REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION		1b. RESTRICTIVE MARKINGS
UNCLASSIFIED		

	3. DISTRIBUTION/AVAILABILITY OF REPORT
SCHEDULE	Approved for Public Release; Distribution Unlimited

AD-A198 273

BER(S)

5. MONITORING ORGANIZATION REPORT NUMBER(S)
AFOSR·TR· 88 - 0 7 6 0

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
Princeton University		AFOSR/NM

6c. ADDRESS (City, State and ZIP Code)	7b. ADDRESS (City, State and ZIP Code)
Princeton, NJ 08544	Building 410 Bolling AFB, DC 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR	NM	AFOSR 87-0050

8c. ADDRESS (City, State and ZIP Code)	10. SOURCE OF FUNDING NOS			
Building 410 Bolling AFB, DC 20332-6448	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.

11. TITLE (Include Security Classification) On Lifetimes Influenced by a Common Environment				
	6.1102F	2304	A5	

12. PERSONAL AUTHOR(S)
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13a. TYPE OF REPORT Interim Journal	13b. TIME COVERED FROM 5-87 TO 4-88	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT 19 pages

16. SUPPLEMENTARY NOTATION

17	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB GR.	
XXXX	XXXXXXXXXXX	XXXX	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
Brian Woodruff	202-767-5028	AFOSR/NM

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE.

ON LIFETIMES INFLUENCED
BY A COMMON ENVIRONMENT

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Abstract

Consider the lifelengths $T_1, \ldots, T_k$ of k components subjected to a randomly varying environment. They are dependent on each other because of their common dependence on the environment. The parameters of the model are the distribution of the random process which describes the environment and a set of rate functions which determine the probability law of $T_1, \ldots, T_k$ as a function of the distribution of the environment. We find conditions on the parameters of the model which imply that $T_1, \ldots, T_k$ are associated. Other conditions which imply that $T_1, \ldots, T_k$ have the multivariate aging properties IHR (increasing hazard rate) and NBU (new better than used) are also described. Also two such models are compared. In particular, we characterize the parameters of these models so that stochastic ordering between the two vectors of resulting lifetimes can be obtained.

1. INTRODUCTION

Consider the lifelengths $T_1, \ldots, T_k$ of k components subjected to a randomly varying environment. They are dependent on each other because of their common dependence on the environment. In the model introduced by Çinlar and Özekici (1987) to handle such dependence, the cumulative hazard functions of the components are made functionals of the environment process and jointly satisfy a differential equation. Therefore, the joint probability law of the lifelengths is specified by the probability law of the environment process X and the intrinsic aging rates $r_i(x, a_1, \ldots, a_k)$, $i \in \{1, \ldots, k\}$, where the latter stands for the instantaneous failure rate of the component i at an instant when the environmental state is x and the intrinsic ages (the cumulative hazards) of the components $1, \ldots, k$ are $a_1, \cdots, a_k$ respectively. We shall make these precise shortly, in Section 2.

Our aim is to explore the dependence of the lifelengths on the function $r = (r_1, \ldots, r_k)$ and the process X. In Section 3, we examine the effects of replacing r and X by another function $\hat{r}$ and another process $\hat{X}$, in both cases seeking results on stochastic dominance. Also in that section is a characterization of "association" (in the sense of Esary, Proschan and Walkup (1967)) for the lifelengths in terms of the association of the process X.

In Section 4, we consider multivariate aging properties of the lifelengths conditioned upon the history $\mathcal{F}_t$ of the environment until t, and also, conditioned upon the history $\mathcal{G}_t$ of the environment and failures during $[0, t]$. In particular, we obtain conditions for the lifelengths to have the "multivariate increasing hazard rate" property with respect to the filtration $(\mathcal{F}_t)$ or $(\mathcal{G}_t)$, and also the "multivariate new better than used" property, again with respect to $(\mathcal{F}_t)$ or $(\mathcal{G}_t)$.

2. PRELIMINARIES

In this section we give an overview of the model introduced by Çinlar and Özekici (1987). Throughout here and the paper, $(\Omega, \mathcal{X}, P)$ is a complete probability space. We write $\boldsymbol{R}_+$ for $[0, \infty)$, call a number or vector a positive [negative] if $a \geq 0 [a \leq 0]$, and call a function f increasing [decreasing] if $f(x) \leq f(y)$ for $x \leq y [x \geq y]$.

