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Technical Report 1224
May 1988

GEOLOCATION VIA SATELLITE:

A Methodology and Error Analysis

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ADMINISTRATIVE INFORMATION

The work reported herein was conducted for the Space and Naval Warfare Systems Command over the period October 1987–May 1988.

Released by
R.H. Moore, Head
Analysis Branch

Under authority of
R.H. Moore, Acting Head
Systems Technology
Division

AS

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SECURITY CLASSIFICATION OF THIS PAGE

ADA197955

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NOSC Technical Report 1224	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Ocean Systems Center 6b. OFFICE SYMBOL (if applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State and ZIP Code) San Diego, California 92152-5000	7b. ADDRESS (City, State and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Space and Naval Warfare Systems Command 8b. OFFICE SYMBOL (if applicable) SPWR-PDW107	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State and ZIP Code) Washington, DC 20363-5100	10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	AGENCY ACCESSION NO.
RDDA	NSA	760-CG48	DN388 590

11. TITLE (include Security Classification)

GEOLOCATION VIA SATELLITE: A Methodology and Error Analysis

12. PERSONAL AUTHOR(S)

M.J. Shensa

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Final	FROM Oct 1987 TO May 1988	May 1988	83

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	satellite communications, least squares estimation, doppler measurement. (JES)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This report is directed towards the problem of geolocating the transmitter or receiver of a satellite-relayed signal; however, its methodology is appropriate to general estimation theory. A unified treatment of nonlinear least squares estimation, dominated by techniques for analyzing the estimate accuracy, is presented. In this context, appropriate equations are derived for doppler measurements as a function of the ground stations and satellite orbit elements. These may be used to determine the satellite orbit as well as ground station locations. A general model for motion on an oblate earth is also developed. (Keywords:)

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL M.J. Shensa	22b. TELEPHONE (include Area Code) (619) 553-1650 22c. OFFICE SYMBOL Code 632

DD FORM 1473, 84 JAN

83 APR EDITION MAY BE USED UNTIL EXHAUSTED
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CONTENTS

1. OVERVIEW

INTRODUCTION

As the title suggests, the work described in this report is motivated by the problem of geolocating the transmitter or receiver of a satellite-relayed signal. The situation is that pictured in Figure 1.1; the doppler shift between a transmitter and receiver is measured over a period of time and utilized to determine the receiver or transmitter location. A number of other parameters, in addition to the unknown location of the transmitter, are intimately involved in the computation. Among these are the transmitter frequency, the satellite's orbital parameters, course and speed of moving stations, etc. Such parameters are not fundamentally different from position variables, and, in fact, any of the entire set of parameters may be considered part of the unknown state to be estimated.[1] The remaining parameters must be known a priori, and their accuracy will affect that of the state estimate.

Although many of the details of the present study are specific to the above problem, our methodology and results are quite general. Consequently, we begin with a brief description of the abstract estimation problem, illustrating it by specializing to geolocation. Let $\mathbf{x}$ be a vector whose components x_s, $s = 1,\ldots, N$ represent a set of parameters to be estimated. We suppose the existence of additional parameters $\mathbf{q}$ which together with the state vector $\mathbf{x}$ determine (in the absence of noise) the measurements. This relationship is described by a set of functions, $m_i(\mathbf{x}, \mathbf{q})$, $i = 1, \ldots, M$, where M is the number of measurements. The actual measurements $\tilde{m}_i$ will be noisy, resulting in errors $\Delta m_i = m_i(\mathbf{x}, \mathbf{q}) - \tilde{m}_i$. We choose as an estimate a state $\mathbf{x}_e$ which minimizes these errors. More precisely, $\mathbf{x}_e$ is the nonlinear least squares estimate determined by minimizing the cost function

$$C(x) = \sum_i \frac{(m_i(\mathbf{x}, \mathbf{q}) - \tilde{m}_i)^2}{\sigma_i^2}. \tag{1.1}$$

$$C(\mathbf{x}_e) = \min_{\mathbf{x}} C(\mathbf{x}). \tag{1.2}$$

The weights σ_i^2 are the variances of the measurement noise.[2] In the above geolocation problem, the state $\mathbf{x}$ would be the transmitter latitude and longitude, and m_i would be the doppler-shifted frequency at the receiver at time t_i, which we write as $f(\mathbf{x}, \mathbf{q}, t_i)$. The weights σ_i^2 correspond to the variances of the measured receiver frequencies $\tilde{f}_i$. It is sometimes also desirable to include other measurement types, such as satellite azimuth and elevation, which may be used to simultaneously determine the orbit and geolocate.

Let us consider measures of the quality of the estimate obtained by (1.2). The components of $\mathbf{x}_e$ are random variables since they are functions of the random variables $\tilde{m}_i$. Thus, an appropriate error measure for a component $(x_e)_s$ of $\mathbf{x}_e$ is its mean squared error,

[1] The measurements may not always support a solution, however.
[2] This particular formulation is appropriate for measurements which are stochastically independent, a restriction which is relaxed in Section 2. If the noise is also Gaussian, then (1.1)-(1.2) is a maximum likelihood estimator.

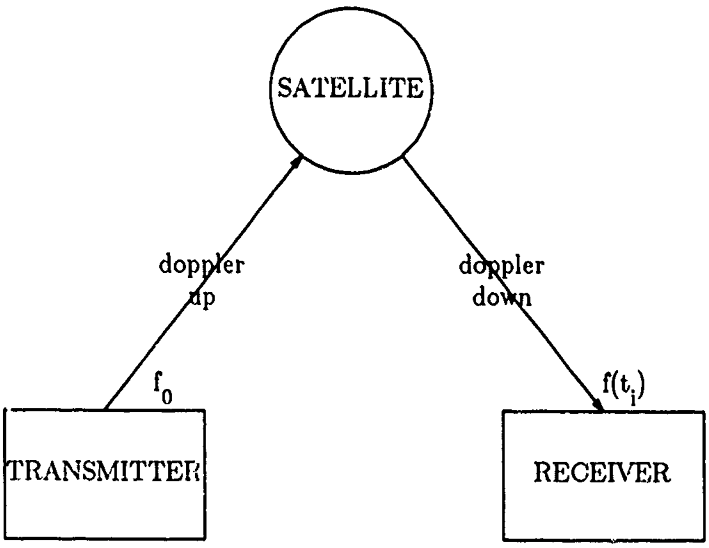

Figure 1.1. Diagram of the geolocation problem.

i.e, E $(\hat{x}_s-(x_e)_s\,)^2$, where E denotes expectation and $\hat{\mathbf{x}}$ is the true state. More generally, we may utilize the covariance matrix

$$\mathbf{X} \triangleq E(\,(\hat{\mathbf{x}} - \mathbf{x}_e)\,(\hat{\mathbf{x}} - \mathbf{x}_e)^\dagger\,) \tag{1.3}$$

as a measure of error. (The dagger † denotes the transpose of the column vector **x**.)

While (1.3) reflects the errors due to measurement noise, it is also important to determine the sensitivity of $\mathbf{x}_e$ to errors in the parameters **q**. This relationship will generally be deterministic (but, see (2.28) - (2.36)). A given error δq_k in the k^{th} parameter will produce an error $\delta\mathbf{x}$ in the state estimate. To the first order, this dependence is linear so that the total error may be expressed in the form

$$\delta\mathbf{x} = \mathbf{G}\,\delta\mathbf{q}\ , \tag{1.4}$$

where **G** is a matrix determined by the true values, i.e., $\mathbf{G} = \mathbf{G}(\hat{\mathbf{x}}, \hat{\mathbf{q}})$. Typical components of **q** for the geolocation problem might include satellite orbit inclination, transmitter frequency, etc.

The parallel between the problem statement and the sensitivity/error analysis is illustrated in Figure 1.2. A given problem is realized by specifying which variables are to be estimated, and which are inputs. For the sensitivity we also need estimates of the accuracies of the inputs. Note the parenthetical quantities in the figure. For the estimation problem, the measurement variances may be used to weight the influence of the inputs on the estimate (cf. equation (1.1)). For the sensitivity, the values of the parameters and measurements are used to determine the relationship of the estimate to inputs[3] and, thereby, the transfer function between the input accuracies and the estimate accuracies.

In the remainder of the report, a weak attempt has been made to place the sections in order of increasing mathematical detail. The current section, although specific to the geolocation problem, contains a minimum of mathematical analysis. The subsection following this introduction specifies the measurement functions, i.e., the dependency of the doppler on the various parameters **x** and **q**. The third subsection contains a qualitative discussion of sensitivity in the same context.

Section 2 is a self-contained development of the sensitivity and error analysis for the general estimation problem. Part of that section also describes an effective numerical technique for the minimization (1.2). Solving these problems requires the computation of the measurement functions and their derivatives. Such computations are detailed in Section 3 for a stationary transmitter and receiver on an oblate earth. This model is extended in Section 4 to include ground station motion. Section 4 (accompanied by Appendix A) may be read independently of the rest of this report. Computations

[3] The values are needed (as well as their accuracies) because the sensitivity is generally a nonlinear function of the state and/or the parameters If the true state is known, as in a simulation, then the measurement values are not needed.

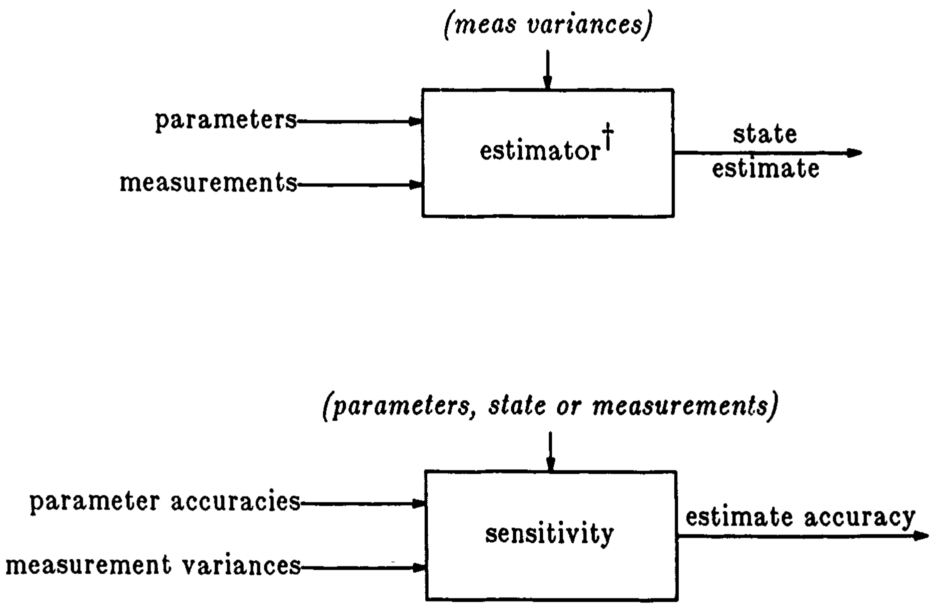

Figure 1.2. Generic diagram of a state estimator and sensitivity analysis. Note that the sensitivity for an unknown state is approximated by using the state estimate obtained from the measurements.

† A nonlinear least squares estimator in the current context.

requiring a specific satellite orbit model are relegated to Appendix B, where they are detailed for Keplerian motion. The actual software can accept an arbitrary orbit model, in which case it computes the derivatives numerically.

DOPPLER MEASUREMENTS

Let us return to Figure 1.1. The arrows marked "doppler up" and "doppler down" are intended to indicate the doppler shift in signal frequency due to the relative motion of the satellite and the transmitter or receiver, respectively. For example, if we denote the transmitter frequency by f_0 and the relative doppler shift on the uplink by b_U, the frequency of the signal received by the satellite will be (ignoring relativistic effects)

$$f = f_0 \, (1 + b_U) \ . \tag{1.5}$$

The relevant variables for the calculation of b are found in Figure 1.3. The vector $\mathbf{R}$ represents the position of the ground station relative to the center of the earth, $\mathbf{r}$ is that of the satellite, and

$$\mathbf{p} \triangleq \mathbf{r} - \mathbf{R} \tag{1.6}$$

is the vector between them. The relative doppler between a source and a receiver is proportional to the time derivative of the distance between them. More precisely, we have

$$b = -\frac{1}{c} \frac{d}{dt} \| \mathbf{p} \| \ , \tag{1.7}$$

where c is the speed of light. Denoting the time derivative and scalar product by dots, we have, from (1.7),

$$b = -\frac{1}{c} \frac{d}{dt} (\mathbf{p} \cdot \mathbf{p})^{1/2} = -\frac{1}{c} \frac{\mathbf{p}}{\| \mathbf{p} \|} \cdot \dot{\mathbf{p}} \ . \tag{1.8}$$

Note that equation (1.8) is coordinate-system-independent.[4] From (1.6)

$$\dot{\mathbf{p}} = \dot{\mathbf{r}} - \dot{\mathbf{R}} \ . \tag{1.9}$$

Equations (1.8) and (1.9) give b as a function of station motion $(\mathbf{R}, \dot{\mathbf{R}})$ and satellite

[4] In contrast, the following equation, (1.9), depends on the coordinate system in the sense that $\dot{\mathbf{R}}$ and $\dot{\mathbf{r}}$ do (i.e., a moving coordinate system will affect the velocities). Of course their difference, $\dot{\mathbf{p}}$, does not.

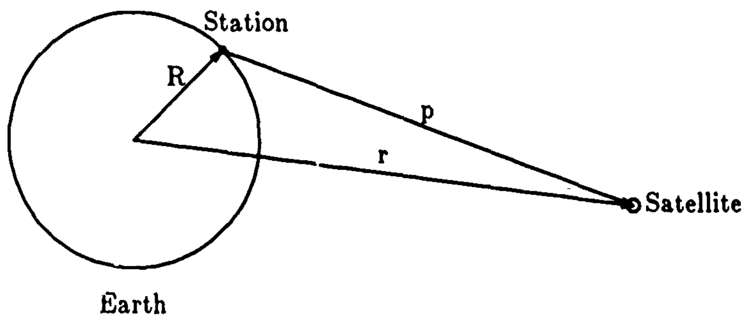

Figure 1.3. Diagram of positional vectors relating a satellite to a ground station.

motion ($\mathbf{r}$, $\dot{\mathbf{r}}$). Typical computations take place in the geocentric coordinate system (so-called IJK inertial coordinates [1]). Consequently, even if a station is stationary with respect to the earth, $\dot{\mathbf{R}}$ is not zero because of the earth's rotation. Then, given the station's latitude, longitude, and altitude at some fixed point in time, as well as additional parameters such as course and speed if there is ground motion, we can compute $\mathbf{R}(t)$ and $\dot{\mathbf{R}}(t)$ at an arbitrary time t. As mentioned in the previous subsection, any of these parameters may be considered either as part of the unknown state $\mathbf{x}$ or as given values $\mathbf{q}$ in the measurement functions $m_i(\mathbf{x}, \mathbf{q})$ of (1.1).

As with $\mathbf{R}$, the satellite position and velocity, $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$, may be computed utilizing a finite set of parameters modeling the satellite's motion. For example, the Kepler model, detailed in Appendix B, contains six parameters: the orbit semimajor axis, its eccentricity, the mean anomaly, inclination, longitude of the ascending node, and the argument of perigee. The epoch (i.e., point in time) at which these hold must also be known. For example, in the geolocation problem, one might take the unknown state $\mathbf{x}$ to be the 2-dimensional vector whose components are transmitter latitude and longitude, and $\mathbf{q}$ to comprise 12 parameters, the transmitter altitude, receiver latitude, longitude, and altitude, the transmitter frequency f_0, the 6 Kepler elements, and a satellite relay (offset) frequency which we denote f_{SO} (in all 12 dimensions).

The above discussion has outlined the functional dependencies $b_{Up}(\mathbf{x}, \mathbf{q})$ and $b_{Down}(\mathbf{x}, \mathbf{q})$ for the uplink and downlink relative doppler shifts. It is now a short step to connect these to f_R, the frequency determined at the receiver. The signal leaves the transmitter with a frequency f_0 and arrives at the satellite with a frequency $f_0\,(1 + b_U)$. This signal is then relayed by the satellite with a possible offset of f_{SO}, resulting in an emitted signal of frequency $f_0\,(1 + b_U) + f_{SO}$. This propagates and, due to the downlink doppler, arrives at the receiver with a frequency $[f_0\,(1 + b_U) + f_{SO}][1 + b_D]$. Finally, the receiver introduces another offset, which we write $-f_{RO}$ (usually set close in value to $-(f_0 + f_{SO})$, in order that the receiver output frequency be as close as possible to the total doppler shift). In summary, the output frequency of the receiver is

$$f_R = [f_0(1 + b_U) + f_{SO}]\,[1 + b_D] - f_{RO} \ . \tag{1.10}$$

We have thus arrived at a point at which we can construct the measurement function(s) $f_R(\mathbf{x}, \mathbf{q}, t_i)$ by substituting the dependencies $b_U(\mathbf{x}, \mathbf{q}, t_i)$ and $b_D(\mathbf{x}, \mathbf{q}, t_i)$ into (1.10). The residuals $m_i(\mathbf{x}, \mathbf{q}) - \widetilde{m}_i$ of (1.1) are given by $f_R(\mathbf{x}, \mathbf{q}, t_i) - \tilde{f}_i$, where the $\tilde{f}_i$ are the frequencies measured at the receiver output during an experiment.

