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1. Introduction

A two-sided matching model is a game in which there are two types of agents, and the essential coalitions are singletons and doubletons containing one agent of each type. Over the years, they have become an important part of economic theory. One reason for this is that they have been deemed worthy models of economic markets with indivisible goods. ~~As far back as 1962,~~ Gale and Shapley modeled college admissions as a matching market. ~~A decade later,~~ Shapley and Shubik ~~[1972]~~ adopted a similar model for their housing market. ~~And still later,~~ Crawford and Knoer ~~[1981]~~ defined labor markets in these terms.

In all of these instances, the relevant solution concept is that of the core, which ~~Simply put, the core~~ is the set of economic allocations where no coalition of agents can improve their lot on their own. Herein lies another reason for the study of these games; the fact that their cores have many "nice" properties. For instance, their cores are always nonempty, ~~[Kaneko, 1982, Quinzii, 1984]. Indeed,~~ much of the literature [Gale & Shapley, Shapley & Shubik, Crawford & Knoer, Kelso & Crawford 1982] relate relatively simple algorithms which calculate core points.

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Another important idea is that in some of these games, the core and the set of economic equilibria are equivalent ~~(Shapley & Shubik)~~, and both are generally sets of positive measure. [This contrasts other economic models (with completely divisible goods) in which the set of equilibria is a proper subset of the core and is of measure zero.[1]] Thus, given a sets of agents' preferences, there is usually some latitude in choosing equilibrium prices. Intuitively, the amount of flexibility would be measured by the core's volume.

Other papers have examined the special structure of the core in these games. (Roth [1984], Demange & Gale [1985]) One thrust in this area is the idea that the core is "elongated". This means that, if restricted to the set of core allocations, the "fortunes of agents on the same side of the market tend to rise and fall together" (Shapley-Shubik).

In this paper, ~~we quantify~~ quantifies this last notion in the setting of Shapley and Shubik's housing market. (KR) We develop an easy-to-calculate, scalar measure, call the competition ratio, or CR, which measures the amount of "competition" in the market. Using economic and geometric arguments, we argue that CR is also a good measure of "elongation". In the process, we obtain simple bounds on the volume of the core, and also necessary and sufficient conditions for the volume to be zero.[2]

Finally, this analysis gives a method for finding bounds for the volume of polytopes of a certain class -- and this method involves only solving four linear programs.

The paper is organized as follows. Section 2 describes the model and gives background results. Section 3 motivates the issue of "elongation" and runs through some attempts to define the concept. In Section 4, we define our CR, give it an economic interpretation, and show how it relates to the core's volume. Finally, in Section 5, we propose areas for future study.

2. The Model[3/]

The Shapley-Shubik housing market is a model of a market for "large" indivisible goods, such as jobs, cars, or, naturally, houses. In such a setting, we assume agents only consider owning zero or one such object. However, they do distinguish amongst the houses, and different consumers may value the same house differently.

To facilitate house trading among the agents, there is a second, completely divisible good called money. An important assumption of the model is that utility is identified with money. In other words, any agent's valuation of any house can be expressed in dollars, and is independent of his income. This has two important ramifications. First, since utility is defined in terms of money, it follows that each agent values money the same. This property we call separability. Second, the above independence clause implies that the utility for money is linear.

In more general models (i.e., like those of Demange & Gale and Crawford & Knoer, where the above property of money is not assumed) it is assumed that money is not completely transferable; instead, it can only be transferred between agents who trade with each other. However, in terms of the core, the linearity and separability assumptions above imply that it does not matter whether or not we restrict monetary transfer in this way.[4/] Hence, to start, we will follow Shapley and Shubik and allow free exchange of money among all agents.

Now let us review Shapley and Shubik's model in detail. Suppose there are n homeowners in a market, and n prospective purchasers. Refer to them as <u>sellers</u> and <u>buyers</u> respectively. The allowable moves in the game are for any seller to sell his house to any buyer. No buyer may buy more than one house. The ith seller values his house at g_i dollars, while the jth buyer values the ith seller's house at h_{ij} dollars.[5/]

It is now possible to define the <u>characteristic function</u> $V(S)$ for any coalition S of buyers and/or sellers in the game:

$$V(S) = \left\{ \begin{array}{l} \text{The maximum net benefit that} \\ \text{players in } S \text{ can attain via} \\ \text{house sales within } S. \end{array} \right.$$

We now proceed to calculate $V(S)$. The simplest case occurs when S consists of one seller i and a buyer j. If $g_i > h_{ij}$, no sale should take place between them; the net benefit for either player is zero. Hence, $V(S) = 0$. However, if $g_i \leq h_{ij}$, then it is better for both i and j if i sells his house to j for price p_i satisfying $g_i \leq p_i \leq h_{ij}$. For any such p_i, seller i's net benefit will be $p_i - g_i \geq 0$, and buyer j's will be $h_{ij} - p_i \geq 0$, so $V(S)$ is equal to $(p_i - g_i) + (h_{ij} - p_i) = h_{ij} - g_i \geq 0$.

In summary, we have V [coalition of a seller i and a buyer j] = $\max(0, h_{ij} - g_i)$. Denote this value by c_{ij}, and note that $c_{ij} \geq 0$.

Upon reflection, we see that for all other coalitions, V will depend directly upon the one-seller-one-buyer case. This is because the best that any coalition can do is to "split up into separate trading pairs and pool the profit". Hence,

$$V(S) = \max_{\text{assignments } \mu} \sum_{\substack{(ij)\varepsilon\mu \\ i,j\varepsilon S}} c_{ij} \tag{2.1}$$

for any coalition S, where an <u>assignment</u> is defined as any sequence μ of seller-buyer pairs $[(i_1 j_1), \ldots, (i_m j_m)]$ containing <u>all</u> distinct players. At this point, a word on notation is important: if (ij) is a pair contained in assignment μ, we write $j = \mu(i)$ and $i = \mu^{-1}(j)$. Also, by a <u>full assignment</u>, or <u>matching</u>, we mean an assignment containing n pairs, i.e., one where every player in the game is used.

For the case where $S = N$, the <u>grand coalition</u> of all $2n$ sellers and buyers, equation (2.1) becomes

$$V(N) = \max_{\text{full assignments } \mu} \sum_{(ij)\varepsilon\mu} c_{ij}, \tag{2.2}$$

the evaluation of which is commonly known as "the assignment problem". We call any matching which solves (2.2) a <u>maximal matching</u>, and denote one by μ^*.

At this point, we are ready to define the core. It is defined as the set of economic allocations under which it is impossible for any coalition on its own to improve the lot of each of its constituents. Hence, the core defines a class of allocations which exhibit a certain kind of stability. Due to the assumption that utility (i.e., money) is completely transferable, we can write that the core is the set of sellers' utilities $(u_1,\ldots,u_n)$ and buyers' utilities $(v_1,\ldots,v_n)$ which satisfy:

$$\sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j = V(N) \tag{2.3}$$

$$\sum_{i\varepsilon S} u_i + \sum_{j\varepsilon S} v_j \geq V(S) \forall S \tag{2.4}$$

$$u_i,\ v_j \geq 0 \ \forall i,j \tag{2.5}$$

However, (2.1) and (2.2) imply this is equivalent to $(u_1,\ldots,u_n)$ and $(v_1,\ldots,v_n)$ satisfying

$$u_i + v_{\mu*(i)} = c_{i\mu*(i)} \tag{2.6}$$

$$u_i + v_j \geq c_{ij} \tag{2.7}$$

$$u_i, v_{\mu*(i)} \geq 0 \ \forall i,j \tag{2.8}$$

for any maximal matching $\mu*$.

Relations (2.6)-(2.8) provide a slightly different interpretation of the problem. Suppose the buyers and sellers have already arranged themselves into pairs $[(i\ \mu^*(i))]_{i=1}^n$. Then (2.6) and (2.8) represent what the pairs $[(i\ \mu^*(i))]$ can achieve for themselves (within the constraints of individual rationality). For this reason, we call (2.6) and (2.8) the <u>feasibility constraints</u>. On the other hand, the inequalities (2.7) describe limits placed on u_i and $v_{\mu^*(i)}$ based on what <u>other</u> pairs are capable of achieving. We call them the <u>stability constraints</u>.

Up to now, we have not tackled the question of how the concepts of core and economic equilibrium compare, or if core vectors even exist. The following theorem, again due to Shapley and Shubik, addresses these issues.

<u>Definition</u>: An <u>equilibrium</u> is a matching μ, a set of prices $[p_i]_{i=1}^n$, and a nonnegative set of utilities[6/] $(u_1,\ldots,u_n)$, $(v_1,\ldots,v_n)$ with

$$v_j = c_{\mu^{-1}(j)j} - p_{\mu^{-1}(j)} = \max_i c_{ij} - p_i \ \forall j \qquad (2.9)$$

$$u_i = \max_j c_{ij} - v_j [=p_i] \ \forall i \qquad (2.10)$$

This puts the usual equilibrium notion of utility maximization into this game's context.

<u>Theorem 2.1</u>: In the above context, an equilibrium exists with μ if and only if μ is a maximal matching. Moreover, the set of utility vectors under μ which satisfy (2.9) and (2.10) is precisely the set of vectors in the core.

Before carrying on with the proof, it is important to note the ramifications of the Theorem. First, by its definition, at least one maximal matching always exists. Hence, the core is always nonempty. The usual case is for there to be exactly one maximal matching. Thus, under only one μ will equilibria exist and under only one μ will the core relations (2.6)-(2.8) hold. Again, we denote this μ by μ^*.

Finally, the second statement in the theorem means that the core and the set of equilibria are equivalent.

<u>Proof</u>: Consider (P), commonly known as the <u>assignment linear program</u>:

$$\max \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} p_{ij} \tag{P}$$

$$\text{s.t.} \sum_{j=1}^{n} p_{ij} \leq 1 \tag{2.11}$$

$$\sum_{i=1}^{n} p_{ij} \leq 1 \tag{2.12}$$

$$p_{ij} \geq 0 \ \forall i,j \tag{2.13}$$

Let $[\bar{p}_{ij}]_{i,j=1}^{n}$ solve (P).