Let $(E, \mathcal{E})$ be a measurable space. Elements of E are called the environmental states. We suppose that, for each $x \in E$, the singleton $\{x\}$ belongs to $\mathcal{E}$. There is a distinguished point in E, denoted by δ, which stands for the state that causes no aging. We let $X = \{X(t); t \in \boldsymbol{R}_+\}$ be a stochastic process with state space $(E, \mathcal{E})$; it represents the environment process.

The set of all components is epresented by $K = \{1, \dots, k\}$. We let $A = \{A(t); t \in \boldsymbol{R}_+\}$ be an increasing continuous process taking values in $\boldsymbol{R}_+^k$; its ith component, namely $A_i = \{A_i(t); t \in \boldsymbol{R}_+\}$, is called the *intrinsic age process* of component i, it plays the role of a random cumulative hazard function.

We let $S_1, \dots, S_k$ be independent of X and of each other and have the standard exponential distribution (with mean 1). The lifelength of component i is modeled by

$$(2.1) \qquad T_i = \inf\{t : A_i(t) > S_i\}, \qquad i \in K,$$

that is, the component i fails when its intrinsic age runs over its "intrinsic lifelength" S_i. We write $S = (S_1, \dots, S_k)$ and $T = (T_1, \dots, T_k)$ for the vectors of intrinsic lifelengths and lifelengths.

In this formulation, the dependencies between the lifelengths and their joint dependence on the environment are reflected via the intrinsic age process A.

Regarding the latter, the main assumption of Çinlar and Özekici (1987) is as follows.

(2.2) HYPOTHESES. (i) For each component i there exists a positive measurable function r_i on $E \times R_+^k$ such that

$$(2.3) \qquad dA_i(t) = r_i(X(t), A(t))dt, \quad t > 0, i \in K.$$

(ii) We have $r_i(x, a) > 0$ for each $i \in K$, $a \in R_+^k$, and all $x \in E$ except $x = \delta$. For $x = \delta$, we have $r_i(x, a) = 0$ for all i and a.

The basic hypothesis is the first one: the intrinsic age process A is a functional of the environment process X. The second hypothesis is a regularity condition, it is meant to ensure that (2.3) has a unique solution A for each starting condition; in particular, it singles out δ as the only state that causes no aging.

Note that A is determined by X and, hence, is independent of the vector S. Therefore, it follows from (2.1) and the independence of the exponential variables S_i from each other that

$$(2.4) \qquad P\{T > t | X\} = \exp\left[-\sum_{i \in K} A_i(t_i)\right], \quad t \in R_+^k.$$

This justifies the term "random cumulative hazard function" for each A_i. However, we prefer to call A_i the intrinsic age process of i. Then (2.1) can be read as follows: each component is endowed with an intrinsic lifelength, the component ages in response to the environmental factors in a manner intrinsic to its own function and nature, it fails when its intrinsic age runs over its allotted intrinsic lifelength. In particular, (2.3) defines $r_i(x, a_1, \ldots, a_k)$ to be the intrinsic aging rate of the component i at a time when the environment is in state x and the

intrinsic ages of the components $1, \ldots, k$ are $a_1, \ldots, a_k$ respectively. It follows from (2.3) and (2.4) that we also have

(2.5) $\quad r_i(x; a_1, \ldots, a_k)$
$$= \lim_{u \downarrow 0} \frac{1}{u} P\{T_i \le t + u | T_i > t, X(t) = x, A_1(t) = a_1, \ldots, A_k(t) = a_k\},$$

that is, $r_i(x, a)$ is the hazard rate for component i as a function of the environmental state x and the intrinsic age vector a. We write $r = (r_1, \ldots, r_k)$ and call it the intrinsic aging rate function. Aside from the probability law of X, it is the only parameter in the model.

If $r(x, a)$ is free of x, then A becomes deterministic and (2.1) shows that $T_1, \ldots, T_k$ are independent. If $r(x, a)$ is free of a, then A becomes a k-dimensional additive functional of X.

A function r from $\boldsymbol{E} \times \boldsymbol{R}_+^k$ into $\boldsymbol{R}_+^k$ will be called an *intrinsic aging rate* function if it satisfies Hypothesis (ii) of (2.2). Given such a function r and the process X, the differential equation (2.3) together with

(2.6) $$A(0) = 0$$

specifies the aging vector $A(t)$ for all $t \ge 0$, and the latter specifies the lifelength vector T via (2.1) from intrinsic lifelength vector S of standard exponentials. Thus, there exists a functional L such that

(2.7) $$T = L(X, r, S).$$

The functional L is defined implicitly via (2.1) and (2.3); it is called the *lifelength functional*. This paper is a study of the dependence of L on its arguments X and r.

