Consider a Brownian motion with a downward drift of rate $a$. Its maximum over all time has the exponential distribution with parameter $2a$. Our aim is to study this maximum as a stochastic process indexed by $a$. That process is related to the convex majorant of the standard Brownian motion and, through the latter, to a Poisson random measure. This connection is exploited to obtain various distributional results. The results are of interest in queueing theory.
SUNSET OVER BROWNISTAN*

by

ERHAN ÇINLAR

Princeton University
Department of Civil Engineering and Operations Research
School of Engineering and Applied Science
Princeton, New Jersey 08544

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Abstract

Consider a Brownian motion with a downward drift of rate $a$. Its maximum over all time has the exponential distribution with parameter $2a$. Our aim is to study this maximum as a stochastic process indexed by $a$. That process is related to the convex majorant of the standard Brownian motion and, through the latter, to a Poisson random measure. This connection is exploited to obtain various distributional results. The results are of interest in queueing theory.
considered by NEWELL [4] and by COFFMAN, KADOTA, and SHEPP [2], the latter viewing
the model as that of storage allocation in computer memory.

Let \( Q_i(n) \) be the random variable that is 0 or 1 according as the stall \( n \) is empty or occupied
at time \( t \). The random vector \( \bar{Q}_t = (Q_1(1), Q_1(2), \ldots) \) is the state of the system at time \( t \). The process \( \sum_n \bar{Q}_t(n); \ t \geq 0 \) is the queue size process in an \( M/M/\infty \) system; it is regenerative, and 0 is
a regeneration state for it. It follows that the vector \( (0, 0, \ldots) \) is a regeneration state for
\( \{Q_i; \ t \geq 0\} \) and that the latter has an equilibrium distribution. Let \( Q \) be a random vector (of
zeros and ones) whose law is that equilibrium distribution.

The distribution of \( \sum_n Q(n) \) is Poisson with mean \( \lambda \). The distribution of \( \sum_n^n Q(n) \) is the
equilibrium distribution of the queue size process in the \( M/M/m/m \) system with arrival rate \( \lambda \) and
service rate 1; thus, that distribution is the conditional distribution of \( \sum_n Q(n) \) given that
\( \sum_n Q(n) \leq m \). Other than these facts and a few conclusions that can be drawn from them by elementar
probabilistic considerations, there is not much known about the distribution of \( Q \).

For \( a > 0 \), let \( \lambda^{1/4} Y(a,t) \) be the number of empty stalls at time \( t \) among those labeled
with \( n < \lambda - a \sqrt{\lambda} \). ALDOUS [1] has shown that the process \( \{Y(a,t); \ a > 0, \ t \geq 0\} \) converges
weakly, as \( \lambda \to \infty \), to a process \( \{Y(a,t); \ a > 0, \ t \geq 0\} \), which he identified and showed that, in
the limit as \( t \to \infty \), converges weakly to the process

\[
Y(a) = \max_{t \geq 0} (\sqrt{2} B_t - a), \quad a > 0,
\]

where \( B \) is the standard Brownian motion. He calls \( \{Y(a); \ a > 0\} \) the exponential process, after
the well-known fact that \( Y(a) \) has the exponential distribution with mean \( 1/a \) for each \( a \).

Our main contribution is to supply the probability law of \( Y \) in simpler terms. For this pur
pose we choose to work with

\[
(1.1) \quad Z_a = \sqrt{\frac{1}{2} a} Y(\sqrt{2} a) = \max_{t \geq 0} (B_t - a t), \quad a > 0,
\]

and let \( D_a \) be the last time \( t \) at which \( B_t - a t \) touches its zenith \( Z_a \), that is,
\[(1.2) \quad D_a = \sup \{ t: B_t - at = Z_a \}, \quad a > 0. \]

It turns out that \( D_a \) is the left-derivative of \( Z \) at \( a \) and, thus, is related to the density of empty stalls in the parking lot, in equilibrium, around \( \lambda - a \lambda^{3/4} \) for large \( \lambda \).