There is substantial value in writing (1.10) in a slightly different form. First, we rearrange the terms

$$f_R = b_U f_0(1 + b_D) + b_D(f_0 + f_{SO}) + (f_{SO} + f_0 - f_{RO}) \ . \tag{1.11}$$

As mentioned above, f_{RO} is chosen to make the last term close to zero. Regardless, all the information lies in the size of the doppler shift, which is of the order of b f_0. An error in f_0 causes a substantial percentage error in this quantity. In fact, since b (b_U and b_D are of the same order) is much less than 1, the percentage error is approximately

$\sigma_{f_0}/bf_0 = \frac{1}{b}\frac{\sigma_{f_0}}{f_0}$. We may remedy this by estimating most of the uncertainty in the base frequency f_0 (as well as that of f_{SO} and f_{RO}) through an additional state variable,

$$f_B \triangleq f_{SO} + f_0 - f_{RO} \,. \tag{1.12}$$

Equation (1.11) becomes

$$f_R = b_U\, f_0(1 + b_D) + b_D\,(f_0 + f_{SO}) + f_B \,. \tag{1.13}$$

Since f_B is now a state variable, the percentage error in f_R of equation (1.13) is approximately $\frac{2\sigma_{f_0} b}{bf_0} \approx \frac{\sigma_{f_0}}{f_0}$, which is much less than $\frac{1}{b}\frac{\sigma_{f_0}}{f_0}$. Of course, the price we pay is the estimation of the additional state variable f_B.

Finally, we may write (1.13) more conveniently by dividing by f_0 to get

$$\begin{aligned} d &\triangleq \frac{f_R}{f_0} \\ &= b_U\,(1 + b_D) + b_D\,(1 + \frac{f_{SO}}{f_0}) + g_B \,, \end{aligned} \tag{1.14}$$

where the (possibly) unknown parameter g_B is

$$g_B \triangleq \frac{f_B}{f_0} = \frac{f_{SO}}{f_0} - \frac{f_{RO}}{f_0} + 1 \tag{1.15}$$

and the corresponding measurements are

$$\tilde{d}_i = \frac{\tilde{f}_i}{f_0} \,. \tag{1.16}$$

Even if f_0 is unknown, but a rough value is available, equations (1.14) and (1.16) will often be a very accurate approximation, with f_0 as the rough a priori value and g_B estimating the true value of f_0 via equation (1.15).

The cost function corresponding to the above measurements is

$$C(\mathbf{x}) = \sum_i \frac{(d(\mathbf{x}, \mathbf{q}, t_i) - \tilde{d}_i)^2}{\sigma_{d_i}^2} \,, \tag{1.17}$$

where, from (1.16), the variance of $\tilde{d}_i$ is $\sigma_{d_i}^2 = \sigma_{f_i}^2/f_0^2$. The minimum of the cost function (1.17) satisfies the set of N nonlinear equations $\nabla_x C(\mathbf{x}_e) = 0$ and may be solved by any number of steepest descent procedures. We recommend the technique described at the end of section 2 (equations (2.37) to (2.40)), which has proved highly effective for the present problem.

SENSITIVITY / ERRORS

In the geolocation problem, as in almost all estimation problems, it is critical to have an an idea of the accuracy of the solution obtained. An unusually poor geometry may produce an ill-conditioned set of equations whose solution is of essentially no practical value (e.g., the transmitter is in Detroit with an average error of ±30° latitude and ±80° longitude). Even in less serious cases, extremely erroneous conclusions may be drawn from using a location estimate without taking into account its accuracy. Furthermore, the geolocation accuracy is not a single fixed value, but varies with the satellite's orbit, transmitter and receiver locations, measurement errors, accuracy of the orbit model, etc. These explicit dependencies are almost always relevant.

Extensive software was developed to determine the sensitivity to input parameters and provide estimates of the geolocation accuracy. The methodology, which is fully described in Section 2, estimates the solution errors analytically without a need for extensive Monte Carlo runs. In fact, if the geolocation estimates are derived by a steepest descent solution of the nonlinear least squares problem, the sensitivities may be provided at negligible additional computation. Regardless, independent of the minimization technique, the sensitivity computation is relatively fast and suitable for operation in real time.

The inputs and outputs of the sensitivity are diagramed in Figures 1.4 through 1.6. The desired output is the expected error (actually the RMS error) in the transmitter position estimate. In Figure 1.4 this is depicted as a function of the orbit errors, the errors in frequency parameters, the transmitter location error, and the receiver measurement errors. In many cases it is also necessary to determine or redetermine the satellite orbit elements. Such a determination represents an inversion of the original problem: the satellite elements (parameters) become the unknown state, and the transmitter location is included as a known input parameter q_k. In that context (Figure 1.5), the sensitivity software outputs the accuracies of the estimates of the orbit elements which, in turn, may be used as inputs to the sensitivity analysis of the geolocation problem (Figure 1.4). Of course, this procedure presupposes the existence of a second transmitter whose location is known and may thus be used in the determination of the orbit elements. It is also theoretically possible to simultaneously determine the transmitter and orbit parameters, thereby avoiding the need for a second transmitter (Figure 1.6); however, this requires a relatively amicable geometry and/or good azimuth and elevation data.[5]

We now describe in somewhat more detail the operation of the sensitivity software. Table 1.1 contains a listing of the relevant variables. As briefly indicated in the introduction, we distinguish between what we term a parameter q, which has a fixed error for

[5] Satellite azimuth and elevation relative to a given ground station represent additional measurements ($m_j = m(\mathbf{x}, \mathbf{q}, t_j)$) which are useful when the orbit is part of the state $\mathbf{x}$.

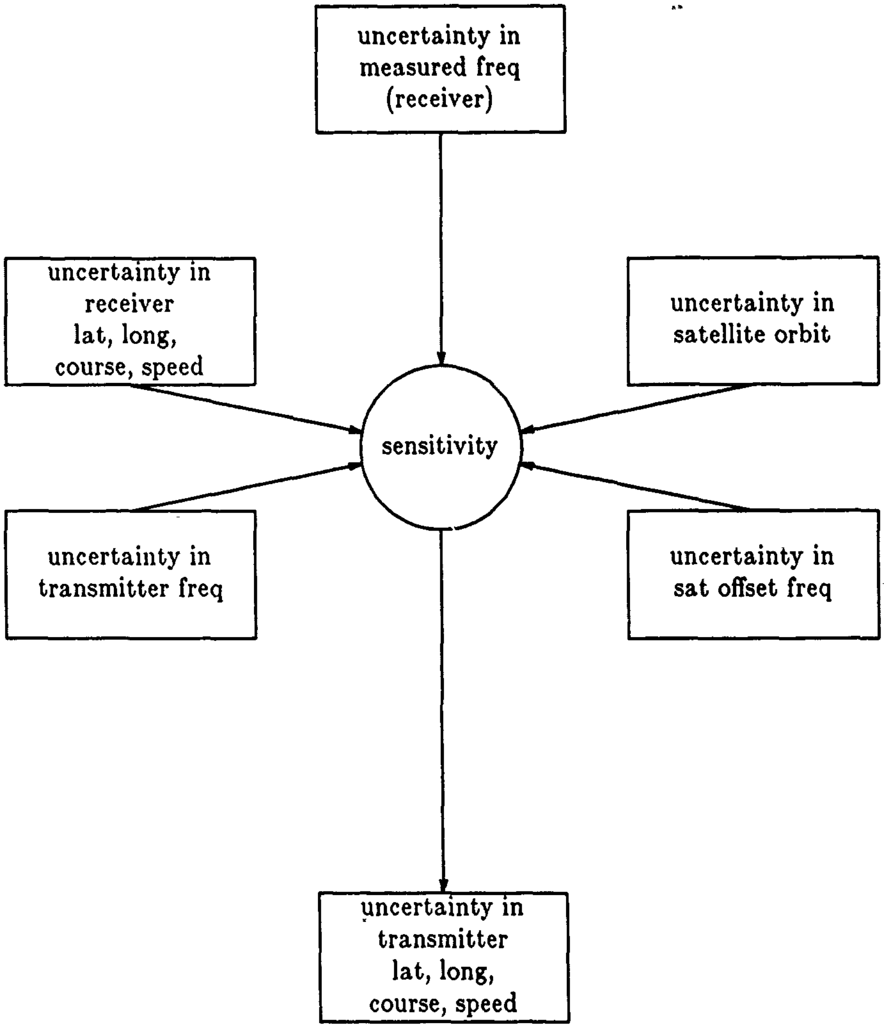

Figure 1.4. Inputs and ouputs of the sensitivity for the problem of determining receiver location and/or motion.

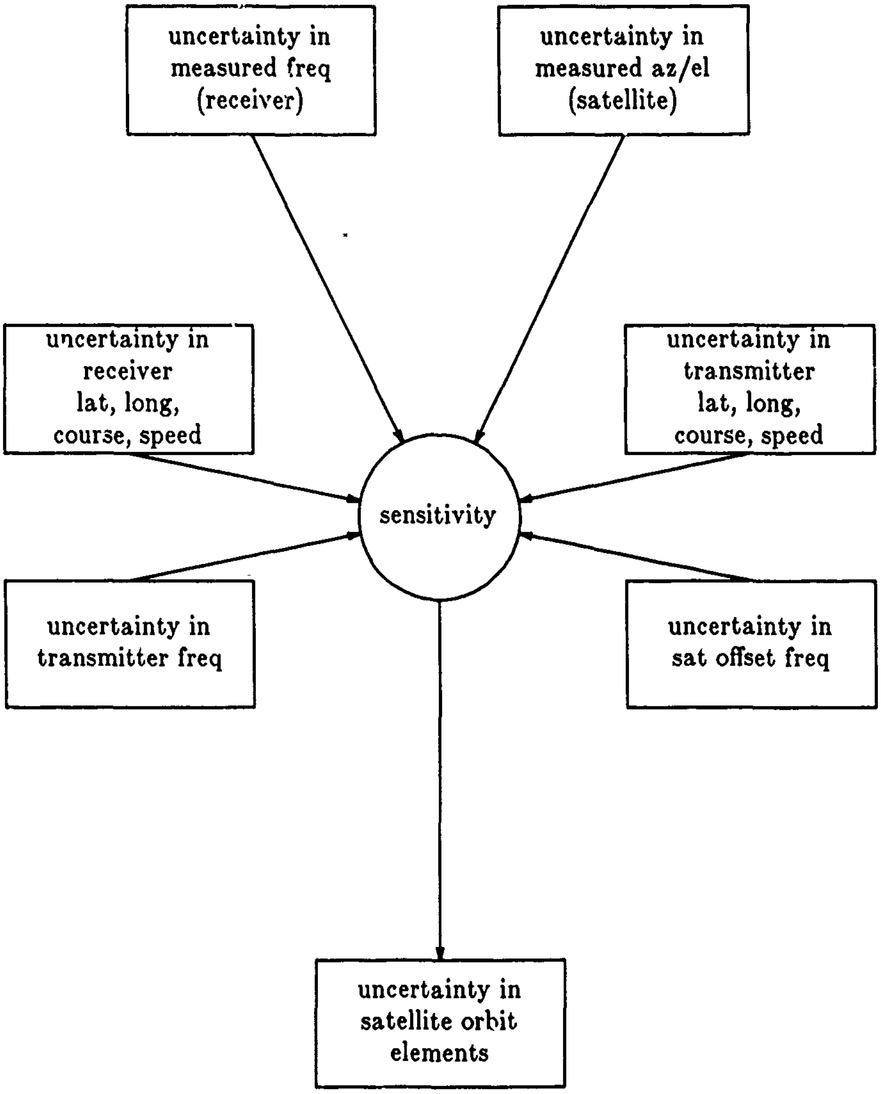

Figure 1.5. Inputs and ouputs of the sensitivity for the problem of determining satellite orbit elements.

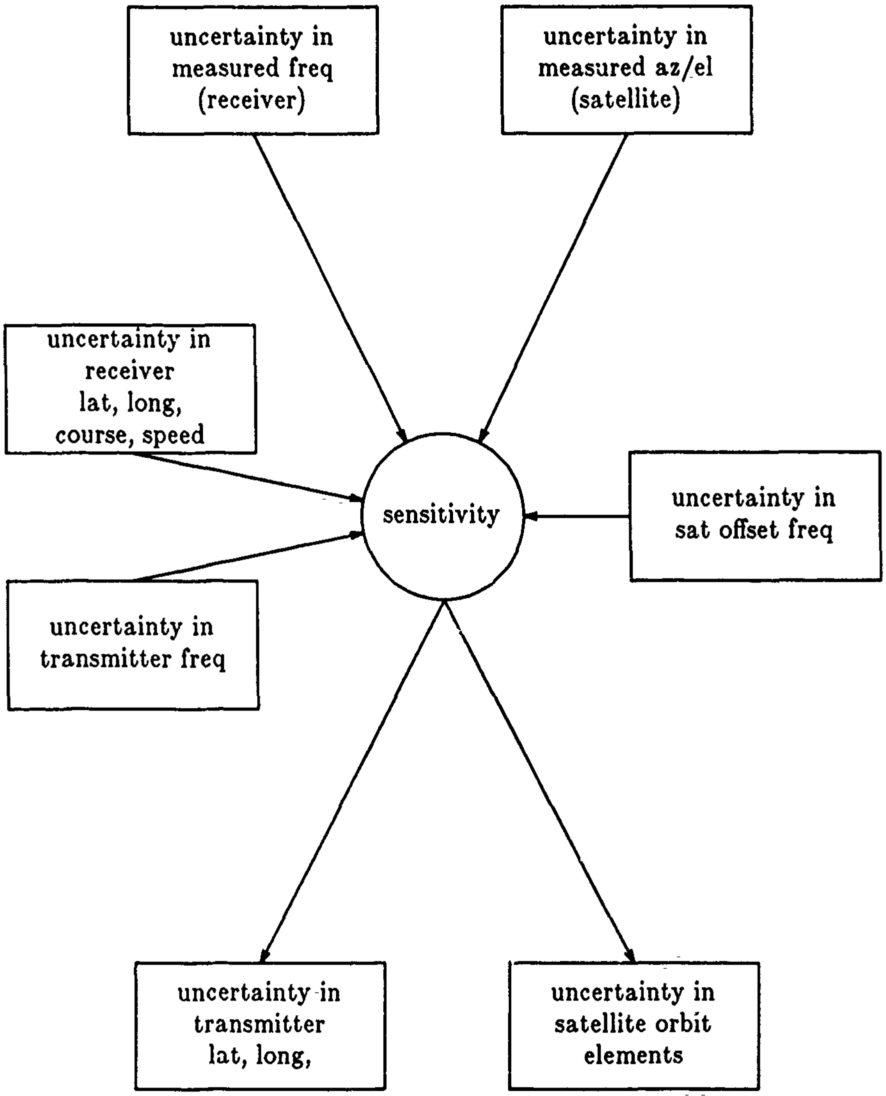

Figure 1.6. Inputs and ouputs of the sensitivity for the problem of determining receiver location and satellite orbit elements.

Table 1.1. **Sensitivity Parameters**

Variable	Type
transmitter	
latitude, longitude	parameters
altitude	parameter
course, speed	parameters
frequency	parameter
receiver	
latitude, longitude, altitude	parameters
course, speed	parameters
frequency(t_i)†	measurements
satellite	
elements††	parameters
offset frequency	parameter
azimuth(t_i), elevation(t_i)†	measurements
position(t_i), velocity(t_i)†††	(stochastic) parameters

† Here, t_i is the time of the i^{th} frequency measurement f_i. Actually, t_i is also a measured quantity, so that there are two measurements, f_i and t_i, for each i.

†† mean motion, eccentricity, mean anomaly, inclination, ascending node, perigee, etc.

††† Observe that these represent six numbers at each time t_i since position and velocity are vectors.

any one geolocation problem, and a measurement variable, which is a random variable representing a set of noisy measurements $\tilde{m}_i$, whose errors share a common probability distribution. An example of the former is the inclination of the satellite orbit, while the set of receiver frequency measurements falls in the latter category. More precisely, measurements are those variables whose residuals are minimized in a cost function in order to estimate the state (cf., equation (1.1)). Sometimes the errors of a set of parameters will also have a common distribution; we term these "stochastic parameters."[6] The usefulness of this concept is discussed below.

Now that we have identified the variables and their roles, let us be a little more specific about what we mean by their "accuracies." The error in a parameter is considered to be fixed throughout an experiment. Thus, if the transmitter frequency is recorded as 10 Mhz when it is actually 9.999 Mhz, then the parameter error is 0.001 Mhz. Corresponding to such an error, the sensitivity analysis will supply an error for the output estimate; for example, a latitude error of 0.02°. If there are no measurement errors (i.e., just parameter errors), then the output error is deterministic and a linear function (to the first order) of the input error. In the present example, an error of -0.002 Mhz in transmitter frequency would produce a latitude error of -0.04°.