3. DEPENDENCE ON ENVIRONMENT AND AGING RATES

In this section we discuss the dependence of the lifelength vector $T = L(X, r, S)$ on the environment process X and the intrinsic aging rate function r. Here, and for the remainder of the section, we assume that the state space E is a complete separable metric space.

(3.1) THEOREM. Let r and $\hat{r}$ be intrinsic aging rate functions and let $T = L(X, r, S)$ and $\hat{T} = L(X, \hat{r}, S)$. Assume that $t \to X(t)$ is piecewise continuous and that either r or $\hat{r}$ is continuous on $E \times \mathbf{R}^k$. Suppose that, for each $i \in K$ and $x \in E$,

(3.2) $$a, \hat{a} \in \mathbf{R}_+^k, a \geq \hat{a}, a_i = \hat{a}_i \Rightarrow r_i(x, a) \geq \hat{r}_i(x, \hat{a}).$$

Then, $T \leq \hat{T}$.

(3.3) REMARK. Suppose that $r \geq \hat{r}$ and that either $a \to r(x, a)$ or $a \to \hat{r}(x, a)$ is increasing for every x. Then, the condition (3.2) of the preceding theorem is satisfied and $T \leq \hat{T}$. For instance, if $r \geq \hat{r}$ and $a \to r(x, a)$ is increasing for each x, then $r(x, a) \geq r(x, \hat{a}) \geq \hat{r}(x, \hat{a})$ for all $a \geq \hat{a}$, and hence (3.2) holds.

(3.4) REMARK. Suppose that r and $\hat{r}$ are as in the preceding theorem and (3.2) is satisfied. Suppose that $T = L(X, r, S)$ and $\hat{T} = L(\hat{X}, \hat{r}, \hat{S})$ where X and $\hat{X}$ have the same probability law, and so do S and $\hat{S}$, and $\hat{S}$ is independent of $\hat{X}$ (as S is of X). Then, the conclusion of the preceding theorem is that T is dominated by $\hat{T}$ stochastically, that is, $Ef(T) \leq Ef(\hat{T})$ for every increasing function f from $\mathbf{R}_+^k$ into $\mathbf{R}_+$.

Proof of Theorem (3.1). Fix r and $\hat{r}$. Suppose first that (3.2) holds with a strict inequality: $r_i(x, a) > \hat{r}_i(x, \hat{a})$. Let A be the solution of (2.3) and let $\hat{A}$ be

the solution of (2.3) with $\hat{r}$ replacing r, both with $A(0) = \hat{A}(0) = 0$. Now, T is defined by (2.1), and $\hat{T}$ is defined by (2.1) with $\hat{A}$ replacing A. Thus, to show that $T \leq \hat{T}$, it is sufficient to show that $A \geq \hat{A}$. Or, equivalently, it is sufficient to show that the random variable

$$(3.5) \qquad \tau = \inf\{t : A_i(t) < \hat{A}_i(t) \text{ for some } i\}$$

is equal to $+\infty$ identically.

Since $A(0) = \hat{A}(0) = 0$, we have $\tau \geq 0$. Suppose for the moment that $\tau(\omega) = t$ (where $t < \infty$) for some outcome $\omega \in \Omega$. Fix that ω and simplify the notation by putting

$$(3.6) \qquad x = X_t(\omega), a = A(\omega, t), \hat{a} = \hat{A}(\omega, t).$$

In view of (2.3), the processes A and $\hat{A}$ are continuous, and the assumed finiteness of $t = \tau(\omega)$ implies the existence of $i \in K$ and of a decreasing sequence $(t_n) \subset \mathbf{R}_+$ with limit t such that

$$(3.7) \qquad A_i(\omega, t_n) < \hat{A}_i(\omega, t_n) \text{ for all } n.$$

Moreover, by the continuity of A and $\hat{A}$, we must have

$$(3.8) \qquad a \geq \hat{a}, \quad a_i = \hat{a}_i$$

The differentiability of A_i and $\hat{A}_i$ ensured by (2 3) implies that, since $a_i = \hat{a}_i$,

$$(3.9) \quad \begin{aligned} &\lim_{n \to \infty} \frac{1}{t_n - t}[A_i(\omega, t_n) - \hat{A}_i(\omega, t_n)] \\ &= \lim_{n \to \infty} \frac{1}{t_n - t}[A_i(\omega, t_n) - a_i] - \lim_{n \to \infty} \frac{1}{t_n - t}[\hat{A}_i(\omega, t_n) - u_i] \\ &= r_i(x, a) - \hat{r}_i(x, \hat{a}). \end{aligned}$$

In view of (3.8) and the assumed strictness in condition (3.2), this is strictly positive. But, the first member of (3.9) must be negative in view of (3.7). This contradiction shows that $\tau(\omega)$ cannot be finite.

