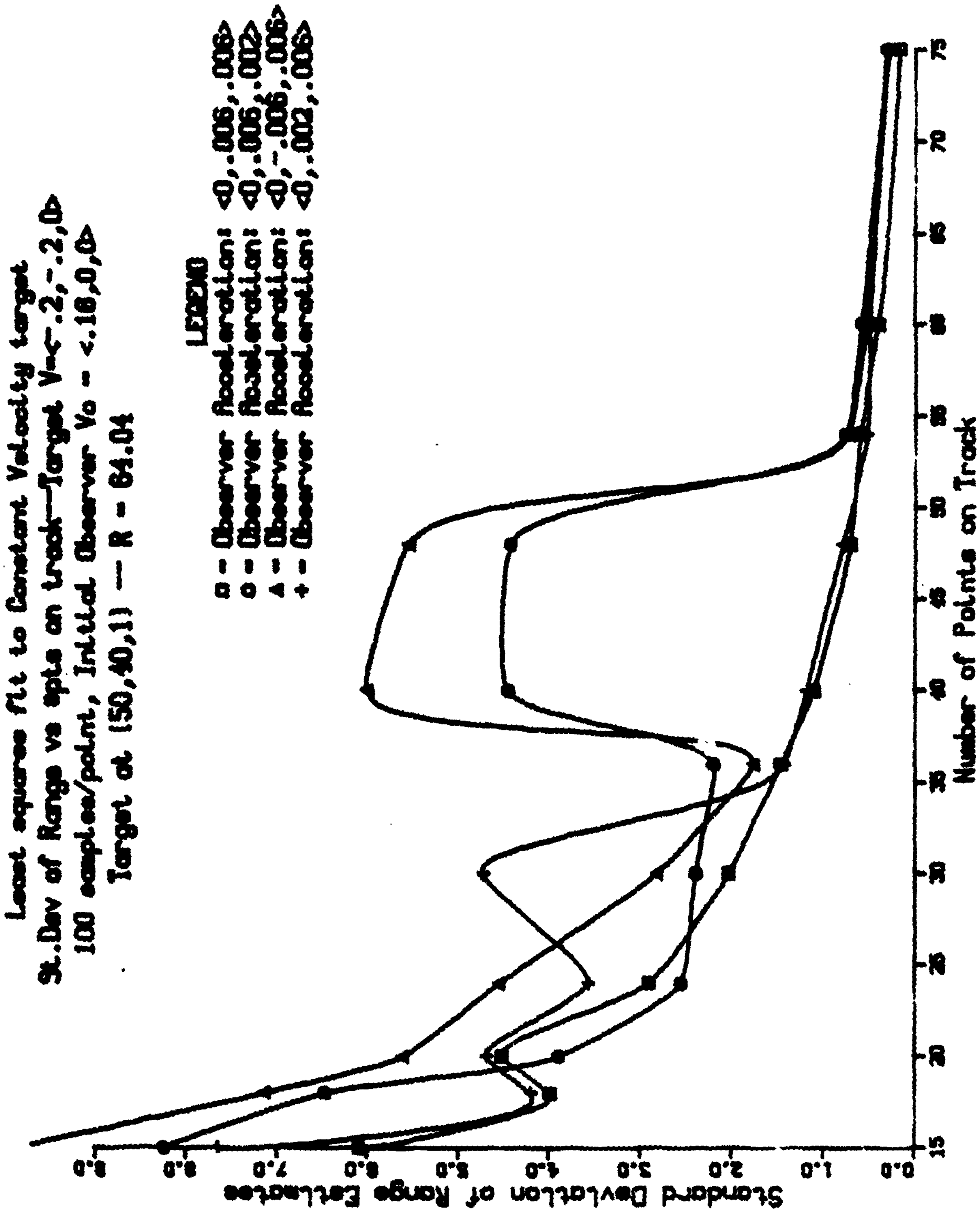
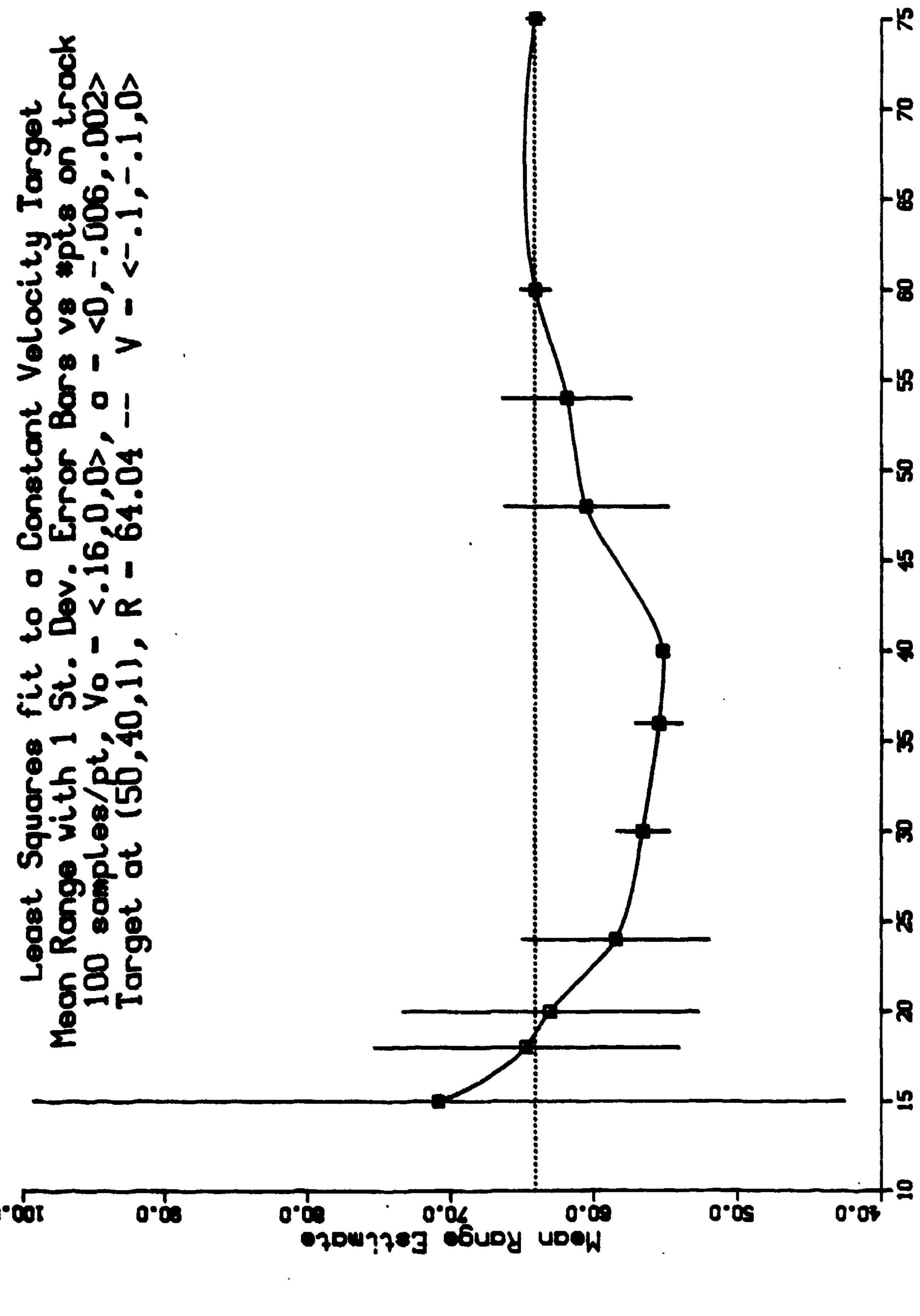

Least squares fit to Constant Velocity target
St.Dev of Range vs npts on track—Target V=<-.2,-.2,0>
100 samples/point, Initial Observer Vo = <.16,0,0>
Target at (50,40,1) — R = 64.04
LEGEND
□ = Observer Acceleration: <0,.006,.006>
o = Observer Acceleration: <0,.006,.002>
Δ = Observer Acceleration: <0,-.006,.006>
+ = Observer Acceleration: <0,.002,.006>
Standard Deviation of Range Estimates
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
15
20
25
30
35
40
45
50
55
60
65
70
75
Number of Points on Track

Figure 21

with acceleration $<0, -.006, .006>$ nm/s^2. Ten such percentile distributions were constructed, and in each it was found that there appear to be two distinct clusters of estimates, with approximately 15% of the range estimates being near 49 nm and the remainder scattered within a few nautical miles of 65 nm. The sample data are listed in Table A20.

