Asymptotic Growth of Cumulative and Regenerative Beam Break-Up Instabilities in Accelerators

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The analysis is extended to include the transition from the cumulative to the regenerative type, both in the presence and absence of a focusing magnetic field.
ASYMPTOTIC GROWTH OF CUMULATIVE AND REGENERATIVE BEAM BREAK-UP INSTABILITIES IN ACCELERATORS

The beam break-up instability (BBU) continues to be a critical factor which places a limit on the current and on the pulse length in both rf and induction accelerators.\(^1\)\(^-\)\(^8\) A transverse displacement of the particle beam excites a non-axisymmetric mode in the accelerating structure (cavity). This non-axisymmetric mode further deflects the beam sideways, reinforcing the mode itself. Depending on whether a wave with negative group velocity is present to provide feedback, BBU may either be regenerative or cumulative.\(^2\),\(^4\),\(^6\)

Much theoretical effort on BBU in the past twenty years has been devoted to the cumulative type,\(^2\),\(^3\),\(^5\),\(^7\),\(^8\) where the accelerating units are assumed to be decoupled from each other electromagnetically. Information is carried only by the beam. Under this assumption, Panofsky and Bander\(^2\) found that the transverse displacement of the beam grows asymptotically like \(\exp(at)^{1/3}\), at a given distance downstream, when the focusing magnetic field is absent. The model including a general focusing magnetic field was laid down in Ref. 2, but it was stated there that, except for the case of weak focusing, it is not possible to obtain the asymptotic growth analytically. In a pioneering paper published somewhat later, Neil, Hall and Cooper\(^5\) used an entirely different approach and found that the asymptotic growth of the cumulative BBU behaves instead like \(\exp(ht)^{1/2}\) in the presence of a strong solenoidal magnetic field. Here, \(a, b\) are parameters proportional to the beam current. These peculiar time dependences, at first sight, are not expected from the usual experiences with beam-circuit interaction. However, the asymptotic growth is firmly established, at least in the case of zero focusing magnetic field, both in the "continuum" limit, (where coupled partial differential equations are solved\(^2\),\(^3\)), and in the discrete model (where the individual beam-cavity
interaction is passed onto the subsequent ones through multiplication of matrices. From the above asymptotic growths, scaling laws on the beam current were established.

In this paper, we use the continuum model and adopt a mode coupling analysis. The asymptotic growth is calculated analytically, for both cumulative and regenerative BBU, in the presence of a general focusing magnetic field. This work was motivated by an attempt to understand the origin of the asymptotic dependences mentioned above, and by the need to assess the importance of the BBU in the two beam accelerator concept currently explored at the Naval Research Laboratory. The present analytic theory yields three specific results which hitherto were not given in the literature. First, the asymptotic growth \( \exp(bt)^{1/2} \) exhibited in the cumulative BBU in the presence of a strong solenoidal magnetic field is a result of the coupling between the slow beam-cyclotron mode and the cavity mode. Second, this growth is reduced to \( \exp(at)^{1/3} \) as \( t \to \infty \). That is, the asymptotic growth of the cumulative BBU is independent of the focusing magnetic field - as if the focusing magnetic field were absent. Third, the treatment of the cumulative BBU is extended to the regenerative type with the inclusion of a negative group velocity \( v_g \). The exponentiation factor is modified by a quantity which depends only on the group velocity, but is independent of the other properties of the structure. Here, we shall present the model and the results. The implication will be discussed but the details will be given elsewhere.

For simplicity, consider a continuous beam with coasting velocity \( v \), relativistic mass factor \( \gamma \), and current \( I \) streaming in a focusing magnetic field \( B \) inside a series of identical accelerating units. Let \( q(z,t) \) be the transverse displacement of the beam from the axis. Let \( q(z,t) \) be a measure of the deflecting force produced by the non-axisymmetric mode (with \( e^{i\omega t} \))
dependence) in the individual accelerating units. In the continuum treatment of the cumulative BBU, the governing equations for $\xi$ and $q$ may be written as

$$
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right) \left[ \gamma \left( \frac{\partial \xi}{\partial t} + v \frac{\partial \xi}{\partial z} \right) \right] + \gamma \omega_c^2 \xi = q(z,t),
$$

(1)

$$
\frac{\partial q(z,t)}{\partial t} - i \omega_0 q(z,t) = -i \gamma \omega_0^3 \xi(z,t),
$$

(2)

where $\omega_c = |e|B/\gamma m_c$ and $\xi$ is the dimensionless coupling constant proportional to the beam current. As usual, Eq. (1) expresses the deflection of the beam by the mode, whereas Eq. (2) describes the excitation of the mode by the beam's transverse displacement $\xi$.

