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THE NATURE OF THE AIRCRAFT COMPONENT FAILURE PROCESS: A WORKING NOTE

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LMI

Executive Summary

THE NATURE OF THE AIRCRAFT COMPONENT FAILURE PROCESS: A WORKING NOTE

The physics of component failures is normally assumed to follow a Poisson process. However, many studies have shown that the component demands in the U.S. Air Force supply system have a variance-to-mean ratio (VMR) much higher than 1.0, the VMR of a Poisson process. This apparent contradiction is resolved by modeling component failures as a Poisson process whose demand rate is not fixed, but rather is itself a stochastic process, wandering over time as a result of various causes such as weather, flying intensity, reliability growth, and, presumably, other unknown factors.

Component and program data for the F-16 and A-10 aircraft show that demand over short time periods is Poisson. This is even more apparent when demands per flying hour are used instead of demands per day. However, when demands are aggregated over longer periods of time or more flying hours, the VMR increases. A gamma-Poisson model for future demands as a function of past observations fits the measured data. We call this process "planetary Poisson" to distinguish it from the general class of nonstationary Poisson models. (SDW)

CONTENTS

CHAPTER 1

INTRODUCTION

BACKGROUND

Models of logistics supply/resupply performance are essential tools in the budgeting and procurement decisions made by the managers of large logistics systems. The Logistics Management Institute (LMI) Aircraft Availability Model (AAM) is used by Headquarters, U.S. Air Force, in the evaluation and justification of reparable spares funding requirements in budget submissions, while the Air Force Logistics Command (AFLC) is incorporating a version of the AAM into the requirements determination process. Other supply models, such as SESAME[1], ACIM[2], and Dyna-METRIC[3] are widely used by the Army, Navy, and Air Force in spares budgeting and procurement.

To project supply performance accurately, such models need data on the variability of component demand as well as on the mean demand rate. Typically, this information is given in terms of the variance-to-mean ratio (VMR) of the demand distribution. It is critical to estimating the need for and effectiveness of various levels of safety stock.

Studies have shown that Air Force statistics on the component demand process are different from those of a Poisson process with a fixed mean [1, 2]. While a Poisson process with a fixed mean has a VMR of 1, observed VMRs often exceed 5. Furthermore, the observed VMR is normally higher for high-demand components than for low-demand ones, which suggests an underlying process that varies the demand rate by some percentage. For example, suppose that the demand process is Poisson but that the instantaneous demand rate at time t [$\lambda(t)$] has a seasonal dependency, with the demand rate higher in the winter. In such a case, the number of events in the time interval T from T_1 to T_2 is Poisson-distributed with mean equal

[1]SESAME – Selected Essential Item Stockage for Availability Method.

[2]ACIM – Availability Centered Inventory Model.

[3]Dyna-METRIC – Dynamic Multi-Echelon Technique for Recoverable Item Control.

to the integral of $\lambda(t)dt$ from T_1 to T_2. But suppose that we do not know $\lambda(t)$ or even that there is such a function describing the mean. Observation of demand over a long period would allow us to develop a histogram of demand. Not knowing of any functional relationship between the demand rate and time, we might assume that the number of demands in the time period T is Poisson-distributed with an unknown mean whose probability distribution is specified by our histogram of observations.

For simplicity, assume that our histogram leads us to believe that our observation of the "true" mean follows a gamma distribution with mean, M, and a standard deviation of 10 percent of the mean. That distribution is given by

$$f(\lambda) = \frac{e^{-\lambda/\beta}\lambda^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \qquad \text{[Eq. 1-1]}$$

where,

- The mean $M = \alpha\beta$.
- The variance $= \alpha\beta^2$.

Since the standard deviation ($\beta\sqrt{\alpha}$) is 0.1M, $\alpha = 100$.

Our model then is a demand process that is Poisson with a mean conditioned on a gamma prior distribution. The number of demands in a time period, T, has a negative binomial distribution with mean MT and VMR $1 + MT/\alpha = 1 + MT/100$.

This trend of increasing VMR with increasing demands is characteristic of a Poisson process with a gamma prior distribution whose standard deviation is a fixed percentage of the mean. This trend is also typical of observed aircraft component failure processes. Thus, we believe that the demand process can be usefully modeled as a Poisson process whose mean varies over time within a gamma prior distribution.

THE PLANETARY POISSON PROCESS

The Poisson distribution, named for Simeon Denis Poisson (1781 – 1840), a French mathematician known for his work on the application of mathematics to physics, is defined as:

$$p(n) = \frac{e^{-M}M^{n}}{n!} \qquad n = 0, 1, 2 \qquad \text{[Eq. 1-2]}$$

where M is the mean.

The first military application of the Poisson distribution was in modeling death by horsekick in the Prussian army. The Poisson distribution received little attention until 1907 when W. S. Gosset ("Student") showed that the sampling error in counting yeast cells with a hemocytometer was Poisson-distributed and provided a theoretical argument to support that model. Now, because of its close relationship to many physical phenomena, the Poisson distribution is widely used.