<u>Claim</u>: Evaluating (2) is equivalent to solving program (P).

<u>Proof</u>: It is clear that the Claim follows if (P) always solves with p_{ij}'s all equal to 0 or 1. However, this holds via a simple

perturbation argument, or the recognition that the matrix defined by (2.11) and (2.12) is totally unimodular.[7/]

Now consider any full assignment μ^* which satisfies

$$(ij)\varepsilon\mu^* \Rightarrow \bar{p}_{ij} = 1 \tag{2.14}$$

Clearly the set of μ's which satisfy (2.14) is just the set of maximal matchings. The next step, then, is to find utility vectors which satisfy (2.9) and (2.10). However, when we take the dual of (P), we obtain program (D):

$$\min \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \tag{D}$$

$$\text{s.t. } u_i + v_j \geq c_{ij} \tag{2.15}$$

$$u_i, v_j \geq 0 \ \forall i,j \tag{2.16}$$

Let $[\bar{u}_i]$ and $[\bar{v}_j]$ solve (D). (In general, there will be many such vectors.) Then, if $\mu^*(i) = j$,

$$\begin{aligned} \mu^*(i) = j &\Rightarrow \bar{p}_{ij} = 1 \\ &\Rightarrow \text{constraint (2.13) is "loose"} \\ &\Rightarrow \bar{u}_i + \bar{v}_j = c_{ij} \end{aligned} \tag{2.17}$$

by the complementary slackness theorem of linear programming.[8/]

Relations (2.15) and (2.17) respectively imply:

$$\bar{u}_i \geq c_{ij} - \bar{v}_j \quad \text{for} \quad j \neq \mu^*(i)$$

$$\bar{u}_i = c_{ij} - \bar{v}_j \quad \text{for} \quad j = \mu^*(i)$$

which together give (2.10). Similarly,

$$\bar{v}_j \geq c_{ij} - \bar{u}_i \quad \text{for} \quad i \neq \mu^{*-1}(j)$$

$$\bar{v}_j = c_{ij} - \bar{u}_i \quad \text{for} \quad i = \mu^{*-1}(j)$$

which implies (2.9). This concludes the "if" part of the theorem.

To prove the converse, suppose matching $\hat{\mu}$ supports equilibrium $[\hat{u}_i]_{i=1}^n$, $[\hat{v}_j]_{j=1}^n$. Then (2.9) and (2.10) together imply that (2.15) and (2.16) hold. Hence, by the weak duality theorem,[9]

$$\sum_{i=1}^{n} \hat{u}_i + \sum_{j=1}^{n} \hat{v}_j \geq \sum_{(ij)\varepsilon\mu} c_{i\mu(i)} \qquad (2.18)$$

for <u>any</u> matching μ. The left hand side of (2.18) represents the total net utility attained by the grand coalition under $\hat{\mu}$, while the right hand side represents the same quantity under μ. Hence, $\hat{\mu}$ is a maximal matching.

Finally, it is easy to see (and we have already pointed out) that (2.9) and (2.10) are equivalent to (2.6)-(2.8); thus the equivalence between core and equilibrium follows easily.

At this point it is helpful to present an example and see what the core looks like. Let the <u>joint utility matrix</u> (JUM) C be the $n \times n$ matrix defined by the c_{ij}'s. From the previous discussion, C is necessary and sufficient to define the core:

$$u_i + v_{\mu*(i)} = c_{i\mu*(i)} \tag{2.6}$$

$$u_i + v_j \geq c_{ij} \tag{2.7}$$

$$u_i, v_{\mu*(i)} \geq 0 \ \forall i,j \tag{2.8}$$

Note that (2.6) implies we can define the core solely in terms of the sellers' utilities $(u_1,\ldots,u_n)$:

$$0 \leq u_i \leq c_{i\mu*(i)} \tag{2.19}$$

$$u_i + c_{\mu*^{-1}(j)j} - u_j \geq c_{ij} \ \forall i,j \tag{2.20}$$

Hence, the core can be represented as an n-dimensional region in "u-space".[10/] As an example, suppose

$$C = \begin{pmatrix} 3 & 1 & 3 \\ 5 & 4 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

Clearly, $\mu* = [(11),(22),(33)]$, and the core is defined by the inequalities

$$\begin{array}{lll} 0 \leq u_1 \leq 3 & u_1 - u_2 \geq -3 & u_2 - u_3 \geq 2 \\ 0 \leq u_2 \leq 4 & u_1 - u_3 \geq 1 & u_3 - u_1 \geq -2 \\ 0 \leq u_3 \leq 2 & u_2 - u_1 \geq 2 & u_3 - u_2 \geq -4 \end{array}$$

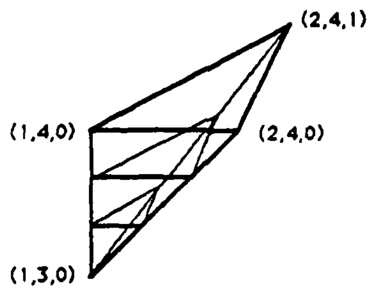


Figure 1

which is a polytope with four vertices (figure 1). Its volume is $\frac{1}{6}$.

3. Elongation and the Volume Ratio

Shapley and Shubik also made some interesting observations on the structure of the core. First, as previously noted, the core is an n-dimensional polytope. However, they observed further that it is "elongated, with its long axis orientated in the direction of market-wide price trends. There is a 'high price corner'...[where] every seller gets his top profit and every buyer his bottom. There is also a 'low price corner', where the reverse is true." All this is evident in Figure 1. The "high price corner", or sellers' optimum, is at $\underline{u} = (2,4,1)$, while the buyers' optimum is at (1,3,0).

The underlying mathematical concept here is that of sublattice. First, define the join of two vectors $\underline{x}$ and $\underline{y}$ by $\underline{z}$ where $z_i = \max(x_i, y_i)$, and the meet by $\underline{w}$ where $w_i = \min(x_i, y_i)$. Then, a sublattice is any region U in R^n where, if $\underline{u}'$ and $\underline{u}''$ are any two vectors in U, their meet and join are also.

<u>Theorem 3.1</u>: (Shapley-Shubik) The core is a sublattice.

<u>Proof</u>: Suppose $\underline{u}'$ and $\underline{u}''$ are in the core, and let $\underline{v}'$ and $\underline{v}''$ be the vectors of buyers' utilities defined by $\underline{u}'$ and $\underline{u}''$ respectively. Also let $\underline{u}^H$ and $\underline{u}^L$ be the join and meet respectively of $\underline{u}'$ and $\underline{u}''$. We need to show that $\underline{u}^H$ and $\underline{u}^L$ are in the core, and do this by demonstrating that $(\underline{u}^H,\underline{v}^L)$ and $(\underline{u}^L,\underline{v}^H)$ both satisfy (2.6)-(2.8). But for all i and j,

$$
\begin{aligned}
u_i^H &= \max(u_i',u_i'') \\
&= \max(c_{i\mu*(i)} - v'_{\mu*(i)}, c_{i\mu*(i)} - v''_{\mu*(i)}) \\
&= c_{i\mu*(i)} - \min(v'_{\mu*(i)}, v''_{\mu*(i)}) \\
&= c_{i\mu*(i)} - v^L_{\mu*(i)}, \\
u_i^L &= c_{i\mu*(i)} - v^H_{\mu*(i)} \quad \text{similarly,} \\
u_i^H + v_j^L &= \min(u_i^H + v_j', \ u_i^H + v_j'') \\
&\geq \min(u_i' + v_j', \ u_i'' + v_j'') \\
&\geq c_{ij}, \\
u_i^L + v_j^H &\geq c_{ij} \quad \text{similarly, and, finally,}
\end{aligned}
$$

u_i^H, v_j^L, u_i^L, and v_j^H are all nonnegative.

A note: an alternative to this proof exists through the realization that relations (2.19)-(2.20) satisfy the requirements for a theorem of Veinott's.[11]

So the core is a polytope and a sublattice. Intuitively, such a region "goes up and to the right". Hence, polytopes A, B, and C are all sublattices; polytope D is not because the joints of points

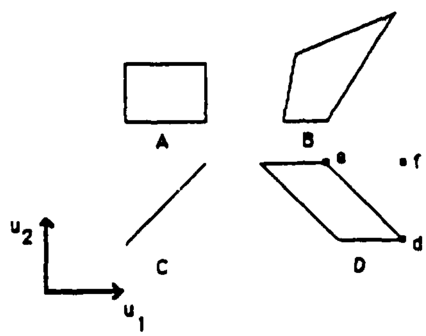


Figure 2

d and e is f, which is not contained in D. Finally, for a case in three dimensions, the reader can verify that the core in Figure 1 is also a sublattice.

What are the economic remaifications of this? First, recall the theorem from lattice theory which states that any compact sublattice has a <u>greatest point</u> and a <u>least point</u>.[12] This implies the existence of the sellers' and buyers' optima described above. From now on, denote these by $\underline{u}^H$ and $\underline{u}^L$ respectively.

Another important observation is that the "fortunes of all the sellers tend to rise and fall together". This makes intuitive sense -- if seller i' is selling his house for a high price, then seller i" is likely to be doing the same because he doesn't face stiff competition from i'. On the other hand, suppose i''s selling price is low. Then, if i" keeps his price high, he is likely to get left out in the cold, because i' will strike up a deal with i"'s buyer. Hence, i"'s price must be low as well.

We can sense this effect by examining level sets in Figure 1. For instance, if $u_2 = 3$, (u_1, u_3) must be equal to (1,0); if u_2 is raised to $\frac{7}{2}$, (u_1, u_3) can lie in the triangle defined by (1,0), $(\frac{3}{2}, 0)$, and $(\frac{3}{2}, \frac{1}{2})$ -- all of whose points are preferred over (1,0) by sellers 1 and 3. Finally, if $u_2 = 4$, the triangle is (1,0), (2,0), and (2,1).