The next section contains a few simple geometric observations. First we relate the process \((D, Z)\) to the convex majorant of the Brownian motion \( B \). Using the hard results of GROENEBOOM [3] and PITMAN [5] about the latter, we are able to express \( D \) and \( Z \) in terms of a Poisson random measure on \((0, \infty) \times (0, \infty)\). It follows, in particular, that \( D \) has non-stationary independent increments, and that \((D, Z)\) is a non-homogeneous Markov process.

The process \( a \rightarrow Z_a \) is continuous, concave, and decreases from its limit \(+\infty\) at \( a = 0^+ \) to its limit \( 0 \) at \( \infty \). Therefore, its "hitting time" process

\[(1.3) \quad A_z = \inf \{ a : Z_a < z \}, \quad z > 0, \]

is the functional inverse of \( Z \). It turns out that the process \( A \) has the same probability law as \( Z \). This observation is also put in the next section.

The last section is devoted to computational issues. We compute the joint distribution of \( D_a \) and \( Z_a \) and also the transition function of the Markov process \((D, Z)\).
2. ZENITH PROCESS

The problem with the definition (1.1) of $Z_a$ is that it suggests re-drawing the path $t \rightarrow B_t - at$ if we wish to vary $a$. The following observation circumvents the problem:

\[
Z_a = \inf \{x > 0 : x + at > B_t \text{ for all } t \geq 0\}.
\]

Obviously, this is a re-wording of (1.1), but the mental picture it suggests is much more convenient for manipulating $a$: the line $t \rightarrow Z_a + at$ is the infimum of all lines of slope $a$ that never touch $B$. This picture is drawn in Figure 1 below.

![Figure 1](image-url)
Let $C$ denote the convex majorant of $B$, that is, the minimal convex path that dominates $B$ (see Figure 1 again). Note that $Z_a$ and $D_a$ are determined by $C$: the line $t \to Z_a + at$ is the infimum of all lines of slope $a$ that never touch $C$, and $D_a$ is the last time at which that line touches $C$. In fact, for fixed $a > 0$, $D_a$ is almost surely the only time $t$ with $C_t = Z_a + at$.

It is known (see GROENEBOOM [3] for instance) that $C$ is continuous and piecewise linear. The countable collection of its vertices has, almost surely, only one accumulation point, namely $(0,0)$. Fix an $a > 0$; note that $(D_a, Z_a + aD_a)$ is a vertex; let $T_0, T_1, \ldots$ be the lengths of successive intervals of linearity going to the right from $D_a$; let $T_{-1}, T_{-2}, \ldots$ be those to the left; and let $S_i$ be the slope of $C$ over the interval whose length is denoted by $T_i$. The following major result was obtained by GROENEBOOM [3]; a simpler proof using the excursions of $B$ may be found in PITMAN [5].

(2.2) THEOREM. The pairs $(S_i, T_i), i \in Z$, form a Poisson random measure $N$ on $(0,\infty) \times (0,\infty)$ whose mean measure is

\begin{equation}
\nu(ds, dt) = \frac{ds}{s} \gamma_s(dt),
\end{equation}

where $\gamma_s$ is the gamma distribution with shape index $1/2$ and scale parameter $s^{3/2}$ (the corresponding mean is $1/s^2$).

The probability law of a Poisson random measure is determined by its mean measure. Thus, the following specifies the probability law of $(D, Z)$. For computational purposes, the representations given here for $D_a$ and $Z_a$ are the key starting points.

(2.4) PROPOSITION. For each $a > 0$,

\begin{equation}
D_a = \int_{[a,\infty) \times (0,\infty)} N(ds, dt) \, t,
\end{equation}

\begin{equation}
Z_a = \int_0^a D_s \, ds = \int_{[a,\infty) \times (0,\infty)} N(ds, dt) \, (s - a) \, t.
\end{equation}
The process $D$ has non-stationary independent increments. The process $(D, Z)$ is a temporally non-homogeneous Markov process.

**PROOF.** First note that (see Figure 1)

$$D_a = \sum_i T_i \mathbf{1}_{[a, \infty)}(S_i),$$

$$Z_a + a D_a = B(D_a) = \sum_i S_i T_i \mathbf{1}_{[a, \infty)}(S_i).$$

Expressed in terms of the Poisson random measure $N$, these become (2.5) and (2.6). The remaining statements are immediate from the independence of the restrictions of $N$ to disjoint Borel sets.