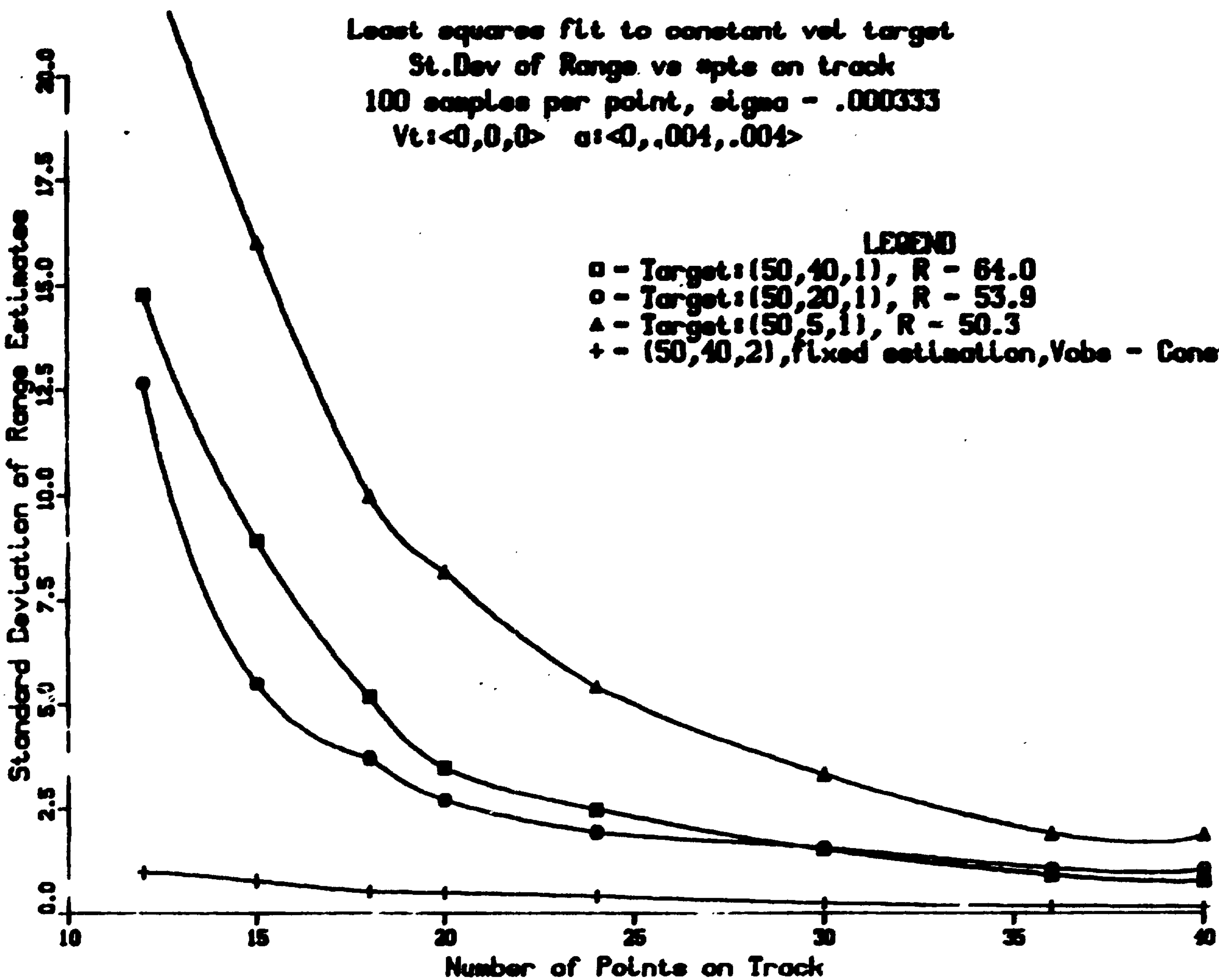
Figures 18 through 21 are the usual graphs of sample standard deviations versus track lengths, and while they indicate irregularity in the estimation process, they do not explain its source. Figure 22, which is typical of what can arise in non-linear least squares estimation, provides some insight into the nature of the irregularities in Figs. 18 thorough 21. In Figure 22 the mean of the 100 estimates is plotted as a function of track length, together with error bars corresponding to one sample standard deviation. Here the mean range drifts away from the true range for a rather long interval of track lengths. Thus a bias, which can be much larger than the sample standard deviation and which depends on track length as well as target and observer motion parameters, is introduced when target velocity must also be estimated. As in Table A20, there may be two (or more) well separated values around which estimates cluster; this accounts for the unusually large sample standard deviations found for long tracks. This phenomenon is most likely due to the fact that in a non-linear estimation problem in six dimensions, rather different combinations of target location and velocity may yield very similar sequences of bearings relative to the specified observer trajectory. Measurement errors then result in random selection, according to some as yet undetermined rules, of one of the possible choices.

Further analysis is being performed to gain a more thorough understanding of the combined effects of target location and velocity and observer motion on the performance of least squares estimators, both in the case of a constant velocity target and, in what promises to be much more complicated, the situation of an accelerating target. A final graph, Figure 23, compares the results when a fixed target is known to be fixed, and so can be ranged using the methods of section 4, versus when it is not assumed to be stationary, and so its lack of motion must be inferred from its bearings. This



Least Squares fit to a Constant Velocity Target
Mean Range with 1 St. Dev. Error Bars vs #pts on track
100 samples/pt, Vo = <.16,0,0>, a = <0,-.006,.002>
Target at (50,40,1), R = 64.04 -- V = <-.1,-.1,0>
Mean Range Estimate
Number of Points on Track

Figure 22



Least squares fit to constant vel target
St.Dev of Range vs #pts on track
100 samples per point, sigma - .000333
Vt:<0,0,0> a:<0,.004,.004>
LEGEND
□ - Target:(50,40,1), R - 64.0
○ - Target:(50,20,1), R - 53.9
△ - Target:(50,5,1), R - 50.3
+ - (50,40,2),fixed estimation,Vobs - Const
Standard Deviation of Range Estimates
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
10
15
20
25
30
35
40
Number of Points on Track

Figure 23

suggests that a comparative study of the ranging of slowly moving targets by both methods might be fruitful.

6. SUMMARY AND CONCLUSIONS

The performance of various air-to-air passive ranging techniques has been examined as a function of target location and motion, observer motion, and length of track. Bearings are obtained at one-second intervals and there are independent Gaussian errors in azimuth and altitude.

For fixed targets, three range estimation methods have been considered, each generally giving accurate estimates of target location for track lengths of 30 seconds or less:

1) Least Squares Range Estimation—This method provides accurate range estimates within a short tracking time, but requires computationally complex software and extensive computer processing time; it does not update target location estimates using sequential bearings.

2) Azimuth Only Least Squares Range Estimation—This method is nearly as accurate as the previous method but is computationally much more efficient. Target location estimates are readily updated as new azimuths are obtained. This method is the best of the three methods for the fixed target case but is not directly adaptable to the moving target case.

3) Range Estimation Based on Minimal Estimators—This method uses only a small number of bearings so that a closed form solution for the target location can be found. Consequently, range estimates are not as accurate as in the previous methods, but explicit information can be obtained on the effect of target and observer parameters on the estimator statistics.

For moving targets, a track length of 60 seconds is typically required for good estimates of target location and velocity, and this figure is highly dependent on the observer/target geometry. Least Squares Range Estimation has been investigated for the constant velocity target case. However, it was found that in some cases range estimates have a cluster point separate from that at the true range. Further study will be required to develop a technique to resolve this ambiguity.

7. REFERENCES

1. W.H. Foy, "Position-Location Solutions by Taylor-Series Estimation," IEEE Trans. Aerospace and Electronic Systems, AES-12(2), 187 (1976).

2. A.G. Lindgren and K.F. Gong, "Position and Velocity Estimation Via Bearing Observations," IEEE Trans. Aerospace and Electronic Systems, AES-14(4), 564 (1978).

3. V.J. Aidala, "Kalman Filter Behavior in Bearings-Only Tracking Applications," IEEE Trans. Aerospace and Electronic Systems, AES-12(2), 29 (1979).

4. W.R. Nunn, "Position Finding with Prior Knowledge of Covariance Parameters," IEEE Trans. Aerospace and Electronic Systems, AES-15(3), 204 (1979).

5. F.H. Hollister, "Bearings-only passive ranging using Kalman-Bucy and Moore-Penrose methods," SPIE Vol. 302 Infrared Technology for Target Detection and Classification 152 (1981).

APPENDIX

Table A1
Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = 0.333 mrad
Target Location: (50,50,2) Range = 70.74 nm
Constant Observer Velocity: <.16,0,0> nm/s

	Time Between Bearings (sec)				
# of Bearings	1	2	4	6	8
2	83.63 61.80	73.09 12.65	70.94 5.23	71.84 3.62	71.21 2.71
3	73.26 12.31	70.68 5.41	70.66 2.45	70.85 1.73	70.74 1.23
4	71.67 6.76	70.86 3.23	70.98 1.57	70.69 1.19	70.62 0.83
5	70.81 4.75	70.76 2.23	70.93 1.23	70.76 0.88	70.69 0.54
6	70.52 3.45	70.70 1.65	70.70 0.81	70.74 0.57	70.71 0.41
8	70.87 1.96	70.74 0.97	70.66 0.55	70.70 0.32	70.73 0.27
9	70.71 1.85	70.77 0.90	70.71 0.42	70.77 0.32	
10	70.99 1.59	70.79 0.71	70.75 0.39	70.77 0.24	
12	70.73 1.17	70.79 0.59	70.74 0.26		
15	70.67 0.96	70.73 0.34	70.71 0.20		
16	70.88 0.73	70.75 0.44			
18	70.72 0.60	70.76 0.31			
20	70.69 0.52	70.73 0.26			
24	70.73 0.37				
25	70.71 0.40				
30	70.74 0.31				

The upper entry in each cell is the sample mean, while the lower entry is the sample standard deviation.

Table A2

Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = 0.333 mrad
Target Location: (50,40,2) Range = 64.06 nm
Initial Observer Velocity: <.16,0,0> nm/s
Observer Acceleration: <0,-.006,0> nm/s²

# of Bearings	Time Between Bearings (sec) 1	2	4	6	8
2	75.85	66.11	64.53	64.98	64.42
	62.07	9.71	4.91	2.93	2.11
3	66.32	64.00	63.98	64.14	64.06
	10.88	4.58	1.88	1.24	0.83
4	64.87	64.14	64.23	64.02	63.99
	5.89	2.63	1.13	0.77	0.49
5	64.08	64.07	64.19	64.08	64.04
	4.02	1.73	0.83	0.53	0.30
6	63.90	64.03	64.05	64.06	64.04
	2.86	1.23	0.51	0.31	0.12
8	64.15	64.06	64.01	64.04	64.06
	1.55	0.68	0.31	0.16	0.12
9	64.04	64.07	64.05	64.08	
	1.44	0.60	0.23	0.15	
10	64.25	64.09	64.06	64.08	
	1.21	0.46	0.20	0.11	
12	64.06	64.09	64.07		
	0.86	0.36	0.13		
15	64.01	64.06	64.05		
	0.67	0.19	0.09		
16	64.16	64.07			
	0.50	0.24			
18	64.05	64.08			
	0.39	0.16			
20	64.03	64.06			
	0.34	0.13			
24	64.06				
	0.23				
25	64.04				
	0.24				
30	64.06				
	0.17				

The upper entry in each cell is the sample mean, while the lower entry is the sample standard deviation.