However, all of the statements in this section so far have been qualitative. The theme of this paper is the question of just <u>how</u> "elongated" the core is, or to <u>what degree</u> the sellers' utilities correlate. For instance, if it resembles sublattice A, the core doesn't seem "elongated" at all. Knowledge of u_1 tells nothing about u_2, and vice versa. We say that the sellers' utilities are <u>uncorrelated</u>. A more typical case would be B, which is somewhat "elongated". Finally, C represents the opposite extreme from A -- the core has been so "stretched out" that is has become a line. In this <u>completely correlated</u> case, u_1 completely determines u_2 and vice versa.

Geometrically, there is a natural way to measure this effect. Let R be the rectangle defined via the following:

$$R = \{(u_1, \ldots, u_n): \; u_i^L \leq u_i \leq u_i^H \text{ for every } i\}.$$

Due to the nature of the sellers' and buyers' optima, it is easy to see that the core is completely contained in R. For this reason, we call R the enclosing rectangle.

The more "elongated" the core, the less room it takes up in its enclosing rectangle. So, define the volume ratio, or VR, by:

$$VR = \frac{\text{volume of core}}{\text{volume of enclosing rectangle}} \tag{3.1}$$

Clearly $0 \leq VR \leq 1$, and, from the above discussion, VR is a good inverse measure of "elongation". In the sublattice A, the lattice is the same as its enclosing rectangle; hence, VR attains its maximal value of one. In case B, VR is somewhere between 0 and 1. In completely correlated case C, VR = 0 because the two-dimensional volume of a line is equal to zero. Finally, in the example presented at the end of the last section, the reader may wish to verify that VR is equal to $\frac{1}{6}$. 13/

However, there are two problems with VR. First, consider how difficult it is to calculate. One must first find the volume of the core; this in turn means determining the intersection of $n^2 + n$ halfplanes, and then on top of that, finding the volume of the intersection. Second, VR has no direct economic meaning in terms of the given data (the c_{ij}'s) of the problem.

So we seek an easy to calculate, economically meaningful measure which at the same time is highly correlated with VR. Is this too tall an order? Our answer lies in the "next best alternative", i.e., the matching $\tilde{\mu}$ which for the entire economy is "next best" after, or, "completes the most with", μ^*. The idea is this: if $\tilde{\mu}$ yields a small utility, each of the members in each of the pairs of μ^* has a lot of "bargaining room"; hence, the core should be "fat" and resemble A. On the other hand, if $\tilde{\mu}$ yields a utility close to μ^*'s, the opposite effect occurs, and the core gets "skinny" and approaches C.

A two-by-two example illustrates the point:

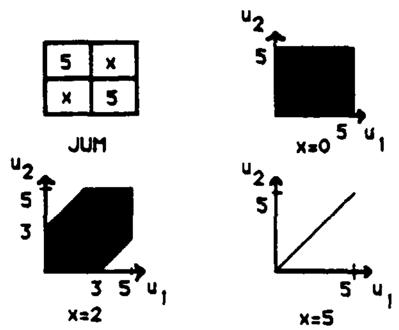


Figure 3

Consider the JUM in the upper left, and suppose $0 \leq x \leq 5$. $\mu^* = [(11),(22)]$, yielding a total utility of ten. The next best alternative (indeed, the only alternative) is $\tilde{\mu} = [(12),(21)]$, which gives $2x$. Now, if $x = 0$, $\tilde{\mu}$'s yield is minimal, and the core is a "fat" square. VR = 1. However, as x increases, $\tilde{\mu}$'s yield of $2x$ rises, and the core and VR both get smaller. Finally, when $x = 5$, $\tilde{\mu}$ has become a second maximal matching, and VR is equal to zero.

Thus, a good "competition ratio" would be

$$CR = \frac{\text{total utility from } \tilde{\mu}}{\text{total utility from } \mu^*} \tag{3.2}$$

Defined this way, CR appears to have the following properties

1) VR = 0 <=> CR = 1. (3.3)

2) VR = 1 <=> CR = 0. (3.4)

These seem plausible based on the above discussion, and we prove them formally later.

3) As the payoff for $\tilde{\mu}$ is [raised, lowered], VR [decreases, increases] and CR [increases, decreases]. (3.5)

Again, we prove a formal version of this in the next section after we define CR for higher dimensions.

Also, CR should satisfy another property. Suppose the cores of two games C_1 and C_2 are the same <u>shape</u> [but possibly displaced from one another in R^{n+}]. Obviously, $VR(C_1) = VR(C_2)$. Hence, we would like the measure CR to satisfy $CR(C_1) = CR(C_2)$. If so, CR is said to have <u>translation invariance</u>.

Translation invariance is easily accomplished by considering the <u>adjusted joint utility matrix</u>, or AJUM, denoted by C^A, where

$$c_{ij}^A = c_{ij} - u_i^L - v_j^L.$$

It is easy to check that $[u_i]_{i=1}^n$ and $[v_j]_{j=1}^n$ satisfy (2.6)-(2.8) with JUM C if and only if $[u_i - u_i^L]_{i=1}^n$ and $[v_j - v_j^L]_{j=1}^n$ satisfy (2.6)-(2.8) with JUM C^A.

Thus the shape of the core is the same for C^A as for C. C^A describes the unique case where $\underline{u}^L$ and $\underline{v}^L$ are "anchored" at $\underline{0}$. Any such C_1 and C_2 as described above will have the same AJUM. Hence, if we define CR for any matrix C in terms of C^A, we automatically accomplish translation invariance.

For this reason, from now assume <u>any</u> JUM is already an AJUM, and drop the superscript "A". Note that since $\underline{u} = \underline{0}$ and $\underline{v} = \underline{0}$ are in the core,

$$c_{ij} \leq c_{\mu^{*-1}(j)j}, \quad \text{and} \tag{3.6}$$

$$c_{ij} \leq c_{i\mu^*(i)} \quad \text{respectively} \tag{3.7}$$

for all i and j. Also, since the enclosing rectangle is now $\times_{i=1}^{n}[0, c_{i\mu^*(i)}]$, we can interpret it as the set of feasible $\underline{u}$'s. Thus, VR is measuring the proportion of feasible allocations which are stable equilibria. Hence, we can interpret it as a measure of "flexibility" for the economy. Finally, note that the volume of the enclosing rectangle is $\Pi_{i=1}^{n} c_{i\mu^*(i)}$. 14/

The next task is to generalize CR to n dimensions. This is not as simple as it seems, because there are a lot of choices for $\tilde{\mu}$ once $n > 2$. We end this section by considering two "intuitive" ways to define CR and show why they are unacceptable.

Based on the above discussion of the 2×2 case, it is logical to try:

$$CR^1 = \frac{\max_{\mu \neq \mu^*} \sum_{i=1}^{n} c_{i\mu(i)}}{\sum_{i=1}^{n} c_{i\mu^*(i)}}$$

CR^1 is the ratio of the global utilities of μ^* and the matching which attains the next highest global utility. But consider the following C_1:

$$C_1 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

[Note that C_1 is already an AJUM because it satisfies (3.6) and (3.7).] It is easy to see that C_1's core is the rectangle with $\underline{u}^L = (0,0,0)$ and $\underline{u}^H = (6,5,2)$, so CR should be equal to 0. However, $CR^1(C_1) = \frac{6}{13}$.

To account for this case, it might be tempting to try:

$$CR^2 = \frac{\max_{\mu:\mu(i)\neq\mu^*(i)\forall i} \sum_{i=1}^n c_{i\mu(i)}}{\sum_{i=1}^n c_{i\mu^*(i)}}$$

In this case $\tilde{\mu}$ is limited to matchings where everyone has a different partner from μ^*. Indeed, $CR^2(C_1) = 0$. However, suppose

$$C_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

From (2.20), any vector in the core must satisfy $u_1 + 1 - u_2 \geq 1$ and $u_2 + 1 - u_1 \geq 1$, which implies $u_1 = u_2$. Hence, the core is two dimensional, VR = 0, and so CR should be equal to one. But $CR^2(C_2) = \frac{1}{4}$.

4. The Competition Ratio

In this section we define our CR and prove some interesting properties. So let C be an $n \times n$ AJUM. In this entire section, and without loss of generality, relabel indices so that $\mu^*(i) = i$ for every i, i.e., the maximal matching is the "long diagonal". Also, abbreviate $c_{i\mu^*(i)}$ $\left(\text{or } c_{ii}\right)$ to c_i.

Now define the competition ratio by:

$$CR = \max_{\mu \neq \mu^*} \frac{\sum_{i:\mu(i) \neq \mu^*(i)} c_{i\mu(i)}}{\sum_{i:\mu(i) \neq \mu^*(i)} c_i} \tag{4.1}$$

Denote any argmax of (4.1) by $\tilde{\mu}$. Let the degree of $\tilde{\mu}$, denoted by d, be the cardinality of the set $\{i: \tilde{\mu}(i) \neq \mu^*(i)\}$. Observe $d \leq n$.

It is important to describe in words what CR is. First note that we are back to considering all matchings in the search for the next best alternative $\tilde{\mu}$. However, this time the search criterion is different: instead of minimizing the percentage loss in global utility from μ^* to $\tilde{\mu}$, we are minimizing the percentage loss in utility restricted to those who switch partners.

Immediately, we see that $0 \leq CR \leq 1$, and that in the 2×2 case, (4.1) reduces to (3.2). Also, we are relieved to find that $CR(C_1) = 0$ and $CR(C_2) = 1$, where C_1 and C_2 are the AJUMs defined at the end of the last section. Finally, CR of the matrix defined at the end of Section 2 is equal to $\frac{2}{3}$. 15/

Perturbation Effects on VR and CR

The following result defines a sense in which VR and CR are inversely related:

Theorem 4.1: Suppose C is an AJUM with competition ratio CR.