Now relax the assumption of strictness in (3.2) but assume for a moment that $t \to X(t)$ is continuous. Then $t \to r(X(t), A(t))$ or $t \to \hat{r}(X(t), A(t))$ is continuous by the hypothesis that either r or $\hat{r}$ is continuous. Suppose the former. Fix an $\varepsilon > 0$ in $[0, \infty)^k$ and define $r^{(n)} \equiv r + \varepsilon/n, n = 1, 2, \dots$. Let $A^{(n)}$ be the solution of (2.3) with $r^{(n)}$ replacing r and with $A^{(n)}(0) = 0$. Then by the previous argument $A^{(n)} \geq \hat{A}$. By the continuity of $r, A^{(n)} \to A$. Therefore $A \geq \hat{A}$ in this case.

The proof for the case, in which $t \to \hat{r}(X(t), A(t))$ [rather than $t \to r(x(t), A(t))$] is continuous, is similar. The above argument can be used except that $\hat{r}$ is replaced by $\hat{r}^{(n)} = \hat{r}(1 - \varepsilon/n)$ where $\varepsilon < 1$ is fixed. This definition of $\hat{r}^{(n)}$ ensures that $\hat{r}^{(n)} > 0$ as required in (2.2) (ii).

If $t \to X(t)$ is piecewise continuous, then let $t_1, t_2, \dots$ be the successive jump times of X. On each interval $[t_\ell, t_{\ell+1})$, X is continuous and the previous argument can be applied to each such interval to show that $A \geq \hat{A}$. ||

Dependence on environment

For the remainder of this section, we assume that the state space E is a partially ordered Polish space (a complete separable metric space with a closed partial ordering). Then, the space $D = D(R_+, E)$ of right-continuous left-limited functions from R_+ into E is again a partially ordered Polish space. A functional $g : D \to R_+$ is said to be increasing if $w \underset{\sim}{<} \hat{w}$ implies $g(w) \leq g(\hat{w})$ for all paths $w, \hat{w} \in D$, where $\underset{\sim}{<}$ denotes the partial ordering.

Let X and $\hat{X}$ be processes with paths in $\boldsymbol{D}$. Then, X is said to dominate $\hat{X}$ stochastically provided that

$$(3.10) \qquad\qquad Eg(X) \geq Eg(\hat{X})$$

for every Borel measurable increasing functional g on $\boldsymbol{D}$. This is obviously the case if $X(\omega,t) \geq \hat{X}(\omega,t)$ for all $\omega \in \Omega$ and $t \in \boldsymbol{R}_+$. More generally, if X stochastically dominates $\hat{X}$, then it follows from Theorem 1 of Kamae, Krengel, and O'Brien (1977) that X and $\hat{X}$ can be "put on the same probability space so that one dominates the other path by path". More precisely, it is possible to construct a new probability space $(W, \mathcal{G}, Q)$ and stochastic processes Y and $\hat{Y}$ defined on $(W, \mathcal{G}, Q)$ and having paths in $\boldsymbol{D}$ such that $Y(\omega,t) \geq \hat{Y}(\omega,t)$ for all $\omega \in W$ and $t \in \boldsymbol{R}_+$, X and Y have the same probability law, and $\hat{X}$ and $\hat{Y}$ have the same probability law. Of course, the new probability space can be enlarged to accommodate k independent standard exponential variables independent of Y and $\hat{Y}$. These remarks will be useful in simplifying the proof of the following theorem, which reduces to Theorem (3.1) when $X = \hat{X}$.

Let $\hat{\boldsymbol{D}} = \hat{\boldsymbol{D}}(\boldsymbol{R}_+, E)$ be the set of functions in $\boldsymbol{D}(\boldsymbol{R}_+, E)$ which are piecewise continuous.

(3.11)   THEOREM. Let X and $\hat{X}$ be processes with paths in $\hat{\boldsymbol{D}}$, let r and $\hat{r}$ be intrinsic aging rate functions, and let S and $\hat{S}$ be k-vectors of independent standard exponential variables independent of X and $\hat{X}$ respectively. Assume that r and $\hat{r}$ are continuous on $E \times \boldsymbol{R}_+^K$. Suppose that

i) X dominates $\hat{X}$ stochastically,

ii) $x \to r(x,a)$ is increasing for every a (or $x \to \hat{r}(x,a)$ is increasing for every a), and the condition (3.2) holds for every $i \in K$ and $x \in E$.