TABLE A3

Least Squares Estimates of Fixed Target Position
100 Samples per Cell Various σ
Target Location: (50,40,2) Range = 64.06 nm
Initial Observer Velocity: <.16,0,0> nm/s
Observer Acceleration: <0,-.006,0> nm/s^2

# Points on Track	σ = .1 mrad		σ = .333 mrad		σ = .666 mrad	
	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
4	64.27	1.77	64.18	5.58	67.97	16.78
5	64.23	1.22	63.74	4.03	66.19	9.18
6	64.00	0.95	65.28	2.93	64.19	6.54
8	64.05	0.54	63.81	1.80	64.14	3.73
9	64.10	0.43	64.12	1.39	64.43	2.92
10	64.13	0.37	64.10	1.20	64.55	2.53
12	64.03	0.27	64.04	0.86	63.87	1.79
15	64.06	0.19	63.95	0.60	64.07	1.24
16	64.03	0.16	63.95	0.54	63.87	1.04
18	64.07	0.13	64.11	0.49	64.10	0.86
20	64.03	0.11	64.05	0.33	63.84	0.74
24	64.05	0.07	64.04	0.27	64.02	0.50
25	64.06	0.07	64.07	0.23	64.02	0.47
30	64.05	0.05	64.05	0.15	63.99	0.35
32	64.06	0.04	64.06	0.15	64.08	0.25
35	64.06	0.03	64.07	0.12	64.05	0.23
36	64.06	0.03	64.06	0.11	64.07	0.23
40	64.06	0.03	64.08	0.08	64.07	0.20
45	64.06	0.02	64.06	0.08	64.07	0.13
48	64.06	0.02	64.06	0.07	64.07	0.12
50	64.06	0.02	64.06	0.06	64.08	0.12

Table A4

Least Squares Estimates of Fixed Target Position
100 Samples per Cell   Various σ
Target Location: (50,10,2)   Range = 51.03 nm
Constant Observer Velocity: <.16,0,0> nm/s

# Points on Track	σ = .1 mrad		σ = .333 mrad		σ = .666 mrad	
	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
4	51.65	3.50	55.57	16.03	186.7	799.7
5	51.18	2.56	52.53	8.86	57.88	23.03
6	51.12	1.85	51.90	6.99	56.42	29.67
8	50.99	1.28	51.11	4.27	51.97	8.97
9	51.00	1.08	51.14	3.69	51.82	7.93
10	51.07	0.70	51.24	2.40	51.68	5.07
12	51.10	0.57	51.30	1.93	51.74	4.00
15	50.97	0.46	50.87	1.52	50.82	3.08
16	51.12	0.41	51.35	1.39	51.74	2.83
18	51.00	0.35	50.96	1.16	50.94	2.33
20	51.03	0.28	51.05	0.95	51.11	1.90
24	51.07	0.22	51.17	0.72	51.33	1.44
25	51.03	0.19	51.05	0.65	51.08	1.30
30	51.04	0.16	51.08	0.52	51.15	1.04
32	51.01	0.13	50.98	0.44	50.93	0.89
35	51.03	0.12	51.04	0.41	51.05	0.81
36	51.02	0.13	51.00	0.42	50.98	0.85
40	51.03	0.09	51.04	0.29	51.05	0.58
45	51.04	0.08	51.07	0.25	51.12	0.50
48	51.03	0.06	51.04	0.20	51.05	0.39
50	51.03	0.07	51.02	0.22	51.01	0.44

Table A5

Least Squares Estimates of Fixed Target Position
100 Samples per Cell   Various σ
Target Location: (50,40,2)   Range = 64.06 nm
Constant Observer Velocity: <.16,0,0> nm/s

# Points on Track	σ = .1 mrad $\bar{x}$	σ = .1 mrad s	σ = .333 mrad $\bar{x}$	σ = .333 mrad s	σ = .666 mrad $\bar{x}$	σ = .666 mrad s
4	63.84	1.70	64.74	6.31	64.48	12.48
5	64.20	1.40	63.11	4.18	66.17	9.94
6	64.18	0.92	63.95	2.83	65.42	6.62
8	64.08	0.61	64.09	2.28	64.38	4.06
9	64.08	0.49	63.91	1.65	64.30	3.28
10	64.00	0.40	64.30	1.50	63.70	2.65
12	64.05	0.30	63.86	1.01	64.04	2.00
15	64.07	0.24	64.16	0.79	64.17	1.60
16	64.11	0.22	64.06	0.66	64.42	1.46
18	64.05	0.16	64.02	0.59	64.01	1.09
20	64.05	0.16	64.12	0.54	64.02	1.03
24	64.06	0.13	64.03	0.35	64.08	0.86
25	64.08	0.10	64.02	0.39	64.17	0.69
30	64.06	0.08	64.08	0.29	64.05	0.52
32	64.06	0.06	64.07	0.22	64.05	0.43
35	64.06	0.06	64.04	0.22	64.08	0.41
36	64.05	0.06	64.05	0.21	64.00	0.38
40	64.06	0.05	64.08	0.16	64.04	0.33
45	64.06	0.04	64.06	0.14	64.03	0.29
48	64.06	0.04	64.06	0.13	64.05	0.23
50	64.07	0.03	64.06	0.11	64.09	0.22

Table A6

Least Squares Estimates of Fixed Target Position
100 Samples per Cell Various σ
Target Location: (50,10,2) Range = 51.03 nm
Initial Observer Velocity: <.16,0,0> nm/s
Observer Acceleration: <0,-.006,0> nm/s²

# Points on Track	σ = .1 mrad $\bar{x}$	σ = .1 mrad s	σ = .333 mrad $\bar{x}$	σ = .333 mrad s	σ = .666 mrad $\bar{x}$	σ = .666 mrad s
4	51.47	3.02	51.27	10.87	106.0	379.1
5	50.86	1.75	52.46	6.11	52.58	15.96
6	51.06	1.34	51.30	4.89	52.77	10.67
8	50.95	0.71	51.10	2.46	50.91	4.86
9	51.12	0.57	51.13	2.15	51.91	3.90
10	51.07	0.48	50.99	1.37	51.44	3.19
12	51.00	0.33	51.15	1.08	50.92	2.23
15	51.00	0.21	51.08	0.61	50.84	1.40
16	51.05	0.18	51.06	0.62	51.20	1.18
13	51.03	0.13	51.04	0.46	51.04	0.86
20	51.02	0.10	51.02	0.32	50.97	0.68
24	51.03	0.06	51.06	0.23	51.02	0.43
25	51.02	0.06	51.04	0.19	51.00	0.42
30	51.03	0.04	51.05	0.14	51.04	0.29
32	51.03	0.03	51.05	0.12	51.06	0.22
35	51.03	0.03	51.03	0.09	51.03	0.20
36	51.03	0.03	51.03	0.09	51.03	0.17
40	51.03	0.02	51.03	0.06	51.02	0.14
45	51.03	0.01	51.04	0.05	51.02	0.09
48	51.03	0.01	51.03	0.04	51.02	0.10
50	51.03	0.01	51.03	0.04	51.03	0.07

Table A7

Least Squares Estimates of Fixed Target Position

100 Samples per Cell $\sigma = 0.333$ mrad

Constant Observer Velocity: <.16,0,0> nm/s

# Points on Track	T=(20,16,2) $\bar{R}$ = 25.69 $\bar{x}$	s	T=(30,24,2) $\bar{R}$ = 38.47 $\bar{x}$	s	T=(40,32,2) $\bar{R}$ = 51.26 $\bar{x}$	s	T=(50,40,2) $\bar{R}$ = 64.06 $\bar{x}$	s
4	25.83	0.99	38.77	2.10	51.40	3.75	63.72	5.76
5	25.63	0.63	38.55	1.54	51.75	2.89	64.76	4.72
6	25.70	0.52	38.53	1.10	51.18	2.26	64.58	3.15
8	25.65	0.31	38.44	0.77	51.26	1.34	64.15	2.02
9	25.74	0.26	38.45	0.66	51.39	1.08	64.14	1.62
10	25.71	0.23	38.49	0.43	51.43	0.96	63.86	1.33
12	25.68	0.17	38.52	0.34	51.18	0.72	64.04	1.00
15	25.67	0.12	38.44	0.27	51.26	0.52	64.11	0.80
16	25.71	0.11	38.53	0.25	51.18	0.44	64.23	0.73
18	25.69	0.08	38.45	0.21	51.28	0.38	64.03	0.55
20	25.68	0.07	38.47	0.17	51.16	0.33	64.04	0.52
24	25.69	0.05	38.50	0.13	51.24	0.24	64.07	0.43
25	25.69	0.05	38.47	0.12	51.24	0.23	64.12	0.35
30	25.69	0.04	38.48	0.10	51.23	0.18	64.06	0.26
32	25.70	0.03	38.46	0.08	51.28	0.13	64.06	0.21
35	25.69	0.03	38.47	0.07	51.26	0.12	64.07	0.21
36	25.69	0.03	38.47	0.08	51.26	0.12	64.03	0.19
40	25.69	0.02	38.47	0.05	51.27	0.11	64.05	0.17
45	25.69	0.02	38.48	0.05	51.27	0.08	64.05	0.15
48	25.69	0.02	38.47	0.04	51.27	0.07	64.06	0.12
50	25.69	0.01	38.47	0.04	51.27	0.07	64.08	0.11