Figure 2 below shows the qualitative features of $D$: it is piecewise constant, left-continuous, and decreases from its limit $+\infty$ at $a = 0+$ to its limit 0 at $+\infty$. It follows from (2.6) that $Z$ is continuous, concave, piecewise linear, and decreases from its limit $+\infty$ at $a = 0+$ to 0 at $+\infty$.

![Figure 2](image-url)
It was noted by ALDOUS [1] that, for each \( c > 0 \), the process \((c^2 D_{c_0}, cZ_{c_0})_{x > c}\) has the same probability law as \((D, Z)\). This can be seen from the preceding characterization: \( a \to c^2 D_{c_0}\) jumps at the points \( S_i/c\) by the amounts \( c^2 T_i\); the pairs \((S_i/c, c^2 T_i)\) form a Poisson random measure that has the same mean measure as \(N\); hence, \( a \to c^2 D_{c_0}\) has the same law as \(D\).

We end this section with an observation on the process

\[(2.7)\]
\[A_z = \inf \{ a : Z_a < z \}, \quad z > 0.\]

Obviously, \( z \to A_z \) is the functional inverse of the one-to-one mapping \( a \to Z_a \) of \((0, \infty)\) onto \((0, \infty)\). It follows that the qualitative picture of \( A \) is exactly that of \( Z \). In particular, \( A \) is piecewise linear and

\[(2.8)\]
\[\hat{D}_z = \lim_{\varepsilon \to 0} \frac{A_z + \varepsilon - A_z}{\varepsilon} = \frac{1}{D(A_z)}, \quad z > 0.\]

The process \(\hat{D}\) is piecewise constant, right-continuous, decreasing.

\[(2.9)\] **PROPOSITION.** The process \((\hat{D}, A)\) has the same probability law as the process \((D, Z)\). In particular, the collection \(\{Z(S_i) ; i \in Z\}\) has the same law as the collection \(\{S_i ; i \in Z\}\); they form Poisson random measures on \((0, \infty)\) with mean measure \(ds/s\).

**PROOF.** We put (2.7) and (2.1) together and manipulate:

\[A_z = \inf \{ a : \inf \{ x : x + at > B_t \text{ for all } t \} < z \}\]

\[= \inf \{ a : z + at > B_t \text{ for all } t \}\]

\[= \inf \{ a : a + sz > u B_{1/u} \text{ for all } u \}.\]

This shows that \( A \) is the zenith process associated with the process \(\{uB_{1/u} ; u \geq 0\}\), just as \( Z \) is
the zenith process associated with $B$. Since $(uB_t^w)$ is a standard Brownian motion like $B$, it follows that $A$ has the same probability law as $Z$. This proves the first statement, since $\dot{D}$ is the derivative of $-A$ and $D$ is the derivative of $-Z$.

The points $S_i$ are the jump locations of $D$, and the points $Z(S_i)$ are those of $\dot{D}$. This proves the second statement.
3. ENTRANCE LAW AND TRANSITION FUNCTION

We derive the distribution of the random variable \((D_a, Z_a)\) and the transition function of the process \((D_t, Z_t)\). The computations rest on the characterization given by Proposition (2.4) and on the well-known formula for the Laplace functional of the Poisson random measure \(N\) with mean measure \(\nu\):

\[
E \exp - \int N(dx) f(x) = \exp - \int \nu(dx) (1 - e^{-f(x)})
\]

for every positive Borel function \(f\) on \((0, \infty) \times (0, \infty)\).

(3.2) PROPOSITION. For each \(a > 0\),

\[
E \exp (-pD_a - qZ_a) = \frac{2a}{a + q + \sqrt{a^2 + 2p}}, \quad p \geq 0, \quad q \geq 0;
\]

(3.4) \(P \{D_a \in dt, Z_a \in dz\} = dt dz \frac{2a e^{-(t+a)^2/2t}}{\sqrt{2\pi t}^3}, \quad t > 0, \quad z > 0\).