1) Suppose $i \neq j$. Then

   A) CR is nondecreasing in c_{ij}.[16/]

   B) VR is nonincreasing in c_{ij}.

2) For any i,

   A) CR is nonincreasing in c_i.

   B) VR is nondecreasing in c_i.

This is the promised analogue for (3.5) of the 2×2 case.

Proof: 1A) and 2A) are evident from the definition of CR (4.1). Now suppose first that C is given and then c_{ij} is raised, $i \neq j$. Consider the core inequalities (2.19)-(2.20). Since $i \neq \mu^*(j)$ and $j \neq \mu^*(i)$, only the right hand side of (2.20) is affected, so the volume of the core decreases (or stays the same). Since the enclosing rectangle remains the same, VR decreases also.

For the proof of 2B), let $i = 1$ without loss of generality. For any $\tilde{u} = (u_2,\ldots,u_n)$, let $[a_{\tilde{u}},b_{\tilde{u}}]$ be the interval $\{u_1: (u_1,\tilde{u})$ is in the core$\}$.[17/] Next, let

$$VR_{\tilde{u}} = \frac{\text{length of } [a_{\tilde{u}}, b_{\tilde{u}}]}{c_1}$$

$VR_{\tilde{u}}$ measures the proportion of u_1's in $[0, c_1]$ under which $(u_1, \tilde{u})$ is in the core. We have

$$VR = \frac{\text{volume of core}}{\text{volume of enclosing rectangle}}$$

$$= \frac{c_1 * \int_{\tilde{u}} VR_{\tilde{u}} d\tilde{u}}{\Pi_{i=1}^{n} c_i}$$

$$= \frac{\int_{\tilde{u}} VR_{\tilde{u}} d\tilde{u}}{\Pi_{i \neq 1} c_i}$$

Only the numerator is dependent on c_1, so it will suffice to show that an increase in c_1 causes an increase in $VR_{\tilde{u}}$ for every $\tilde{u}$.

So suppose c_1 is raised to $c_1 + \alpha$, resulting in a new game C^{α}. Let $(u_1, \tilde{u})$ be any vector in the core of C, i.e., solving (2.19) and (2.20). Then this implies $(u_1 + x, \tilde{u})$ solves (2.19)-(2.20) for C^{α}, where x is any number satisfying $0 \leq x \leq \alpha$. Hence $(a^C_{\tilde{u}}, \tilde{u})$ and $(b^C_{\tilde{u}} + \alpha, \tilde{u})$ lie in C^{α}'s core, so[18/]

$$VR^{C^{\alpha}}_{\tilde{u}} \geq \frac{b^C_{\tilde{u}} + \alpha - a^C_{\tilde{u}}}{c_1 + \alpha} \geq \frac{b^C_{\tilde{u}} - \alpha^C_{\tilde{u}}}{c_1} = VR^C_{\tilde{u}} \tag{4.2}$$

An Upper Bound for VR

The next step is to establish a quantitative link between VR and CR. One would think this is improbable, given that VR is dependent on n^2 variables and CR is a scalar. Nevertheless, we shall see that it is possible to establish meaningful bounds for VR in terms of CR.

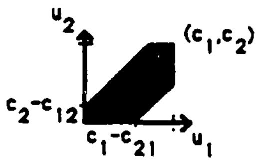
<u>Theorem 4.2</u>: $VR \leq 1 - CR^n$

<u>Proof</u>: We prove this by induction on the number of sellers n. If $n = 2$,

$$\begin{array}{|c|c|} \hline c_1 & c_{12} \\ \hline c_{21} & c_2 \\ \hline \end{array}$$

AJUM

$c_1 \geq c_{12}, c_{21}$
$c_2 \geq c_{12}, c_{21}$ by (26)-(27).



The shaded area at the left represents the core.

<u>Figure 4</u>

We have:

$$VR = 1 - \frac{c_{12}^2 + c_{21}^2}{2c_1c_2}$$

$$CR = \frac{c_{12} + c_{21}}{c_1 + c_2}$$

We need to show:

$$1 - \frac{c_{12}^2 + c_{21}^2}{2c_1c_2} \leq 1 - \left(\frac{c_{12} + c_{21}}{c_1 + c_2}\right)^2,$$

or, equivalently, that

$$(c_{12}^2 + c_{21}^2)(c_1 + c_2)^2 - 2c_1c_2(c_{12} + c_{21})^2 \geq 0.$$

But the left hand side of this is equal to

$$(c_1c_{21} - c_2c_{12})^2 + (c_1c_{12} - c_2c_{21})^2,$$

which is clearly nonnegative.

To prove the result for higher dimensions, consider two cases:

Case 1: $\tilde{\mu}(i) = \mu^*(i)$ for some seller i.

Without loss of generality, let $i = 1$, and consider the game C^- without seller 1 and buyer 1. Since we have not "disturbed" $\tilde{\mu}$, CR for this game is the same as for the larger one. Now let U be the n-dimensional region defined by

$$U = \{u_1 \in [0,c_1],(u_2,\ldots,u_n) \in \mathrm{core}(C^-)\}$$

Now,

$$\mathrm{vol}(U) = c_1 * \mathrm{vol}(\mathrm{core}(C^-))$$

$$= c_1 * \mathrm{VR}(C^-) * \prod_{i=2}^{n} c_i$$

It is also clear that the core of C is contained in U. Hence,

$$\mathrm{vol}(\mathrm{core}(C)) \leq \prod_{i=1}^{n} c_i * \mathrm{VR}(C^-)$$

$$\Rightarrow \mathrm{VR}(C) \leq \mathrm{VR}(C^-)$$

Finally, using the inductive hypothesis and the fact that $\mathrm{CR} \leq 1$,

$$\mathrm{VR}(C) \leq \mathrm{VR}(C^-) \leq 1 - \mathrm{CR}^{n-1} \leq 1 - \mathrm{CR}^n. \quad (4.3)$$

<u>Case 2</u>: $\tilde{\mu}(i) \neq \mu^*(i)$ for every i.

In order to prove the result for this case, it is first necessary to state two simple lemmas from the theory of convex functions.[19]

<u>Lemma 1</u>: Let $x_i \geq 0$ for all i. Then, for any integer $n > 0$,

$$n^n * \prod_{i=1}^{n} x_i \leq \left(\sum_{i=1}^{n} x_i \right)^n$$

<u>Lemma 2</u>: Let $x_i \geq 0$ for all i. Then, for any integer $n > 0$,

$$\sum_{i=1}^{n} x_i^n \geq n * \left(\frac{\sum_{i=1}^{n} x_i}{n} \right)^n$$

The logic of the proof for Case 2 is as follows:

1) Define n disjoint regions which are within the enclosing rectangle R but are <u>not</u> in the core.

2) Show that the ratio of the total area of these regions to that of R is at least CR^n.

<u>Lemma</u>: Let $\underline{u} = (u_1, \ldots, u_n)$ be any vector in R satisfying

$$u_i > u_{\tilde{\mu}^{-1}(i)} + c_i - c_{\tilde{\mu}^{-1}(i)i} \tag{4.4}$$

for some i. Then $\underline{u}$ is not in the core.

Proof: Rearranging the terms in (4.4) gives

$$u_{\tilde{\mu}^{-1}(i)} + c_i - u_i < c_{\tilde{\mu}^{-1}(i)i},$$

which is a direct contradiction of (2.20).

Now define the n regions $[R_i]$ by:

$$R_i = \left\{ \underline{u} \varepsilon R: \quad u_i > \left(\max_{k \neq i} u_k \right) + c_i - c_{\tilde{\mu}^{-1}(i)i} \right\}$$

By the preceding lemma, no point in R_i is in the core. Furthermore, because $c_i - c_{\tilde{\mu}^{-1}(i)i} \geq 0$, [by (3.6)], all of the R_i's are disjoint.

The next step is to find the area of the R_i's. To do this, we calculate R_1 as an example:

$$R_1 = \left\{ \underline{u} \varepsilon R: \quad u_1 > \left(\max_{k \neq 1} u_k \right) + c_1 - c_{\tilde{\mu}^{-1}(1)1} \right\}$$

Well, for any particular u_1, the $n - 1$ dimensional volume of

$$\left\{ (u_2, \ldots, u_n): \quad \left(\max_{k \neq 1} u_k \right) < u_1 - c_1 + c_{\tilde{\mu}^{-1}(1)1} \right\}$$

is obviously $(u_1 - c_1 + c_{\tilde{\mu}^{-1}(1)1})^{n-1}$. To get the volume of R_1, we need to integrate over possible values of u_1:

$$\text{vol}(R_1) = \int_{c_1 - c_{\tilde{\mu}^{-1}(1)1}}^{c_1} (u_1 - c_1 + c_{\tilde{\mu}^{-1}(1)1})^{n-1} du_1$$

$$= \frac{(c_{\tilde{\mu}^{-1}(1)1})^n}{n}$$

So the volume of all the R_i's is:

$$\sum_{i=1}^{n} \frac{(c_{\mu^{-1}(i)i})^n}{n}$$

Thus,

$$VR \leq 1 - \frac{\sum_{i=1}^{n} \frac{(c_{\mu^{-1}(i)i})^n}{n}}{\Pi_{i=1}^{n} c_i}$$

Applying lemma 1 gives

$$VR \leq 1 - \frac{n^n * \sum_{i=1}^{n} \frac{(c_{\mu^{-1}(i)i})^n}{n}}{(\sum_{i=1}^{n} c_i)^n}$$

Finally, applying lemma 2 gives

$$VR \leq 1 - \frac{(\sum_{i=1}^{n} c_{\mu^{-1}(i)i})^n}{(\sum_{i=1}^{n} c_i)^n}$$

which is indeed $VR \leq 1 - CR^n$.