A stochastic process is said to be Poisson with intensity λ if it generates a sequence of events over time such that for any time interval $(t, t+\Delta t)$ the probability distribution for the number of events in the interval is Poisson with mean $\lambda\Delta t$. The necessary and sufficient conditions for this to occur, which are often used as the definition of a Poisson process, are:

- The number of events in nonoverlapping intervals is independent.
- As Δt approaches zero, the probability of one event in $[t, t+\Delta t]$ approaches $\lambda\Delta t$.
- As Δt approaches zero, the probability of more than one event in $[t, t+\Delta t]$ approaches zero faster than Δt (i.e., $p(n>1)/\Delta t$ approaches 0).

The instantaneous probability of an event is $\lambda\Delta t$. The key to the process being Poisson is that this instantaneous probability is always the same irrespective of the recent occurrence or nonoccurrence of events. This process is stationary; that is, the probability distribution of the number of demands in $[t, t+\Delta t]$ is a function only of Δt.

The definition can be generalized to the case where the intensity is not a constant. If the intensity is a function of time, $\lambda(t)$, we interpret $\lambda(t)\Delta t$ as the instantaneous probability of an event. If Λ is the integral of $\lambda(t)dt$ from T_1 to T_2, then the number of events in the interval T_1 to T_2 is Poisson-distributed with mean Λ. This dynamic Poisson process is at the heart of the pipeline calculations in wartime models such as the Aircraft Sustainability Model [3] and Dyna-METRIC [4].

However, even in a peacetime/steady-state environment, the intensity may not be constant. The intensity could have a seasonal dependency or could be higher in the daytime than at night. Further generalizing the concept of fluctuating intensity to include an intensity that is a stochastic process yields a "planetary Poisson" process.

In the logistics application of interest here – the demand process for an aircraft component – the intensity corresponds to the instantaneous component demand or failure rate. Because the demand rate is affected by such unpredictable factors as weather, the mood and health of the pilots, enemy action, or any number of imaginable influences, on any given day the actual instantaneous probability of an event (i.e., a failure) will not be known with certainty. Thus, the planetary Poisson process is different from a dynamic Poisson process, in which, even though the demand rate is not constant, it is known with certainty at all times.

The planetary Poisson also differs from the commonly used Poisson process with a prior distribution on the intensity [5]. In that case, the intensity is a random variable but is time invariant. The probability distribution (the prior) on λ represents our lack of knowledge about what the actual demand rate is, but whatever it is, it is not a function of time.

It is essential to make a clear distinction between this prior distribution, which represents our lack of knowledge about the mean of a stationary Poisson process, and a distribution on a randomly varying mean. In computing the VMR of a resupply pipeline for use in a model, we must separately consider both the natural, observable VMR and the additional variance attributable to our uncertainty (i.e., ignorance) about the true mean of the pipeline. These are two very different sources of variance, and they must not be confused.

By keeping these two types of variance (which we call *natural* and *forecasting* variance) separate, we can better model the demand process. Furthermore, only when the *natural* variance is well understood can we build the correct Bayesian formulas for updating total mean and variance.

Proper forecasting depends on the nature of what you are forecasting. An excellent procedure for forecasting a standard Poisson process may be very poor for forecasting a planetary Poisson process. Past studies of forecasting failures have focused on the total variance rather than on the separate *natural* and *forecasting* components. This paper focuses only on computing the *natural* variance of a planetary Poisson process. Once that has been accomplished, we will be in a position to rethink the way we have forecast pipeline means and variance as a function of engineering estimates and observed data.

The *natural* VMR of a pipeline will depend on the distribution of $\lambda(t)$ and on the autocorrelation function of $\lambda(t)$. The *natural* VMR of a pipeline as a function of the pipeline length (T) and the components' overall average demand rate can be computed directly from the data, but it is also of interest to know the distribution of $\lambda(t)$ and its autocorrelation function for use in future demand forecasting studies.

For now, we will treat $\lambda(t)$ as stationary; that is, we will assume that the probability distribution for $\lambda(t)$ does not change from day to day. Presumably, for some components, the failure rates will actually have the seasonal dependency referred to earlier, but we will not treat that case here.

CHAPTER 2

ANALYSIS

We wish to find a mathematical formulation for the demand process as it evolves over time that is consistent with the "physics" and what we have already found in our studies on demand prediction [1, 2, 6]. The fundamental question is whether historical demand can be described as Poisson over short periods of time or whether some other demand model is required. Our objective is to describe historical demand patterns; the more difficult problem of demand prediction will be addressed in a later report.

We need to be able to represent the demand process over arbitrary periods of time because our inventory models require demand over repair/resupply lead-times of various lengths.

Data for the analysis come from a variety of sources. The characteristics are quite different, and thus the information that can be obtained is also different. For example, we have:

- A-10 aircraft daily demand data from England Air Force Base (AFB), Louisiana, for a 2-year period. Those data include a number of fairly high demand items, which is useful for analysis. While the overall flying program was fairly stable, we do not have the flying-hour data for each day. Thus, some microanalyses relating demand to program cannot be performed.

- F-16 aircraft daily demand and flying-hour data from Hill AFB, Utah; MacDill AFB, Florida; and Nellis AFB, Nevada. We have flying hours by day, and we can distinguish line replaceable units (LRUs) from shop replaceable units (SRUs). However, the data for a given base are only for about a year, and most items have fairly low demand. For that reason we have analyzed total demand for LRUs, SRUs, or combined units. We also have some monthly aggregate data for longer periods of time — on the order of 2½ years.