On the other hand, the measurement errors are stochastic. Thus, we might say that the set of received frequencies has a root mean square (RMS) error of $\sigma_f = 3$Hz, meaning that if we average the squares of the errors of the measurements and take the square root of that average, we get 3 Hz. Correspondingly, the output estimate will have a random component, and its error will be expressed in terms of an expected value; for example, the RMS error in latitude. Note that if there are no parameter errors, the average error of the latitude estimate over many repetitions of the experiment is zero[7] since it will randomly fluctuate in the positive and negative direction. Of course, the mean square error, which is the expectation of a positive quantity, is still nonzero. Also, just as in estimating a noisy variable by averaging over a set of samples, for a single measurement type the standard deviation of the state estimate will vary inversely with the square root of the number of measurements. When the state estimate contains more than one variable, the sensitivity outputs a covariance matrix, i.e., $E(\Delta x_s \Delta x_r)$, where Δx_s is the error in the estimate to the s^{th} variable. For example, if x_1 and x_2 represent the latitude and longitude, respectively, then we obtain $E(\Delta\text{lat}\ \Delta\text{long})$ as well as $E(\Delta\text{lat})^2$ and $E(\Delta\text{long})^2$. When both deterministic and stochastic errors are present, the total expected (RMS) error of a state variable is computed by summing the deterministic errors and then combining that result with the other errors by taking the square root of the sum of their squares.

Figure 1.7 contains an example of the output of the software package SENS for a fixed transmitter and a moving receiver. For completeness, we have also included a listing of the SENS input requests in Figure 1.8. In the case illustrated, the unknown state is the transmitter latitude and longitude, indicated by the state type STATEISLOC. The orbit model is the NORAD deep space model; a Kepler model has also been

[6] Stochastic parameters differ from measurements only in their functional role in the cost function. They enter implicitly, as parameters, and thus, conceptually, are not as central to the estimation criterion.

[7] This is rigorously true only to the first order since our nonlinear estimates are only asymptotically unbiased.

```
doppler type is: RELDOPPLER
angle type is: NONE
element estimates type is: NONE
State type is:    STATEISLOC
Orbit model is: NORAD

Here is the covariance matrix:
  component 1 is lat, component2 is long
   7.420718742300576E+00   6.942098880222244E+01
   6.942098880222244E+01   7.753117053419069E+02

RMS lat error=   2.724099620480238E+00
RMS long error=   2.784441964455188E+01

Here are the orbit parameter sensitivities:
Units are degrees, km, and seconds of time
                                             LATITUDE (deg)           LONGITUDE (deg)
semi-maj axis: error lat/long =    2.653789595519299E-02   2.871645812066904E-01
eccentricity:  error lat/long =   -2.061625284638502E+04  -1.092716754410192E+05
epoch(2-body rslt):
               error lat/long =    5.423958191187525E-04   4.273828602526120E-03
mean anomaly:  error lat/long =   -1.298190677558637E-01  -1.022914309755793E+00
inclination:   error lat/long =    5.973604795742479E-01  -8.025665754087354E+00
ascending node:error lat/long =   -2.559585668129556E-01  -1.563732042288754E+00
arg perigee:   error lat/long =    6.899827141345592E-02   1.692882756185627E+00

Sensitivity to freq params per Hz:
                                             LATITUDE (deg)           LONGITUDE (deg)
sat offset:    error lat/long =   -5.720171350036819E+00  -6.170486123683860E+01
receiv offset: error lat/long =    5.720171702619600E+00   6.170486225655878E+01
transmit freq: error lat/long =   -5.720171587202191E+00  -6.170486150408782E+01

Sensitivity to receiver location:
gives error for an error of 1 degree in indicated parameter
rcv latitude (1 deg):
   tr  lat (deg)=   6.819453815607811E-02
   tr long (deg)=  -3.260144317799733E-01
rcv longitude (1 deg):
   tr  lat (deg)=   2.402443807119458E-01
   tr long (deg)=   2.420232834061336E+00

Sensitivity to receiver motion:
gives state error for an error of 1 km/hr in spd or of 1 degree in course
great circle speed (1 km/hr):
   tr lat (deg) =   1.199627807273553E+00
   tr long (deg)=   1.285460873073998E+01
course (1 deg) :
   tr lat (deg) =   2.334775376994807E-01
   tr long (deg)=   2.556731597113552E+00
```

Figure 1.7a. Example of the output of the sensitivity software (SENS) where the estimated state is the transmitter position.

Sensitivity to satelite position per RMS meter
(coords are in geocentric inertial system)

		LATITUDE (deg)	LONGITUDE (deg)
xcoord:	error lat/long =	1.809036298711036E-04	1.344364853049826E-03
ycoord:	error lat/long =	3.182725136278286E-04	3.493877932361680E-03
zcoord:	error lat/long =	1.611327330902054E-05	1.124117784462831E-04

Sensitivity to satelite velocity per RMS meter/sec
(coords are in geocentric INERTIAL system)

		LATITUDE (deg)	LONGITUDE (deg)
xcoord:	error lat/long =	4.350484472181783E+00	4.775662164106116E+01
ycoord:	error lat/long =	2.490454480418472E+00	1.850237658893264E+01
zcoord:	error lat/long =	3.487122438460089E-01	3.858319826910118E+00

Figure 1.7 continued.

```
SENSITIVITY INPUT : datafile

give element epoch (yr,day, fractday)
give major axis (km), eccentricity, meananom (deg)
give inclination, ascend node, argum of perigee  (deg)
give sn20 and sn60 (revs/powerofday)                          SATELLITE
give bstar (drag term)
give iexp, ibexp, exponents base 10 for sn60 and bstar
 give offset freqency (MHz) and mixing sign (+- 1)

give station epoch (yr,day, fractday)
give station base freq (MHz)
 (this is transm freq or receiver offset)                     TRANSMITTER
give geodetic latitude and longitude(deg)
give altitude above sea (km)
give motion type: 0 = fixed; 1 = greatcircle motion
give course (deg) and speed (km/hr)                           if motiontype = 1
give minimum and max elevation (deg)

 for the receiver input:
give station epoch (yr,day, fractday)
give station base freq (MHz)
 (this is transm freq or receiver offset)                     RECEIVER
give geodetic latitude and longitude(deg)
give altitude above sea (km)
give motion type: 0 = fixed; 1 = greatcircle motion
give course (deg) and speed (km/hr)                          if motiontype = 1
give minimum and max elevation (deg)
give standard dev of receiver output (Hz:  note unit)

for doppler measurements give (yr day fractday):
  start time
  upper bound (eg stop) time for meas                        MEAS GENERATION
give interval in hrs between mesurements and max no of pts
for azel measurements give (yr day fractday):
  start time
  upper bound (eg stop) time for meas
give interval in hrs between mesurements and max no of pts
```

Figure 1.8. Input requests for the sensitivity software (SENS).

implemented, and other models may be interfaced with a minimum of programming. The type RELDOPPLER indicates that the received frequency is included among the input measurements in the form of equation (1.16), and it is seen that its standard deviation is set to 1.0 Hz. Angle type NONE implies that there are no satellite azimuth or elevation measurements. The satellite orbit information may be in the form of parameters (an element set) or "stochastic parameters" (position and velocity at each time). In other words, the estimation problem corresponds to Figure 1.4.

Each of the entries indicates the sensitivity for a single error.[8] To evaluate performance when there is more than one, these errors must be combined in the appropriate manner as described at the end of the previous subsection. Note that when only a single type of input error is present, multiplying that error by a factor will multiply the state estimate error by the same factor; however, in summing different types of errors, the stochastic errors must be combined in an RMS sense.

The first group of results shows the covariance of the latitude, longitude for a 1.0-Hz standard deviation in the measured receiver frequencies. We see, for example, that the standard deviation of the estimate of the latitude is 2.7°. Note that the RMS errors of the state are simply the square roots of the diagonal elements of the covariance matrix. The next group gives the parameter sensitivities. It shows the errors of the latitude and longitude estimates for a given error in the orbit elements or in the frequency parameters. The input errors are considered to be a single unit of the indicated type. Thus, we see displayed respectively the lat/long error (not RMS!) for a 1.0-km error in the semimajor axis; for a 1.0 error in eccentricity (this quantity is dimensionless); for a 1.0-second error in the epoch at which the elements are supplied; for a 1.0° error in the mean anomaly, inclination, ascending node, or argument of perigee; and for a 1.0-Hz error in each of the frequency parameters. The next two groups give the sensitivities of the state to errors in receiver location and receiver motion respectively. For example, an error in receiver latitude of 1.0° results in an error in the estimated transmitter longitude of -0.3°.

The last group, shown in Figure 1.7b, gives the RMS errors in latitude and longitude as a function of errors in the satellite position or velocity. In other words, if instead of computing the satellite's orbit via an orbit model driven by six predetermined parameters (elements),[9] we supply a set of satellite location and velocity "measurements" (assumed to be corrupted by zero-mean identically distributed noise), then the transmitter location will have the indicated RMS errors. This is particularly useful in determining how accurate an orbit model one must have to obtain a given state estimate accuracy. (In contrast, the element sensitivities cannot take into account any errors or approximations which may reside in the model itself.) The output is arranged analogously to that above. Thus, for example, an RMS error of 1.0 meter/sec in the velocity of the satellite's y-coordinate (i.e., the values supplied for v_y are noisy and that noise is zero-mean with a standard deviation of 1.0 meter/sec) will result in an RMS error in latitude of 2.5° and longitude of 18.5°. Actually, the entire covariance matrix is

[8] An exception in the current implementation is the case in which both doppler and azimuth/elevation information are present. In that situation, the covariance matrix combines both (actually three) types of errors.

[9] In the NORAD model there are several parameters in addition to the six elements, but we do not currently compute their sensitivities.

computed but not displayed.

SUMMARY OF EQUATIONS

We make the following definitions:

$\mathbf{x} \triangleq$ N-dimensional vector whose s^{th} component is the s^{th} state variable.

$\hat{\mathbf{x}} \triangleq$ true state vector.

$\mathbf{x}_e \triangleq$ estimated state vector.

$\Delta\mathbf{x} \triangleq \mathbf{x}_e - \hat{\mathbf{x}}$ the error in the state estimate.

$q \triangleq$ an arbitrary parameter. (That is, any component of $\mathbf{q}$. The remaining components are suppressed for notational convenience.)

$\Delta q \triangleq$ error in q.

$m(\mathbf{x}, q) \triangleq$ M-dimensional vector function whose i^{th} component is the i^{th} measurement when there is no noise.

$\tilde{m} \triangleq$ vector whose components are the actual measured values.

We note that the cost function for M measurements takes the form

$$C(\mathbf{x}) = \sum_{i=1}^{M} \frac{(m_i(\mathbf{x}, q) - \tilde{m}_i)^2}{\sigma_i^2}, \tag{1.18}$$

and the estimate $\mathbf{x}_e$ is determined by

$$C(\mathbf{x}_e) = \min_{\mathbf{x}} C(\mathbf{x}). \tag{1.19}$$

We define the N by N information matrix $\mathbf{A}$ by

$$A_{rs} = \sum_{i=1}^{M} \frac{\partial m_i}{\partial x_r} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial x_s}, \tag{1.20}$$

the parameter vector $\mathbf{b}$ by

$$b_r = - \sum_{i=1}^{M} \frac{\partial m_i}{\partial x_r} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial q}, \tag{1.21}$$

and the parameter matrix $\mathbf{B}$ by

$$B_{rr'} = \sum_{i=1}^{M} \frac{1}{\sigma_i^4} \frac{\partial m_i}{\partial x_r} \left(\frac{\partial m_i}{\partial q} \right)^2 \frac{\partial m_i}{\partial x_{r'}} \ . \tag{1.22}$$

A numerical procedure for the solution of (1.18) consists of the iterations (over an index k)

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \lambda\, \mathbf{A}^{-1} \nabla_x C(\mathbf{x}^k)\,, \tag{1.22b}$$

where, when feasible, $\lambda = 1$ (cf., Section 2).

The covariance of the estimate error due to measurement errors (alone) of standard deviation σ_i is computed by

$$\mathbf{X} \approx \mathbf{A}^{-1}\ . \tag{1.23}$$

Also, for a single parameter error of Δq, we have the corresponding state estimate error

$$\Delta \mathbf{x} \approx \mathbf{A}^{-1}\mathbf{b}\ \Delta q, \tag{1.24}$$

and for errors in satellite position or velocity (stochastic parameters) with standard deviation σ_q, we have

$$\mathbf{X} \approx \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\sigma_q^2\ . \tag{1.25}$$

Observe that the matrix $\mathbf{A}$ is a sum of M similar terms (equation (1.20)). Thus, as the number of measurements increases, holding the geometry approximately constant, the entries of $\mathbf{X}$ in equation (1.23) will vary inversely as M. We may interpret this as a rule of thumb: excluding parameter errors, for a single measurement type, the standard deviations of the errors in the state estimate will vary inversely as the square root of the number of measurements $(1/\sqrt{M})$. Note, also, that the matrix in equation (1.25) reduces to products of the components of (1.24) if we substitute Δq for σ_q. The approximation signs "$\approx$" are used rather than equalities because these equations are only correct to the first order. They are derived in Section 2 by linearization; hence, the use of the term sensitivity.

2. SENSITIVITY AND ERROR ANALYSIS

DEFINITIONS

In the previous section, we discussed a nonlinear least squares estimator with a cost function of the following form:

$$C(\mathbf{x}) = \sum_{i=1}^{M} \frac{(m_i(\mathbf{x}) - \tilde{m}_i)^2}{\sigma_i^2} , \tag{2.1}$$

where $\mathbf{x}$ is the state vector, $m_i(\mathbf{x})$ is the i^{th} measurement function, $\tilde{m}_i$ is the i^{th} measurement, and σ_i is its variance. The state estimate $\mathbf{x}_e$ was determined by

$$C(\mathbf{x}_e) = \min_{\mathbf{x}} C(\mathbf{x}) \ . \tag{2.2}$$

Expression (2.1) is appropriate for uncorrelated measurements; if the noise also happens to be Gaussian, then $\mathbf{x}_e$ is the maximum likelihood estimate (MLE). However, since generalizing to correlated measurements does not complicate our derivations, we consider the cost function

$$C(\mathbf{x}) = (\Delta\mathbf{m})^{\dagger} R^{-1}(\Delta\mathbf{m}) \ , \tag{2.3}$$

where

$\dagger$ = matrix transpose.
$\mathbf{x}$ = N-dimensional vector of state variables.
$\mathbf{m} = (m_1(\mathbf{x}), m_2(\mathbf{x}), \cdots, m_M(\mathbf{x}))^{\dagger}$ column vector of measurements
$\tilde{\mathbf{m}}$ = vector of noisy measurements
= $\mathbf{m}(\hat{\mathbf{x}}) + \mathbf{noise}$ where $\hat{\mathbf{x}}$ is the true state
$\Delta\mathbf{m} = \mathbf{m}(\mathbf{x}) - \tilde{\mathbf{m}}$
$\mathbf{R}$ = correlation matrix of $\Delta\mathbf{m}$.

Note that in the uncorrelated case, the components of $\mathbf{m}$ are the m_i of equation (2.1), the correlation matrix $\mathbf{R}$ is diagonal with entries σ_i^2, and (2.3) reduces to (2.1).

The above definitions carry the implicit restriction on the cost function that $\mathbf{R}$ be the correlation matrix of $\Delta\mathbf{m}$. This insures two properties: (i) that the cost function be positive semidefinite and (ii) that the expected value $E(\Delta\mathbf{m}^{\dagger}\,\Delta\mathbf{m}) = \mathbf{R}$. The second condition will be used in our derivation of an expression for the sensitivity. It implies, for example, that the a priori variances σ_i^2 in expression (2.1) be chosen with reasonable accuracy. The resulting simplification is well worth this restriction. For most applications, this requirement is not a serious problem. The fundamental formula (2.16), which we shall derive, may be modified by scaling to correct approximately for a least squares weighting which is not exactly the inverse of the covariance. Note that if σ_i is a constant (i.e., if the measurement noise is independent and stationary), then the weighting is irrelevant to the solution of (2.2).

A necessary condition for (2.2) to be satisfied is that the gradient of (2.3) be zero, i.e., that $\mathbf{x}_e$ be a solution of the set of N nonlinear equations $\nabla C(\mathbf{x}_e)=0$,

$$\Delta \mathbf{m}^{\dagger} \mathbf{R}^{-1} \frac{\partial \mathbf{m}}{\partial x_r} = 0; \quad r = 1, ..., N. \tag{2.4}$$

(We have used the symmetry of $\mathbf{R}$ and $\frac{\partial \Delta \mathbf{m}}{\partial x_r} = \frac{\partial \mathbf{m}}{\partial x_r}$ in deriving (2.4) from (2.3).)