Actually, there is an easy improvement to Theorem 4.2:

Corollary: Let d be the degree of μ. Then $VR \leq 1 - CR^d$.

Proof: Let $D = \{(i,j) : \mu(i) \neq \mu^*(i), j = \mu^*(i)\}$. Then consider the subgame C^D containing only the players in D. Using Case 2 above, $VR^{C^D} \leq 1 - CR^d$. One by one, add back into the model the players in C/D. Now the corollary follows by using the argument of Case

1, except without the final "$\leq$ " in (4.3).

A Lower Bound for VR

Theorem 4.3:

$$VR \geq (1 - CR)^n \tag{4.5}$$

Proof: This proof has three basic steps:

1) Define a class of Γ of matrices for which

$$VR \geq (1 - CR)^n \quad \text{for all} \quad C\varepsilon\Gamma$$

$$\Longrightarrow VR \geq (1 - CR)^n \quad \text{for all} \quad \text{AJUMs.}$$

2) Determine the structure of matrices in this class.

3) Using 2), define an area in the enclosing rectangle with a volume of $(1 - CR)^n * \Pi_{i=1}^n c_i$, all of whose points are in the core.

For the first part of the proof, it is necessary to define the concept of tightness. Suppose C is an AJUM with competition ratio CR, i.e.,

$$\sum_{i:\mu(i)\neq\mu^*(i)} c_{i\mu(i)} \leq \sum_{i:\mu(i)\neq\mu^*(i)} c_i CR \;\; \forall\mu \neq \mu^*, \text{ and} \tag{4.6}$$

$$\sum_{i:\mu(i)\neq\mu^*(i)} c_{i\mu(i)} = \sum_{i:\mu(i)\neq\mu^*(i)} c_i CR \quad \text{for some} \quad \tilde{\mu} \neq \mu^*. \tag{4.7}$$

Choose any $\hat{i}$ and $\hat{j}$, $\hat{i} \neq \hat{j}$, and let $M^{\hat{i}\hat{j}}$ be the set of full assignments in which $\mu(\hat{i}) = j$. If (4.6) holds with equality for some $\mu \epsilon M^{\hat{i}\hat{j}}$, it means that $c_{\hat{i}\hat{j}}$ cannot be raised without destroying the validity of (4.6)-(4.7), i.e., without raising CR. In this case, we say that $c_{\hat{i}\hat{j}}$ is "at its limit". On the other hand, if (4.6) holds with strict inequality for all $\mu \epsilon M^{\hat{i}\hat{j}}$, $c_{\hat{i}\hat{j}}$ can be raised without affecting CR. Perturb C by raising $c_{\hat{i}\hat{j}}$ until it reaches its limit, even if it means violating (3.6) or (3.7).

Continue by raising every c_{ij}, $i \neq j$, to its limit, and denote the resulting matrix by $\hat{C}$. We have:

$$\hat{c}_{ij} \geq 0 \ \forall i,j \tag{4.8}$$

$$\sum_{i:\mu(i) \neq \mu^*(i)} \hat{c}_{i\mu(i)} \leq \sum_{i:\mu(i) \neq \mu^*(i)} c_i CR \ \forall \mu \neq \mu^* \tag{4.6}$$

Also, for every $\hat{i}$ and $\hat{j}$, $\mu^{\hat{i}\hat{j}} \epsilon\, M^{\hat{i}\hat{j}}$ such that

$$\sum_{i:\mu^{\hat{i}\hat{j}}(i) \neq \mu^*(i)} \hat{c}_{i\mu^{\hat{i}\hat{j}}(i)} = \sum_{i:\mu^{\hat{i}\hat{j}}(i) \neq \mu^*(i)} c_i CR \tag{4.9}$$

Any $\hat{C}$ which satisfies (4.6), (4.8), and (4.9) is said to be <u>tight</u>. Alternatively, a tight matrix is one where no c_{ij}, $i \neq j$, can be raised without raising CR. In other words, given $[c_i]_{i=1}^n$ and CR, the partnerships in μ^* face a "maximal amount of competition." At any rate, note that $\mu^*(\hat{C}) = \mu^*(C)$, and of course that $CR(\hat{C}) = CR(C)$.

At this point it is useful to present an example:

$$C = \begin{pmatrix} 6 & 4 & 5 \\ 0 & 4 & 2 \\ 0 & 3 & 6 \end{pmatrix}$$

$$\hat{C} = \begin{pmatrix} 6 & 5 & 5 \\ 0 & 4 & 2 \\ 1 & 3 & 6 \end{pmatrix}$$

For both matrices, CR = 1/2. AJUM C is not tight because c_{31} and c_{12} have not reached their limits. By raising these coefficients to their limits, we arrive at $\hat{C}$, which is tight. Note that $\hat{C}$ is no longer an AJUM, because it violates (3.6) for $i=1$ and $j=2$.

Next, consider what has happened to VR as C was being transformed into $\hat{C}$. Through repeated applications of 1B) of Theorem 4.1, it has either stayed constant or decreased. Hence, if (4.5) holds for $\hat{C}$, then it holds for C. And if it holds for all $\hat{C} \varepsilon \Gamma$, Where Γ is the class of all tight matrices, it will hold for the entire set of AJUMs.

The next step is to get an idea of the structure of tight matrices. Given $[c_i]_{i=1}^n$, and CR, it turns out that Γ is an $n - 1$ dimensional object, easily parameterized by the values in $n - 1$ of a matrix's cells.

First, though, we need to state a lemma:

<u>Lemma</u>: Let η be a sequence of integer pairs $[(i_p\ j_p)]_p$, i, $j \varepsilon 1,\ldots,n$. Define:

$$I_k^p = \begin{cases} 1, & \text{if } i_p = k \\ 0, & \text{otherwise} \end{cases}$$

$$J_k^p = \begin{cases} 1, & \text{if } j_p = k \\ 0, & \text{otherwise} \end{cases}$$

Suppose for every $k \varepsilon 1,\ldots,n$

$$\sum_{\text{pairs } p} I_k^p = \sum_{\text{pairs } p} J_k^p. \qquad (4.10)$$

[Note that (4.10) means that for any $k \varepsilon 1,\ldots,n$, k appears equally many times as a first coordinate in η as it does a second coordinate.]

Then, by adding pairs of the form $(k\ k)$ to η we can form η^*, which is partionable into one or more matchings on $1,\ldots,n$.

<u>Example</u>: $n = 4$, $\eta = [(12), (12), (23), (31), (21)]$,
$\eta^* = [(12), (23), (31), (44) || (12), (21), (33), (44)]$.

The proof of this lemma is in Appendix 1.

<u>Lemma 1</u>: If $\hat{C}$ is tight, and $i \neq j$,

$$\hat{c}_{ij} + \hat{c}_{ji} = c_i CR + c_j CR. \qquad (4.11)$$

<u>Proof</u>: Suppose not. Without loss of generality, assume (4.11) fails with $i=1$ and $j=2$. In light of (4.6), this means

$$\hat{c}_{12} + \hat{c}_{21} < c_1 CR + c_2 CR. \qquad (4.12)$$

Next, since $\hat{C}$ is tight, matchings μ^{12} and μ^{21} such that [see (4.9)]

$$\mu^{12}(1) = 2$$

$$\sum_{i:\mu^{12}(i)\neq\mu^*(i)} \hat{c}_{i\mu^{12}(i)} = \sum_{i:\mu^{12}(i)\neq\mu^*(i)} c_i^{CR} \tag{4.13}$$

$$\mu^{21}(2) = 1$$

$$\sum_{i:\mu^{21}(i)\neq\mu^*(i)} c_{i\mu^{21}(i)} = \sum_{i:\mu^{21}(i)\neq\mu^*(i)} c_i^{CR} \tag{4.14}$$

Let $\eta = \{(ij):j=\mu^{12}(i)\neq\mu^*(i),\ i\neq 1\} \mathbin{++} \{(ij):j=\mu^{21}(i)\neq\mu^*(i),\ i\neq 2\}$, the symbol ++ meaning "take the union but save repeats". Clearly η satisfies (4.10), so let $\eta_1,\ldots,\eta_m$ be matchings partitioning η^* (see previous lemma). By (4.6),

$$\sum_{i:\eta_q(i)\neq\mu^*(i)} \hat{c}_{i\eta_q(i)} \leq \sum_{i:\eta_q(i)\neq\mu^*(i)} c_i^{CR} \quad \text{for } q\varepsilon 1,\ldots,m. \tag{4.15}$$

Adding inequalities (4.15) gives

$$\sum_{\text{pairs}\varepsilon\eta^*:j_p\neq\mu^*(i_p)} \hat{c}_{i_p j_p} \leq \sum_{\text{pairs}\varepsilon\eta^*:j_p\neq\mu^*(i_p)} c_{i_p}^{CR}, \quad \text{or}$$

$$\sum_{\substack{i:\mu^{12}(i)\neq\mu^*(i)\\ i\neq 1}} \hat{c}_{i\mu^{12}(i)} + \sum_{\substack{i:\mu^{21}(i)\neq\mu^*(i)\\ i\neq 2}} \hat{c}_{i\mu^{21}(i)}$$

$$\leq \sum_{\substack{i:\mu^{12}(i)\neq\mu^*(i)\\ i\neq 1}} c_i^{CR} + \sum_{\substack{i:\mu^{21}(i)\neq\mu^*(i)\\ i\neq 2}} c_i^{CR}. \tag{4.16}$$

But (4.13), (4.14), and (4.16) together imply

$$\hat{c}_{12} + \hat{c}_{21} \geq c_1 CR + c_2 CR,$$

which contradicts (4.12).

<u>Lemma 2</u>: If $\hat{C}$ is tight, (4.6) holds with equality for all μ, i.e.,

$$\sum_{i:\mu(i)\neq\mu^*(i)} \hat{c}_{i\mu(i)} = \sum_{i:\mu(i)\neq\mu^*(i)} c_i CR \quad \text{for every} \quad \mu\neq\mu^*. \qquad (4.17)$$

Note that this is much stronger than (4.9). Also, note that (4.11) is just the special case of (4.17) where $\mu=[(ij),(ji),\{(kk)\}_{k\neq i \text{ or } j}]$.