Assuming a dependence $\exp(i\omega t - ikz)$ for the solutions, Eqs. (1) and (2) yield the dispersion relation

$$
D(\omega, k) = \left[ \left( \omega - kv \right)^2 - \omega_c^2 \right] \left( \omega - \omega_0 \right) - \omega_0^3 \xi = 0.
$$

(3)

In the terminology of mode coupling, this dispersion relation describes the interaction between the cavity mode ($\omega = \omega_0$) and the fast and slow beam-cyclotron mode ($\omega - kv = \pm \omega_c$).

From the general stability theory, the dominant asymptotic growth of disturbances may be determined from the Green's function

$$
G(z,t) = \int_{\Gamma} dw \int_{\Gamma} dk \exp(i\omega t - ikz) / D(k,\omega) \sim \int_{\Gamma} dw \exp(i\omega t - ik(\omega)z),
$$

(4)

where the Bromwitch contour $\Gamma$ lies sufficiently far in the lower half complex $\omega$ plane, and $k(\omega)$ is the meaningful solution obtained from the dispersion relation $D(\omega,k) = 0$. 

3
Let us first recover the previously established results from Eq. (4).

When the focusing magnetic field is absent, $\omega_c = 0$ and

$$k(\omega) = \omega/v \pm (\omega_0/v) [\omega_0 \varepsilon/(\omega-\omega_0)]^{1/2}.$$ Substitution of this $k(\omega)$ in (4) yields the following asymptotic formula (from a saddle point calculation$^2$):

$$\left|G_n(z,t)\right| \sim \exp\left(1.64W^{1/3}\right),$$

(5)

where

$$W = \varepsilon \left(\omega_0^3z^2/v^2\right) (t - z/v), \quad (t > z/v > 0).$$

(6)

The asymptotic solution (5) was first obtained by Panofsky and Bander [cf. Eq. (33) of Ref. (2)] and was confirmed by Neil and Cooper$^3$ and by Gluckstern et al.$^7$ in two rather different analyses. In the other limit, where a strong focusing magnetic field is present, the dominant interaction is expected to be between the (positive energy) cavity mode

$(\omega = \omega_0)$ and the negative energy beam-cyclotron mode,$^{13,14}$ for which

$\omega - kv = -\omega_c$. In this case, the dispersion relation (3) may be approximated by $(\omega - kv + \omega_c)(\omega - \omega_0) = -\omega_0^3\varepsilon/2\omega_c$, yielding

$$k(\omega) = (\omega + \omega_c)/v + \varepsilon \omega_0^3/(2v \omega_c(\omega - \omega_0)).$$ Substituting this $k(\omega)$ in Eq. (4) and performing a saddle point calculation similar to that given in Ref. 2, one obtains the asymptotic solution

$$\left|G(z,t)\right| = \left|G_s(z,t)\right| \sim \exp\left[(2\rho W/z)^{1/2}\right],$$

(7)

where $\rho = v/\omega_c$ and $W$ is given by Eq. (6). With the use of the appropriate coupling constant$^{11}$ in $W$, the asymptotic formula (7) is easily shown to be identical to the growth factor given in Eq. (5.13) of Neil et al.$^5$ The simple interpretation, in terms of mode coupling, of the asymptotic dependence $\exp(bt)^{1/2}$ evident in (7) is given here for the first time. It is not readily extracted from the original analyses in Ref. 5.
For general values of $\omega_c$, Eq. (3) gives

$$k(\omega) = \frac{\omega}{v} + \frac{1}{v} \left[ \omega_c^2 + \frac{\epsilon \omega_0^3}{(\omega-\omega_0)} \right]^{1/2},$$

(8)

and the saddle point contribution may also be calculated analytically (contrary to the assertion of Ref. 2, at least for a coasting beam). The dominant contribution to (4) gives

$$|G(z,t)| \sim \exp \left\{ \Re \left( \frac{iz \epsilon}{\omega_s} (\omega_s - 1 - \frac{1}{2 \omega_s^3}) \right) \right\},$$

(9)

where the dimensionless time $\tau$ is

$$\tau = \mathcal{V}(\rho/z)^3,$$

(10)

and $\omega_s$ is the root of the fourth degree polynomial:

$$\omega_s^3 (1+\omega_s) = (1/2\tau)^2.$$

(11)

It is easy to show that there is one and only one root of $\omega_s$ in Eq. (11) with $\Im \omega_s < 0$ for all values of $\tau \neq 0$, and we should use that root of $\omega_s$ in (9).