Then. $T = L(X, r, S)$ is stochastically dominated by $\hat{T} = L(\hat{X}, \hat{r}, \hat{S})$.

Proof. In view of the foregoing remarks. by moving onto a new probability space if necessary. we may and do assume that $S = \hat{S}$ and $X(\omega, t) \underset{\sim}{\geq} \hat{X}(\omega, t)$ for all ω and t.

Let A be as before, and define $\hat{A}$ as the solution of (2.3) with X and r replaced by $\hat{X}$ and $\hat{r}$, $A(0) = \hat{A}(0) = 0$. As in the proof of Theorem (3.1), it is sufficient to show that $A \geq \hat{A}$, or equivalently, that τ defined by (3.5) is equal to $+\infty$ identically.

First assume that (3.2) holds with a strict inequality. Let τ be defined by (3.5) and suppose again that $\tau(\omega) = t (t < \infty)$ for some $\omega \in \Omega$. Pick $i \in K$ and $(t_n) \subset \mathbf{R}_+$ so that (t_n) decreases to t and (3.7) holds. With the notations (3.6) supplemented by $\hat{x} = \hat{X}(\omega, t)$, (3.9) becomes

$$(3.12) \qquad \lim_{n \to \infty} \frac{1}{t_n - t} [A_i(\omega, t_n) - \hat{A}_i(\omega, t_n)] = r_i(x, a) - \hat{r}_i(\hat{x}, \hat{a}).$$

Since X dominates $\hat{X}$, we have $x = X(\omega, t) \underset{\sim}{\geq} \hat{X}(\omega, t) = \hat{x}$. Thus, the condition (ii) implies that

$$r_i(x, a) \geq r_i(\hat{x}, a) > \hat{r}_i(\hat{x}, \hat{a})$$

if $x \to r(x, a)$ is increasing (and $r_i(x, a) > \hat{r}_i(x, \hat{a}) \geq \hat{r}_i(\hat{x}, \hat{a})$ if $x \to \hat{r}(x, a)$ is increasing). It follows that the right side of (3.12) is strictly positive. But from (3.7) it is seen that it is negative. Hence $\tau(\omega)$ cannot be finite.

The extension of the above argument to the case in which strictness in (3.2) is not assumed can be done as in Theorem (3.1). $\parallel$

The preceding proof, with $r = \hat{r}$, yields the following technical result regarding the lifelength functional L.

(3.13)   COROLLARY. Suppose that $a \to r(x,a)$ is increasing for every $x \in E$, that $x \to r(x,a)$ is increasing (respectively, decreasing) for every $a \in \boldsymbol{R}_+^k$ and that r is continuous on $E \times \boldsymbol{R}_+^k$. Then, $w \to L(w,r,s)$, is decreasing (respectively, increasing) in $w \in \hat{\boldsymbol{D}}$ for fixed r and s.

4. ASSOCIATION OF LIFELENGTHS

Let $Z_1, \ldots Z_m$ be random variables taking values in $\boldsymbol{R}^n$. Then, they are said to be *associated* provided that the vector $Z = (Z_1, \ldots, Z_m)$ satisfy

$$Cov(g(Z), h(Z)) \geq 0$$

for all increasing functions $g, h : \boldsymbol{R}^{m \times n} \to \boldsymbol{R}$ for which the covariance exists. A stochastic process $Z = \{Z(t); t \in \boldsymbol{R}_+\}$ with state space $\boldsymbol{R}^n$ is said to be *associated in tin. :* if $Z(t_1), \ldots, Z(t_m)$ are associated for all integers $m \geq 1$ and times $t_1, \ldots, t_m \in \boldsymbol{R}_+$. Our aim in this section is to show that, if the environment process X is associated in time and certain conditions hold for the aging rate function r, then the lifelengths $T_1, \ldots, T_k$ are associated. We refer to Esary, Proschan, and Walkup (1967), Barlow and Proschan (1975), Arjas and Norros (1984), Shaked and Shanthikumar (1987) and references therein for the usefulness of the concept of association for lifelengths, and to Barlow and Proschan (1976) and Harris (1977) for examples of processes associated in time.

For the purposes of this section we assume that the environment process X takes values in $E = \boldsymbol{R}^n$ and its paths belong to $\hat{\boldsymbol{D}}$ as in the preceding section.