- F-16 and A-10 quarterly demand data from the Recoverable Consumption Item Requirements System (D041) for 4 years. These data have program element information by item.

EMPIRICAL FINDINGS

Data Set 3 [1] was used to find that the variance-to-mean ratio of recoverable item demand over a year is best described by:

$$VMR = 1 + 0.14M^{0.5} \qquad \text{1-year period} \qquad \text{[Eq. 2-1]}$$

where M is the estimate of mean annual demand. This relationship is similar to the one developed in an earlier study [2], also based on D041 data, for 1,020 items on various weapon systems. Over a 2-year period, some evidence indicated that the relationship should be:

$$VMR = 1 + 0.355M^{0.55} \qquad \text{2-year period} \qquad \text{[Eq. 2-2]}$$

This relationship should be applied to demand/flying hour or demand/quarter depending on which was more stable during the first 2-year period used as history. For about half of the items, demand/quarter was more stable or demand/flying hour could not be computed because of missing data.

Analyses of Data Sets 1 and 3 [1, 2] (and the earlier 1,020 items) show that demand in adjoining time periods is more highly correlated and the correlation decreases smoothly as the interval of time between the periods increases. An example of a process that is consistent with these data would be a Poisson process with a time-varying mean. The problem is to find a probabilistic mechanism to describe the variation of the mean. Even though we believe the wandering mean is consistent with the data, we have been unable to predict the trend (except in the sense that exponential smoothing, the recommended technique for predicting the mean, gives more weight to more recent data).

THEORETICAL CONSIDERATIONS

The primary reason that we believe a Poisson process with wandering mean is a good model is that the Poisson is the most general "independent increments" model in which demands do not occur in clumps (the compound Poisson leads to clusters of demand, but they are rarely observed when transaction data are analyzed over short time periods such as a day). By "independent increments" we mean that future demand is independent of the times at which earlier demands occurred.

The simplest type of wandering Poisson is a Markov-type process, in which the knowledge of the demand rate today contains all the information about the process (i.e., if we know the true mean last week as well, it adds no information). However, the analysis of Data Sets 1 and 3 refutes this simple model.

- Exponential smoothing with a constant of 1.0 would be the appropriate technique if the process were Markov. However, exponential smoothing with a constant of 0.4 applied to quarterly data was better on every data set [1, 2].
- If the correlation between demand in Period 1 and Period 2 is r and the correlation between demand in Period 2 and Period 3 is r, then in a Markov process the correlation between demand in Period 1 and Period 3 should be about r^2. In our previous study [1], the correlations between demand in adjoining quarters for the F-16 were 0.201 and separated by a quarter were 0.085 – much higher than the value of $0.04 = 0.201 \times 0.201$. Similarly for the A-10, the correlations were 0.207 and 0.095, respectively. Also, for biweekly data, the values were 0.034 and 0.027, respectively.

Another model of a Poisson process with time-varying mean that is similar to the Markovian model is a process in which the true mean demand has a probability distribution (e.g., gamma). Demand is assumed to be Poisson with that (unknown) mean. Then after some time has elapsed, that true mean demand is used as the mean of a new probability distribution of true mean demand. Demand is again Poisson for some period of time, and the branching continues.

Three such models are discussed in the Appendix, the last of which appears to be a useful model for a planetary Poisson process.

Statistical Tests

To analyze data statistically, we must hypothesize some probability model and invoke statistical tests to evaluate the adequacy of the probability model. As described earlier, a Poisson process is the most appealing physical model, although the assumption of a constant mean is not supported. How do we determine that the constant-mean Poisson does not fit?

Chi-Square Goodness of Fit

The best-known goodness-of-fit test is the chi-square. To test whether some data are Poisson with a constant mean, we compute the average demand per period and then compare the number of periods with 0, 1, 2, . . . N observed demands with the theoretical probabilities from a Poisson distribution with that mean.

Let

O(i) = The number of periods with i demands observed.

E(i) = The number of periods with i demands expected.

Then:

$$\mathrm{CHI} = \sum_{i=0}^{N} \frac{[O(i) - E(i)]^2}{E(i)} \qquad \text{[Eq. 2-3]}$$

has a chi-square distribution with N degrees of freedom. If the computed value exceeds the 95-percent level (i.e., if there is less than a 5-percent probability that such a result could occur by chance if the distribution were really Poisson), the Poisson hypothesis is rejected.

One problem with using this test is that the expected number of observations in each of the (N+1) cells above should have an expected value of 5 or more. This means that the tail probabilities for large values of i must be combined, but those values are the most likely ones to provide evidence that the data are not Poisson.

A much more sensitive test of goodness-of-fit for the Poisson is known as the Poisson index of dispersion [7]. Since no combinations of data are required as above, a few very large observations will tend to result in rejection of the Poisson hypothesis. Let

X = The average observed demand/period.

X(j) = The number of demands observed in period j.

M = The number of periods.