# Points on Track	T=(60,48,2) $\bar{R}$ = 76.86 $\bar{x}$	s	T=(80,64,2) $\bar{R}$ = 102.47 $\bar{x}$	s	T=(100,80,2) $\bar{R}$ = 128.08 $\bar{x}$	s	T=(120,96,2) $\bar{R}$ = 153.69 $\bar{x}$	s
4	78.00	9.21	100.96	17.23	128.57	24.88	159.52	41.31
5	76.97	6.04	103.70	11.24	132.02	17.11	154.62	29.96
6	76.73	4.08	104.03	8.42	129.21	14.71	163.26	20.83
8	76.93	3.30	102.10	5.26	128.45	8.43	152.62	12.15
9	76.66	2.38	103.06	4.28	128.60	7.95	154.37	9.72
10	77.22	2.18	102.66	3.83	128.04	4.91	154.26	8.62
12	76.58	1.46	102.67	2.93	128.70	4.59	153.63	6.22
15	77.00	1.15	102.58	2.17	128.30	2.99	152.90	4.62
16	76.85	0.95	102.58	1.93	128.22	3.20	152.85	4.29
18	76.81	0.86	102.42	1.43	128.13	2.47	154.14	4.05
20	76.95	0.78	102.43	1.34	128.06	1.96	153.62	2.82
24	76.81	0.52	102.63	1.03	128.31	1.57	153.56	2.42
25	76.81	0.57	102.45	0.97	128.16	1.30	153.75	2.18
30	76.89	0.42	102.41	0.72	128.29	1.12	153.57	1.57
32	76.87	0.32	102.47	0.64	128.23	0.98	153.68	1.48
35	76.83	0.32	102.57	0.52	128.11	0.91	153.79	1.27
36	76.84	0.31	102.48	0.62	128.13	0.85	153.67	1.19
40	76.88	0.24	102.50	0.42	128.05	0.61	153.86	0.92
45	76.87	0.21	102.48	0.39	128.21	0.63	153.71	0.92
48	76.86	0.19	102.47	0.34	128.14	0.53	153.66	0.81
50	76.86	0.17	102.49	0.29	128.12	0.52	153.68	0.68

Table A8

Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = 0.333 mrad
Initial Observer Velocity: <.16,0,0> nm/s
Observer Acceleration: <.006,0,0> nm/s²

# Points on Track	T=(20,16,2) R = 25.69 $\bar{x}$	s	T=(30,24,2) R = 38.47 $\bar{x}$	s	T=(40,32,2) R = 51.26 $\bar{x}$	s	T=(50,40,2) R = 64.06 $\bar{x}$	s
4	25.82	0.94	38.75	1.98	51.37	3.52	63.72	5.42
5	25.64	0.59	38.54	1.44	51.72	2.67	64.66	4.39
6	25.69	0.48	38.52	1.00	51.17	2.06	64.52	2.86
8	25.66	0.27	38.44	0.68	51.25	1.18	64.14	1.78
9	25.73	0.23	38.45	0.57	51.36	0.94	64.13	1.42
10	25.71	0.20	38.49	0.36	51.40	0.81	63.87	1.14
12	25.68	0.14	38.51	0.28	51.19	0.59	64.04	0.82
15	25.68	0.10	38.45	0.21	51.26	0.40	64.10	0.63
16	25.70	0.08	38.46	0.19	51.20	0.34	64.19	0.56
18	25.69	0.06	38.47	0.16	51.27	0.28	64.04	0.41
20	25.69	0.05	38.49	0.12	51.19	0.24	64.04	0.37
24	25.69	0.03	38.47	0.09	51.25	0.16	64.07	0.29
25	25.69	0.03	38.48	0.08	51.25	0.15	64.10	0.23
30	25.69	0.02	38.47	0.06	51.24	0.11	64.06	0.16
32	25.69	0.02	38.47	0.05	51.27	0.08	64.06	0.13
35	25.69	0.02	38.47	0.04	51.26	0.07	64.07	0.12
36	25.69	0.01	38.47	0.04	51.26	0.07	64.04	0.11
40	25.69	0.01	38.47	0.03	51.27	0.06	64.06	0.09
45	25.69	0.01	38.47	0.02	51.26	0.04	64.06	0.07
48	25.69	0.01	38.47	0.02	51.27	0.03	64.06	0.06
50	25.69	0.00	38.47	0.02	51.27	0.03	64.07	0.05

# Points on Track	T=(60,48,2) R = 76.86 $\bar{x}$	s	T=(80,64,2) R = 102.47 $\bar{x}$	s	T=(100,80,2) R = 128.08 $\bar{x}$	s	T=(120,96,2) R = 153.69 $\bar{x}$	s
4	77.92	8.69	100.94	16.41	128.62	25.21	159.01	38.09
5	76.92	5.61	103.51	10.38	131.57	15.69	154.26	26.74
6	77.74	3.73	103.76	7.86	128.89	13.20	162.40	18.83
8	76.91	2.90	102.09	4.62	128.35	7.46	152.59	10.66
9	76.67	2.06	102.97	3.71	128.48	6.84	154.24	8.41
10	77.17	1.85	102.62	3.23	128.02	4.26	154.10	7.31
12	76.63	1.20	102.63	2.41	128.55	3.76	153.64	5.18
15	76.97	0.90	102.55	1.72	128.26	2.35	153.04	3.64
16	76.85	0.74	102.54	1.50	128.19	2.48	153.03	3.32
18	76.82	0.65	102.44	1.08	128.12	1.87	154.02	3.01
20	76.93	0.57	102.44	0.99	128.05	1.42	153.62	2.06
24	76.82	0.35	102.57	0.71	128.23	1.08	153.58	1.67
25	76.83	0.38	102.45	0.66	128.13	0.89	153.71	1.45
30	76.88	0.26	102.43	0.45	128.21	0.72	153.60	0.97
32	76.86	0.19	102.46	0.39	128.18	0.61	153.68	0.92
35	76.84	0.18	102.53	0.30	128.09	0.53	153.74	0.76
36	76.85	0.18	102.48	0.36	128.11	0.49	153.67	0.70
40	76.88	0.13	102.49	0.24	128.07	0.33	153.78	0.50
45	76.86	0.10	102.48	0.20	128.15	0.32	153.69	0.47
48	76.86	0.09	102.48	0.17	128.11	0.26	153.67	0.41
50	76.86	0.08	102.47	0.14	128.09	0.25	153.68	0.34

Table A9

Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = 0.333 mrad
Constant Observer Velocity: <.16,0,0> nm/s

	T=(20,4,2) R = 20.49		T=(30,6,2) R = 30.66		T=(40,8,2) R = 40.84		T=(50,10,2) R = 40.03	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
4	20.85	1.84	31.73	4.34	41.72	7.79	52.43	14.69
5	20.40	1.14	30.95	2.98	42.29	5.42	53.64	10.34
6	20.57	0.93	30.82	2.21	40.88	4.60	52.39	6.59
8	20.45	0.52	30.64	1.46	40.95	2.59	51.49	3.94
9	20.58	0.47	30.68	1.22	41.13	2.13	51.19	3.14
10	20.51	0.44	30.72	0.80	41.23	1.86	50.68	2.64
12	20.48	0.30	30.73	0.65	40.69	1.35	50.97	1.90
15	20.47	0.22	30.59	0.51	40.88	0.96	51.14	1.54
16	20.52	0.19	30.75	0.46	40.72	0.84	51.38	1.40
18	20.49	0.15	30.63	0.38	40.86	0.69	51.00	1.04
20	20.48	0.12	30.66	0.30	40.67	0.61	51.02	1.00
24	20.50	0.08	30.70	0.24	40.80	0.44	51.06	0.83
25	20.49	0.08	30.66	0.21	40.80	0.42	51.14	0.66
30	20.50	0.06	30.68	0.16	40.78	0.32	51.01	0.48
32	20.50	0.04	30.64	0.14	40.87	0.24	51.02	0.41
35	20.49	0.04	30.66	0.13	40.83	0.23	51.04	0.38
36	20.49	0.04	30.65	0.13	40.83	0.22	50.97	0.34
40	20.49	0.03	30.66	0.09	40.85	0.19	51.01	0.31
45	20.49	0.02	30.67	0.07	40.85	0.14	51.00	0.26
48	20.49	0.02	30.66	0.06	40.85	0.12	51.02	0.22
50	20.49	0.02	30.66	0.06	40.86	0.12	51.06	0.20

	T=(60,12,2) R = 61.22		T=(80,16,2) R = 81.61		T=(100,20,2) R = 102.00		T=(120,24,2) R = 122.39	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
4	67.16	22.79	114.11	228.2	139.77	214.9	275.42	825.5
5	63.25	14.24	90.03	30.21	120.45	53.90	164.73	232.6
6	61.76	8.41	87.36	20.08	112.51	50.76	157.78	71.48
8	61.85	7.10	81.91	11.84	104.50	17.73	123.10	24.89
9	60.93	4.61	83.36	8.78	104.73	18.15	125.84	21.23
10	62.08	4.56	82.36	8.29	102.57	10.11	125.09	18.37
12	60.72	2.86	82.29	5.85	103.77	9.49	123.08	12.99
15	61.54	2.28	82.00	4.35	102.68	6.02	121.22	9.14
16	61.25	1.94	81.96	3.82	102.56	6.54	121.17	8.56
18	61.13	1.70	81.55	2.91	102.28	4.95	123.65	8.13
20	61.41	1.53	81.59	2.61	102.08	3.80	122.39	5.63
24	61.12	0.98	81.95	2.05	102.46	3.14	122.22	4.84
25	61.11	1.08	81.64	1.89	102.19	2.58	122.62	4.30
30	61.27	0.80	81.48	1.43	102.41	2.19	122.25	3.09
32	61.23	0.62	81.60	1.24	102.29	1.92	122.44	2.91
35	61.16	0.59	81.82	1.01	102.10	1.77	122.66	2.46
36	61.18	0.58	81.66	1.20	102.11	1.65	122.37	2.35
40	61.26	0.45	81.67	0.83	101.96	1.18	122.76	1.81
45	61.23	0.38	81.65	0.73	102.26	1.21	122.43	1.81
48	61.22	0.34	81.62	0.64	102.14	1.03	122.33	1.57
50	61.21	0.31	81.64	0.56	102.07	0.99	122.38	1.34