ERROR SENSITIVITY

We wish to determine the accuracy of the solution $\mathbf{x}_e$ to (2.4) in terms of the accuracy of the measurements $\tilde{m}_i$ (the i^{th} component of $\tilde{\mathbf{m}}$) and the parameters which enter into the function $\mathbf{m}(\cdot)$. Let us first consider just the effect of measurement errors. To do so we write

$$\begin{aligned} \tilde{m}_i &= \hat{m}_i + n_i \\ &= m_i(\hat{\mathbf{x}}) + n_i \ , \end{aligned} \tag{2.5}$$

where n_i is the the noise of the i^{th} measurement, and $\hat{m}_i$ is the value of the measurement when there is no noise. We also assume that the measurements are unbiased; i.e., $E(n_i) = 0$. Define a vector $\mathbf{F}$ whose components F_r are the left-hand side of (2.4),

$$F_r(\mathbf{x}_e, \tilde{\mathbf{m}}) \triangleq \Delta \mathbf{m}(\mathbf{x}_e, \tilde{\mathbf{m}})^{\dagger} \mathbf{R}^{-1} \frac{\partial \mathbf{m}}{\partial x_r}(\mathbf{x}_e); \quad r = 1, ..., N. \tag{2.6}$$

Then, linearizing about $(\hat{\mathbf{x}}, \hat{\mathbf{m}})$, we have

$$\begin{aligned} F_r(\mathbf{x}_e, \tilde{\mathbf{m}}) \approx\ & F_r(\hat{\mathbf{x}}, \hat{\mathbf{m}}) + \sum_{s=1}^{N} \frac{\partial F_r}{\partial x_s}(\hat{\mathbf{x}}, \hat{\mathbf{m}})\,(x_e - \hat{x})_s \\ & + \sum_{i=1}^{M} \frac{\partial F_r}{\partial \tilde{m}_i}(\hat{\mathbf{x}}, \hat{\mathbf{m}})\, n_i \ . \end{aligned} \tag{2.7}$$

Also, since $\Delta \mathbf{m}(\hat{\mathbf{x}}, \hat{\mathbf{m}}) = 0$,

$$F_r(\hat{\mathbf{x}}, \hat{\mathbf{m}}) = 0 \ , \tag{2.8a}$$

$$\frac{\partial F_r}{\partial x_s}(\hat{\mathbf{x}}, \hat{\mathbf{m}}) = \frac{\partial \Delta \mathbf{m}^{\dagger}}{\partial x_s} \mathbf{R}^{-1} \left. \frac{\partial \mathbf{m}}{\partial x_r} \right|_{x=\hat{x}} , \tag{2.8b}$$

$$\frac{\partial F_r}{\partial \tilde{m}_i} = - \sum_j R_{ij}^{-1} \frac{\partial m_j}{\partial x_r} \ . \tag{2.8c}$$

Thus, to the first order, (2.4) becomes

$$\sum_{s=1}^{N} \frac{\partial \mathbf{m}^{\dagger}}{\partial x_s} \mathbf{R}^{-1} \frac{\partial \mathbf{m}}{\partial x_r} (x_e - \hat{x})_s = \mathbf{n}^{\dagger} \mathbf{R}^{-1} \frac{\partial \mathbf{m}}{\partial x_r}, \quad r = 1, \ldots, N \ , \tag{2.9}$$

where the partial derivatives are understood to be evaluated at $\hat{\mathbf{x}}$.

For convenience, define the nxn symmetric matrix $\mathbf{A}$ by

$$A_{sr} \triangleq \frac{\partial \mathbf{m}^{\dagger}}{\partial x_s} \mathbf{R}^{-1} \frac{\partial \mathbf{m}}{\partial x_r} \ . \tag{2.10}$$

Letting r assume two values, r and r', and multiplying equation (2.9) by itself, we obtain

$$(\mathbf{A}\,(\mathbf{x}_e - \hat{\mathbf{x}}))_r\,(\mathbf{A}\,(\mathbf{x}_e - \hat{\mathbf{x}}))_{r'} = \sum_{iji'j'} \frac{\partial m_i}{\partial x_r} R_{ij}^{-1}\, n_j n_{i'}\, R_{i'j'}^{-1} \frac{\partial m_{j'}}{\partial x_{r'}} \ . \tag{2.11}$$

We then take the expectation of (2.11), noting that $E(n_j n_{i'}) = R_{ji'}$. The right-hand side of (2.11) becomes

$$E(\mathrm{RHS}) = \sum_{ij} \frac{\partial m_i}{\partial x_r} R_{ij}^{-1} \frac{\partial m_j}{\partial x_{r'}}$$

$$= A_{rr'} \ , \tag{2.12}$$

where E denotes expectation.

The expression $(x_e - \hat{x})_r$ is simply the r^{th} component of the error in the estimate $\mathbf{x}_e$. Thus, $\mathbf{X}$, the covariance matrix of the error is

$$X_{rs} \triangleq E\,(x_e - \hat{x})_r (x_e - \hat{x})_s \ , \tag{2.13}$$

and substituting (2.12) into the expectation of (2.11), we have

$$\sum_{ss'} A_{rs} A_{r's'} X_{ss'} = A_{rr'} \ , \tag{2.14}$$

or

$$\mathbf{AXA} = \mathbf{A} \tag{2.15}$$

since A is symmetric. Solving for $\mathbf{X}$, we find

$$\mathbf{X} = \mathbf{A}^{-1} \ . \tag{2.16}$$

Equation (2.16) along with definition (2.10) gives a simple formula for the covariance of the estimate $\mathbf{x}_e$. (Actually, since it was derived for the expected squared errors it is a covariance only to the extent that the estimate is unbiased. It is easy to see, by taking the expectation of (2.9), that $\mathbf{x}_e$ is unbiased up to the first order.) If the noise is Gaussian, then the expression (2.16) represents the Cramer-Rao bound [3]. We also note that the diagonal components of $\mathbf{A}^{-1}$ are the variances of the components of the state estimates. In practical cases, where we may not have exact knowledge of the noise covariance, it is often a reasonable approximation to assume that we know it up to some unknown positive factor α^{-2}, i.e., $E(n_i n_j) = R'_{ij}/\alpha^2$, so that $\mathbf{R} = \mathbf{R}'/\alpha^2$. Such a factor does not affect the solution $\mathbf{x}_e$, and (2.16) becomes

$$\mathbf{X} = \alpha^2 \mathbf{A}^{-1} \ . \tag{2.17}$$

The expected cost is then $E(\ C(\hat{\mathbf{x}})\ ) = \mathrm{trace}(\mathbf{R}'\,\mathbf{R}^{-1}) = M\alpha^2$; thus, we may approximate α^2 by examining the average of the residuals. More precisely, the sample expectation for C is given by

$$\widetilde{C} = \frac{M}{M-N}\,(\mathbf{m}(\mathbf{x}_e)-\widetilde{\mathbf{m}})^{\dagger}\,\mathbf{R}^{-1}\,(\mathbf{m}(\mathbf{x}_e)-\widetilde{\mathbf{m}}) \ , \tag{2.18a}$$

where M - N is the number of degrees of freedom in the Gaussian case. (For Gaussian noise, $\widetilde{C}(\hat{\mathbf{x}})$ is a χ^2 distribution.) We then estimate α by

$$\alpha^2 \approx \widetilde{C}/M. \tag{2.18b}$$

Finally, we note that if the measurements are independent (i.e., $R_{ij} = \delta_{ij} \ / \ \sigma_i^{\,2}$), then (2.10) becomes

$$A_{rs} = \sum_{i=1}^{M} \frac{\partial m_i}{\partial x_r}\,\frac{\partial m_i}{\partial x_s}\,\frac{1}{\sigma_i^{\,2}} \tag{2.19}$$

or

$$\mathbf{A} = \sum_{i=1}^{M} \frac{\nabla m_i}{\sigma_i} \otimes \frac{\nabla m_i}{\sigma_i} \ , \tag{2.20}$$

where $\otimes$ stands for the matrix tensor product.

Next we compute the sensitivity to deterministic parameter errors. The procedure is almost identical to that which led to equation (2.16). Let q stand for some parameter, so that the measurement *functions* $m_i(\mathbf{x}, q)$ are dependent on the state $\mathbf{x}$ and a parameter q. Previously the dependence on q was left implicit. Similarly, we now consider only one parameter q at a time and set all other errors equal to zero. (Since the sensitivity is a linearization, the first-order error in **x** due to all the errors will be the sum of the individual contributions.) In the absence of measurement noise we write (cf., equation (2.5))

$$\widetilde{m}_i = m_i(\hat{\mathbf{x}}, \hat{q}) \ , \tag{2.21}$$

where $\hat{q}$ is the true value of q. As in (2.6) we define

$$F_r(\mathbf{x}_e, q) \triangleq \Delta\mathbf{m}(\mathbf{x}_e, q)^\dagger \, \mathbf{R}^{-1} \, \frac{\partial \mathbf{m}}{\partial x_r}(\mathbf{x}_e, q). \tag{2.22}$$

Then, linearizing about $(\hat{\mathbf{x}}, \hat{q})$, we have

$$\begin{aligned} F_r(\mathbf{x}_e, q) \approx \ & F_r(\hat{\mathbf{x}}, \hat{q}) + \sum_{s=1}^{N} \frac{\partial F_r}{\partial x_s}(\hat{\mathbf{x}}, \hat{q}) \, (x_e - \hat{x})_s \\ & + \frac{\partial F_r}{\partial q}(\hat{\mathbf{x}}, \hat{q}) \, (q - \hat{q}) \ . \end{aligned} \tag{2.23}$$

Also, since $\Delta\mathbf{m}(\hat{\mathbf{x}}, \hat{q}) = 0$,

$$F_r(\hat{\mathbf{x}}, \hat{q}) = 0 \tag{2.24a}$$

$$\frac{\partial F_r}{\partial x_s}(\hat{\mathbf{x}}, \hat{q}) = \frac{\partial \Delta\mathbf{m}^\dagger}{\partial x_s} \, \mathbf{R}^{-1} \, \frac{\partial \mathbf{m}}{\partial x_r} \bigg|_{\substack{q=\hat{q} \\ x=\hat{x}}} \tag{2.24b}$$

$$\frac{\partial F_r}{\partial q}(\hat{\mathbf{x}}, \hat{q}) = \frac{\partial \Delta\mathbf{m}^\dagger}{\partial q} \, \mathbf{R}^{-1} \, \frac{\partial \mathbf{m}}{\partial x_r} \bigg|_{\substack{q=\hat{q} \\ x=\hat{x}}} \ . \tag{2.24c}$$

Thus, to the first order, (2.4) becomes

$$\sum_{s=1}^{N} \frac{\partial \mathbf{m}^\dagger}{\partial x_s} \, \mathbf{R}^{-1} \, \frac{\partial \mathbf{m}}{\partial x_r} (x_e - \hat{x})_s = - \frac{\partial \mathbf{m}^\dagger}{\partial q} \, \mathbf{R}^{-1} \frac{\partial \mathbf{m}}{\partial x_r} \, (q - \hat{q}), \quad r = 1, \ldots, N \tag{2.25}$$

or

$$\mathbf{A} \, (\mathbf{x}_e - \mathbf{x}) = \mathbf{b} \, \Delta q \ , \tag{2.26a}$$

where

$$b_r \triangleq - \frac{\partial \mathbf{m}^\dagger}{\partial q} \, \mathbf{R}^{-1} \, \frac{\partial \mathbf{m}}{\partial x_r} \tag{2.26b}$$

$$\Delta q \triangleq q - \hat{q} \ . \tag{2.26c}$$

Note that if $\mathbf{R}$ is diagonal, then (2.26b) reduces to (cf., equation (2.19))

$$b_r = -\sum_{i=1}^{M} \frac{\partial m_i}{\partial q} \frac{\partial m_i}{\partial x_r} \frac{1}{\sigma_i^2} \quad . \tag{2.26d}$$

Solving (2.26a) for $\Delta \mathbf{x} \triangleq \mathbf{x}_e - \mathbf{x}$, the error in $\mathbf{x}$, we get

$$\Delta \mathbf{x} = \mathbf{A}^{-1} \mathbf{b} \, \Delta q \quad . \tag{2.27}$$

Equation (2.27) is an expression for the error (to the first order) in the estimate $\mathbf{x}_e$ caused by a single parameter error Δq, whereas (2.16) gives the *mean* squared error $E(\Delta x_r)^2 = A_{ii}^{-1}$ due to an entire set of noisy measurements. The derivation of (2.27) does not require that the least square weights $\mathbf{R}^{-1}$ be the actual covariance of the measurements $\tilde{m}_i$. In contrast, expression (2.16) was derived under the assumption that $E(n_i n_j) = R_{ij}$.

Finally, let us consider the case in which some of the parameters are themselves random variables, i.e., where the measurement function is of the form $m_i(\mathbf{x}, q_a)$, where

$$q_a = \hat{q}_a + n_a \; ; \qquad E(n_a) = 0 \; ; \qquad a = 1, 2, \cdots \tag{2.28}$$

are a set of noisy parameter estimates. The prototype for this in the current study is a set of satellite positions and velocities. Note that if these are supplied for each measurement time $\tilde{m}_i$, then the index a will run from 1 to 6M, and $m_i(\mathbf{x}, q_a)$ will only be a function of q_a for the six parameters relevant to that time, e.g., $a = 6(i-1) + 1$ to $a = 6i$. This property implies

$$\frac{\partial m_i}{\partial q_a} \frac{\partial m_{i'}}{\partial q_a} = 0 \qquad \text{if } i \neq i' \, . \tag{2.29}$$

Observe that the only difference between the "stochastic parameters" of equation (2.28) and the noisy measurements of equation (2.5) is the functional manner in which they enter the cost function.

Continuing with our development, we note that equation (2.26) remains valid except that q is replaced with a sum over q_a .

$$(\mathbf{A}(\mathbf{x}_e - \mathbf{x}))_r = -\sum_{aij} \frac{\partial m_i}{\partial q_a} R_{ij}^{-1} \frac{\partial m_j}{\partial x_r} \Delta q_a \; ; \qquad r = 1, \ldots, N \tag{2.30}$$

As in the derivation of (2.14), we take two values of r, multiply (2.30) by itself, and take expectations. If we assume that the noisy parameters are independent and identically distributed,

$$E(\Delta q_a \Delta q_{a'}) = \sigma_q^2 \, \delta_{aa'} \; , \tag{2.31}$$

we get

$$\sum_{ss'} A_{rs} A_{r's'} X_{ss'} = \sigma_q^2 \sum_a \sum_{iji'j'} \frac{\partial m_i}{\partial q_a} R_{ij}^{-1} \frac{\partial m_j}{\partial x_r} \; \frac{\partial m_{i'}}{\partial q_a} R_{i'j'}^{-1} \frac{\partial m_{j'}}{\partial x_{r'}} \; . \tag{2.32}$$

This may be written as

$$\mathbf{AXA}^\dagger = \mathbf{B} \, \sigma_q^2 \; , \tag{2.33}$$

with **B** defined by (2.32). If **R** is diagonal, **B** simplifies to

$$B_{rr'} = \sum_a \sum_{ii'} \frac{\partial m_i}{\partial x_r} \frac{1}{\sigma_i^2} \frac{\partial m_i}{\partial q_a} \; \frac{\partial m_{i'}}{\partial q_a} \frac{1}{\sigma_{i'}^2} \frac{\partial m_{i'}}{\partial x_{r'}} \; . \tag{2.34}$$

Furthermore, if equation (2.29) holds, then

$$B_{rr'} = \sum_a \sum_i \frac{1}{\sigma_i^4} \frac{\partial m_i}{\partial x_r} \left(\frac{\partial m_i}{\partial q_a} \right)^2 \frac{\partial m_i}{\partial x_{r'}} \; . \tag{2.35}$$

In any case, the solution to (2.33) is given by

$$\mathbf{X} = \mathbf{A}^{-1} \mathbf{B} \mathbf{A}^{-1} \sigma_q^2 \tag{2.36}$$

since all the matrices involved are symmetric.

Inasmuch as the above computations are first-order approximations (and $E(n_i) = 0$), it is easy to see that if we assume the various stochastic types of errors are independent, then

(1) Parameter errors add when more than one is present.
(2) The total stochastic error is the RMS error of the individual errors.
(3) The total error is the RMS sum of the errors in (1) and (2).

SOLVING FOR THE STATE: COMPUTATIONAL CONSIDERATIONS

The major computational steps in the above sensitivity analysis involve finding (a) the gradients of the measurement function, $\partial m_i/\partial x_r$ and $\partial m_i/\partial q$, (b) the matrix **A** which equals their outer product, and (c) the inverse of **A**. Typically, (b) and (c) are inexpensive in comparison with (a). In light of this, and noting that the most common methods for solving (2.2) are steepest descent methods utilizing the gradient of the cost function ($\nabla C = \Delta \mathbf{m}^\dagger \mathbf{R}^{-1} \nabla \mathbf{m}$), we see that the same computations are involved in determining the estimate $\mathbf{x}_e$ and the sensitivity. In other words, the sensitivity can be computed at essentially no additional cost.