In particular,

\[
P \{Z_a \in dz\} = dz 2ae^{-2az}, \quad P \{D_a \in dt\} = dt \int du \frac{ae^{-a^2u/2}}{\sqrt{2\pi u}}.
\]

PROOF. Fix \(a > 0, p \geq 0, \quad q \geq 0\). In view of (2.5) and (2.6),

\[
pD_a + qZ_a = \int_{(0, \infty)} N(ds, dt) (pt + q(s - a)t).
\]

Using (3.1) and the form of the mean measure \(\nu\) given by (2.3), the Laplace transform (3.3) is obtained via elementary calculus. To invert the Laplace transform, first write it as
\[
\int_0^\infty dz \ e^{-\sqrt{2}z} \ 2ae^{-\alpha z} = e^{-\sqrt{a^2 + 2p}}
\]

and then recall that \(e^{-r\sqrt{z}}\), \(r \geq 0\), is the Laplace transform of \(H_z\), the first time a standard Brownian motion hits the level \(z\), that is,

\[
e^{-\sqrt{a^2 + 2p}} = \int_0^\infty dt \ \frac{t e^{-\frac{3}{2}t}}{\sqrt{2\pi t^3}} \ e^{-\left(p + \alpha \frac{3}{2} t\right)}
\]

The rest is trivial.

(3.6) REMARK. Although (3.4) is explicit and shades of exponential and stable distributions can be felt, it does not seem well-suited for probabilistic thinking. The following representation is better, especially for Monte-Carlo methods. For \(a > 0\),

\[
a^2 D_a = X (1 - \sqrt{U})^2, \quad \alpha Z_a = X \sqrt{U} (1 - \sqrt{U}),
\]

where \(X\) and \(U\) are independent, \(U\) has the uniform distribution on \((0,1)\), and \(X\) has the gamma distribution with shape index \(3/2\) and scale parameter \(1/2\).

The following specifies the joint Laplace transform of any number of increments of \(Z\) (upon taking \(f = p_1 1_{A_1} + \ldots + p_n 1_{A_n}\) with \(A_1, \ldots, A_n\) disjoint intervals).

(3.7) PROPOSITION. For any positive Borel function \(f\) on \((0,\infty)\),

\[
E \exp \left( \int_0^\infty f(a) \ dZ_a \right) = \exp - \int_0^\infty ds \ \left( \frac{1}{s} - \frac{1}{\sqrt{s^2 + 2f(s)}} \right)
\]

where \(\overline{f}(s)\) is the Lebesgue integral of \(f\) over \((0,s)\).
PROOF. Note that
\[ \int f(a) \, dZ_a = -\int f(a) \, D_a \, da = -\int N(ds, dt) \, \tilde{f}(s) \, \, , \]
and use (3.1).

As mentioned in Proposition (2.4), the process \( D \) has non-stationary independent increments, and the process \((D, Z)\) is a non-homogeneous parameter Markov process. Let

\[ P_{ab}(t; x; du, dy) = P\{D_b \in du, Z_b \in dy \mid D_a = t, Z_a = x\} \]
for \( 0 < b < a, 0 < t < u, 0 < x < y \) (in our zeal to deal with positive random variables, we choose to work with the parameters in decreasing order). In view of (2.5) and (2.6),

\[ P_{ab}(t; x; du, dy) = P\{t + U \in du, x + (a - b)t + Y \in dy\} \]
where

\[ U = \int_{[b,a) \times (0,\infty)} N(ds, dt) \, t, \quad Y = \int_{[b,a) \times (0,\infty)} N(ds, dt) \, (s - b)t \, . \]

The joint Laplace transform of \( U \) and \( Y \) can be obtained from (3.1) as in the first step of the proof of (3.2):

\[ E e^{-u -qY} = \frac{b}{a} \cdot \frac{a + q + \sqrt{a^2 + 2p + 2(a - b)a}}{b + q + \sqrt{b^2 + 2p}} \]
\[ = \frac{b}{a} + \frac{a - b}{a} \cdot \frac{1}{2} \cdot \frac{2b}{b + q + \sqrt{b^2 + 2p}} \]
\[ + \frac{a - b}{a} \cdot \frac{1}{2} \cdot \frac{2b}{b + q + \sqrt{b^2 + 2p}} \cdot \frac{1}{a - b} \int \frac{(c + q) \, dc}{\sqrt{(c + q)^2 + 2p - 2bq - q^2}} . \]