Table A10

Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = 0.333 mrad
Initial Observer Velocity: <.16,0,0> nm/s
Observer Acceleration: <.006,0,0> nm/s²

# Points on Track	T=(20,4,2) $\underline{R}$ = 20.49 $\bar{x}$	s	T=(30,6,2) $\underline{R}$ = 30.66 $\bar{x}$	s	T=(40,8,2) $\underline{R}$ = 40.84 $\bar{x}$	s	T=(50,10,2) $\underline{R}$ = 40.03 $\bar{x}$	s
4	20.82	1.72	31.64	4.06	41.59	7.26	52.18	13.67
5	20.40	1.05	30.91	2.77	42.18	5.42	53.22	9.42
6	20.56	0.85	30.80	2.01	40.84	4.17	52.19	5.94
8	20.46	0.46	30.65	1.29	40.91	2.26	51.40	3.46
9	20.57	0.40	30.67	1.04	41.07	1.83	51.16	2.73
10	20.51	0.37	30.70	0.67	41.16	1.57	50.69	2.25
12	20.48	0.25	30.71	0.53	40.71	1.11	50.97	1.55
15	20.47	0.16	30.61	0.39	40.87	0.75	51.11	1.20
16	20.51	0.14	30.73	0.35	40.74	0.64	51.28	1.07
18	20.49	0.10	30.64	0.28	40.84	0.50	51.00	0.78
20	20.49	0.08	30.66	0.21	40.73	0.43	51.01	0.71
24	20.50	0.05	30.68	0.16	40.81	0.29	51.05	0.55
25	20.49	0.05	30.66	0.13	40.81	0.27	51.10	0.43
30	20.50	0.03	30.67	0.09	40.81	0.19	51.01	0.28
32	20.50	0.02	30.65	0.07	40.85	0.13	51.02	0.24
35	20.49	0.02	30.66	0.06	40.84	0.12	51.04	0.21
36	20.49	0.02	30.65	0.06	40.84	0.11	51.00	0.18
40	20.49	0.01	30.66	0.04	40.84	0.09	51.02	0.15
45	20.49	0.01	30.66	0.03	40.84	0.06	51.02	0.12
48	20.49	0.00	30.66	0.02	40.84	0.05	51.02	0.09
50	20.49	0.00	30.66	0.02	40.85	0.05	51.04	0.08

# Points on Track	T=(60,12,2) $\underline{R}$ = 61.22 $\bar{x}$	s	T=(80,16,2) $\underline{R}$ = 81.61 $\bar{x}$	s	T=(100,20,2) $\underline{R}$ = 102.00 $\bar{x}$	s	T=(120,24,2) $\underline{R}$ = 122.39 $\bar{x}$	s
4	66.41	20.54	138.64	419.9	145.12	313.4	317.74	*
5	62.80	12.78	88.74	26.72	117.43	44.81	150.68	148.8
6	61.65	7.60	86.63	17.96	109.83	39.41	150.72	55.97
8	61.68	6.08	81.63	10.02	103.82	15.25	122.38	21.42
9	60.92	3.96	83.02	7.54	104.04	15.14	125.08	18.06
10	61.92	3.81	82.11	6.88	102.34	8.67	124.37	15.27
12	60.81	2.37	82.13	4.78	103.28	7.71	122.84	10.66
15	61.46	1.76	81.89	3.40	102.49	4.67	121.32	7.15
16	61.24	1.49	81.84	2.93	102.39	4.97	121.38	6.58
18	61.15	1.27	81.56	2.18	102.17	3.71	123.23	6.00
20	61.36	1.10	81.57	1.92	102.02	2.73	122.36	4.09
24	61.14	0.66	81.82	1.38	102.29	2.14	122.23	3.28
25	61.15	0.72	81.61	1.27	102.12	1.75	122.50	2.82
30	61.24	0.49	81.53	0.88	102.26	1.37	122.26	1.88
32	61.22	0.37	81.60	0.74	102.19	1.17	122.41	1.77
35	61.18	0.32	81.72	0.57	102.04	1.00	122.51	1.44
36	61.20	0.32	81.64	0.67	102.06	0.92	122.36	1.35
40	61.25	0.23	81.65	0.45	101.99	0.63	122.59	0.97
45	61.23	0.18	81.62	0.36	102.13	0.58	122.39	0.90
48	61.22	0.15	81.62	0.30	102.06	0.49	122.36	0.77
50	61.22	0.13	81.61	0.25	102.02	0.45	122.39	0.65

Table A11

Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = .333 mrad
Initial Observer Velocity: <.16,0,0>
Observer Acceleration: <0,-.006,0>

# Points on Track	T=(20,4,2) R_ = 20.49 $\bar{x}$	s	T=(30,6,2) R_ = 30.66 $\bar{x}$	s	T=(40,8,2) R_ = 40.84 $\bar{x}$	s	T=(50,10,2) R_ = 40.03 $\bar{x}$	s
4	20.74	1.50	31.38	3.34	41.37	5.97	51.50	10.28
5	20.40	0.86	30.84	2.21	41.82	4.19	52.36	7.22
6	20.53	0.67	30.75	1.50	40.77	3.09	51.81	4.31
8	20.46	0.34	30.65	0.90	40.84	1.59	51.21	2.39
9	20.54	0.28	30.65	0.72	40.96	1.22	51.15	1.85
10	20.51	0.24	30.68	0.44	41.04	1.02	50.78	1.46
12	20.48	0.16	30.70	0.33	40.75	0.68	51.00	0.95
15	20.48	0.10	30.63	0.23	40.84	0.43	51.08	0.67
16	20.50	0.08	30.70	0.20	40.77	0.36	51.15	0.59
18	20.49	0.06	30.65	0.16	40.85	0.27	51.01	0.42
20	20.49	0.05	30.66	0.11	40.78	0.23	51.01	0.36
24	20.49	0.03	30.67	0.08	40.83	0.14	51.04	0.26
25	20.49	0.03	30.66	0.07	40.83	0.13	51.06	0.20
30	20.49	0.02	30.67	0.05	40.82	0.09	51.02	0.13
32	20.50	0.01	30.66	0.04	40.85	0.06	51.03	0.11
35	20.49	0.01	30.66	0.03	40.84	0.06	51.03	0.09
36	20.49	0.01	30.66	0.03	40.84	0.05	51.01	0.08
40	20.49	0.01	30.66	0.02	40.84	0.04	51.03	0.06
45	20.49	0.01	30.66	0.02	40.84	0.03	51.02	0.05
48	20.49	0.01	30.66	0.01	40.84	0.02	51.03	0.04
50	20.49	0.00	30.66	0.01	40.84	0.02	51.03	0.04

# Points on Track	T=(60,12,2) R_ = 61.22 $\bar{x}$	s	T=(80,16,2) R = 81.61 $\bar{x}$	s	T=(100,20,2) R_ = 102.00 $\bar{x}$	s	T=(120,24,2) R_ = 122.39 $\bar{x}$	s
4	64.68	15.93	90.41	78.83	130.47	230.0	214.16	644.3
5	61.90	9.40	85.28	18.25	111.74	30.19	138.15	119.8
6	61.30	5.54	84.55	12.28	105.27	22.72	139.72	33.28
8	61.39	4.01	81.37	6.48	102.79	10.22	121.63	14.31
9	60.98	2.65	82.37	4.92	102.80	9.24	123.70	11.24
10	61.64	2.35	81.80	4.14	101.98	5.58	123.12	9.25
12	60.96	1.42	81.86	2.82	102.59	4.50	122.46	6.25
15	61.34	0.96	81.72	1.87	102.24	2.53	121.69	3.90
16	61.21	0.79	81.70	1.58	102.16	2.63	121.75	3.50
18	61.18	0.67	81.59	1.11	102.05	1.91	122.73	3.00
20	61.29	0.55	81.57	0.97	101.97	1.35	122.34	1.99
24	61.18	0.32	81.70	0.64	102.13	0.98	122.30	1.50
25	61.19	0.34	81.59	0.59	102.04	0.80	122.41	1.25
30	61.23	0.22	81.58	0.37	102.11	0.60	122.31	0.78
32	61.22	0.16	81.61	0.32	102.09	0.50	122.39	0.75
35	61.20	0.14	81.65	0.24	102.01	0.41	122.43	0.59
36	61.21	0.14	81.62	0.28	102.03	0.38	122.37	0.54
40	61.23	0.10	81.63	0.18	102.00	0.25	122.46	0.38
45	61.22	0.08	81.61	0.15	102.05	0.23	122.39	0.34
48	61.22	0.07	81.61	0.12	102.02	0.19	122.38	0.30
50	61.22	0.06	81.61	0.10	102.01	0.17	122.39	0.24