(4.1)   THEOREM. Suppose that X is associated in time. If $x \to r(x,a)$ is increasing for every $a \in \boldsymbol{R}_+^k$ (or decreasing for every $a \in \boldsymbol{R}_+^k$), $a \to r(x,a)$ is increasing for every $x \in E = \boldsymbol{R}^n$, and r is continuous on $E \times \boldsymbol{R}_+^k$, then the lifelengths $T_1, \ldots, T_k$ are associated.

Proof. Fix r, suppose that $r(x,a)$ is increasing in both x and a. Then, by Corollary (3.13), the mapping $w \to L(w,r,s)$ from $\hat{D}$ into R_+^k is decreasing. Thus, if g and h are increasing functions from R_+^k into R, then $-g \circ L(w,r,s)$ and $-h \circ L(w,r,s)$ are increasing functions of $w \in \hat{D}$ and we have

$$(4.2) \qquad Eg \circ L(X,r,s)h \circ L(X,r,s) \geq Eg \circ L(X,r,s)Eh \circ L(X,r,s)$$

by the assumption that X is associated in time. The same is true for the case where $x \to r(x,a)$ is decreasing, by Corollary (3.13) and the association applied directly to $g \circ L$ and $h \circ L$.

Let μ denote the k-dimensional standard exponential (that is, the distribution of S). By the independence of X and S, the integral of the left side of (4.2) with respect to $\mu(ds)$ is equal to $Eg(T)h(T)$. Thus, (4.2) gives,

$$(4.3) \qquad Eg(T)h(T) \geq \int \mu(ds)Eg \circ L(X,r,s)Eh \circ L(X,r,s).$$

On the other hand, it is obvious that $s \to L(w,r,s)$ is increasing, which implies that $s \to Eg \circ L(X,r,s)$ and $s \to Eh \circ L(X,r,s)$ are increasing. Since $S_1, \ldots, S_k$ are independent, they are associated. This in turn implies that the right-side of (4.3) is greater than or equal to

$$\int \mu(ds)Eg \circ L(X,r,s) \int \mu(ds')Eh \circ L(X,r,s')$$
$$= Eg \circ L(X,r,S)Eh \circ L(X,r,S) = Eg(T)Eh(T).$$

This completes the proof. $\|$

In the preceding theorem, the condition that X be associated in time is satisfied for processes X that have independent positive increments (e.g. increasing compound Poisson processes, gamma processes, etc.). More generally, in the

case of real-valued processes X, association in time holds if X is stochastically monotone, that is, if

$$(4.4) \qquad E[g(X(t))|X(0) = x] \leq E[g(X(t))|X(0) = y]$$

for $x \leq y$ and g increasing Borel measurable (see Barlow and Proschan (1976) and Harris (1977) for this). Thus, the preceding theorem remains true if X is a real valued, stochastically monotone Markov process.

5. MULTIVARIATE AGING PROPERTIES

In this section θ_t will denote an operator that shifts the time origin to t. In particular,

$$(5.1) \qquad \theta_t T_i = \max(0, T_i - t), \quad i \_ K.$$

The following properties were defined in Arjas (1981).

(5.2)   DEFINITION. Let $(\mathcal{H}_t)$ be a filtration. The lifelength vector T is said to have a *multivariate increasing hazard rate* with respect to $(\mathcal{H}_t)$ (abbreviated as $(\mathcal{H}_t)$-MIHR) if

$$(5.3) \qquad E[f(\theta_t T) \mid \mathcal{H}_t] \geq E[f(\theta_u T) \mid \mathcal{H}_u]$$

for all $t < u$ and all positive increasing Borel functions f on $\mathbf{R}_+^k$. It is said to have the *multivariate new better than used* property with respect to $(\mathcal{H}_t)$ (abbreviated as $(\mathcal{H}_t)$-MNBU) if

$$(5.4) \qquad E[f(T) \mid \mathcal{H}_0] \geq E[f(\theta_t T) \mid \mathcal{H}_t]$$

for all $t > 0$ and all positive increasing Borel functions f on $\mathbf{R}_+^k$.

Two special filtrations of interest to serve as $(\mathcal{H}_t)$ above are defined by

$$(5.5) \qquad \mathcal{F}_t = \sigma(A(0), X(s) : s \le t),$$

$$(5.6) \qquad \mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_{\{T_i \le s\}} : s \le t, i \in K),$$

which are, respectively, the history of environment and age processes until t and the complete history of environment, ages, and failures until t. Note that we allow the initial ages $A_i(0)$ to be non-zero random variables.

Our aim in this section is to discuss MIHR and MNBU properties of T with respect to $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ assuming that X is a Markov process with certain properties. We start by some computations in the Markovian case.