The solution (9) implies that, given a focusing magnetic field, the asymptotic growth is independent of the strength of the magnetic field. To see this, consider a time long enough so that $2\tau \gg 1$. Then Eq. (11) gives

$$\omega_s = (1/2\tau)^{2/3} e^{-i2\pi/3}$$

and the solution (9) reduces to (5), the formula corresponding to zero focusing magnetic field. This is a rather surprising result, obtained directly from the model of Panofsky and Bander,² but is, at first sight, contradictory to the findings of Neil et al.⁵

The above paradox may be resolved by noting that in the "strong focusing" regime, defined as $2\tau \ll 1$, Eq. (11) gives

$$\omega_s = -i(1/2\tau)^{1/2} [1-i(1/4)(2\tau)^{1/2}].$$

Substitution of this $\omega_s$ into (9)
yields (7) in the limit $2\tau << 1$. In fact, the condition $2\tau << 1$ may easily be shown to be similar to the one imposed by Neil et al when they derived Eq. (7) using an entirely different approach. Thus, the dimensionless time $\tau - 1/2$ marks the transition from the strong focusing to the weak focusing regime [cf Eq. (11)]. From Eqs. (10) and (6), the time ($t_T$) about which this transition occurs is $t_T = (z/v)(\omega_c/\omega_0)^3/2\varepsilon$.

As an example, consider a 1 kA, 30 MeV electron beam coasting in a solenoidal magnetic field of 3 kG inside a disk-loaded waveguide whose individual cavities support a deflecting TM$_{110}$ mode with $\omega_0/2\pi = 1$ GHz. Suppose that a misalignment, say, produces an "excitation" at some axial location and we examine the BBU in response to this excitation 3 m downstream. For these parameters, $\varepsilon = 4.13 \times 10^{-4}$ [cf. Ref. 11] and $t_T = 33$ ns. Thus, if the pulse length substantially exceeds $t_T$, the cumulative BBU would evolve at the later stage as if the focusing magnetic field is absent.

The present model may readily be generalized to treat the regenerative BBU with the inclusion of a negative group velocity $v_g$. The asymptotic formulas given below explicitly show the change of the character in the BBU growth with the sign of $v_g$. The simplest way to include the effects of a non-zero group velocity is to replace the factor $(\omega - \omega_0)$ in Eq. (3) by $(\omega - \omega_0) - v_g (k - k_0)$, where $(\omega_0, k_0)$ may now be taken as the point of intersection of the dispersion curves in the $(\omega, k)$ plane between the "beam line" and that of the slow wave structure formed by the accelerating units. With this replacement, the Green's function (4) may again be re-evaluated. In the case of no focusing magnetic field, it gives

$$|G_n(z, t)| = \exp \left\{ \frac{1.84 v^{1/3}}{1 - v_g^2/z} \left( 1 - \frac{1}{1 - v_g^2/z} \right)^{2/3} \right\}, \quad (12)$$
whereas in the case of strong focusing, it reads

\[
|G_s(z,t)| = \exp \left\{ \left( \frac{2\alpha W z}{\beta_g} \right)^{1/2} \left( \frac{1}{1-\beta_g} \right) \left( 1 - \frac{v}{\beta_g} \frac{t}{z} \right)^{1/2} \right\}, \tag{13}
\]

where \( \beta_g = v/\beta \). It is obvious that (12) and (13) reduce to (5) and (7), respectively, as \( v \to 0 \). Since \( v < 0 \) for regenerative BBU, it is easily seen from Eqs. (12) and (13) that both \( G_n \) and \( G_s \) grow like a simple exponential function of time [i.e., \( \exp(\alpha t) \)] as \( t \to \infty \) when \( v < 0 \). This, of course, is consistent with what is expected from the outset when a backward wave interacts with an electron beam\(^{12,13} \). It also reaffirms the potential danger of the regenerative BBU, as \( \exp(\alpha t) \) grows considerably faster than either \( \exp(\beta t)^{1/3} \) or \( \exp(\gamma t)^{1/2} \) for large \( t \).