Then the following statistic is distributed as chi-square with M−1 degrees of freedom.

$$\mathrm{CHI} = \sum_{j=1}^{M} \frac{[X(j) - X]^2}{X} \qquad \text{[Eq. 2-4]}$$

Since M is much larger than N (at least 5 times as large), the test is much more sensitive. That is, a set of data not rejected by the chi-square test of Equation 2-3 may be rejected by the more sensitive test of Equation 2-4.

Essentially this test adds up VMRs in each period. The VMR for a Poisson is 1, so it will reject a set of data if chi in Equation 2-4 is much greater than the number of periods, M.

EMPIRICAL EVIDENCE

Our empirical data analyses consistently show a significant autocorrelation between demand in neighboring periods. Furthermore, the assumption of a constant mean Poisson over several periods is rejected by both the tests of Equations 2-3 and 2-4. Because the Poisson process is the most general "independent increments" process in which demand in nonoverlapping time periods is independent (and does not occur in clusters), we would like to retain that model. The obvious solution is to look for processes that are Poisson but over shorter periods of time.

That solution requires us to develop a new goodness-of-fit test for a time-varying Poisson mean. Equation 2-3 is no help because we do not have enough periods (because of the restriction on the number of observations per cell to at least five). Equation 2-4 is the obvious candidate for modification. Suppose that we compute the mean for each group of three periods (i.e., $1+2+3$, $2+3+4$, $3+4+5, \ldots$) and take as observed values the value from the middle period [i.e., X(2), X(3), X(4), . . .]. More generally, let K periods be used, where K is odd. Then we assert that the following quantity should be distributed approximately as chi-square with $(M-K+1)(K-1)/K$ degrees of freedom:

$$\mathrm{CHI} = \sum_{j=(K+1)/2}^{M-(K-1)/2} \frac{[X(j) - X(j)]^2}{X(j)} \qquad \text{[Eq. 2-5]}$$

To justify the number of degrees of freedom, recall that for K periods and a constant mean, we would subtract 1 degree of freedom in Equation 2-4 for estimating the mean or $(1/K)$ degrees of freedom for each term. This leaves $(K-1)/K$ degrees of freedom for each term added in Equation 2-5, and there are $M-K+1$ terms.

At the other extreme, when $K=1$, the chi-square test should be meaningless since nothing is being tested. The value of the test statistic in Equation 2-5 is zero, but so are the degrees of freedom. At this time, we will not attempt a formal proof that this statistic is chi-squared with the stated degrees of freedom.

The individual terms used in computing the chi-square statistic are identical to those used in estimating the VMR when the mean is changing. The latter is our primary interest, and the chi-square test is used only to determine whether the data could have come from a Poisson distribution.

ANALYSIS OF A-10 DATA AS A POISSON PROCESS WITH A CONSTANT MEAN

In Tables 2-1, 2-2, and 2-3, we provide summaries of the analyses on 14 high-demand A-10 items with 2 years of transaction data from England AFB. We begin by considering the possibility that a constant value for mean demand on each of the 14 items is adequate over a 2-year period. As noted earlier, we do not have flying hours by day, but the overall pattern of flying hours was fairly stable. Table 2-1 shows the VMR for the group of 14 items using a constant mean for each item and time periods of various lengths.

For periods longer than a day, several different starting points are selected for combining days into a period. For example, when the period is 7 days long, seven computations are made and the results averaged. The periods for the first computation are composed of days $1-7$, $8-14$, $15-21$, etc; for the second computation, days $2-8$, $9-15$, $16-22$, etc. Any partial period at the end of the data series is excluded to remove extraneous variance.

Table 2-1 demonstrates that VMRs tend to increase as the number of days in the period increases, which shows that there is positive autocorrelation between data in neighboring time periods. The one surprise is that the effect of weekends does not make the VMRs for 7 days and 14 days smaller. Although the data in Table 2-1 are aggregated over all 14 items, individual item results even for periods of 1 day show sample VMRs that all exceed 1 (ranging from 1.12 to 4.89).

ANALYSIS OF A-10 DATA AS A PLANETARY POISSON PROCESS

Now that we have established the inadequacy of a constant mean assumption, we turn to an analysis of time-varying means. The VMR is the sum of terms in Equation 2-5 multiplied by $K/(K-1)$ to give an unbiased statistic and divided by the

TABLE 2-1

A-10 VMRs AS A FUNCTION OF NUMBER OF DAYS IN A PERIOD

Number of days in a period	VMR	Number of days in a period	VMR
1	1.61	21	2.37
2	1.72	28	2.50
3	1.77	35	2.63
4	1.77	42	2.76
5	1.77	49	2.89
6	1.80	56	2.99
7	1.86	63	3.08
8	1.93	70	3.18
9	2.01	77	3.25
10	2.07	84	3.29
11	2.11	91	3.35
12	2.13	98	3.42
13	2.16	105	3.47
14	2.18		
15	2.22		

number of terms. We note that the VMR, which would be 1 for a Poisson, increases consistently with K, the number of weeks in Table 2-2.

To eliminate any weekend effect, we aggregated the data into 7-day periods and computed VMRs as a function of the number of weeks used in computing mean demand to obtain Table 2-3. Note that the values for 7 days in Table 2-2 are the first four entries in Table 2-3.