<u>Proof</u>: Again, suppose (4.17) does not hold, i.e.,

$$\sum_{i:\mu(i)\neq\mu^*(i)} \hat{c}_{i\mu(i)} < \sum_{i:\mu(i)\neq\mu^*(i)} c_i CR \qquad (4.18)$$

for some matching μ.

Define the "complementary matching" μ^c by

$$\mu^c(i) = j \iff \mu(j) = i.$$

By (4.6),

$$\sum_{i:\mu^c(i)\neq\mu^*(i)} \hat{c}_{i\mu^c(i)} \leq \sum_{i:\mu^c(i)\neq\mu^*(i)} c_i CR. \qquad (4.19)$$

Observe that the set of pairs $[(i,\mu^c(i))]_{i=1}^n$ is the same as $[(\mu(i),i)]_{i=1}^n$. Thus, we can rewrite (4.19) as

$$\sum_{i:\mu(i)\neq\mu^*(i)} \hat{c}_{\mu(i)i} \leq \sum_{i:\mu(i)\neq\mu^*(i)} c_{\mu(i)}CR. \tag{4.20}$$

Adding (4.18) and (4.20) gives

$$\sum_{i:\mu(i)\neq\mu^*(i)} \hat{c}_{i\mu(i)} + \hat{c}_{\mu(i)i} < \sum_{i:\mu(i)\neq\mu^*(i)} c_i CR + c_{\mu(i)}CR. \tag{4.21}$$

However, this is a contradiction because Lemma 1 implies (4.21) should hold with equality.

It is important to dwell for a bit on the ramifications of (4.17). Suppose $\hat{C}$ is a tight matrix, $\mu^*(i) = i$ for all i, and we are given CR and the diagonal elements $[c_i]_{i=1}^{n}$. Suppose further that we specify the elements $[a_i]_{i=1}^{n-1}$, where $a_i = \hat{c}_{i,i+1}$.

$$\hat{C} = \begin{pmatrix} c_1 & a_1 & & \cdots & & \\ b_1 & c_2 & a_2 & \cdots & & \\ & b_2 & c_3 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & \cdots & c_{n-1} & a_{n-1} \\ & & & \cdots & b_{n-1} & c_n \end{pmatrix}$$

Immediately (4.11) determines the values of the b_i's, $i=1,\ldots,n-1$. From there, we can use (4.17) to determine the rest of the $\hat{c}_{ij}$'s. For instance, we can calculate $\hat{c}_{13}$ from

$$\hat{c}_{13} + b_1 + b_2 = c_1CR + c_2CR + c_3CR.$$

In general it turns out that:

$$\text{If } j > i,\ c_{ij} = \sum_{k=1}^{j-1} a_k - \sum_{k=i+1}^{j-1} c_k CR \tag{4.22}$$

$$\text{If } j < i,\ \hat{c}_{ij} = \sum_{k=j}^{i} c_k CR - \sum_{k=j}^{i-1} a_k \tag{4.23}$$

As an exercise, the reader may wish to consider $\hat{C}$ from the last example, and verify (4.22)-(4.23) using $a_1 = 5$ and $a_2 = 2$.

Thus, Γ consists of the matrices which satisfy (4.22) and (4.23), with the further stipulation that all $\hat{c}_{ij}$'s so defined are nonnegative.

The next lemma is a curious result of the way we define CR.

<u>Lemma 3</u>: Suppose C is an AJUM with competition ratio CR. Then a column $\bar{j}$ for which

$$c_{i\bar{j}} \leq c_i CR \quad \text{for all } i \neq \bar{j}. \tag{4.24}$$

Suppose seller i considers selling his house to someone other than buyer i. Since CR is a measure of the dropoff in utility of next best alternatives, he should expect a yield of $c_i CR$ if he does this. Thus, if $c_{ij} \leq c_i CR$, buyer j will probably not be attractive to him. In fact, (4.24) means that poor buyer $\bar{j}$ won't appeal in this way to anyone. For this reason we call $\bar{j}$ a <u>weak buyer</u>.[20]

Proof: Again, suppose the lemma is false. Then for every column j, $i \neq j$ with

$$c_{ij} > c_i CR. \tag{4.25}$$

Now consider column $j=1$. Without loss of generality, suppose (4.25) holds with $i=2$, i.e., $c_{21} > c_2 CR$. Then c_{12} is not greater than $c_1 CR$, because, in that case,

$$c_{12} + c_{21} > c_1 CR + c_2 CR,$$

which contradicts (4.6).

Now let $j = 2$. Because $c_{12} \leq c_1 CR$, we know (4.25) does not hold with $i = 1$. So, again without loss of generality, suppose it holds with $i = 3$, i.e., $c_{32} > c_3 CR$. Then,

$$c_{13} \not> c_1 CR, \text{ because otherwise,}$$

$$c_{21} + c_{32} + c_{13} > c_1 CR + c_2 CR + c_3 CR,$$

$$\text{and } c_{23} \not> c_2 CR, \text{ because otherwise,}$$

$$c_{23} + c_{32} > c_2 CR + c_3 CR.$$

So, for $j=3$, (4.25) does not hold for $i=1$ or 2. So, without loss of generality, suppose $c_{43} > c_4 CR$...

Continuing to define row and column numbers in this fashion, we see that in general, for the kth column,

$$c_{ik} \not> c_i CR \text{ for all } i < k.$$

Thus, by the time k finally reaches n, we have that $c_{in} \geq c_i CR$ for all $i < n$, which is indeed (4.24) for $\bar{j} = n$.

<u>Corollary</u>: Suppose $\hat{C}$ is a tight matrix with competition ratio CR. Then row $\bar{i}$ which satisfies

$$\hat{c}_{\bar{i}j} \geq CR \quad \text{for all} \quad j \neq \bar{i}. \tag{4.26}$$

<u>Proof</u>: Let $\bar{j}$ be the column specified in the previous lemma, i.e.,

$$\hat{c}_{k\bar{j}} \leq c_k CR \quad \text{for any} \quad k \neq \bar{j} \tag{4.27}$$

Now let $\bar{i} = \mu^{*-1}(\bar{j}) = \bar{j}$. By (4.11),

$$\hat{c}_{\bar{i}k} + \hat{c}_{k\bar{j}} = c_{\bar{j}} CR + c_k CR \tag{4.28}$$

Finally, (4.27) and (4.28) together imply $\hat{c}_{\bar{i}k} \geq c_k CR$.

The last colollary states that if $\hat{C}$ is tight, there is a <u>strong seller</u> $\bar{i}$, for whom <u>all</u> buyers can provide more than c_i - CR. Interestingly, in μ^* he is paired with the weak buyer $\bar{j}$ described in the last lemma.

Thus, in the search for core allocations, it is natural to try to give $\bar{i}$ "a lot" and $\bar{j}$ "little". In fact, this is precisely what we do.

So let $\hat{C}$ be any tight matrix, defined by $[c_i]_{i=1}^{n}$, CR, and $[a_i]_{i=1}^{n-1}$. Without loss of generality, assume further that $\bar{i}$ (and thus $\bar{j}$) is equal to 1. Thus,

$$\hat{c}_{i1} \leq c_i CR \tag{4.29}$$

$$\hat{c}_{1j} \geq c_1 CR \tag{4.30}$$

for all $i,j \varepsilon 2,\ldots,n$.

Now let $0 \leq x_i \leq c_i - c_i CR$, $i=1,\ldots,n$, and let the point $\underline{u}$ be defined by

$$u_1 = c_1 CR + x_1$$

$$u_i = \sum_{k=1}^{i} c_k CR - \sum_{k=1}^{i-1} a_k + x_i$$

$$= \hat{c}_{i1} + x_i \text{ [see (4.23)]}, \ i=2,\ldots,n.$$

Notice that $U \equiv$ [set of $\underline{u}$'s which can be defined this way] has a volume of $\Pi_{i=1}^{n}(c_i - c_i CR)$. Thus, if we can show that any point in U is in the core, we'll have

$$VR \geq \frac{\Pi_{i=1}^{n}(c_i - c_i CR)}{\Pi_{i=1}^{n} c_i} = (1 - CR)^n,$$

which is, of course, the Theorem.

Before demonstrating that this is so, note how U's definition makes sense in light of the previous discussion concerning $\bar{i}$ and $\bar{j}$. In general, u_1 and v_1 must lie in $I_1 = [0,c_1]$. In U, u_1 (which is the utility for $\bar{i}$) is constrained to lie in $[c_1 CR, c_1]$, which is the high end of I_1. Menawhile, v_1 (the utility for $\bar{j}$) is in $[0, c_1 - c_1 CR]$, which is the low end.

At any rate, suppose $\underline{u}$ is any point in U. To show $\underline{u}$ is in the core, we first need to prove feasibility, i.e., that $0 \le u_i \le c_i$ for every i.

Case 1: i = 1. It is easy to see that u_1 ranges from $c_1 CR$ to c_1.

Case 2: i > 1. The lowest possible value for u_i, occurring when $x_i = 0$, is equal to $\hat{c}_{i1}$, which is of course nonnegative. The highest possible value is $\hat{c}_{i1} + c_i - c_i CR$, which is less than or equal to c_i because of (4.29).