Finally, we note that a particularly effective steepest descent algorithm is the modified Newton-Raphson algorithm

$$\mathbf{x}(i+1) = \mathbf{x}(i) - \mathbf{A}^{-1}(i)\ \nabla C(i) \tag{2.37}$$

or the more general

$$\mathbf{x}(i+1) = \mathbf{x}(i) - \lambda(i)\ \mathbf{A}^{-1}(i)\ \nabla C(i)\ . \tag{2.38}$$

Since $\mathbf{A}$ is positive definite unless the state $\mathbf{x}$ is unobservable (i.e., unless the system of equations (2.4) is singular), the direction $\mathbf{A}^{-1}\ \nabla C$ is one of decreasing cost, and (2.38) will converge for suitable choices of the step size $\lambda(i)$. Recursion (2.37) is simply an N-dimensional Newton-Raphson iteration for solving the nonlinear system of equations (2.4) but with the Jacobian of that system,

$$J_{rs} = \frac{\partial \Delta \mathbf{m}^\dagger}{\partial x_r}\ \mathbf{R}^{-1}\ \frac{\partial \Delta \mathbf{m}}{\partial x_s} + \Delta \mathbf{m}^\dagger\ \mathbf{R}^{-1}\ \frac{\partial^2 \mathbf{m}}{\partial x_r \partial x_s}\ , \tag{2.39}$$

replaced by

$$\frac{\partial \Delta \mathbf{m}^\dagger}{\partial x_r}\ \mathbf{R}^{-1}\ \frac{\partial \Delta \mathbf{m}}{\partial x_s}\ . \tag{2.40}$$

That is, we use the matrix $\mathbf{A}$ in place of the Jacobian. Such an approximation is intuitively reasonable since for $\mathbf{x} = \hat{\mathbf{x}}$ with zero noise, $\Delta\mathbf{m} = 0$ and the Jacobian actually equals $\mathbf{A}$. Furthermore, even if it happens that the cost does not decrease during an iteration of the form (2.37), it is still guaranteed to do so (unless one has arrived at the minimum $\mathbf{x}_e$) for some $\lambda < 1$ in (2.38).

COMPOSITION OF EXPERIMENTS

Suppose that two experiments are performed in succession. The first is used to determine a set of satellite orbit elements y_s, where $s = 1, ..., N_1$ (e.g., $N_1 = 6$). Let its covariance matrix be represented by $\mathbf{Y}$. The second experiment employs these elements as parameters in performing a localization, that is, to determine a position x_r where $r = 1, ..., N_2$ (e.g., $N_2 = 2$). We wish to determine the covariance of $\mathbf{x}$ due to the orbit errors of the first experiment. From (2.27), the errors in $\mathbf{x}$ are (deterministically) related to each of those in $\mathbf{y}$ by

$$\Delta\mathbf{x}(\text{due to } y_s) = \mathbf{A}^{-1}\mathbf{b}^s\ \Delta y_s\ , \tag{2.41}$$

where $\mathbf{b}^s$ is defined to be $\mathbf{b}$ of equation (2.26b) with q replaced by y_s. If we let

$$D_{rs} \triangleq (A^{-1}b^s)_r \tag{2.42}$$

be the matrix whose columns are the sensitivity of $\mathbf{x}$ to y_s, then the sensitivity to all the errors is given by the their sum, i.e., by the sum over s of the right-hand side of (2.41). This yields

$$\Delta\mathbf{x} = \mathbf{D}\Delta\mathbf{y} \ . \tag{2.43}$$

Finally, we have for the covariance of $\mathbf{x}$ *due only to the errors in* $\mathbf{y}$

$$E(\Delta\mathbf{x}\Delta\mathbf{x}^\dagger) = E(\mathbf{D}\Delta\mathbf{y}(\Delta\mathbf{y})^\dagger\mathbf{D}^\dagger)$$

$$= \mathbf{DYD}^\dagger \ . \tag{2.44}$$

It should be emphasized that if the two experiments are not exactly the same, the measurement functions used in determining Y and those in D may not be the same.

Let us examine (2.44) a little more closely. Suppose that the measurement spaces of the two experiments coincide. Define a set of N_1 vectors $\boldsymbol{\kappa}^s$ and a set of N_2 vectors χ^r in Euclidian M-dimensional space by

$$\boldsymbol{\kappa}^s = \frac{\partial \mathbf{m}}{\partial y_s} \quad s = 1, ..., N_1; \qquad \chi^r = \frac{\partial \mathbf{m}}{\partial x_r} \quad r = 1, \ldots, N_2 \ . \tag{2.45}$$

Thus, for example, the i^{th} component of $\boldsymbol{\kappa}^s$ is equal to $\partial m_i/\partial y_s$. We define an inner product on that space by

$$\mathbf{m}\cdot\mathbf{m}' \ \triangleq \sum_{i,j=1}^{M} m_i \mathbf{R}_{ij}^{-1} m'_j \ . \tag{2.46}$$

Then we have

$$A_{rr'} = \chi^r \cdot \chi^{r'} \tag{2.47a}$$

$$(b^s)_r = -\chi^r \cdot \boldsymbol{\kappa}^s \tag{2.47b}$$

$$Y_{ss'}^{-1} = \boldsymbol{\kappa}^s \cdot \boldsymbol{\kappa}^{s'} \ . \tag{2.47c}$$

The matrix $\mathbf{D}$ becomes

$$D_{rs} = \sum_k A_{rk}^{-1} (-\chi^k \cdot \boldsymbol{\kappa}^s), \tag{2.48}$$

and

$$(\mathbf{DYD}^\dagger)_{rr'} = \sum_{kmni} A_{rk}^{-1}(\chi^k \cdot \boldsymbol{\kappa}^m) Y_{mn} (\boldsymbol{\kappa}^n \cdot \chi^i) A_{ir'}^{-1} . \tag{2.49}$$

Let **P** be the operator defined on an arbitrary vector α by

$$\mathbf{P}\alpha \triangleq \sum_{mn} \boldsymbol{\kappa}^m Y_{mn} (\boldsymbol{\kappa}^n \cdot \alpha). \tag{2.50}$$

Then **P** is linear and it projects vectors onto the space spanned by the $\boldsymbol{\kappa}^s$, s = 1, ..., N_1. To see this we note that $\mathbf{P}\boldsymbol{\kappa}^j = \sum_{mn} \boldsymbol{\kappa}^m Y_{mn}(\boldsymbol{\kappa}^n \cdot \boldsymbol{\kappa}^j) = \sum_{mn} \boldsymbol{\kappa}^m Y_{mn} Y_{nj}^{-1} = \boldsymbol{\kappa}^j$, and if $\alpha \cdot \boldsymbol{\kappa}^m = 0$ for all m, then $\mathbf{P}\alpha = 0$. To take advantage of the operator **P** in equation (2.49), we decompose the vectors χ^r into

$$\chi^r = \chi_+^r + \chi_-^r \ , \tag{2.51}$$

where χ_+^r is contained in the space spanned by the $\boldsymbol{\kappa}'$ s, and χ_-^r is orthogonal to it. Note that the orthogonality produces a corresponding decomposition in **A**:

$$\begin{aligned} A_{rr'} &= \chi_+^r \cdot \chi_+^{r'} + \chi_-^r \cdot \chi_-^{r'} \\ &\triangleq (A_+)_{rr'} + (A_-)_{rr'} \ . \end{aligned} \tag{2.52}$$

Equation (2.49) becomes

$$\begin{aligned} (\mathbf{DYD}^\dagger)_{rr'} &= \sum_{ki} A_{rk}^{-1} (\chi^k \cdot \mathbf{P}\chi^i) A_{ir'}^{-1} \\ &= \sum_{ki} A_{rk}^{-1} (\chi_+^k \cdot \chi_+^i) A_{ir'}^{-1} \end{aligned} \tag{2.53}$$

or

$$\begin{aligned} \mathbf{DYD}^\dagger &= \mathbf{A}^{-1} (\mathbf{A} - \mathbf{A}_-) \mathbf{A}^{-1} \\ &= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{A}_-\mathbf{A}^{-1}. \end{aligned} \tag{2.54}$$

Since $\mathbf{A}$ and $\mathbf{A}_-$ are positive semidefinite[10] and since $\mathbf{A}^{-1} = (\mathbf{A}^{-1})^\dagger$, we conclude that

$$\mathbf{DYD}^\dagger \leq \mathbf{A}^{-1} \tag{2.55}$$

as quadratic forms. In other words, if the measurement functions are the same in the two experiments, then the influence of the errors in the first experiment on the covariance of the output of the second is at most as great as the influence of the measurement errors in the second experiment on its output:

$$\mathrm{E}(\Delta\mathbf{x}\Delta\mathbf{x}^\dagger) \leq \mathbf{A}^{-1}\,. \tag{2.56}$$

The total error, if both experiments are noisy, satisfies

$$\mathrm{E}_{\mathrm{total}}(\Delta\mathbf{x}\Delta\mathbf{x}^\dagger) \leq 2\mathbf{A}^{-1}\,. \tag{2.57}$$

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10 To see this, note that they are sums of outer products of vectors. For example, $\sum_{rr'} z_r A_{rr'} z_{r'} = \sum_{rr'} z_r \chi^r \cdot \chi^{r'} z_{r'} = |\,\mathbf{w}\,|^2$ where $\mathbf{w} = \sum_r z_r \chi^r$ for an arbitrary N_z-dimensional vector $\mathbf{z}$. Hence, $\mathbf{A}$ is a nonnegative quadratic form.

3. COMPUTATIONS IN GEOCENTRIC COORDINATES

In order to implement the sensitivity equations of Section 2, as well as to solve the estimation problem via (2.38), we require the gradients of the measurement functions $f_R(\mathbf{x}, \mathbf{q}, t)$ or $d(\mathbf{x}, \mathbf{q}, t)$ (cf. equations (1.13) and (1.14)) with respect to the variables $\mathbf{x}$ and $\mathbf{q}$. Since distinguishing a variable as being part of the state variable $\mathbf{x}$ or one of the a priori parameters $\mathbf{q}$ is irrelevant to such a computation, we simply employ the notation q for the independent variable in the derivative. On the other hand, the difference between a ground station parameter and a satellite parameter will prove useful, and for these we shall write q_G and q_S, respectively.

We begin Section 3 with a computation of the derivatives of the relative doppler function b (cf., equation (1.8)) with respect to q_G and q_S. This brief derivation is at an intermediate level of detail, almost independent of the coordinate system. In the next subsection we introduce standard geophysical coordinates and describe the computations leading from latitude, longitude, and velocity on an oblate earth to the vectors $\mathbf{R}$, $\mathbf{r}$, $\dot{\mathbf{R}}$, and $\dot{\mathbf{r}}$ of Figure 1.3. We also provide equations for satellite azimuth and elevation measurements. The extension to ground station motion (i.e., great circle motion parameterized by initial course and speed) is given in Section 4.

DOPPLER DERIVATIVES

We proceed to compute the derivatives of the relative doppler function b of equation (1.8). Noting that

$$\frac{\partial}{\partial q}\frac{1}{\| \mathbf{p} \|} = \frac{\partial}{\partial q}(\mathbf{p}\cdot\mathbf{p})^{-1/2} = -\frac{\mathbf{p}\cdot\dfrac{\partial \mathbf{p}}{\partial q}}{\| \mathbf{p} \|^3}, \tag{3.1}$$

we have

$$\frac{\partial}{\partial q}\left(\frac{\mathbf{p}}{\| \mathbf{p} \|}\cdot\dot{\mathbf{p}}\right) = \frac{\partial \dot{\mathbf{p}}}{\partial q}\cdot\frac{\mathbf{p}}{\| \mathbf{p} \|} + \frac{\partial \mathbf{p}}{\partial q}\cdot\left(\frac{\dot{\mathbf{p}}}{\| \mathbf{p} \|} - \frac{(\mathbf{p}\cdot\dot{\mathbf{p}})}{\| \mathbf{p} \|^3}\,\mathbf{p}\right). \tag{3.2}$$

Thus, if we define the normal vector $\mathbf{n}$ by

$$\mathbf{n} = \frac{\mathbf{p}}{\| \mathbf{p} \|}, \tag{3.3}$$

then

$$\begin{aligned}\frac{\partial b}{\partial q} &= -\frac{1}{c}\frac{\partial \dot{\mathbf{p}}}{\partial q}\cdot\mathbf{n} - \frac{1}{\| \mathbf{p} \|}\frac{\partial \mathbf{p}}{\partial q}\cdot\left(\frac{\dot{\mathbf{p}}}{c} - \frac{(\mathbf{p}\cdot\dot{\mathbf{p}})}{c\,\| \mathbf{p} \|}\mathbf{n}\right)\\ &= -\frac{1}{c}\frac{\partial \dot{\mathbf{p}}}{\partial q}\cdot\mathbf{n} - \frac{1}{\| \mathbf{p} \|}\frac{\partial \mathbf{p}}{\partial q}\cdot\left(\frac{\dot{\mathbf{p}}}{c} + b\,\mathbf{n}\right).\end{aligned} \tag{3.4}$$

As a consequence of (1.6) and (1.9), equation (3.4) is a function of $\mathbf{r}$, $\dot{\mathbf{r}}$, $\mathbf{R}$, and $\dot{\mathbf{R}}$. We can remove the dependency on $\dot{\mathbf{R}}$ by taking into account the earth's motion. Let us denote by Ω the vector

$$\Omega = \omega_0 \mathbf{K} \ , \tag{3.5}$$

where $\mathbf{K}$ is a unit vector pointing north along the earth's axis, and ω_0 is the rate of the earth's rotation (e.g., radians/sec). Then, if a ground station is motionless relative to the earth with position vector $\mathbf{R}$ (cf. Figure 1.3), its velocity with respect to any *geocentric inertial coordinate system* (cf. next subsection) is $\dot{\mathbf{R}} = \boldsymbol{\Omega} \times \mathbf{R}$, where x indicates the vector product. Finally, if the station is moving relative to the earth with a "ground" velocity of $\mathbf{v}$, we have

$$\dot{\mathbf{R}} = \boldsymbol{\Omega} \times \mathbf{R} + \mathbf{v} \ . \tag{3.6}$$

That is, $\mathbf{v}$ denotes the velocity the station would have if we instantaneously stopped the earth's rotation. Equation (1.9) becomes

$$\dot{\mathbf{p}} = \dot{\mathbf{r}} - \Omega \times \mathbf{R} - \mathbf{v} \ . \tag{3.7}$$

The derivatives of b simplify if we consider satellite parameters and ground station parameters separately. For the satellite $\dfrac{\partial \mathbf{R}}{\partial q_S} = \dfrac{\partial \mathbf{R}}{\partial q_S} = 0$ so that

$$\frac{\partial b}{\partial q_S} = -\frac{1}{c}\frac{\partial \dot{\mathbf{r}}}{\partial q_S} \cdot \mathbf{n} - \frac{1}{\| \mathbf{p} \|}\frac{\partial \mathbf{r}}{\partial q_S} \cdot \left(\frac{\dot{\mathbf{p}}}{c} + b\,\mathbf{n} \right) . \tag{3.8}$$

For the ground station $\dfrac{\partial \mathbf{r}}{\partial q_G} = \dfrac{\partial \dot{\mathbf{r}}}{\partial q_G} = 0$ and using (3.6) with (3.7),

$$\begin{aligned}
\frac{\partial b}{\partial q_G} &= \frac{1}{c}\left(\Omega \times \frac{\partial \mathbf{R}}{\partial q_G} + \frac{\partial \mathbf{v}}{\partial q_G} \right) \cdot \mathbf{n} + \frac{1}{\| \mathbf{p} \|}\frac{\partial \mathbf{R}}{\partial q_G} \cdot \left(\frac{\dot{\mathbf{p}}}{c} + b\,\mathbf{n} \right) \\
&= \frac{\partial \mathbf{R}}{\partial q_G} \cdot \left(\frac{\mathbf{n} \times \Omega}{c} + \frac{\dot{\mathbf{p}}}{c \| \mathbf{p} \|} + \frac{b\,\mathbf{n}}{\| \mathbf{p} \|} \right) + \frac{1}{c}\frac{\partial \mathbf{v}}{\partial q_G} \cdot \mathbf{n} \\
&= \frac{1}{\| \mathbf{p} \|}\frac{\partial \mathbf{R}}{\partial q_G} \cdot \left(\frac{\mathbf{p} \times \Omega}{c} + \frac{\dot{\mathbf{p}}}{c} + b\,\mathbf{n} \right) + \frac{1}{c}\frac{\partial \mathbf{v}}{\partial q_G} \cdot \mathbf{n} \ .
\end{aligned} \tag{3.9}$$

(We have used the vector indentity $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$.