The number of degrees of freedom starts to increase with K because of the factor $(K-1)/K$ and then decreases because of the factor for the number of periods, $M-K+1$. When the number of degrees of freedom, D, is more than 30, the chi-square statistic is normally distributed with mean D and variance 2D. The number of standard normal deviates is used for significance testing of the Poisson hypothesis. A 95-percent level of significance and a one-sided critical region is 1.64 standard deviations. The number of standard deviations for the group of 14 items in Table 2-3

TABLE 2-2

A-10 VMRs AS A FUNCTION OF NUMBER OF DAYS IN A PERIOD AND NUMBER OF PERIODS, K, USED FOR MEAN

Number of days in a period	Variance/Mean			
	K = 3	K = 5	K = 7	K = 9
1	1.13	1.17	1.23	1.29
2	1.26	1.38	1.27	1.25
3	1.41	1.25	1.33	1.39
4	1.33	1.28	1.35	1.40
5	1.22	1.35	1.36	1.40
6	1.22	1.31	1.40	1.46
7	1.26	1.36	1.46	1.50
8	1.31	1.43	1.52	1.54

TABLE 2-3

A-10 VMRs AS A FUNCTION OF NUMBER OF WEEKS USED IN COMPUTING MEAN DEMAND

K	VMR	Number of DF	K	VMR	Number of DF
3	1.26	818	15	1.56	1,083
5	1.36	1,011	17	1.59	1,065
7	1.46	1,072	19	1.63	1,046
9	1.50	1,091	21	1.68	1,025
11	1.50	1,093	31	1.82	906
13	1.51	1,086	41	2.00	777

Note: DF = degrees of freedom.

exceeds 5.0 at each value of K. Therefore, the Poisson hypothesis is strongly rejected everywhere.

However, on an individual item basis, 9 of the 14 items pass the chi-square test of significance for the Poisson (values less than 1.64 standard deviations) at $K=3$; this drops to 5 of 14 at $K=5$. Furthermore, the planetary Poisson model only predicts that the trend as $K \rightarrow 1$ would be a VMR of 1.0. No claims are made for particular $K > 1$. Tables 2-2 and 2-3 are mostly encouraging with respect to the trend as $K \rightarrow 1$. For example, Table 2-3 shows a VMR trending to 1.16 at $K = 1$.

Nevertheless, these values still fail the test. This is not surprising since we could not incorporate daily flying-hour data into the A-10 data base. The flying-hour program does not necessarily "wander slowly" from day to day the way we hypothesize the failure rate does.

DATA ANALYSIS: F-16 WITH CONSTANT MEAN

Now we repeat the analyses above for the F-16 with one important difference: we have the daily flying hours as well as daily demands over 58 weeks. Table 2-4 shows the results on VMR of various period lengths in days for the assumption of constant demand by item (actually 16 Work Unit Code aggregations). Those results confirm the results shown in Table 2-1 for the A-10 although the increase in VMR is even more dramatic with the F-16 because of the flying-hour program that began low and increased substantially.

TABLE 2-4

F-16 VMRs AS A FUNCTION OF NUMBER OF DAYS AND NUMBER OF FLYING HOURS

Number of days in a period	VMR	Number of flying hours in a period	VMR
1	2.8	40	1.70
2	3.8	80	2.12
3	4.4	120	2.45
4	4.8	160	2.73
5	5.1	200	3.10
6	5.4	240	3.18
7	5.9	280	3.59
8	6.4	320	3.73
9	7.0	360	3.74

The average number of flying hours per day for the 58 weeks was about 40. Using multiples of 40 as an aggregation criterion, we computed VMRs. The VMRs on any given line are thus of comparable period length (e.g., 7 days are comparable to 280 flying hours). Thus, flying hours are seen to be better than calendar time for purposes of aggregation (we examined the possibility that a combination of both would lead to still lower VMRs, but that did not occur).

One technical note should be added. When aggregating into short periods such as 40 or 80 flying hours, a period or even several periods are likely to end during a given day. Since we have only the total flying hours and the total demands for the day, we must split the demand for the day into the flying-hour periods.

Our first procedure was to say that if the 40 hours was a fraction, P, of the flying hours for the day, then demand for the period would be estimated as the fraction P of the demand for the day. However, it is clear that this induces artificial stability into the demands/period for short flying-hour periods. A better procedure, employed in Tables 2-4 (and later in Table 2-6), is to use P as the probability that each demand of the total for the day occurs during a particular flying-hour period. Thus, the total of a day's demand *for a particular flying-hour period* has a binomial distribution.

DATA ANALYSIS: F-16 WITH VARYING MEAN

Table 2-5 is similar to Table 2-2; Table 2-6 differs from those tables in that we show results for the aggregation over flying hours.

The results in Table 2-6 based on flying hours are clearly better than those in Table 2-5 based on time. This is not surprising since the flying hours and demand on the F-16 both increased dramatically during the time period.

In all cases, the VMRs using means computed from three periods are lower than those computed from five. This is not surprising either, since the former allows the mean demand to change more rapidly.

Once again, if we ignore the flying-hour data (Table 2-5), even the trend as $K \rightarrow 1$ is not encouraging. However, when we use the flying-hour data (Table 2-6), the trend is good. Figure 2-1 graphs these data, and we can see the trend as $K \rightarrow 1$.