Table A12

Least Squares Estimates of Fixed Target Position
100 Samples per Cell   σ = 0.333 mrad
Target Range = 40 nm   Various Directions
Constant Observer Velocity: <.16,0,0> nm/s

# Points on Track	T=(28.3,28.3,2) $\bar{x}$	s	T=<30.6,25.7,2> $\bar{x}$	s	T=<34.6,20.0,2> $\bar{x}$	s	T=<37.5,13.7,2> $\bar{x}$	s
5	39.91	1.42	40.07	1.81	39.99	2.26	40.38	3.10
8	40.14	0.74	40.08	0.77	39.99	1.07	40.16	1.56
10	40.02	0.48	39.99	0.57	39.96	0.76	40.06	1.07
12	39.96	0.37	40.02	0.46	40.01	0.52	39.90	0.73
15	40.02	0.26	40.02	0.32	40.01	0.39	39.99	0.54
18	40.02	0.22	40.05	0.25	40.01	0.25	39.99	0.39
20	40.00	0.18	40.01	0.17	40.02	0.21	39.98	0.33
25	39.99	0.11	39.99	0.14	40.02	0.17	40.00	0.26
30	40.00	0.09	40.00	0.10	39.99	0.13	39.98	0.16
35	40.01	0.07	40.00	0.07	39.99	0.10	40.02	0.12
40	40.00	0.05	40.00	0.06	39.98	0.08	39.99	0.10
45	40.00	0.04	40.00	0.05	39.99	0.06	40.00	0.09
50	39.99	0.04	40.00	0.04	40.00	0.05	40.00	0.07

# Points on Track	T=(38.6,10.3,2) $\bar{x}$	s	T=<39.3, 6.9,2> $\bar{x}$	s	T=<39.8, 3.5,2> $\bar{x}$	s
5	40.13	4.30	40.22	6.47	41.44	12.56
8	40.05	1.98	40.54	2.97	40.29	4.40
10	40.29	1.28	39.68	1.73	40.13	3.35
12	40.05	1.02	39.88	1.29	40.17	2.53
15	39.99	0.65	40.03	0.91	40.24	1.84
18	40.01	0.58	40.11	0.72	39.94	1.32
20	40.01	0.42	39.93	0.71	40.21	1.09
25	39.99	0.30	39.99	0.41	39.96	0.77
30	39.96	0.21	40.02	0.33	40.09	0.55
35	40.03	0.18	39.96	0.21	39.93	0.39
40	40.01	0.12	40.00	0.21	40.01	0.35
45	39.99	0.12	40.00	0.17	39.99	0.30
50	40.02	0.09	39.98	0.12	40.02	0.23

Table A13

Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = 0.333 mrad
Target Range = 40 nm Various Directions
Initial Observer Velocity: <.16,0,0> nm/s
Observer Acceleration: <0,-.006,0> nm/s²

# Points on Track	T=(28.3,28.3,2) $\bar{x}$	T=(28.3,28.3,2) s	T=<30.6,25.7,2> $\bar{x}$	T=<30.6,25.7,2> s	T=<34.6,20.0,2> $\bar{x}$	T=<34.6,20.0,2> s	T=<37.5,13.7,2> $\bar{x}$	T=<37.5,13.7,2> s
5	40.28	1.31	39.98	1.41	40.45	1.56	40.16	2.28
8	40.18	0.53	40.04	0.69	39.93	0.74	39.92	0.96
10	39.94	0.40	39.89	0.47	39.97	0.51	39.96	0.75
12	40.05	0.28	40.00	0.29	40.07	0.36	40.00	0.48
15	40.05	0.21	39.99	0.21	40.00	0.23	40.00	0.31
18	39.98	0.14	40.00	0.16	39.99	0.19	39.99	0.19
20	40.01	0.12	39.97	0.13	39.98	0.14	39.98	0.17
25	40.01	0.09	39.99	0.09	39.99	0.10	40.02	0.11
30	40.01	0.05	39.99	0.06	40.00	0.06	40.01	0.06
35	40.00	0.05	39.99	0.04	40.01	0.05	40.00	0.05
40	40.00	0.03	40.00	0.04	40.00	0.04	40.00	0.03
45	40.00	0.03	39.99	0.03	40.00	0.03	40.00	0.03
50	40.00	0.02	40.00	0.02	40.00	0.02	40.00	0.02

# Points on Track	T=(38.6,10.3,2) $\bar{x}$	T=(38.6,10.3,2) s	T=<39.3, 6.9,2> $\bar{x}$	T=<39.3, 6.9,2> s	T=<39.8, 3.5,2> $\bar{x}$	T=<39.8, 3.5,2> s
5	40.14	3.10	41.04	4.88	40.10	6.04
8	39.79	1.34	40.12	1.61	40.59	2.37
10	39.93	0.80	39.91	0.98	40.10	1.24
12	39.95	0.52	39.99	0.73	39.92	0.87
15	39.96	0.37	40.00	0.43	40.03	0.51
18	39.98	0.25	39.99	0.26	40.02	0.36
20	40.01	0.19	40.00	0.21	39.99	0.24
25	40.01	0.12	39.99	0.14	40.02	0.13
30	40.00	0.08	40.01	0.07	40.00	0.09
35	39.99	0.05	39.99	0.06	40.00	0.06
40	40.00	0.05	40.00	0.03	40.00	0.04
45	40.00	0.03	39.99	0.03	40.00	0.03
50	40.00	0.02	40.00	0.02	40.00	0.03

Table A14

Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = 0.333 mrad
Target Location: (50,40,2) Range = 64.06 nm
Initial Observer Velocity: <.16,0,0> nm/s

# Points on Track	a=<.006,0,0> $\bar{x}$	s	a=<0,0,0> $\bar{x}$	s	a=<0,.006,0> $\bar{x}$	s	a=<0,-.006,0> $\bar{x}$	s
4	64.32	5.58	64.74	6.31	63.12	6.87	64.18	5.58
5	64.82	4.24	64.11	4.18	64.48	4.71	63.74	4.03
6	63.97	3.26	63.95	2.83	64.63	3.83	65.28	2.93
8	64.05	1.87	64.09	2.28	63.86	2.43	63.81	1.80
9	64.22	1.48	63.91	1.65	64.33	2.01	64.12	1.39
10	64.29	1.29	64.30	1.50	64.15	1.87	64.10	1.20
12	63.95	0.93	63.86	1.01	64.15	1.51	64.04	0.86
15	64.06	0.64	64.16	0.79	64.12	1.21	63.95	0.60
16	63.96	0.54	64.06	0.66	64.13	1.12	63.95	0.54
18	64.08	0.44	64.02	0.59	64.01	0.88	64.11	0.49
20	63.95	0.38	64.12	0.54	64.05	0.85	64.05	0.33
24	64.04	0.26	64.03	0.35	64.19	0.78	64.04	0.27
25	64.04	0.24	64.02	0.39	64.09	0.75	64.07	0.23
30	64.03	0.17	64.08	0.29	63.98	0.72	64.05	0.15
32	64.07	0.13	64.07	0.22	64.05	0.69	64.06	0.15
35	64.06	0.11	64.04	0.22	64.18	0.66	64.07	0.12
36	64.06	0.11	64.05	0.21	64.08	0.69	64.06	0.11
40	64.07	0.09	64.08	0.16	63.95	0.60	64.08	0.08
45	64.06	0.06	64.06	0.14	64.10	0.52	64.06	0.08
48	64.07	0.05	64.06	0.13	64.01	0.40	64.06	0.07
50	64.07	0.05	64.06	0.11	64.08	0.30	64.06	0.06

Table A15

Least Squares Estimates of Fixed Target Position

100 Samples per Cell σ = 0.333 mrad

Target Location: (50,10,2) Range = 51.03 nm

Initial Observer Velocity: <.16,0,0> nm/s

# Points	a=<.006,0,0>		a=<0,0,0>		a=<0,.006,0>		a=<0,-.006,0>	
on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
4	56.52	23.67	55.57	16.03	56.92	35.39	51.27	10.87
5	51.20	8.09	52.53	8.86	57.59	18.41	52.46	6.11
6	51.77	6.48	51.90	6.99	54.29	12.98	51.30	4.89
8	50.82	3.42	51.11	4.27	53.92	12.09	51.10	2.15
9	51.68	2.93	51.14	3.69	50.85	8.21	51.13	2.15
10	51.30	2.56	51.24	2.40	52.64	10.01	50.99	1.37
12	50.95	1.82	51.30	1.93	51.19	5.91	51.15	1.08
15	50.88	1.26	50.87	1.52	50.86	3.21	51.08	0.61
16	51.21	1.09	51.35	1.39	51.03	2.38	51.06	0.62
18	51.04	0.82	50.96	1.16	51.21	1.59	51.04	0.46
20	50.96	0.66	51.05	0.95	51.16	1.06	51.02	0.32
24	51.03	0.43	51.17	0.72	51.01	0.61	51.06	0.23
25	51.00	0.45	51.05	0.65	50.99	0.46	51.04	0.19
30	51.04	0.32	51.08	0.52	51.05	0.24	51.05	0.14
32	51.06	0.23	50.98	0.44	51.04	0.20	51.05	0.12
35	51.03	0.22	51.04	0.41	51.02	0.16	51.03	0.09
36	51.02	0.19	51.00	0.42	51.05	0.14	51.03	0.09
40	51.02	0.15	51.04	0.29	51.03	0.10	51.03	0.06
45	51.02	0.10	51.07	0.25	51.04	0.08	51.04	0.05
48	51.02	0.10	51.04	0.20	51.03	0.06	51.03	0.04
50	51.02	0.07	51.02	0.22	51.02	0.05	51.03	0.04

TABLE A16

Azimuth Only Least Squares Estimates of Fixed Target Position
100 Samples per Cell σ = 0.333 mrad
Constant Observer Velocity: <.16,0,0> nm/s