Finally, we need to show stability, i.e., (2.20):

Case 1: i > j. Then,

$$u_i + c_j - u_j = \sum_{k=1}^{i} c_k CR - \sum_{k=1}^{i-1} a_k + x_i$$

$$+ c_j - \left(\sum_{k=1}^{j} c_k CR - \sum_{k=1}^{j-1} a_k + x_j \right)$$

$$= c_j + \sum_{k=j+1}^{i} c_k CR - \sum_{k=j}^{i-1} a_k + x_i - x_j$$

$$= \sum_{k=j}^{i} c_k CR - \sum_{k=j}^{i-1} a_k + c_j - c_j CR + x_i - x_j$$

$$\ge \sum_{k=j}^{i} c_k CR - \sum_{k=j}^{i-1} a_k \quad \text{by def. of } x_i, x_j$$

$$= \hat{c}_{ij} \quad [\text{see } (4.23)].$$

<u>Case 2</u>: $i < j$. Then,

$$u_i + c_j - u_j = \sum_{k=1}^{i} c_k CR - \sum_{k=1}^{i-1} a_k + x_i$$

$$+ c_j - \left(\sum_{k=1}^{j} c_k CR - \sum_{k=1}^{j-1} a_k + x_j \right)$$

$$= \sum_{k=1}^{j-1} a_k - \sum_{k=i+1}^{j} c_k CR + c_j + x_i - x_j$$

$$= \sum_{k=1}^{j-1} a_k - \sum_{k=i+1}^{j-1} c_k CR + c_j - c_j CR + x_i - x_j$$

$$\geq \sum_{k=i}^{j-1} a_k - \sum_{k=i+1}^{j-1} c_k CR$$

$$= \hat{c}_{ij}$$

This concludes the proof of the Theorem.

<u>Corollary</u>: The volume of the core is zero if and only if $CR = 1$.

<u>Proof</u>: Follows directly from the inequalities $(1 - CR)^n \leq VR \leq 1 - CR^n$.

Remember that $CR = 1$ is equivalent to the linear program (P) [section 2] being degenerate. Also, the same proof suffices to show that the core is the same as the enclosing rectangle $(VR = 1)$ iff $CR = 0$ (i.e., all of the off-diagonal elements of C are zeroes).

Mathematical and Computational Ramifications

Next, consider the issue of computational complexity. Given a Shapley-Shubik game C, let P(C) be the polytope which is the core (in "u-space") of C. Looking back over our work, we have defined a method for determining bounds on the volume of P(C) for any C. The first step is to solve linear program (P) [Section 2] and find μ^*. Next, we calculate C^A, which in turn necessitates finding $\underline{u}^L$ and $\underline{v}^L$. However, $\underline{u}^L$ is the solution to:

$$\min \sum_{i=1}^{n} u_i \qquad \text{(LPUL)}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j = \sum_{i=1}^{n} c_{i\mu^*(i)}$$

$$u_i + v_j \geq c_{ij} \quad \forall i,j$$

$$u_i, v_j \geq 0 \ \forall i,j.$$

Similarly, by changing the objective function from $\sum u_i$ to $\sum v_j$, we can present a linear program which solves with $\underline{v}^L$.

Hence, we can find C^A merely by solving three linear programs. In addition,

<u>Theorem 4.4</u>: The calculation of

$$CR = \max_{\mu \neq \mu^*} \frac{\sum_{i:\mu(i)\neq\mu^*(i)} c_{i\mu(i)}}{\sum_{i:\mu(i)\neq\mu^*(i)} c_i} \qquad (4.1)$$

can be done by solving a linear program.

<u>Proof</u>: Consider the <u>minimum cost/time cycle problem</u> (MCCP). This problem, presented in Dantzig, Blattner, and Rao (1967), is as follows. We're given a directed graph G, with node set $N = (1,\ldots,n)$. With each arc $(i \rightarrow j)$ is associated a <u>profit</u> π_{ij} and a <u>time</u> t_{ij}. The problem is to find a "min-cost/time ratio" cycle, i.e., a simple cycle Z which maximizes

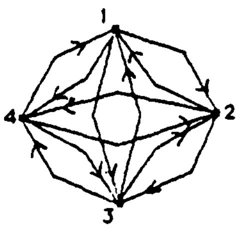
$$f(Z) \equiv \frac{\sum_{(i\rightarrow j)\varepsilon Z}\pi_{ij}}{\sum_{(i\rightarrow j)\varepsilon Z}t_{ij}}$$

<u>Lemma</u>: Suppose we solve (MCCP), where

1) $N = 1,\ldots,n$.

2) G is the directed graph where arc $(i \rightarrow j)$ exists if and only if $i \neq j$. For example, if $n = 4$,

G =



Note that arcs
i-->i (loops)
not allowed.

3) $\pi_{ij} = c_{ij}$ for all i,j.

4) $t_{ij} = c_i [= c_{ii} = c_{i\mu^*(i)}]$ for all i,j.

and suppose $\tilde{Z}$ is a solution. Define the matching $\tilde{\mu}$ by:

$$\tilde{\mu}(i) = \begin{array}{ll} j, & \text{if } (i \to j)\varepsilon\tilde{Z} \\ i, & \text{if } (i \to k)\not\varepsilon\tilde{Z} \ \forall k\varepsilon 1,\ldots,n. \end{array}$$

Then $\tilde{\mu}$ is an argmax to (4.1).

<u>Proof</u>: Suppose not. Then let $\hat{\mu}$ be an argmax to (4.1), with

$$\frac{\sum_{i:\hat{\mu}(i)\neq\mu^*(i)} c_{i\hat{\mu}(i)}}{\sum_{i:\hat{\mu}(i)\neq\mu^*(i)} c_i} > \frac{\sum_{i:\tilde{\mu}(i)\neq\mu^*(i)} c_{i\tilde{\mu}(i)}}{\sum_{i:\tilde{\mu}(i)\neq\mu^*(i)} c_i} = f(\tilde{Z})$$

Now define the cycle $\hat{Z}$ by:

$$[\hat{\mu}(i) = j, i\neq j] \implies (i\to j)\varepsilon\hat{Z}.$$

Then,

$$f(\hat{Z}) = \frac{\sum_{i:\hat{\mu}(i)\neq\mu^*(i)} c_{i\hat{\mu}(i)}}{\sum_{i:\hat{\mu}(i)\neq\mu^*(i)} c_i},$$

which in turn implies $f(\hat{Z}) > f(\tilde{Z})$. But this contradicts the definition of $\tilde{Z}$ as a solution to (MCCP).

<u>Lemma</u>: We can solve (MCCP) with a G, N, $\underline{\pi}$ and $\underline{t}$ defined as in 1)-4) above by solving the linear program:

$$\max \sum_{\substack{i,j=1 \\ i \neq j}}^{n} c_{ij} x_{ij} \qquad \text{(LPCR)}$$

$$\text{s.t.} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} c_{ii} x_{ij} = 1$$

$$\sum_{\substack{i=1 \\ i \neq k}}^{n} x_{ik} - \sum_{\substack{j=1 \\ j \neq k}}^{n} x_{kj} = 0 \ \forall k \varepsilon 1,\dots,n$$

$$x_{ij} \geq 0 \ \forall i,j[i \neq j].$$

<u>Proof</u>: See Dantzig, Blattner, and Rao, Theorem 1.

Thus, the linear program (LPCR) can be used to calculate CR in (4.1).

Next, define the sets

P_C^n = Polytopes P : P = P(C) for some n x n matrix C

$\mathbf{P}^n$ = Polytopes P : P = $\{\underline{u} \in R^n$ satisfying

$$u_i - u_j \geq d_{ij} \ \forall i,\ j \varepsilon 1,\dots,n$$

$$0 \leq u_i \leq c_i \ \forall i \varepsilon 1,\dots,n \zeta$$

for some $\underline{c} \varepsilon R^n$, $\underline{d} \varepsilon R^{n^2}$ }.

P_C^n is the class of all polytopes which can arise as cores of Shapley-Shubik games. From (2.19)-(2.20) [and a relabeling of columns so that $\partial^*(i) = i \ \forall i$], it is easy to conclude that $P_C^n \subseteq \mathbf{P}^n$. However, we also have:

<u>Theorem 4.5</u>: $P^n_C = P^n$.

<u>Proof</u>: We need to show $\mathbf{P}^n \subseteq P^n_C$. So, suppose we're given $\tilde{\mathbf{P}} \in \mathbf{P}^n$, defined by

$$u_i - u_j \geq \tilde{d}_{ij} \quad \forall i,j \tag{4.31}$$

$$0 \leq u_i \leq \tilde{c}_i \quad \forall i \tag{4.32}$$

Now define the JUM C by:

$$c_{ii} = \tilde{c}_i \quad \forall i \tag{4.33}$$

$$c_{ij} = \begin{cases} \tilde{c}_j + \tilde{d}_{ij} & \text{if } \tilde{d}_{ij} \geq -\tilde{c}_j; \quad \forall i,j \\ 0 & \text{otherwise} \end{cases} \tag{4.34}$$

<u>Claim</u>: $\mu^* = (1\ 1),\ldots,(n\ n)$ is a maximal matching for C.

<u>Proof</u>: Suppose not, and let $\hat{\mu} \neq \mu^*$ be a maximal matching. Let $D = \{i : \hat{\mu}(i) \neq \mu^*(i)\}$. Since $\hat{\mu}$ is maximal,

$$d \equiv \sum_{i \in D} \tilde{d}_{i\hat{\mu}(i)} > 0. \tag{4.35}$$

Now, since $\tilde{\mathbf{P}}$ exists, $(\tilde{u}_1,\ldots,\tilde{u}_n) \in n$ such that

$$\tilde{u}_i - \tilde{u}_{\hat{\mu}(i)} \geq \tilde{d}_{i\hat{\mu}(i)} \quad \forall i \in D \tag{4.36}$$

Adding inequalities (4.36) gives

$$0 = \sum_{i \in D} (\tilde{u}_i - \tilde{u}_{\hat{\mu}(i)}) \geq \sum_{i \in D} \tilde{d}_{i\hat{\mu}(i)} = d,$$

which contradicts (4.35).