The following point may also be of interest. A comparison of (12) with (5) and (13) with (7) suggests that the modification in the exponentiation due to a non-zero group velocity depends only on \( v \) and is otherwise independent of the accelerating structure. It is expected, then, that the same modification would emerge in a matrix formulation similar to the ones given in Refs. 5 and 8. The asymptotic formulas (12) and (13) would provide an immediate determination of the potential prevalence of the accumulative or regenerative BBU from a knowledge of the structure (e.g., Brillouin diagram\(^{10,15} \)).

Finally, we remark that Eqs. (12) and (13) are also valid when \( \beta_g > 0 \). In that case, the BBU becomes convective\(^{12} \) and there would be an additional restriction on the solutions (12) and (13); namely, disturbance growth is possible only for \( z/v < t < z/v \) at a given position \( z \). In fact, all of the various asymptotic time dependences examined in this paper may easily be understood in terms of the analyticity of the Laplace transform of the Green's function. From Briggs' stability theory,\(^{12} \) it may be shown that the \( \beta_g > 0 \) and the \( \beta_g < 0 \) cases correspond to, respectively, analyticity in
the lower half $\omega$ plane including the real $\omega$ axis, and a branch point somewhere in the lower half $\omega$ plane. The special case $g = 0$ (cumulative BBU) corresponds to analyticity in the lower $\omega$ plane, but with a branch point on the real $\omega$ axis, leading to an exponential growth whose exponent is a fractional power of time, as exhibited in Eqs. (5) and (7).

In addition to the general theory reported above, a similar formulation has been carried out to calculate the excitation of the BBU in the accelerated beam, by the misalignments in the primary beam, in the two beam accelerator configuration recently proposed by Friedman and Serlin. The preliminary conclusion is that, in the parameter regime currently studied, the BBU does not appear to be serious.

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References


10. The quality factor $Q$ associated with the accelerating structure is an extremely important parameter in assessing the potential danger of BBU, once the beam current and the pulse length are given. In this paper, we focus mainly on the growth of BBU and its transition in different regimes, as the stabilizing influence due to a finite $Q$ may be accounted for phenomenologically by simply multiplying the Green's function by $\exp\left(-\omega_0 t/20\right)$.

11. The coupling constant $\varepsilon$ depends on the accelerating structure and on the deflecting mode under consideration. The configuration treated in Ref. 1 consists of a circular waveguide loaded with identical apertured disks along the guide axis. In that case, $\varepsilon = 0.0248 (I/I_kA)B/c$ if the deflecting mode within the individual cavities is the $TM_{110}$ mode. Here, $B = \omega c$, $c$ is the speed of light.
In the more general configuration treated in Refs. 2 and 5, the accelerating units are separated by a distance \( L \) and the \( n \)-th unit is located at \( z = nL \). In that case, \( \epsilon = (v^2 K/\omega_n^2 L) I/(17\beta kA) \) where \( K \) is proportional to the "transverse impedance" of the structure. [\( K \) has a unit of inverse length; it is identical to the \( k \) defined in Eq. (3.11) of Ref. 5]. Note that \( \epsilon \) is independent of the focusing magnetic field but is inversely proportional to \( \gamma \).


15. It should be emphasized that the damping mechanisms associated with phase mixing and a finite \( Q \) might have a decisive influence on the 3BU growth. Nevertheless, from an analytical point of view, it is interesting that the asymptotic growth in an unstable system described by the dispersion relation (3) is independent of \( \omega_c \).

In the case of an accelerating beam, a careful examination of the Green's function would show that the asymptotic growth of the cumulative 3BU is also independent of the focusing magnetic field, in the sense described here. In fact, the analyses and the conclusions given in this paper need not be restricted to a solenoidal focusing magnetic field. For general linear transverse focusing, we simply replace \( \omega_c \) by \( k_b c \), where \( k_b \) is the betatron wave number, and the asymptotic formulas remain valid.
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