TABLE 2-5

F-16 VMRs AS A FUNCTION OF NUMBER OF DAYS IN A PERIOD AND NUMBER OF PERIODS, K, USED FOR MEAN

Number of days in a period	Variance/mean			
	K = 3	K = 5	K = 7	K = 9
1	1.53	1.72	1.89	2.08
2	2.13	2.65	2.28	2.14
3	2.90	2.29	2.41	2.52
4	2.69	2.30	2.45	2.50
5	2.14	2.47	2.36	2.42
6	2.00	2.21	2.34	2.50
7	2.09	2.25	2.44	2.50
8	2.18	2.38	2.59	2.73

TABLE 2-6

F-16 VMRs AS A FUNCTION OF NUMBER OF FLYING HOURS IN A PERIOD AND NUMBER OF PERIODS, K, USED FOR MEAN

Number of flying hours in a period	Variance/mean			
	K = 3	K = 5	K = 7	K = 9
40	1.10	1.19	1.23	1.25
80	1.30	1.38	1.44	1.51
120	1.40	1.51	1.63	1.64
160	1.51	1.66	1.77	1.80
200	1.62	1.83	1.87	1.97
240	1.61	1.83	1.93	2.00
280	1.92	2.03	2.20	2.23
320	1.93	2.12	2.25	2.29

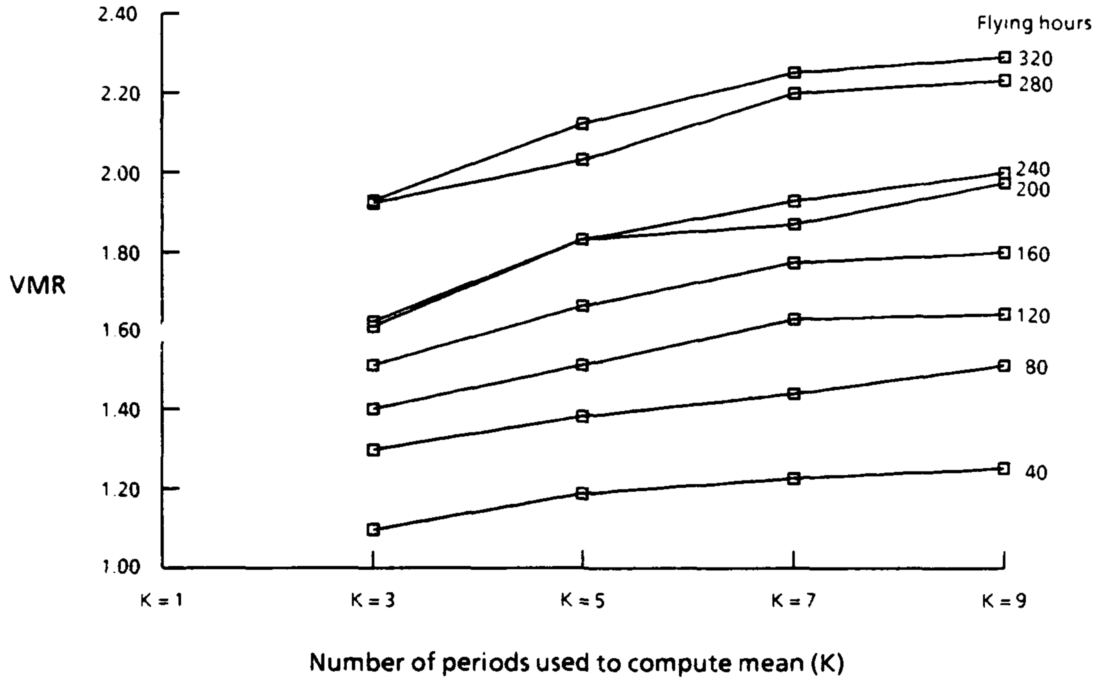

FIG. 2-1. VMR VERSUS PERIOD LENGTH AND NUMBER OF PERIODS

CHAPTER 3

CONCLUSIONS

- The component failure process in the U.S. Air Force is not Poisson with a fixed mean demand rate.
- Mean demand rates drift over time. Both sets of data confirm that demand in neighboring time periods is highly correlated. Thus, the VMRs increase as the number of days or number of flying hours in a period increase (Tables 2-1 and 2-4).
- Demand over short periods of time or flying hours is nearly Poisson. However, for this conclusion to hold, mean demand must be computed over 3- to 5-day time periods (Tables 2-2, 2-5, and 2-6).
- When flying hours change dramatically, as in the F-16 case, it is essential to compute demand on a per flying-hour basis (Table 2-6).
- The component failure process in the U.S. Air Force can be modeled as a nonstationary Poisson process whose demand rate is itself a stochastic process – what we have called a "planetary Poisson" process.

REFERENCES

[1] LMI Report AF601R1. *Evaluation of Demand Prediction Techniques.* Sherbrooke, Craig C. Mar 1987.

[2] Sherbrooke, Craig C. *Estimation of the Variance-To-Mean Ratio for AFLC Recoverable Items.* Potomac, Maryland: Sherbrooke and Associates. Jan 1984.