Lifelengths in a Markovian environment

Let X be a temporally homogeneous Markov process with state space $E = R_+^n$. Suppose that its paths belong to D, the space of all right-continuous left-limited functions from R_+ into E. Let A satisfy the differential equation (2.3), but with the initial condition $A(0)$ unspecified. It follows, then, that the pair (X, A) is a temporally homogeneous Markov process with state space $E \times R_+^k$. As is usual in the theory of Markov processes, we will write

$$P^{xa}\{\cdot\} = P\{\cdot \mid X(0) = x, A(0) = a\}, \quad x \in E, a \in R_+^k,$$

and will write E^{xa} for the corresponding expectation operator. Note that P^{xa} does not put any conditions on the vector S of standard exponentials, except that S is assumed to be independent of $\mathcal{F}_\infty$, that is, of the process (X, A).

The lifelengths T_i are still defined by (2.1), which implies that some of the T_i can be 0 with a strictly positive probability. However, if it is given that $T_i > 0$,

its probability law is the same as that of

$$(5.7) \qquad U_i = \inf\{t : A_i(t) - A_i(0) > S_i\}, \quad i \in K,$$

which fact follows from the independence of S from A and the memorylessness of exponential variables. The following is a precise version of this circle of ideas. Here, and below, for $I \subset K$ and $v \in \mathbf{R}_+^k$ we define $v_I \in \mathbf{R}_+^k$ to be the vector whose i-entry is v_i or 0 according as $i \in I$ or not.

(5.8)   LEMMA. Let f be a positive Borel function on $\mathbf{R}_+^k$ and put

$$(5.9) \qquad g(x,a,I) = E^{xa}[f(T) \mid S_i > a_i \text{ for } i \in I \text{ and } S_i \leq a_i \text{ for } i \in K - I],$$

where $x \in E$, $a \in \mathbf{R}_+^k$, and $I \subset K$. Then,

$$(5.10) \qquad g(x,a,I) = E^{xa} f(U_I).$$

Proof. Under P^{xa} we have $A(0) = a$. Thus, on $\{S_i \leq a_i\}$ we have $T_i = 0$ almost surely, and (5.9) becomes

$$(5.11) \qquad g(x,a,I) = E^{xa}[f(T_I) \mid S_i > a_i, \quad i \in I].$$

On the other hand, on $\{S_i > a_i\}$, we have

$$T_i = \inf\{t : A_i(t) > S_i\} = \inf\{t : A_i(t) - A_i(0) > \hat{S}_i\}$$

where $\hat{S}_i = S_i - a_i$ since $A_i(0) = a_i$ under P^{xa}. By the independence of S from (X, A), and since S_i is exponential, $\hat{S}_i = S_i - a_i$ has the standard exponential distribution as its conditional distribution on $\{S_i > a_i\}$. It follows that the conditional distribution of T_I, given $\{S_i > a_i, i \in I\}$, under P^{xa} coincides with the distribution of U_I under P^{xa}. Hence, the right sides of (5.10) and (5.11) are the same. $\parallel$

(5.12)   LEMMA. Let f and g be as in Lemma (5.8). Let μ be the standard exponential distribution on $\boldsymbol{R}_+^k$, that is, $\mu(ds) = \exp(-s_1 - \cdots - s_k)ds_1 \cdots ds_k$. Then,

$$(5.13) \qquad E^{za}f(T) = \int_{\boldsymbol{R}_+^k} \mu(ds)g(x, a, I_{sa}) \equiv h(x, a)$$

where $I_{sa} = \{i \in K : s_i > a_i\}$.

Proof. It is immediate from Lemma (5.8) by unconditioning. ‖

The proof of the next lemma follows from the Markov property of (X, A). Here,

$$(5.14) \qquad R(t) \equiv \{i \in K : T_i > t\},$$

is the set of components remaining alive at t.

(5.15)   LEMMA. Let X be a temporally homogeneous Markov process. Let f, g, and h be related by (5.9) and (5.13). Then

$$(5.15) \qquad E[f(\theta_t T) \mid \mathcal{G}_t] = g(X(t), A(t), R(t)),$$

$$(5.16) \qquad E[f(\theta_t T) \mid \mathcal{F}_t] = h(X(t), A(t)).$$

Increasing hazard rates

(5.17)   THEOREM. Let X be a temporally homogeneous Markov process with state space $E = \boldsymbol{R}^n$. Suppose that

a) $r(x, a)$ increases in x and in a and is continuous,

b) X is stochastically monotone

c) the paths of X belong to $\hat{D}(\boldsymbol{R}_+, \boldsymbol{R}^n)$ and are increasing.