For completeness, we compute the derivatives of the measurement function $d(\mathbf{x}, \mathbf{q}, t_i)$ with respect to the various parameters. If q is not a frequency parameter then we have

$$\frac{\partial d}{\partial q} = \frac{\partial b_U}{\partial q} b_D + \frac{\partial b_D}{\partial q}(b_U + 1 + \frac{f_{SO}}{f_0}) \quad \text{for } q \neq \text{freq param} . \tag{3.10}$$

Also,

$$\frac{\partial d}{\partial g_B} = 1 , \tag{3.11}$$

$$\frac{\partial d}{\partial f_{SO}} = \frac{1}{f_0}(b_D + \epsilon) , \tag{3.12}$$

and

$$\frac{\partial d}{\partial f_{RO}} = -\frac{\epsilon}{f_0} , \tag{3.13}$$

where

$$\epsilon \triangleq \begin{cases} 0 & \text{for } g_B \text{ a state variable} \\ 1 & \text{otherwise} \end{cases} .$$

The partial derivative with respect to f_0 which is suitable for use in the sensitivity is somewhat more complex, since the modified measurement $\tilde{d}_i$ of (1.16) is a function of f_0. One can either work entirely in terms of the measurement functions $f_R(\mathbf{x}, \mathbf{q}, t_i)$ of (1.13) or utilize the derivative of the entire residual (which appears in (1.17)),

$$\frac{\partial(d - \tilde{d}_i)}{\partial f_0} = -\frac{1}{f_0^2}(b_D f_{SO} + \epsilon(f_{SO} - f_{RO}) - \tilde{f}_i) . \tag{3.14}$$

GEOCENTRIC COORDINATES

Equations (1.6), (1.8), and (3.7)-(3.14) determine the measurements at an arbitrary time t as a function of the vectors $\mathbf{R}(t)$, $\mathbf{r}(t)$, $\dot{\mathbf{r}}(t)$, and $\mathbf{v}(t)$ at that time. However, the parameters q are fixed numbers associated, at least conceptually, with some fixed time t_0. Even for a stationary ground station, it is natural to equate the latitude and longitude at some time t_0 with a fixed vector $\mathbf{R}_0$ and then to compute $\mathbf{R}(t)$ as a function of $\mathbf{R}_0$ and $t - t_0$ by taking into account the earth's rotation.

In order to be more precise, let us introduce some specific coordinate systems. Figure 3.1 illustrates the geocentric, or what we shall call the IJK coordinate system. The K-axis is the axis of the earth's rotation; the other two axes are chosen so that **IJK** forms a right-handed system of mutually perpendicular axes with **I** pointing in the

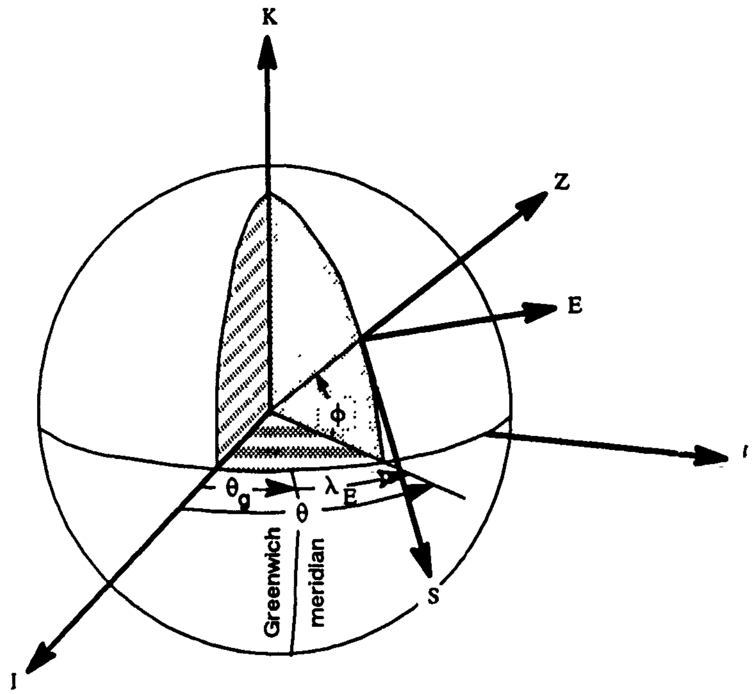

Figure 3.1. Illustration of IJK and SEZ coordinate systems. Note that the longitude λ and θ are measured towards the east.

direction of the vernal equinox[11] [1, 2]. The vernal equinox and the earth's axis both move with respect to the celestial sphere. Thus, to specify an IJK coordinated system we must also supply an epoch (point in time) τ. (Note that t is yet a third time, that of the measurement. It usually is, but need not be, the same as τ.) A typical computation involves the determination of the satellite position $\mathbf{r}(t)$ in IJK coordinates relative to one epoch, τ_S and the location of a ground station $\mathbf{R}(t)$ in a coordinate system of another epoch τ_G. Since vector computations such as $\mathbf{p} = \mathbf{r}(t) - \mathbf{R}(t)$ must be done in the same coordinate system, we must either transform $\mathbf{R}$ from IJK_G to IJK_S, or $\mathbf{r}$ from IJK_S to IJK_G, or both vectors to some other, common coordinate system before subtracting. Such a transformation, between orthogonal coordinate systems, will be a rotation, which may be identified with a matrix $\mathbf{T}$. We use the notation

$$\mathbf{a}^{(S)} = \mathbf{T}(\tau_S, \tau_G)\mathbf{a}^{(G)}$$

$$\triangleq \mathbf{T}_{SG}\mathbf{a}^{(G)} \tag{3.15}$$

for the mapping of the representation of an arbitrary vector $\mathbf{a}$ in IJK_G coordinates to IJK_S coordinates. One should be careful not to confuse vectors as abstract entities with their representations; thus, $\mathbf{a}$ is a single object, $\mathbf{a}^{(S)}$ is its representation as a column of three numbers in the IJK_S coordinate system, while $\mathbf{a}^{(G)}$ is a representation of the same object in the IJK_G coordinate system. The matrix $\mathbf{T}$ is a function only of the two epochs τ_S and τ_G; the details may be found in [2]. Finally, we have

$$\mathbf{p}^{(S)} = \mathbf{r}^{(S)} - \mathbf{R}^{(S)}$$

$$= \mathbf{r}^{(S)} - \mathbf{T}_{SG}\mathbf{R}^{(G)} \tag{3.16}$$

$$\dot{\mathbf{p}}^{(S)} = \dot{\mathbf{r}}^{(S)} - \mathbf{T}_{SG}\left[\boldsymbol{\Omega}^{(G)} \times \mathbf{R}^{(G)} - \mathbf{v}^{(G)}\right] = \mathbf{r}^{(S)} - \mathbf{T}_{SG}\mathbf{R}^{(G)} \ . \tag{3.17}$$

The computation of $\mathbf{R}^{(G)}$ is detailed below; that of $\mathbf{r}^{(S)}$ and $\dot{\mathbf{r}}^{(S)}$ are described in Appendix B.

From Figure 3.2, we see that the IJK coordinates of a station are determined by θ, the angle of its meridian with respect to the I-axis, and by its rectangular coordinates (x, z) in the plane of that meridian. That is,

$$\mathbf{R}^{(G)} = (x \cos\theta, x \sin\theta, z) \tag{3.18}$$

[11] The vernal equinox is the intersection of the equatorial plane (the IJ plane through the geocenter) with the ecliptic.

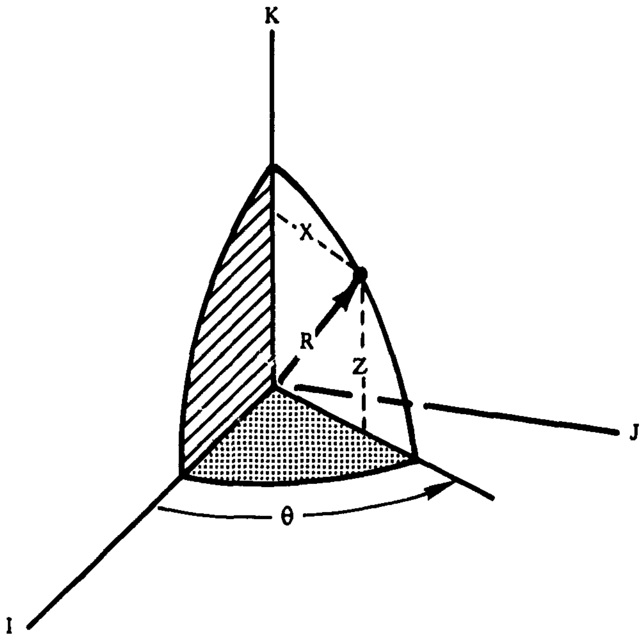

Figure 3.2. A meridial slice through a ground station at R. For an oblate earth this slice is an ellipse. See Figures 3.3 and 3.4.

According to Figure 3.1, θ is simply

$$\theta(t) = \theta_g(t) + \lambda \ , \tag{3.19}$$

where λ is the longitude of the ground station, $\theta_g(t)$ is the angle of the Greenwich meridian with respect to I at time t (i.e., the greenwich sidereal time), and coordinate system IJK_G is that of the epoch $\tau_G = t$.

The earth is more accurately represented by an ellipsoid than the sphere drawn in Figure 3.1. In that case the meridial slice is an ellipse, as pictured in Figure 3.3. The geodetic latitude ϕ shown there is determined by the angle subtended by a line perpendicular to the earth's surface. Some straightforward algebra yields the rectangular coordinates of the ground station [1],

$$\begin{aligned} x &= \left(\frac{a_e}{\sqrt{1 - e^2 \sin^2\phi}} + H \right) \cos\phi \\ z &= \left(\frac{a_e(1 - e^2)}{\sqrt{1 - e^2 \sin^2\phi}} + H \right) \sin\phi \end{aligned} \tag{3.20}$$

where the ellipsoid is taken to lie at mean sea level, H is the altitude (i.e., height of the ground station above sea level), a_e is the equatorial radius of the earth, and e is the earth's eccentricity. Equations (3.18) to (3.20) determine $\mathbf{R}^{(G)}(t)$, given the latitude and longitude ϕ and λ for a motionless ground station (i.e., $\mathbf{v} = 0$).

The computation of $\mathbf{v}^{(G)}(t)$, the instantaneous ground station velocity relative to the earth in IJK coordinates, is not quite so straightforward. If we are given a course $\nu(t)$ and speed s(t) for each time t, then the procedure is a simple one, which we describe below. However, if we wish to model the motion parametrically, for example, in terms of a constant speed and an initial course at some time t_0, then we must specify a curve. We cannot simply use a rectilinear motion of the form $\mathbf{a} = \mathbf{a}_0 + \mathbf{v}_0 (t - t_0)$ if we wish the station to remain on the surface of the earth. A natural choice for a spherical earth is great circle motion, i.e., motion along a great circle with constant speed and an initial course ν_0. Note, that in that case, $\nu(t) \neq \nu_0$. Motion on an oblate earth is even less obvious. These are treated in Section 4.

AZIMUTH AND ELEVATION

In addition to the IJK coordinates, Figure 3.1 illustrates the so-called SEZ coordinate system. The letters S and E stand for south and east. The **Z**-axis is a perpendicular to the earth's surface at the ground station; **S** and **E** are placed in the plane tangent to the earth. (Note that these definitions are valid for an oblate earth even though Figure 3.1 is misleading.) SEZ coordinates are fairly natural for describing local motion. For example, an object moving at a speed s and course ν (clockwise from north) has a velocity with SEZ coordinates

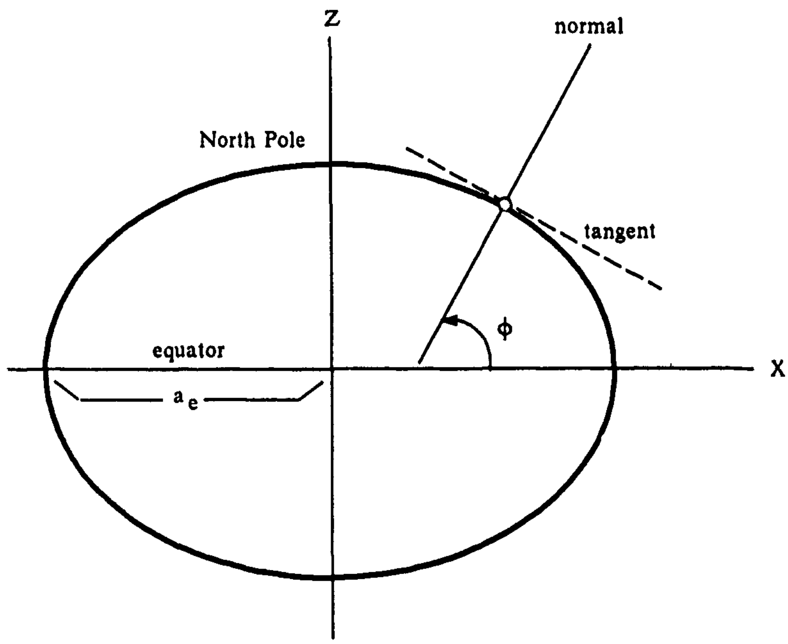

Figure 3.3. Illustration of the definition of geodetic latitude.

$$\mathbf{v}^{(SEZ)} = \begin{bmatrix} -s\cos\nu \\ s\sin\nu \\ 0 \end{bmatrix} . \tag{3.21}$$

This SEZ coordinate system, of course, moves with time as the earth rotates. It also depends on the ground station location, i.e., on ϕ and λ. The transformation from SEZ coordinates at time t to IJK coordinates at the same epoch (i.e., $\tau = t$) is given by [1]

$$\mathbf{a}^{(IJK)} = \mathbf{D}^{-1}\,\mathbf{a}^{(SEZ)} , \tag{3.22a}$$

where

$$\mathbf{D} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & -\cos\phi \\ -\sin\theta & \cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \end{bmatrix} . \tag{3.22b}$$

(The fact that this transformation holds even though it does not reflect the oblateness of the earth is clarified in Figure 3.4. If the origin of the IJK system is translated from A to B, then the transformation is the same as would be obtained in Figure 3.1. Since we are describing only detached vectors, i.e., a length and a direction, in this transformation the translation from A to B has no effect.)

The relationship of the so-called topocentric coordinate system (i.e., azimuth and elevation) to SEZ coordinates is described by Figure 3.5. From that figure,

$$\begin{aligned} AZ &= -\tan^{-1}\frac{p_E}{p_S} \\ EL &= \sin^{-1}\frac{p_Z}{\|\mathbf{p}\|} \end{aligned} \tag{3.23}$$

with the sign of AZ determined by

$$\sin AZ = \frac{p_E}{\sqrt{p_E^2 + p_S^2}} . \tag{3.24}$$

Next we compute the derivatives. Using $(\tan^{-1}x)' = 1/(1 + x^2)$, we have

$$\frac{\partial AZ}{\partial q} = -\frac{p_S^2}{p_E^2 + p_S^2}\frac{\partial}{\partial q}\frac{p_E}{p_S}$$

$$= -\frac{p_S^2}{p_E^2 + p_S^2}\left(\frac{1}{p_S^2}\right)\left(p_S\frac{\partial p_E}{\partial q} - p_E\frac{\partial p_S}{\partial q}\right)$$

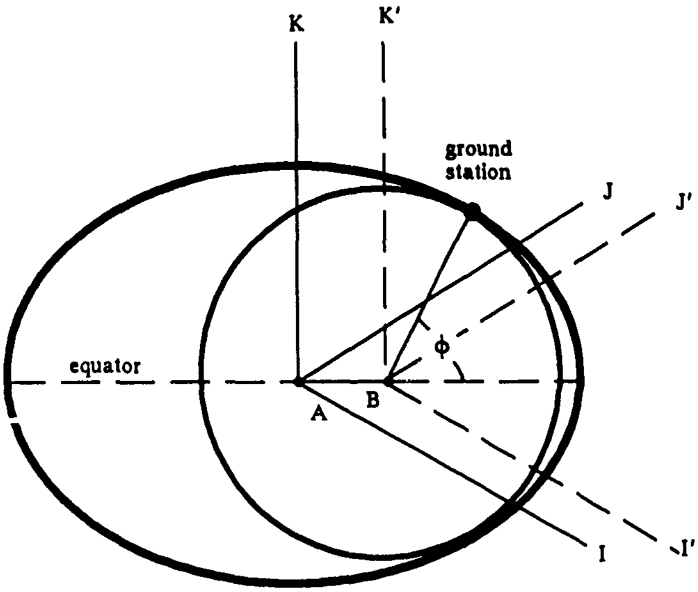

Figure 3.4. Translation of origin of IJK coordinate system.