[3] LMI Report AF601R2. *Prototype Aircraft Sustainability Model.* Slay, F. Michael, and Randall M. King. Mar 1987.

[4] Hillestad, R. J. *Dyna-METRIC: Dynamic Multi-Echelon Technique for Recoverable Item Control.* R-2785-AF. Santa Monica, Calif.: The RAND Corporation, Jul 1982.

[5] LMI Report AF301-3. *VARI-METRIC – An Approach to Modeling Multi-Echelon Resupply when the Demand Process Is Poisson with a Gamma Prior.* Slay, F. Michael. Mar 1984.

[6] LMI Report ML108. *Toward Improved Initial Provisioning Strategies.* Abell, John B., Joan E. Lengel and F. Michael Slay. Apr 1982.

[7] Hoel, Paul G. *Introduction to Mathematical Statistics.* New York: John Wiley & Sons, Inc., 1962.

APPENDIX

GAMMA POISSON MODELS

In this appendix, we examine three alternative Bayesian models in which the prior distribution of mean demand is assumed to be gamma and the distribution of demand about that mean is Poisson. Poisson demand is of interest because our work in the body of the report shows that demand over short time periods tends to be Poisson. A gamma prior distribution is of interest because it tends to have the right shape (skewed to the right) and it has mathematical properties that simplify combination with the Poisson.

Model 1, the simplest model, assumes that demand in each period has the same gamma prior distribution. Because demand in each period is independent of demand in other periods, the mean demand is constant and demands in neighboring periods are uncorrelated. Thus, this model does not conform with our observations of the real world in which mean demand rates do change and demands in neighboring quarters have higher correlations.

Model 2 is also well-known. Here the prior distribution is updated with the observed demand to obtain a new posterior distribution. The Poisson and gamma are called conjugate distributions because the posterior is also a gamma distribution, and that simplifies the analysis. As contrasted with Model 1, a dependency exists between periods. However, the true mean demand is assumed to be constant, and as the number of periods increases, the variance around the mean decreases to zero. This result conflicts with our results about a time-varying mean.

The Model 3 is our attempt to represent the changing mean. As in Model 2, Model 3 has a gamma prior distribution and Poisson demand from which a gamma posterior can be computed. Instead of using that gamma distribution as the prior for the second period mean, we modify it. Our gamma prior distribution will use the *same* mean as used in Model 2, but we do not allow its variance-to-mean ratio (VMR) to decrease as in Model 2. Instead, the VMR is held constant.

The rationale for our model is that the prior distribution for mean demand in a period should be influenced by the most recent demand. However, the VMR does not shrink to zero because other influences act on the demand process – mean demand rates do not stay constant. What we examine below is how the correlational structure of our model compares with that of our observations.

MODEL 1: SAME GAMMA PRIOR DISTRIBUTION IN EACH PERIOD AND POISSON DEMAND

A common model that leads to the negative binomial is to assume that the mean demand for an item over a specified period of time is given by a gamma distribution and the demands are then generated from a Poisson with that mean. The gamma distribution is given by

$$g(y) = \frac{1}{(a-1)!b^a} \exp\left(\frac{-y}{b}\right) y^{a-1} \qquad y>0 \qquad \text{[Eq. A-1]}$$

This function is defined for a>0 and b>0 where the factorial is to be interpreted as a gamma function for nonintegral values of a. If demand is Poisson with mean y, p(v|y), then the distribution of demand v is obtained by integrating over y, and the result is negative binomial.

$$n(v) = \frac{(a+v-1)!}{v!(a-1)!} \left(\frac{1}{b+1}\right)^a \left(\frac{1}{b+1}\right)^v \qquad v=0,1,2\ldots \qquad \text{[Eq. A-2]}$$

Note that this distribution of demand is the same for every period because it is assumed that the mean demand is drawn at random from the same gamma prior distribution each period. Thus, demand is independent from one period to another, in conflict with our observations.

MODEL 2: ONE GAMMA PRIOR DISTRIBUTION IN FIRST PERIOD WITH POISSON UPDATING

Now suppose that v demands are observed over the specified, fixed time period. Under these assumptions, the posterior probability distribution for the mean demand of the item is still gamma distributed, but the parameters a,b now become (a + v) and b/(b + 1). If that is used as the prior distribution for the second period, and the process is allowed to continue over many periods, the gamma distribution eventually degenerates to a single point and demand becomes Poisson about that mean.

MODEL 3: A GAMMA PRIOR DISTRIBUTION IN EACH PERIOD WHOSE MEAN IS OBTAINED FROM BAYES AND WHOSE VARIANCE/MEAN REMAINS CONSTANT

We will consider two periods and denote by v,w the demands observed in Period 1 and Period 2. The means and variances of these distributions are shown in Table A-1 to simplify later references. Our primary interest will be in the VMRs and their behavior as more time elapses.