Then, T has the $(\mathcal{F}_t)$-MIHR and $(\mathcal{G}_t)$-MIHR properties.

<u>Proof</u>. i) Let f be an increasing function on $\boldsymbol{R}_+^k$ and let g and h be defined by (5.9) and (5.13). To show that (5.13) holds with $\mathcal{H}_t = \mathcal{F}_t$ or $\mathcal{G}_t, t \geq 0$, it is sufficient to show that (5.15) and (5.16) are decreasing in t. Since X and A are increasing processes (the assertion on X is via the assumption (c)) and R is decreasing, this amounts to showing that g and h are decreasing in their first two arguments and g is increasing in its last argument.

ii) It is easy to see that $g(x, a, I)$ increases as I increases: if $I \subset J$ then $U_I \leq U_J$ and $f(U_I) \leq f(U_J)$.

iii) Fix a and I. Since $x \to r(x, a)$ is increasing, the random vector U_I is a decreasing functional of X (by (3.13)). By the assumed stochastic monotonicity of X, this implies that $g(x, a, I)$ decreases in x. Further, in view of (5.13), $h(x, a)$ decreases in x.

iv) Fix x and I. Let $a \leq \hat{a}$, and let A and $\hat{A}$ be the solutions of (2.3) starting from a and $\hat{a}$ respectively. As before in Theorem (3.1), using the assumption that $r(x, a)$ is increasing in a, we see that $A \leq \hat{A}$. This implies that

$$(5.18) \qquad \frac{d}{dt} A(t) = r(X(t), A(t)) \leq r(X(t), \hat{A}(t)) = \frac{d}{dt} \hat{A}(t).$$

Integrating over $(0, t]$ we see that $A(t) - A(0) \leq \hat{A}(t) - \hat{A}(0)$. Thus, in view of the definition (5.7) of U, we have $U \geq \hat{U}$ where $\hat{U}$ corresponds to $\hat{A}$ as U does to A. It follows that $f(U_I) \geq f(\hat{U}_I)$ and, since the law of U under $P^{x\hat{a}}$ is the same as that $\hat{U}$ under $P^{x\hat{a}}$, we have that $g(x, a, I) \geq g(x, \hat{a}, I)$. Hence, $g(x, a, I)$ decreases in a.

Finally, fix x and let $a \leq \hat{a}$. Consider the formula (5.13) for h. For any $s \in \mathbf{R}_+^k$, $I_{sa} = \{i : s_i > a_i\} \supset \{i : s_i > \hat{a}_i\} = I_{s\hat{a}}$, and hence, $g(x, a, I_{sa}) \geq g(x, \hat{a}, I_{s\hat{a}})$. It follows from (5.13) that $h(x, a) \geq h(x, \hat{a})$, that is, $h(x, a)$ decreases in a. $\|$

Note that the conditions of Theorem (5.17) imply the conditions of Theorem (4.1) [See the discussion following the proof of Theorem (4.1)]. This is not surprising: Using ideas such as in Norros (1985) it can be shown that if T has the $(\mathcal{G}_t)$-MIHR property then $T_1, \ldots, T_k$ are associated.

In the preceding proof we had the assumption that the paths of X are increasing. For proving the generally weaker property MNBU, we may replace it with something weaker.

(5.19)   THEOREM. Let X be a temporally homogeneous Markov process with state space $E = \mathbf{R}^n$. Suppose that the condition (a) and (b) of Theorem (5.17) hold, and that

(c') $X(0) \leq X(t)$ almost surely for each t and the paths of X belong to $\hat{D}(\mathbf{R}_+, \mathbf{R}^n)$.

Then, T has the properties $(\mathcal{F}_t)$-MNBU and $(\mathcal{G}_t)$-MNBU.

<u>Proof.</u> Here we have $\mathcal{F}_0 = \mathcal{G}_0 = \sigma(X(0)A(0))$. And, by the computations of Lemma (5.8),

$$E[f(T) \mid \mathcal{G}_0] = g(X(0), A(0), K)$$

So, we need to show (by (c')) that

$$g(X(0), A(0), K) \geq g(X(t), A(t), R(t))$$

and that

$$h(X(0), A(0)) \geq h(X(t), A(t)).$$

But these follow from the proof of Theorem (5.17). ‖

Theorem (5.19) applies to "new" components by setting $A(0) = 0$ with probability one.

Note that (c') holds whenever $E = R_+^n$ and $P\{X(0) = 0\} = 1$.

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