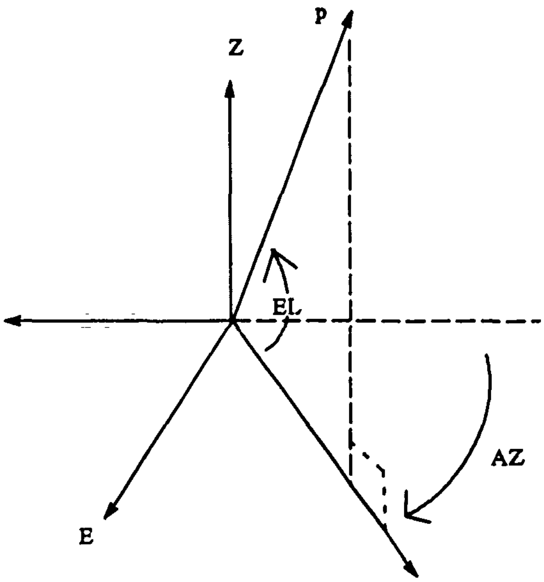

Figure 3.5. Diagram of the relation of the SEZ coordinate system to azimuth and elevation.

$$= \frac{1}{p_E^2 + p_S^2} \left(\frac{\partial \mathbf{p}}{\partial q} \times \mathbf{p} \right)_Z . \tag{3.25}$$

An alternative form obtained by using $\mathbf{n} = \mathbf{p} / \| \mathbf{p} \|$ (definition (3.3)) and $p_S^2 + p_E^2 = \| \mathbf{p} \|^2 (1 - n_Z^2)$ is

$$\frac{\partial AZ}{\partial q} = \frac{\left(\frac{\partial \mathbf{p}}{\partial q} \times \mathbf{n} \right)_Z}{\| \mathbf{p} \| (1 - n_Z^2)} . \tag{3.26}$$

Also, since

$$\sin EL = n_Z , \tag{3.27}$$

$$\frac{\partial EL}{\partial q} \cos EL = \frac{\partial n_Z}{\partial q} . \tag{3.28}$$

Then, $\cos EL = \sqrt{1 - n_Z^2}$ implies

$$\frac{\partial EL}{\partial q} = \frac{\frac{\partial n_Z}{\partial q}}{\sqrt{1 - n_Z^2}} . \tag{3.29}$$

Then, since

$$\frac{\partial n_Z}{\partial q} = \frac{\frac{\partial p_Z}{\partial q}}{\| \mathbf{p} \|} - \frac{p_Z}{\| \mathbf{p} \|^3} \left(\frac{\partial \mathbf{p}}{\partial q} \cdot \mathbf{p} \right)$$

$$= \frac{1}{\| \mathbf{p} \|} \left(\frac{\partial p_Z}{\partial q} - n_Z \left(\frac{\partial \mathbf{p}}{\partial q} \cdot \mathbf{n} \right) \right) , \tag{3.30}$$

we have

$$\frac{\partial EL}{\partial q} = \frac{\frac{\partial p_Z}{\partial q} - n_Z \left(\frac{\partial \mathbf{p}}{\partial q} \cdot \mathbf{n} \right)}{\| \mathbf{p} \| \sqrt{1 - n_Z^2}} . \tag{3.31}$$

TIME AS A PARAMETER

It is readily apparent from the current discussion that timing represents a potential physical source of error in the geolocation computations. Table 3.1 below summarizes the relevant variables.

Table 3.1. Summary of time variables.

	initial state	signal time	pos/vel	coordinate sys
receiver	t_{R0}	t	$\mathbf{R}_R(t)$, $\dot{\mathbf{R}}_R(t)$	τ_R
transm	t_{T0}	t	$\mathbf{R}_T(t)$, $\dot{\mathbf{R}}_T(t)$	τ_T
satellite	t_{S0}	t	$\mathbf{r}(t)$, $\dot{\mathbf{r}}(t)$	τ_S

The parameters t_{R0} and t_{T0} represent the times at which the ground station states are specified. Similarly, t_{S0} stands for the epoch for the satellite elements. t is time of the receiver measurement; hence, effectively, it also equals the time of the signal transmission and of its relay at the satellite. τ_G and τ_S are intermediate variables, not parameters; that is, they are either functions of parameters or errorless constants. In the present case, $\tau_R = \tau_T = t$, while τ_S is the epoch which the satellite model uses for its coordinate system. Usually $\tau_S = t_{S0}$, the epoch of the element set (Kepler) or some fixed astronomical date as in $\tau_S =$ Besselian year 1950 (NORAD).

The parameters t_{R0} and t_{T0} only have significance when a ground station is moving. Even in that case, an error in t_0 may be equated to an appropriate error in the initial latitude, longitude, course, and speed parameters. Similarly, t_{S0} is equivalent to a set of errors in the satellite parameters. For a Kepler model, it is equivalent to an error in the mean anomaly. Of greater interest, perhaps, are errors in t, the measurement times. These may usually be modeled as a bias in the setting of the receiver's clock. More specifically, we write

$$t = t' + c_R \ , \tag{3.32}$$

where c_R is the clock offset. Then an error in c_R corresponds to a bias in the clock. The sensitivity to this bias is a function of the partial derivatives with respect to c_R. From (3.16), (3.17), and (3.22), we have

$$\frac{\partial \mathbf{p}^{(S)}}{\partial c_R} = \frac{\partial \mathbf{r}^{(S)}}{\partial t} - \frac{\partial \mathbf{T}_{SG}}{\partial t}\mathbf{R}^{(G)} - \mathbf{T}_{SG}\frac{\partial \mathbf{R}^{(G)}}{\partial t} \tag{3.33a}$$

$$\frac{\partial \dot{\mathbf{p}}^{(S)}}{\partial c_R} = \frac{\partial \dot{\mathbf{r}}^{(S)}}{\partial t} - \frac{\partial \mathbf{T}_{SG}}{\partial t}\left[\Omega^{(G)} \times \mathbf{R}^{(G)} - \mathbf{v}^{(G)}\right]$$

$$-T_{SG}\left[\boldsymbol{\Omega}^{(G)} \times \frac{\partial \mathbf{R}^{(G)}}{\partial t} - \frac{\partial \mathbf{v}^{(G)}}{\partial t}\right]. \tag{3.33b}$$

The terms $\frac{\partial T_{SG}}{\partial t}$ are generally negligible compared with the other derivatives. We also have

$$\frac{\partial \mathbf{R}^{(G)}}{\partial c_R} = \frac{\partial \mathbf{R}^{(G)}}{\partial \theta_g}\frac{d\theta_g}{dt}$$

$$= \frac{\partial \mathbf{R}^{(G)}}{\partial \theta_g}\,\omega\ . \tag{3.34}$$

If the station is not moving, then $\frac{\partial \mathbf{v}^{(G)}}{\partial c_R}=0$; otherwise, this derivative depends on the ground station motion model. Note that although c_R is a receiver parameter, it appears in both satellite and ground stations computations so that equation (3.4) must be used rather than (3.8) or (3.9).

SUMMARY

We summarize the parameters and the stages which lead from them to the doppler residuals $d(t) - \tilde{d}_t$ of (1.17):

Satellite Parameters:
time, semimajor axis, eccentricity, inclination, mean anomaly, ascending node, argument of perigee, offset freq.

$q_S = (t_{S0}, a_0, e_0, i_0, \nu_0, \Omega_0, \omega_0, f_{S0})$ [12]

Ground Station Parameters:
time, latitude, longitude, course, speed, frequency

receiver
$q_G = (t_{R0}, \phi_{R0}, \lambda_{R0}, \nu_{R0}, s_{R0}, f_{R0})$

transmitter
$q_G = (t_{T0}, \phi_{T0}, \lambda_{T0}, \nu_{T0}, s_{T0}, f_0)$

12 Note that ν and ω as elements should not be confused with ground station course or the earth's angular velocity.

$$q_G(t_{G0}) \xrightarrow{\text{Section 4}} q_G(t) \xrightarrow{(3.6),\ (3.18)\text{ to }(3.20)} \mathbf{R}^{(G)}(t),\ \dot{\mathbf{R}}^{(G)}(t) \xrightarrow{(3.15)} \mathbf{R}^{(S)}(t),\ \dot{\mathbf{R}}^{(S)}(t)$$

$$q_S(t_{S0}) \xrightarrow{\text{Appendix B}} \mathbf{r}^{(S)}(t),\ \dot{\mathbf{r}}^{(S)}(t)$$

$$\begin{matrix} \mathbf{R}_R^{(S)}(t),\ \dot{\mathbf{R}}_R^{(S)}(t) \\ \mathbf{r}^{(S)}(t),\ \dot{\mathbf{r}}^{(S)}(t) \end{matrix} \xrightarrow{(1.6)\text{ to }(1.9)} b_D(t)$$

$$\begin{matrix} \mathbf{R}_T^{(S)}(t),\ \dot{\mathbf{R}}_T^{(S)}(t) \\ \mathbf{r}^{(S)}(t),\ \dot{\mathbf{r}}^{(S)}(t) \end{matrix} \xrightarrow{(1.6)\text{ to }(1.9)} b_U(t)$$

$$b_U,\ b_D,\ f_0,\ f_{RO},\ f_{SO} \xrightarrow{(1.14)\text{ to }(1.16)} d(t) - \tilde{d}_t$$

Derivatives are computed by the chain rule. The relevant equations are (3.10) to (3.14) for d, (3.6) to (3.9) for b_U and b_D, Section 4 for $\mathbf{R}^{(G)}$ and $\dot{\mathbf{R}}^{(G)}$ (note that $\mathbf{T}_{SG}$ is only a linear factor in derivations with respect to the above parameters), and Appendix B for $\mathbf{r}^{(S)}$, $\dot{\mathbf{r}}^{(S)}$.

4. MOTION ON AN OBLATE EARTH

In this section we develop a four-parameter model (i.e., initial latitude, longitude, course, and speed) for ground station motion. The obvious choice of great circle motion is somewhat complicated by the consideration of a nonspherical earth. Some of the relations reduce considerably with the appropriate algebraic manipulation (e.g., (4.18) and (4.19)), but the total number of equations remains large. The less interesting equations are computed in Appendix A and summarized at the end of this section. We remark that, for those only interested in motion on an oblate earth, Section 4 and Appendix A may be read independently of the rest of the report.

STATION COORDINATES TO GEOCENTRIC COORDINATES

The term *station coordinates* refers to geodetic latitude, designated by ϕ, longitude east, λ, and altitude above mean sea level, H (cf., [1] or Section 3). Of course, since the earth is moving, these must also be accompanied by a time. The other system, geocentric coordinates, will be denoted by IJK ($\mathbf{K}$ = axis of rotation, $\mathbf{I}$ = direction of vernal equinox, $\mathbf{J} = \mathbf{K} \times \mathbf{I}$) These, of course, also depend on a specified epoch. If we let θ_g be the angle between the Greenwich meridian and the axis I, i.e., greenwich sidereal time, then the local sidereal time θ is given by

$$\theta = \theta_g + \lambda \, . \tag{4.1}$$

Let us designate the equatorial radius of the earth by a_e and its eccentricity by e, and let

$$q \triangleq \frac{a_e}{\sqrt{1 - e^2 \sin^2\phi}} \tag{4.2a}$$

and

$$q_s \triangleq \frac{\partial q}{\partial(\sin\phi)} = \frac{a_e e^2 \sin\phi}{(1 - e^2\sin^2\phi)^{3/2}} \tag{4.2b}$$

Then, assuming an oblate earth, the position vector in IJK coordinates is given by [1] (or by equations (3.18) - (3.20))

$$\mathbf{R}^{(IJK)} = (x \cos\theta, x \sin\theta, z)^\dagger \tag{4.3a}$$

where

$$x \triangleq (q+H) \cos\phi \tag{4.3b}$$

$$z \triangleq (q\,(1-e^2) + H) \sin\phi \, , \tag{4.3c}$$

and † indicates the transpose. Equations (4.1) to (4.3) describe the transformation from station coordinates to geocentric coordinates.

Computation of the doppler shift, of the sensitivity, or an analytic steepest descent to estimate the state, requires the derivatives of these transformations with respect to various parameters involved in the station location and/or its motion. Since these parameters are more conveniently expressed in terms of functions of the station coordinates (e.g., in terms of $\sin\phi$ and $\cos\phi$ rather than ϕ itself), we retain that dependence in computing the derivatives. Let w be an arbitrary parameter, then

$$\frac{\partial \mathbf{R}_{IJK}}{\partial w} = \left(\frac{\partial x}{\partial w}\cos\theta + x\frac{\partial \cos\theta}{\partial w},\ \frac{\partial x}{\partial w}\sin\theta + x\frac{\partial \sin\theta}{\partial w},\ \frac{\partial z}{\partial w}\right)^{\dagger} \tag{4.4}$$

where

$$\frac{\partial x}{\partial w} = q_s \cos\phi\ \frac{\partial \sin\phi}{\partial w} + (q + H)\ \frac{\partial \cos\phi}{\partial w} \tag{4.5a}$$

and

$$\frac{\partial z}{\partial w} = q_s\,(1 - e^2)\sin\phi\ \frac{\partial \sin\phi}{\partial w} + (q(1 - e^2) + H)\ \frac{\partial \sin\phi}{\partial w}$$

$$= \left[\frac{q(1 - e^2)}{1 - e^2\sin^2\phi} + H\right]\frac{\partial \sin\phi}{\partial w}\,. \tag{4.5b}$$

These, of course, do not include the case where w is a function of H.

STATION MOTION

We model our motion as "great circle" motion on a nearly spherical earth. By this we mean that the geodetic latitude and longitude trace out a great circle on the unit sphere. Thus, for example, in Figure 4.1, P_0 is the initial position, and P is the position after traveling the distance ς (radians) along a great circle specified by the initial course ν_0. This motion composed with equations (4.1) - (4.3) is the assumed motion of the station. In this sense, the local position $\mathbf{R}(t)$ is a function of the initial position ϕ_0, λ_0, of the initial course ν_0, and of the distance traveled $\varsigma(t)$. Of course, $\varsigma(t)$ is related to the initial speed s_0 and the time elapsed $t - t_0$ as diagramed below.

$$t_0, \phi_0, \lambda_0, \nu_0, s_0 \xrightarrow{(4.22)} \phi_0, \lambda_0, \nu_0, \varsigma(t) \xrightarrow{(4.40)\text{-}(4.43)} \phi(t), \lambda(t) \xrightarrow{(4.1)\text{-}(4.3)} \mathbf{R}(t)$$

For lack of a better term, we call this "pseudo great circle motion."

The velocity $\dot{\mathbf{R}}$ is then, of course, the time derivative of this motion. It will simplify our results to separate the part of the motion due to the earth's rotation from that of the station relative to the earth. Let $\boldsymbol{\Omega}$ be a vector along the earth's axis with magnitude equal to the earth's rate of rotation (cf., (3.5)) and $\mathbf{v}$ the velocity the station would have if we instantaneously stopped the earth's rotation. Then

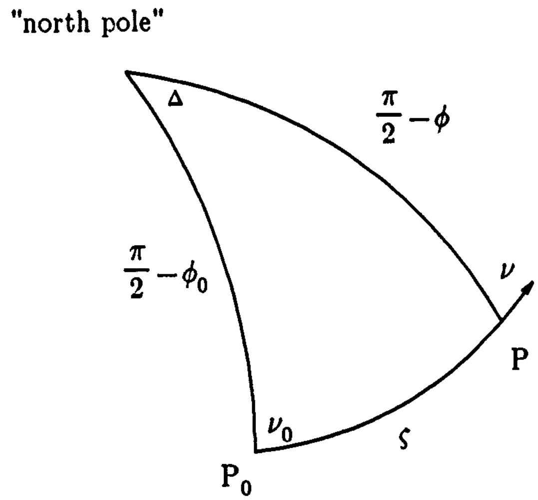

Figure 4.1. Great circle motion.

$$\dot{\mathbf{R}}(t) = \Omega(t) \times \mathbf{R}(t) + \mathbf{v}(t)\,. \tag{4.6}$$

The vector $\Omega(t) \times \mathbf{R}(t)$ is simply $\frac{\partial \mathbf{R}}{\partial \theta}\frac{d\theta}{dt}$.[13] Hence, $\mathbf{v}$ is computed by taking the time derivative of $\mathbf{R}$ but setting $d\theta/dt = 0$.

One must, however, resist the temptation to further simplify by trying to directly convert the course ν and speed s to rectangular coordinates (e.g., north and east) and rotating that to IJK coordinates, thus, apparently obtaining $\mathbf{v}$ without the need to differentiate (4.2) to (4.3). More precisely, let the transformation from SEZ (topocentric: south, east, up) coordinates to IJK be given by the rotation ([1] or equation (3.22))

$$\mathbf{v}^{(\mathrm{IJK})} = \mathbf{D}^{-1}\mathbf{v}^{(\mathrm{SEZ})} \tag{4.7}$$

where

$$\mathbf{D}^{-1} = \begin{bmatrix} \sin\phi\,\cos\theta & -\sin\theta & \cos\phi\,\cos\theta \\ \sin\phi\,\sin\theta & \cos\theta & \cos\phi\,\sin\theta \\ -\cos\phi & 0 & \sin\phi \end{bmatrix} \tag{4.8}$$

and is valid for an ellipsoidal earth. One might expect the velocity vector to lie in the SE plane at an angle of $\pi - \nu$ to the south, i.e.,

$$\mathbf{v}^{(\mathrm{SEZ})} \ne \dot{\varsigma}\ a_{\mathrm{R}} \begin{bmatrix} -\cos\nu \\ \sin\nu \\ 0 \end{bmatrix} \tag{4.9}$$

and

$$\mathbf{v}^{(\mathrm{IJK})} \approx \dot{\varsigma}\ a_{\mathrm{R}}\,\mathbf{D}^{-1} \begin{bmatrix} -\cos\nu \\ \sin\nu \\ 0 \end{bmatrix}, \tag{4.10}$$

where $\dot{\varsigma}$ indicates the time derivative of ς, and a_{R} is the radius of the earth at $\mathbf{R}$.