Inverting this is tedious but manageable. It yields the following for the distribution \( \Phi \) of the pair
(U, Y):

\[
(3.12) \quad \psi = \frac{b}{a} \delta_{(0,0)} + (1 - \frac{b}{a}) \left( \frac{1}{2} \lambda_b + \frac{1}{2} \lambda_b^* \frac{\mu_{bb} - \mu_{ab}}{b - a} \right)
\]

where the asterisque denotes convolution, \( \delta_x \) is the Dirac measure at \( x \), \( \lambda_b \) is the distribution of \((D_b, Z_b)\) specified by (3.4), and

\[
(3.13) \quad \mu_{ab}(dt, dy) = \frac{e^{-\frac{b^2}{2}}}{\sqrt{2\pi t}} dt \delta_{(a-b,y)}(dy), \quad t > 0, \ y > 0.
\]

Putting the distribution \( \psi \) of \((U, Y)\) into (3.9) yields an explicit expression for the transition function \( P_{ab} \). As a by-product, we have the joint distribution of

\[
U = D_b - D_a, \ Y = Z_b - Z_a - (b - a) D_a.
\]

Noting that \( D_a \) is independent of \((U, Y)\), one can obtain the distribution of \((D_b - D_a, Z_b - Z_a)\) among other things.

However, it is clear that such results are of limited use because of their complexity. Overall, the computational complexity is caused by a confluence of two incompatible operations, addition and multiplication: look at the form (2.3) of the mean measure \( \nu \); the Haar measure \( ds \) indicates that the natural group operation on the jump points \( S_i \) is multiplication, whereas the jump amounts \( T_i \) are additive.

Of course, it is easy to transform the Poisson random measure \( N \) into one with a nicer intensity: define \( f \) to be the mapping \((s,t) \rightarrow (\log s, s^2t)\); then the image of \( N \) under \( f \) is the Poisson random measure \( \tilde{N} = Nf^{-1} \) on \((\mathbb{R}, \mathbb{R}_+) \times (0,\infty)\) with mean measure \( du\gamma dv \) where \( \gamma \) is the gamma distribution with shape and scale parameters equal to \( 1/2 \). But, then, expressions for \( D_a \) and \( Z_a \) in terms of \( \tilde{N} \) have to undo the transformation, and there is no gain at the end. Using \( p = \log a \) to index the processes involved (and working with \( \tilde{Z}_p = Z(e^p) \)) does not help either.
On the other hand, it is easy to describe the construction of the path of \((D, Z)\) over the interval \((0, a]\). This may be useful for Monte-Carlo purposes.

First, we observe that the conditional distribution of \(S_{i+1}\) given \((S_i, S_{i-1}, \ldots)\) is the uniform distribution on \((0, S_i)\). Thus, to construct \((D, Z)\) over \((0, a]\), we start with \(U\) and \(X\) described in Remark (3.6) and generate \(D_a\) and \(Z_a\). Then, we let \(U_1, U_2, \ldots\) be i.i.d. uniform on \((0,1)\), let \(X_1, X_2, \ldots\) be i.i.d. Gaussian with mean 0 and variance 1, and set

\[
S_0 = a, \quad S_i = a U_1 U_2 \cdots U_i, \quad T_i = \left(\frac{X_i}{S_i}\right)^2, \quad i = 1, 2, \ldots.
\]

With these, define

\[
D(S_0) = D_a, \quad D(S_i) = D(S_{i-1}) + T_i, \quad i \geq 1,
\]

\[
Z(S_0) = Z_a, \quad Z(S_i) = Z(S_{i-1}) + (S_{i-1} - S_i) \cdot D(S_{i-1}), \quad i \geq 1.
\]

Then, \((D_b)_{b \in (0, a)]\) is the left-continuous piecewise constant path whose value at \(S_i\) is \(D(S_i)\), and \((Z_b)_{b \in (0, a)}\) is the continuous piecewise linear path whose value at \(S_i\) is \(Z(S_i)\). Incidentally, \((S_i), (S_i, D(S_i)), (S_i, D(S_i), Z(S_i))\) are all Markov chains.
REFERENCES