TABLE A-1

PROBABILITY DISTRIBUTION VALUES

Distribution	Mean	Variance	VMR
Gamma prior – Period 1	ab	ab^2	b
N.B. demand (v) – Period 2	ab	$ab + ab^2$	$1 + b$
Gamma prior – Period 2	$(a + v)b/(b + 1)$	$(a + v)b^2/(b + 1)$	b
N.B.|v	$(a + v)b/(b + 1)$	$(a + v)b$	$1 + b$
N.B demand (w) – Period 2	ab	$ab[1 + b + b^2/(b + 1)]$	$1 + b + b^2/(b + 1)$
Total demand (v + w) – both periods	$2ab$	$ab[2 + 4b + b^2/(b + 1)]$	$1 + 2b + b^2/[2(b + 1)]$
Correlation between v and w	$b / \sqrt{(b + 1)^2 + b^2}$		

The means and variances for the gamma and negative binomial in Period 1 were noted above and are well-known. The gamma prior distribution for Period 2 has parameters that depend on the observed demands, v, in Period 1. Note that its variance is b times the mean by our assumptions of Model 3 [under Model 2 the variance would be the same as the gamma prior distribution for Period 1 with a replaced by $(a+v)$ and b replaced by $b/(b+1)$ – this is always smaller than the variance for Model 3 by the factor $(b+1)$ in the denominator].

Note that the VMRs for every variable are independent of the parameter a. That is, the VMR depends only on the original VMR of the gamma, b. Note also that though v and w each have negative binomial distributions, they have different VMRs. As a result, the distribution of $v+w$ is not negative binomial although we can still compute a mean and variance.

Our major interest is the behavior of the VMRs for demand over Period 1, Period 2, and both periods. Note that, even though the gamma prior distribution VMR stays constant from one period to the next at b, the VMR for the period demand increases. Over both periods, it is even larger than over Period 2. This is similar to the behavior we observe in our real-world data.

Lastly, the correlation between demand in the two periods, v and w, depends only on b; it increases from 0 to 0.707 as b increases. Thus, by selecting an appropriate value of b, we can model any desired correlation between periods (see Table A-2). By comparison, the correlation for Model 1, in which demands in each period are independent, is zero, and for Model 2 is b/(b + 1).

We will derive the mean and variance for w, the demand in Period 2:

$$E(w) = E(w|v)n(v) \qquad \text{[Eq. A-3]}$$

The conditional mean on the right-hand side is equal to the mean of the prior distribution for Period 2, found in Table A-1. The mean of the negative binomial distribution for v is found just above it.

$$E(w) = \sum \frac{(a+v)b\,n(v)}{(b+1)} = \frac{ab}{b+1} + \frac{ab^2}{b+1} = ab \qquad \text{[Eq. A-4]}$$

Similarly, by assumption the variance to mean of the prior distribution for Period 2 is b, and this implies, as above, that the variance to mean of the negative binomial distribution for w, conditional on v, is b + 1. Thus,

$$\mathrm{Var}(w|v) = (a+v)b \qquad \text{[Eq. A-5]}$$

$$\begin{aligned}
\mathrm{Var}(w) &= E(w^2|v)n(v) - [E(w)]^2 \\
&= \{\mathrm{Var}(w|v) + [E(w|v)]^2\}n(v) - (ab)^2 \\
&= \left(\frac{(a+v)b + (a^2+2av+v^2)b^2}{(b+1)^2}\right)n(v) - (ab)^2
\end{aligned}$$

$$\mathrm{Var}(w) = ab\left(1 + b + \frac{b^2}{b+1}\right) \qquad \text{[Eq. A-6]}$$

TABLE A-2

CORRELATION OF DEMAND IN PERIODS 1 AND 2 AS FUNCTION OF b

b	VMR			Correlation
	Period 1	Period 2	Both	
0.1	1.1	1.1	1.2	0.091
0.2	1.2	1.2	1.4	0.164
0.3	1.3	1.4	1.6	0.255
0.4	1.4	1.5	1.9	0.275
0.5	1.5	1.7	2.1	0.316
0.6	1.6	1.8	2.3	0.351
0.7	1.7	2.0	2.5	0.381
0.8	1.8	2.2	2.8	0.406
0.9	1.9	2.3	3.0	0.428
1.0	2.0	2.5	3.3	0.447
2.0	3.0	4.3	5.7	0.555
3.0	4.0	6.3	8.1	0.600
4.0	5.0	8.2	10.6	0.625
5.0	6.0	10.2	13.1	0.640
6.0	7.0	12.1	15.6	0.651
7.0	8.0	14.1	18.1	0.659
8.0	9.0	16.1	20.6	0.664
9.0	10.0	18.1	23.1	0.669
10.0	11.0	20.1	25.5	0.673
20.0	21.0	40.0	50.5	0.690
30.0	31.0	60.0	75.5	0.695
40.0	41.0	80.0	100.5	0.698
50.0	51.0	100.0	125.5	0.700
60.0	61.0	120.0	150.5	0.701
70.0	71.0	140.0	175.5	0.702
80.0	81.0	160.0	200.5	0.703
90.0	91.0	180.0	225.5	0.703
100.0	101.0	200.0	250.5	0.704

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A new model of the component failure process is presented that explains why observed failures exhibit a variance-to-mean ratio greater than 1. Called the "planetary Poisson" process, this model of component failures is Poisson but with a demand rate that is itself a stochastic process.

Component data for the F-16 and A-10 aircraft are used to test the model.

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