# Points on Track	T=(20,16,2) R = 25.69 $\bar{x}$	s	T=(30,24,2) R = 38.47 $\bar{x}$	s	T=(40,32,2) R = 51.26 $\bar{x}$	s	T=(50,40,2) R = 64.06 $\bar{x}$	s
4	25.70	1.00	38.55	2.15	50.69	3.58	62.59	5.47
5	25.59	0.65	38.54	1.54	51.35	2.92	64.06	4.69
6	25.65	0.53	38.39	1.09	50.89	2.24	63.85	3.02
8	25.63	0.31	38.36	0.76	51.06	1.38	63.73	2.02
9	25.72	0.26	38.38	0.66	51.21	1.08	63.85	1.63
10	25.70	0.23	38.44	0.43	51.30	0.96	63.58	1.28
12	25.65	0.17	38.48	0.35	51.09	0.71	63.87	0.99
15	25.67	0.12	38.42	0.27	51.20	0.52	63.99	0.80
16	25.70	0.11	38.51	0.25	51.13	0.45	64.12	0.73
18	25.68	0.08	38.44	0.22	51.24	0.38	63.95	0.54
20	25.68	0.07	38.46	0.18	51.13	0.33	63.96	0.52
24	25.69	0.05	38.49	0.13	51.22	0.24	64.02	0.43
25	25.68	0.05	38.46	0.12	51.22	0.22	64.07	0.35
30	25.69	0.04	38.47	0.10	51.21	0.18	64.03	0.26
32	25.69	0.03	38.46	0.08	51.26	0.13	64.03	0.21
35	25.69	0.03	38.47	0.07	51.25	0.12	64.05	0.21
36	25.69	0.03	38.46	0.08	51.26	0.12	64.01	0.19
40	25.69	0.02	38.47	0.05	51.26	0.11	64.04	0.17
45	25.69	0.02	38.48	0.05	51.26	0.08	64.03	0.15
48	25.69	0.02	38.47	0.04	51.27	0.07	64.05	0.12
50	25.69	0.01	38.47	0.04	51.27	0.07	64.07	0.11

# Points on Track	T=(60,48,2) R = 76.86 $\bar{x}$	s	T=(80,64,2) R = 102.47 $\bar{x}$	s	T=(100,80,2) R = 128.08 $\bar{x}$	s	T=(120,96,2) R = 153.69 $\bar{x}$	s
4	76.51	8.70	96.90	15.80	120.07	21.27	145.50	37.44
5	75.49	5.70	100.89	10.89	125.27	16.16	143.75	23.39
6	75.76	4.15	101.71	8.42	124.04	13.60	153.66	19.79
8	76.24	3.21	100.42	5.10	125.58	8.04	147.52	11.36
9	76.11	2.40	101.98	4.33	126.07	7.98	150.42	9.66
10	76.83	2.16	101.79	3.73	126.05	5.06	150.46	8.79
12	76.27	1.43	102.00	2.90	127.30	4.50	151.05	6.18
15	76.80	1.16	102.10	2.18	127.34	3.03	151.22	4.47
16	76.66	0.95	102.11	1.95	127.39	3.21	151.29	4.20
18	76.66	0.85	102.08	1.41	127.49	2.41	152.98	4.04
20	76.84	0.78	102.13	1.33	127.45	1.95	152.62	2.80
24	76.73	0.51	102.42	1.03	127.93	1.58	152.88	2.44
25	76.73	0.57	102.26	0.98	127.78	1.28	153.07	2.24
30	76.84	0.42	102.29	0.72	128.05	1.12	153.15	1.60
32	76.83	0.32	102.35	0.64	128.01	0.98	153.27	1.51
35	76.79	0.32	102.48	0.53	127.95	0.90	153.48	1.26
36	76.80	0.31	102.39	0.63	127.95	0.86	153.35	1.20
40	76.85	0.24	102.42	0.42	127.90	0.63	153.59	0.92
45	76.84	0.21	102.43	0.39	128.10	0.63	153.51	0.92
48	76.84	0.19	102.43	0.33	128.05	0.53	153.49	0.81
50	76.84	0.17	102.44	0.29	128.03	0.52	153.52	0.64

TABLE A17

Median Based Estimates of Fixed Target Position

Medians of all Pairs of Two-point Estimates

100 Samples per Cell σ = 0.333 mrad

Constant Observer Velocity: <.16,0,0>

# Points on Track	T = (50,10,2) R = 51.03 $\bar{x}$	s	T = (50,40,2) R = 64.06 $\bar{x}$	s
4	51.53	15.55	65.09	6.30
5	49.69	10.53	64.23	4.46
6	50.03	7.99	64.20	3.59
8	50.38	4.32	63.99	2.29
9	50.11	3.82	64.02	1.92
10	50.93	3.10	64.15	1.38
12	50.49	2.31	64.24	0.95
15	50.52	1.56	64.01	0.85
16	51.19	1.43	64.20	0.75
18	50.84	1.12	64.02	0.63
20	50.82	0.96	64.05	0.53
24	50.93	0.69	64.13	0.38
25	50.91	0.74	64.06	0.36
30	50.97	0.54	64.09	0.28
32	51.07	0.33	64.04	0.25
35	51.01	0.42	64.07	0.23
36	50.97	0.36	64.05	0.24
40	50.99	0.32	64.06	0.15
45	50.98	0.25	64.09	0.14
48	51.01	0.26	64.07	0.11
50	51.02	0.20	64.06	0.12

TABLE A18

Least Squares Estimates of Constant Velocity Target Parameters

100 Samples per Cell σ = 0.333 mrad

Target: Initial Position = (50,10,1) Range = 51.00

Various Velocities

Initial Observer Velocity: <.16,0,0>

Various Observer Accelerations

# Points on Track	a = <0,-.002,.006> V = <-.1,-.1,0> Range		Vx		a = <0,-.006,.002> V = <-.1,-.1,0> Range		Vx	
	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
15	50.77	5.37	-0.05	0.26	50.45	5.05	0.12	0.55
18	50.25	4.39	-0.07	0.18	49.03	4.68	0.10	0.35
20	50.66	3.10	-0.08	0.12	49.02	5.31	0.06	0.36
24	50.94	1.61	-0.09	0.08	50.35	3.28	-0.06	0.16
30	50.40	2.59	-0.08	0.07	50.36	2.83	-0.08	0.09
36	50.68	1.55	-0.09	0.03	50.77	1.93	-0.10	0.05
40	50.97	0.95	-0.10	0.02	50.80	1.64	-0.10	0.04
48	50.98	0.64	-0.10	0.01	51.03	0.72	-0.10	0.01
54	50.97	0.53	-0.10	0.01	50.81	1.28	-0.10	0.02
60	50.96	0.34	-0.10	0.00	51.08	0.45	-0.10	0.01
75	50.91	0.17	-0.10	0.00	51.01	0.26	-0.10	0.00

# Points on Track	a = <0,-.006,.002> V = <0,-.16,0> Range		Vy		a = <0,-.002,.006> V = <0,-.16,0> Range		Vy	
	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
15	53.92	9.81	-0.14	0.05	51.38	4.31	-0.17	0.02
18	50.71	3.80	-0.16	0.02	51.06	2.66	-0.16	0.01
20	51.08	2.88	-0.16	0.02	51.22	2.22	-0.16	0.01
24	51.16	1.27	-0.16	0.01	50.74	1.40	-0.16	0.00
30	50.96	0.82	-0.16	0.01	50.94	0.87	-0.16	0.00
36	51.16	0.58	-0.16	0.00	50.84	0.64	-0.16	0.00
40	50.96	0.49	-0.16	0.00	50.93	0.45	-0.16	0.00
48	50.99	0.45	-0.16	0.00	51.05	0.30	-0.16	0.00
54	50.87	0.35	-0.16	0.00	50.98	0.28	-0.16	0.00
60	50.88	0.30	-0.16	0.00	50.96	0.22	-0.16	0.00
75	50.99	0.19	-0.16	0.00	50.95	0.14	-0.16	0.00

TABLE A19

Least Squares Estimates of Constant Velocity Target Parameters
100 Samples per Cell = 0.333 mrad
Target: Initial Position = (50,40,1) Range = 64.04
Velocity:<-.1,-.1 ,0>
Initial Observer Velocity: <.16,0,0>
Various Observer Accelerations

	a = <0,.006,.006>				a = <0,-.006,.006>			
	Range		Vx		Range		Vx	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
15	65.33	5.61	-0.16	0.17	66.23	8.80	-0.16	0.19
18	65.08	3.72	-0.15	0.10	66.97	7.38	-0.15	0.14
20	64.80	3.05	-0.14	0.08	64.91	5.18	-0.12	0.09
24	64.14	2.24	-0.10	0.06	64.70	3.79	-0.11	0.06
30	64.47	1.55	-0.11	0.04	64.36	2.51	-0.11	0.04
36	63.99	1.15	-0.10	0.02	63.86	1.75	-0.10	0.02
40	63.91	1.11	-0.10	0.02	60.25	4.38	-0.05	0.05
48	63.99	0.70	-0.10	0.01	63.85	1.01	-0.10	0.01
54	64.09	0.64	-0.10	0.00	64.10	0.80	-0.10	0.01
60	63.98	0.44	-0.10	0.00	64.07	0.57	-0.10	0.01
75	63.96	0.20	-0.10	0.00	63.88	0.37	-0.10	0.00

	a = <0,.002,.006>				a = <0,.006,.002>			
	Range		Vx		Range		Vx	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
15	64.09	7.06	0.00	0.09	65.80	8.67	-0.02	0.05
18	63.11	5.10	-0.03	0.09	65.68	6.03	-0.02	0.06
20	62.74	4.24	-0.05	0.08	65.09	4.47	-0.01	0.03
24	64.39	2.98	-0.11	0.06	63.70	2.60	-0.02	0.06
30	64.02	1.92	-0.10	0.03	62.32	1.72	-0.01	0.05
36	64.19	1.39	-0.10	0.02	60.74	0.86	0.02	0.01
40	64.18	1.01	-0.10	0.01	61.42	2.08	-0.03	0.06
48	64.04	0.79	-0.10	0.01	63.76	1.00	-0.10	0.02
54	63.81	1.34	-0.10	0.02	61.72	3.34	-0.06	0.05
60	64.01	0.44	-0.10	0.00	64.08	0.52	-0.10	0.01
75	63.85	1.25	-0.10	0.01	64.02	0.29	-0.10	0.00