Thus, the inequalities for the core P(C) are

$$u_i + \tilde{c}_j - u_j \geq [\tilde{c}_j + \tilde{d}_{ij}]^+ \ \forall i,j \tag{4.37}$$

$$0 \leq u_i \leq \tilde{c}_i \ \forall_i. \tag{4.38}$$

We now prove the Theorem by showing that $\underline{u}\varepsilon\tilde{\mathbf{P}} \iff \underline{u}\varepsilon P(C)$.

First, suppose $\underline{u}\varepsilon\tilde{\mathbf{P}}$. Then obviously (4.32) implies (4.38). To show that $\underline{u}$ also satisfies (4.37), consider two cases:

<u>Case 1</u>: $\tilde{c}_j + \tilde{d}_{ij} \geq 0$.
Then $u_i - u_j \geq \tilde{d}_{ij} \implies u_i + \tilde{c}_j - u_j \geq \tilde{c}_j + \tilde{d}_{ij} = [\tilde{c}_j + \tilde{d}_{ij}]^+$.

<u>Case 2</u>: $\tilde{c}_j + \tilde{d}_{ij} < 0$.
Then $u_i + \tilde{c}_j - u_j \geq 0$ because $u_i \geq 0$ and $u_j \leq \tilde{c}_j$.

Now suppose $\underline{u}\varepsilon P(C)$. Again, (4.38) implies (4.32). And again, in proving $\underline{u}$ also satisfies (4.31), there are two cases:

<u>Case 1</u>: $\tilde{c}_j + \tilde{d}_{ij} \geq 0$.
Then $u_i + \tilde{c}_j - u_j \geq [\tilde{c}_j + \tilde{d}_{ij}]^+ \implies u_i - u_j \geq \tilde{d}_{ij}$ by the converse argument to the previous Case 1.

<u>Case 2</u>: $\tilde{c}_j + \tilde{d}_{ij} < 0$.
Then, since $u_i \geq 0$ and $u_j \leq \tilde{c}_j$, $u_i - u_j \geq -\tilde{c}_j \geq \tilde{d}_{ij}$. This concludes the proof of the Theorem.

Thus, we can get the bounds $(1 - CR)^n \leq VR \leq 1 - CR^d$ for the volume of any poloyope $\tilde{P}$ in P^n in the following fashion. First, apply (4.33) and (4.34) to get "$C(\tilde{P})$", i.e., the Shapley-Shubik game in which $\tilde{P}$ is the core. Then, simply apply the four linear programs outlined previously to find CR and the bounds.

Also, note that since all four linear programs have on the order of (or less than) n^2 constraints, the amount of work done to get the bounds <u>is polynomial in the amount of problem data</u>.[21/]

Of course, the next logical question is "How sharp are the bounds?" The answer is "Not very, but they <u>do</u> tell us something." For results on a random sample of 15 problems, turn to Appendix 2.

5. Conclusion

In this paper, we have attempted to relate the mathematical and economic concepts of core, competition, correlation, and volume. And, as may be expected, we have raised many questions in doing so. The most obvious question is that of how good the bounds $(1 - CR)^n \leq VR \leq 1 - CR^d$ are, especially as n increases. The upper bound seems more informative for two reasons. First is the fact that d rises more slowly than n. (An interesting question is just how fast an "average" d does rise.) Second, since the maximum in (4.1) is taken over more and more μ's one would expect CR to increase toward one. Our results in Appendix 2 seem to corrborate this.

The lower bound $(1 - CR)^n$ appears to be worse on both of these counts. However, we still believe Theorem 4.3 is very significant.

This is because we proved the theorem by constructing U, a rectangular region wholly contained in the core. Remember that the length in the ith dimension of U is $(1 - CR)*c_i$. In other words, there exist intervals $\{I_i\}$ of length $\{(1 - CR)*c_i\}$, where, so long as $p_i \varepsilon I_i \; \forall_i, \underline{p}$ is an equilibrium price vector. Since by feasibility $p_i \varepsilon [0,c_i]$, the factor $(1 - CR)$ expresses a measure of (a lower bound of) "flexibility" for p_i. Indeed, this was one of the ideas expressed way back in the Introduction.

Nevertheless, a meaningful problem is that of improving the lower bound. This we hope to accomplish through further study of either the geometry of the core or the set of tight matrices. Another way is to try and define how individual players' welfare correlate, both to other individuals and to the set of other agents as a whole. This would be an improvement over CR, which is a global measure of the maximal amount of competition in the game. The new version of "CR" would be a vector, and the increased detail would probably give a more precise range for VR.

Finally, we would like to extend the above analysis to the more general case where linearity and separability assumptions (see section 2) do not hold. There is some hope here -- for instance, Demange and Gale have proved that the core is still a sublattice. Such work we hope to do in future research.

APPENDIX 1

Lemma: Let η be a sequence of integer pairs $[(i_p\ j_p)]_p$, $i, j \varepsilon 1,\ldots,n$. Define:

$$I_k^p = \begin{cases} 1, & \text{if } i_p = k; \\ 0, & \text{otherwise} \end{cases}$$

$$J_k^p = \begin{cases} 1, & \text{if } j_p = k; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose for every $k \varepsilon 1,\ldots,n$

$$\sum_{\text{pairs } p} I_k^p = \sum_{\text{pairs } p} J_k^p \qquad \text{(A1)}$$

Then, by adding pairs of the form (k,k) to η, we can form η^*, which is partitionable into one or more matchings on $1,\ldots,n$.

Proof: Use the following algorithm:

1). Set $\eta^* = \emptyset$, and consider all pairs $p \varepsilon \eta$ as unused.

2). Set $K = \emptyset$ and consider any unused pair $p = (ij) \varepsilon \eta$. If there are no unused pairs, STOP.

3). Let $K = K \quad \{i\}$, and let $\eta^* = \eta^* {+}{+}\, p$. (The symbol ++ means 'take the union but save repeats.') The pair p is now used. If $j \varepsilon K$, go to 5). If not, continue to 4).

4). Now find an unused pair $p^* = (i^*\ j^*)$ with $i^* = j$ and $j^* \neq j$. p^* exists by virtue of (A1). Now set $p = p^*$, $i = i^*$, and $j = j^*$, and return to 3).

5). Add (kk) to η^* for all $k \notin K$. Go to 2).

It is easy to see that this algorithm actually forms the set of assignments specified in the lemma's statement. [The number of such assignments in η^* is just the number of times we pass through step 5)].

APPENDIX 2

We generated fifteen JUMs by choosing random integers from the interval [0,999] for each cell in each JUM.

n	CR	d	$(1-CR)^n$	$1-CR^d$
2	.32	2	.46	.90
2	.64	2	.13	.59
2	.64	2	.13	.59
2	.51	2	.24	.74
2	.18	2	.67	.97
4	.95	2	$(.05)^4$	.10
4	.57	4	.03	.89
4	.71	2	.007	.49
4	.90	3	.0001	.28
4	.74	2	.006	.47
8	.93	5	$(.07)^8$	.32
8	.9994	3	$(.0006)^8$	.00
2				
8	.94	2	$(.06)^8$	.13
8	.92	4	$(.08)^8$	.27
8	.86	2	$(.14)^8$	.26

Source: Handbook of Tables for Mathematics, 4th edition, Cleveland: Chemical Rubber Co., 1970, p. 976.

FOOTNOTES

1/ See for instance, Debreu, Chapters 11 and 14.

2/ Operations researchers will recognize the core as the set of dual solutions to the "assignment" linear program.

3/ For a more thorough discussion of the model, the reader is referred to Shapley and Shubik, 1972.

4/ This is borne out when we reduce the core equations to (2.6)-(2.8).

5/ The model here extends without loss of generality to the case where there are m homeowners and n prospective purchasers, $m \neq n$, because we can add "dummy" sellers (i.e., those with houses which all buyers value at zero) or "dummy" buyers (those who value all houses at zero) to the model. See also footnote 14.

6/ By utilities here we of course mean "net utilities", which for sellers i are their true utility minus g_i.

7/ Hoffman and Kruskal, p. 225.

8/ Dantzig, pp. 135-6.

9/ Dantzig, p. 130.

10/ We could of course also represent the core as an n-dimensional region in "v-space". In fact, all of the subsequent analysis holds for the "v-space core" as well.

11/ Veinott, Section 2.4, "Finite Meet Representation of Sublattices".

12/ Veinott, Section 1.2, "Partially Ordered Sets".

13/ The enclosing rectangle R is the cube with vertices (1,3,0), (1,3,1), (1,4,0), (2,3,0), (2,4,0), (2,3,1), (1,4,0), and (2,4,1), which has a volume of 1. Hence, $VR = \frac{1/6}{1} = \frac{1}{6}$.

14/ Suppose the market originally had m sellers and n buyers, $m < n$. When "dummy" players were added, $c_{i\mu^*(i)} = 0$ for $n - m$ i's, say for $i \varepsilon m + 1,\dots,n$. Then (3.6) and (3.7) imply that $c^A_{ij} = 0 \; \forall(i,j)$: $i \varepsilon m + 1,\dots,n$ and $\forall(ij)$: $j \varepsilon \mu^*(m + 1),\dots,\mu^*(n)$.

Thus removing players $\{(i\ \mu^*(i)\}_{i=m+1}^{n}$ from the game does not affect the core in m-dimensional space, and our results concerning VR and CR can apply for this smaller, $m \times m$ game.

15/ This is evident once we realize the AJUM is

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

16/ In all four parts of this theorem, we assume the perturbations are small enough so as not to change μ^*.

17/ This region is an interval because the core is a polytope.

18/ Discerning readers will note that the first "$\geq$" relation in (4.2) actually holds with equality.

19/ Hardy, Littlewood, and Polya, p. 17 and 26.

20/ The idea of "ranking" the buyers and sellers, in a different context, has been raised by Shapley [1962] and more recently by Mo [1986].

21/ The fact that the linear programming problem is polynomial in the amount of problem data is due to Khachian (1979) and is discussed in Papadimitriou and Steiglitz (1982), Section 8.7.

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