TABLE A19 (Continued)

Least Squares Estimates of Constant Velocity Target Parameters
100 Samples per Cell = 0.333 mrad
Target: Initial Position = (50,40,1) Range = 64.04
Velocity:<-.1,-.1 ,0>
Initial Observer Velocity: <.16,0,0>
Various Observer Accelerations

	a = <0,-.002,.006>				a = <0,-.006,.002>			
	Range		Vx		Range		Vx	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
15	67.07	9.70	-0.18	0.22	70.88	28.36	[illegible]	0.49
18	64.75	5.81	-0.11	0.11	64.74	10.64	-0.11	0.16
20	65.82	5.49	-0.13	0.10	63.04	10.28	-0.08	0.16
24	64.31	3.07	-0.11	0.05	58.48	6.47	-0.02	0.09
30	63.98	2.13	-0.10	0.03	56.56	1.83	-0.01	0.02
36	64.17	1.52	-0.10	0.02	55.47	1.59	0.00	0.02
40	63.98	1.35	-0.10	0.02	55.21	3.25	0.00	0.03
48	64.06	0.82	-0.10	0.01	60.50	5.69	-0.07	0.06
54	64.04	0.62	-0.10	0.01	61.86	4.49	-0.08	0.04
60	63.98	0.65	-0.10	0.01	64.03	1.04	-0.10	0.01
75	63.97	0.31	-0.10	0.00	64.00	0.62	-0.10	0.00

	a = <0,.004,.004>			
	Range		Vx	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s
15	65.16	8.70	-0.16	0.15
18	64.82	5.54	-0.16	0.11
20	64.91	4.04	-0.12	0.09
24	64.05	2.54	-0.11	0.06
30	63.85	1.76	-0.10	0.04
36	64.00	1.25	-0.10	0.02
40	64.04	0.98	-0.10	0.02
48	62.14	3.33	-0.07	0.05
54	64.05	0.64	-0.10	0.01
60	64.08	0.52	-0.10	0.01
75	63.95	0.31	-0.10	0.00

TABLE A19 (Continued)

Least Squares Estimates of Constant Velocity Target Parameters
100 Samples per Cell = 0.333 mrad
Target: Initial Position = (50,40,1) Range = 64.04
Velocity:<0,-.16 ,0>
Initial Observer Velocity: <.16,0,0>
Various Observer Accelerations

	a = <0,-.006,.002>				a = <0,-.002,.006>			
	Range		Vy		Range		Vy	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
15	70.62	35.56	-0.41	1.33	73.84	15.63	-0.41	0.43
18	66.85	6.95	-0.26	0.19	67.57	7.98	-0.26	0.22
20	66.86	6.00	-0.23	0.15	68.16	8.98	-0.25	0.20
24	64.72	2.47	-0.16	0.04	64.55	5.83	-0.17	0.12
30	62.60	2.34	-0.14	0.04	63.95	3.41	-0.16	0.05
36	61.26	2.50	-0.12	0.04	63.76	1.36	-0.16	0.02
40	61.68	2.65	-0.13	0.03	63.02	1.35	-0.15	0.02
48	61.83	3.77	-0.14	0.04	63.18	1.35	-0.15	0.01
54	64.90	0.96	-0.17	0.01	64.94	0.94	-0.16	0.01
60	63.16	0.82	-0.15	0.01	63.36	0.51	-0.16	0.00
75	63.29	2.61	-0.16	0.02	63.73	0.38	-0.16	0.00

TABLE A19 (Continued)

Least Squares Estimates of Constant Velocity Target Parameters
100 Samples per Cell σ = 0.333 mrad
Target: Initial Position = (50,40,1) Range = 64.04
Velocity:<-.2,-.2 ,0>
Initial Observer Velocity: <.16,0,0>
Various Observer Accelerations

	a = <0,.006,.006>				a = <0,-.006,.006>			
	Range		Vx		Range		Vx	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
15	65.42	6.09	-0.26	0.23	65.57	9.88	-0.25	0.25
18	64.44	3.98	-0.20	0.16	65.81	7.14	-0.24	0.15
20	63.95	4.51	-0.22	0.14	63.80	5.61	-0.20	0.12
24	64.45	2.90	-0.21	0.08	64.56	4.54	-0.21	0.08
30	64.34	2.03	-0.21	0.04	64.07	2.82	-0.20	0.04
36	64.20	1.46	-0.20	0.02	63.94	1.76	-0.20	0.02
40	63.97	1.10	-0.20	0.02	61.24	6.01	-0.16	0.08
48	64.03	0.70	-0.20	0.01	62.91	5.55	-0.19	0.07
54	64.18	0.59	-0.20	0.01	63.92	0.73	-0.20	0.01
60	64.03	0.39	-0.20	0.00	64.14	0.52	-0.20	0.01
75	63.94	0.17	-0.20	0.00	64.02	0.32	-0.20	0.00

	a = <0,.002,.006>				a = <0,.006,.002>			
	Range		Vx		Range		Vx	
# Points on Track	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
15	59.88	7.65	0.03	0.03	64.38	8.25	-0.03	0.06
18	59.86	4.20	-0.05	0.13	65.52	6.47	0.00	0.02
20	60.61	4.69	-0.07	0.13	63.68	3.89	-0.03	0.07
24	63.12	3.56	-0.17	0.08	61.91	2.55	-0.04	0.10
30	61.37	4.73	-0.14	0.10	59.81	2.39	-0.02	0.08
36	64.02	1.43	-0.20	0.02	56.13	2.20	0.02	0.06
40	63.87	1.18	-0.20	0.02	61.07	4.45	-0.13	0.10
48	64.12	0.78	-0.20	0.01	62.28	4.42	-0.17	0.07
54	64.08	0.51	-0.20	0.01	64.04	0.76	-0.20	0.01
60	64.00	0.48	-0.20	0.00	64.13	0.58	-0.20	0.01
75	63.95	0.28	-0.20	0.00	64.07	0.32	-0.20	0.00

TABLE A20

Least Squares Estimates of Constant Velocity Target Position
1000 Samples per Cell σ = 0.333 mr 40 Points/Track 10 Replicates
Target -- Location:(50,40,2) Range: 64.04 Velocity: <-.20,-.20,0>
Observer -- Initial Velocity:<.16,0,0> Acceleration:<0,-.006,.006>

$\bar{R}$	61.76	62.00	61.88	61.87	61.73	61.50	61.87	61.73	61.69	61.8
s	5.69	5.56	5.64	5.76	5.82	5.96	5.71	5.77	5.83	5.7
min	48.43	48.46	48.54	48.50	48.39	48.51	48.42	48.56	48.47	[illegible]
1%	48.66	48.67	48.69	48.64	48.70	48.68	48.73	48.70	48.67	48.7
5%	48.92	48.96	48.94	48.90	48.92	48.92	48.95	48.88	48.92	48.9
10%	49.12	49.16	49.08	49.09	49.09	49.08	49.12	49.08	49.08	49.1
15%	49.41	59.60	49.49	49.41	49.34	49.26	49.42	49.34	49.35	49.3
20%	62.03	62.25	62.16	62.17	62.08	61.53	62.13	61.80	61.78	62.0
25%	62.61	62.70	62.63	62.64	62.69	62.48	62.64	62.53	62.45	62.6
30%	62.94	63.04	62.95	62.93	63.00	62.78	62.99	63.02	62.77	63.0
35%	63.20	63.26	63.26	63.25	63.30	63.09	63.28	63.28	63.17	63.3
40%	63.41	63.48	63.53	63.51	63.45	63.35	63.48	63.45	63.49	63.5
45%	63.65	63.73	63.74	63.76	63.69	63.59	63.68	63.68	63.70	63.7
50%	63.85	63.94	63.92	63.93	63.88	63.79	63.92	63.85	63.88	63.9
55%	64.03	64.11	64.10	64.07	64.05	63.95	64.09	63.99	64.04	64.0
60%	64.18	64.27	64.28	64.26	64.29	64.15	64.27	64.15	64.24	64.2
65%	64.37	64.47	64.44	64.45	64.47	64.36	64.52	64.42	64.44	64.5
70%	64.54	64.71	64.64	64.79	64.66	64.60	64.77	64.59	64.71	64.7
75%	64.89	64.95	64.92	65.05	64.89	64.87	65.02	64.88	64.93	[illegible]
80%	65.12	65.20	65.17	65.34	65.19	65.17	65.24	65.24	65.21	65.2
85%	65.46	65.48	65.50	65.66	65.49	65.45	65.57	65.57	65.52	65.4
90%	65.74	65.90	65.86	66.04	65.87	65.87	65.95	65.92	65.93	65.9
95%	66.25	66.44	66.33	66.59	66.39	66.48	66.49	66.36	66.43	66.5
99%	67.39	67.52	67.26	67.57	67.45	67.59	67.53	67.49	67.46	67.4
max	68.78	68.56	68.73	69.22	69.75	68.50	69.26	68.35	68.62	69.1