However, this is only approximately correct. Due to the earth's oblateness, the actual curve traced on the earth's surface is not the great circle of a sphere, and the SEZ velocity vector is only approximately given by (4.9). The exact equations are computed below. We also note that we must specify whether $\dot{\varsigma}$ is a constant or whether the speed, $\|\mathbf{v}\|$ is a constant. Although these are approximately equivalent, once again, because of the oblateness of the earth, they are not exactly the same. The final result is

$$\mathbf{v}^{(\mathrm{IJK})} = \dot{\varsigma}\ \mathbf{D}^{-1} \left(a \begin{bmatrix} -\cos\nu \\ \sin\nu \\ 0 \end{bmatrix} + b \begin{bmatrix} \cos\phi\,\cos\nu \\ 0 \\ 0 \end{bmatrix} \right) \tag{4.11}$$

[13] Ω depends on t since the earth's axis precesses. $\dot{\Omega}$, which is negligible, is effectively set to zero if we use equation (4.6); however, if one is fussy, one may replace the first term in (4.6) with $\frac{\partial \mathbf{R}}{\partial \theta}\frac{d\theta}{dt}$

where

$$a \triangleq q + H, \tag{4.12a}$$

and

$$b \triangleq \frac{e^2 q^3}{a_e^{\,2}} \cos\phi = q_s \frac{\cos\phi}{\sin\phi}. \tag{4.12b}$$

We proceed to derive (4.11). Using definitions (4.12), equations (4.3) and (4.5) may be rewritten

$$x = a \cos\phi \tag{4.13a}$$

$$z = a \sin\phi - e^2 q \sin\phi \tag{4.13b}$$

$$\frac{\partial x}{\partial w} = a \frac{\partial \cos\phi}{\partial w} + b \sin\phi \frac{\partial \sin\phi}{\partial w} \tag{4.13c}$$

$$\frac{\partial z}{\partial w} = a \frac{\partial \sin\phi}{\partial w} - b \cos\phi \frac{\partial \sin\phi}{\partial w} \tag{4.13d}$$

In order to determine the velocity, we wish to set $w = \varsigma$ and use $\mathbf{v} = (\partial \mathbf{R}/\partial \varsigma)\, \dot{\varsigma}$. Equations (A.15a) and (A.15b) give

$$\frac{\partial \sin\phi}{\partial \varsigma} = \cos\phi \cos\nu \tag{4.14a}$$

$$\frac{\partial \cos\phi}{\partial \varsigma} = -\sin\phi \cos\nu, \tag{4.14b}$$

while from (A.38a) and (A.38b)

$$\frac{\partial \sin\theta}{\partial \varsigma} = \cos\theta \frac{\sin\nu}{\cos\phi} \tag{4.15a}$$

and

$$\frac{\partial \cos\theta}{\partial \varsigma} = -\sin\theta \frac{\sin\nu}{\cos\phi} \tag{4.15b}$$

Combining (4.13c) and (4.13d) with (4.14a) and (4.14b) yields

$$\frac{\partial\, x}{\partial \varsigma} = -a \sin\phi \cos\nu + b \sin\phi \cos\phi \cos\nu \tag{4.16a}$$

$$\frac{\partial\, z}{\partial \varsigma} = a \cos\phi \cos\nu - b \cos^2\phi \cos\nu \tag{4.16b}$$

Substituting (4.15) - (4.16) into (4.4), we obtain

$$\frac{\partial\, \mathbf{R}^{(IJK)}}{\partial \varsigma} = \begin{bmatrix} -a \sin\phi \cos\nu \cos\theta - a \sin\theta \sin\nu \\ -a \sin\phi \cos\nu \sin\theta + a \cos\theta \sin\nu \\ a \cos\phi \cos\nu \end{bmatrix} + \begin{bmatrix} b \sin\phi \cos\phi \cos\nu \cos\theta \\ b \sin\phi \cos\phi \cos\nu \sin\theta \\ -b \cos^2\phi \cos\nu \end{bmatrix} \tag{4.17}$$

Comparing this with (4.8), we find

$$\frac{\partial\, \mathbf{R}^{(IJK)}}{\partial \varsigma} = \mathbf{D}^{-1} \left(a \begin{bmatrix} -\cos\nu \\ \sin\nu \\ 0 \end{bmatrix} + b \begin{bmatrix} \cos\phi \cos\nu \\ 0 \\ 0 \end{bmatrix} \right). \tag{4.18}$$

Since

$$\mathbf{v} = \dot{\varsigma}\ \frac{\partial\, \mathbf{R}}{\partial \varsigma}, \tag{4.19}$$

this proves (4.11).

It remains to describe the relationship of $\dot{\varsigma}$ to speed. Since speed is the magnitude of the velocity, we have

$$\dot{\varsigma} = \text{speed} / \left\| \frac{\partial \mathbf{R}}{\partial \varsigma} \right\| \ , \tag{4.20}$$

where

$$\left\| \frac{\partial \mathbf{R}}{\partial \varsigma} \right\|^2 = a^2 + b^2 \cos^2\phi \cos^2\nu - 2\, a\, b \cos\phi \cos^2\nu \tag{4.21}$$

from (4.18) and the orthogonality of the matrix $\mathbf{D}$. Thus, we see that we may postulate either motion with a constant speed, or with a constant rate along the great circle (i.e., constant $\dot{\varsigma}$), but not both. Although speed is almost always the measured parameter, constant-speed motion would require a complex integration of the function $\| (\partial \mathbf{R}/\partial\varsigma)^{-1} \|$ along the great circle path in order to determine ς as a function of time (the integral of (4.20)). Consequently, it seems advisable to model the motion by

$$\varsigma = s_0 (t - t_0) \tag{4.22}$$

where s_0 is a constant set equal to the true $\dot{\varsigma}$ at time t_0. In general, this is an excellent approximation (it would be exact for a spherical earth), and the error involved is less than that involved in the assumption of pseudo great circle motion. (In most cases, the actual motion is only approximated by a set of constant-course segments. Even for great circle motion on a spherical earth, the course is always changing.) Furthermore, for computation of the doppler, we may still use the exact value of $\dot{\varsigma}$. More precisely, at a constant speed *spd*, (4.20) at time t_0 yields

$$s_0 \triangleq \dot{\varsigma}(t_0) = \frac{\text{spd}}{\| \frac{\partial \mathbf{R}}{\partial \varsigma}(t_0) \|}, \tag{4.23}$$

and at time t, we have

$$\dot{\varsigma}(t) = \frac{\text{spd}}{\| \frac{\partial \mathbf{R}}{\partial \varsigma}(t) \|} = s_0 \frac{\| \frac{\partial \mathbf{R}}{\partial \varsigma}(t_0) \|}{\| \frac{\partial \mathbf{R}}{\partial \varsigma}(t) \|}. \tag{4.24}$$

In summary, we propose two possible models, both of which avoid the integration of (4.20) with respect to time. In the first model, we assume that $\dot{\varsigma} = s_0$, a constant.

(i):

$$\varsigma(t) = s_0(t - t_0) \tag{4.25a}$$

$$\dot{\varsigma}(t) = s_0. \tag{4.25b}$$

This model is physically consistent, i.e., $\dot{\varsigma}$ is the time derivative of ς. However, it is clear from (4.20) that in this case the speed will not be a constant. For the second model, we *define* ς and $\dot{\varsigma}$ by

(ii):

$$\varsigma(t) = s_0(t - t_0) \tag{4.26a}$$

$$\dot{\varsigma}(t) \triangleq s_0 \frac{\| \frac{\partial \mathbf{R}}{\partial \varsigma}(t_0) \|}{\| \frac{\partial \mathbf{R}}{\partial \varsigma}(t) \|}. \tag{4.26b}$$

In this case $\dot{\varsigma}$ is not the time derivative of ς, and $\mathbf{v}$ of (4.19) will not be the velocity corresponding to the motion (4.26a). However, $\mathbf{v}$ will be the velocity at the position $\varsigma(t)$ given that the speed is $s_0 \| \frac{\partial \mathbf{R}}{\partial \varsigma}(t_0) \|$. Thus, if the true motion is constant speed, model (ii) will give a (slightly) incorrect position but a correct velocity $\mathbf{v}$ *for that position.*

DERIVATIVES OF STATION MOTION

From (4.19) we have, for an arbitrary parameter w,

$$\frac{\partial v}{\partial w} = \dot{\varsigma}\ \frac{\partial}{\partial w}\ \frac{\partial R}{\partial \varsigma} + \frac{\partial \dot{\varsigma}}{\partial w}\ \frac{\partial R}{\partial \varsigma}\ . \tag{4.27}$$

If $w = s_0$, then in both models (i) and (ii) equation (4.27) becomes

$$\frac{\partial v}{\partial s_0} = \dot{\varsigma}\ \frac{\partial}{\partial s_0}\ \frac{\partial R}{\partial \varsigma} + \frac{\dot{\varsigma}}{s_0}\ \frac{\partial R}{\partial \varsigma}$$

$$= \dot{\varsigma}\,(t - t_0)\frac{\partial^2 R}{\partial \varsigma} + \frac{\dot{\varsigma}}{s_0}\ \frac{\partial R}{\partial \varsigma}\ . \tag{4.28}$$

For model (i) these simplify to

(i): $$\frac{\partial v}{\partial w} = s_0\ \frac{\partial}{\partial w}\frac{\partial R}{\partial \varsigma} \qquad \text{for } w \neq s_0 \tag{4.29a}$$

(i): $$\frac{\partial v}{\partial s_0} = s_0(t - t_0)\ \frac{\partial^2 R}{\partial \varsigma} + \frac{\partial R}{\partial \varsigma}. \tag{4.29b}$$

On the other hand, model (ii) requires considerably more computation. Since, from (4.24),

$$\frac{\partial \dot{\varsigma}}{\partial w} = - \frac{\dot{\varsigma}}{\left\| \frac{\partial R}{\partial \varsigma} \right\|}\ \frac{\partial}{\partial w} \left\| \frac{\partial R}{\partial \varsigma} \right\|, \tag{4.30}$$

we have for model (ii)

(ii): $$\frac{\partial v}{\partial w} = \dot{\varsigma}\ \left[\frac{\partial}{\partial w}\frac{\partial R}{\partial \varsigma}\ -\ \frac{\partial R}{\partial \varsigma}\ \left\| \frac{\partial R}{\partial \varsigma} \right\|^{-1} \frac{\partial}{\partial w} \left\| \frac{\partial R}{\partial \varsigma} \right\| \right] \tag{4.31a}$$

(ii): $$\frac{\partial v}{\partial s_0} = \dot{\varsigma}\ \left((t - t_0)\frac{\partial^2 R}{\partial \varsigma^2} + \frac{1}{s_0}\ \frac{\partial R}{\partial \varsigma} \right). \tag{4.31b}$$

To a very high approximation, we could drop the second term in (4.31a); however, if we use the partials in a steepest descent, then convergence to the true minimum of a cost function can only be guaranteed if the second term is retained.

Let us derive a slightly more explicit form for $\frac{\partial}{\partial w}(\partial \mathbf{R}/\partial \varsigma)$. From (4.18),

$$\frac{\partial}{\partial w}\frac{\partial \mathbf{R}}{\partial \varsigma} = \frac{\partial}{\partial w}\left(\mathbf{D}^{-1}\begin{bmatrix}(-a + b\cos\phi)\cos\nu \\ a\sin\nu \\ 0\end{bmatrix}\right). \tag{4.32}$$

Definitions (4.12) and (4.2) give us

$$-a + b\cos\phi = -q - H + \frac{e^2}{a_e^2}q^3\cos^2\phi$$

$$= -H + q\left(-1 + \frac{e^2}{1 - e^2\sin^2\phi}\cos^2\phi\right)$$

$$= -H - \frac{(1-e^2)}{a_e^2}q^3. \tag{4.33}$$

Then, indicating $\partial/\partial w$ by subscripting w, we have

$$\frac{\partial}{\partial w}\frac{\partial \mathbf{R}}{\partial \varsigma} = (\mathbf{D}^{-1})_w\begin{bmatrix}(-a + b\cos\phi)\cos\nu \\ a\sin\nu \\ 0\end{bmatrix}$$

$$+ \mathbf{D}^{-1}\begin{bmatrix}(-a + b\cos\phi)_w\cos\nu + (-a + b\cos\phi)(\cos\nu)_w \\ a_w\sin\nu + a(\sin\nu)_w \\ 0\end{bmatrix}, \tag{4.34a}$$

where, from (4.12a) and (4.33),

$$a_w = q_s\frac{\partial\sin\phi}{\partial w} \tag{4.34b}$$

$$(-a + b\cos\phi)_w = -\frac{3(1-e^2)}{a_e^2}q^2 q_s\frac{\partial\sin\phi}{\partial w}. \tag{4.34c}$$

SUITABLE STATE VARIABLES FOR STEEPEST DESCENT

In many cases, the gradients computed in the previous section will be used in a steepest descent or sensitivity calculation for which the state variables include course and speed. It can be be seen from (4.19), (4.22) and the fact that $\sin\varsigma$ appears as a factor in $\mathrm{d}\mathbf{R}/\mathrm{d}\nu_0$ (substitute successively (4.57), (4.50), (4.46), and (4.5) into (4.4)), that at $s_0 = 0$, the partials of $\mathbf{R}$ and $\mathbf{v}$ with respect to ν_0 are zero, and the resulting "A" matrix (cf., (2.10)) will be singular. This is essentially an artifact of the "polar" (i.e., course, speed) coordinate system and may be alleviated by changing state variables.

A natural choice for new state variables is the rectangular velocity coordinates defined by

$$\begin{aligned} \beta_{n0} &\triangleq s_0 \cos\nu_0 \\ \beta_{e0} &\triangleq s_0 \sin\nu_0 \ , \end{aligned} \tag{4.35}$$

where the subscripts n and e are intended to indicate velocity north and east respectively. We can transform to this system using the Jacobian

$$\frac{\partial\beta_{n0}\ \partial\beta_{e0}}{\partial s_0\ \partial\nu_0} = \begin{bmatrix} \cos\nu_0 & \sin\nu_0 \\ -\,s_0 \sin\nu_0 & s_0 \cos\nu_0 \end{bmatrix}, \tag{4.36}$$

or, more practically, its inverse

$$\frac{\partial s_0\ \partial\nu_0}{\partial\beta_{n0}\ \partial\beta_{e0}} = \begin{bmatrix} \cos\nu_0 & -\dfrac{\sin\nu_0}{s_0} \\ \sin\nu_0 & \dfrac{\cos\nu_0}{s_0} \end{bmatrix}. \tag{4.37}$$

For example, from the chain rule, (4.29) and (4.37) yield

$$\text{(i):} \quad \begin{bmatrix} \dfrac{\partial\mathbf{v}}{\partial\beta_{n0}} \\ \dfrac{\partial\mathbf{v}}{\partial\beta_{e0}} \end{bmatrix} = \begin{bmatrix} \cos\nu_0 & -\dfrac{\sin\nu_0}{s_0} \\ \sin\nu_0 & \dfrac{\cos\nu_0}{s_0} \end{bmatrix} \begin{bmatrix} s_0(t - t_0)\dfrac{\partial^2\mathbf{R}}{\partial\varsigma} + \dfrac{\partial\mathbf{R}}{\partial\varsigma} \\ s_0\,\dfrac{\partial}{\partial\nu_0}\,\dfrac{\partial\mathbf{R}}{\partial\varsigma} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\nu_0 & -\sin\nu_0 \\ \sin\nu_0 & \cos\nu_0 \end{bmatrix} \begin{bmatrix} s_0(t - t_0)\dfrac{\partial^2\mathbf{R}}{\partial\varsigma} + \dfrac{\partial\mathbf{R}}{\partial\varsigma} \\ \dfrac{\partial}{\partial\nu_0}\,\dfrac{\partial\mathbf{R}}{\partial\varsigma} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\nu_0 & -\sin\nu_0 \\ \sin\nu_0 & \cos\nu_0 \end{bmatrix} \begin{bmatrix} \varsigma\dfrac{\partial^2\mathbf{R}}{\partial\varsigma} + \dfrac{\partial\mathbf{R}}{\partial\varsigma} \\ \dfrac{\partial}{\partial\nu_0}\,\dfrac{\partial\mathbf{R}}{\partial\varsigma} \end{bmatrix}. \tag{4.38}$$

The same coordinate changes circumvent the vanishing of $\partial \mathbf{R}/\partial \nu_0$ at $s_0 = 0$. To see this, we first note that the partials of $\mathbf{R}$ with respect to β_{no} and β_{eo} take the form of (4.38) with the vector on the right-hand side having components $\partial \mathbf{R}/\partial s_0 = (t - t_0)\, d\mathbf{R}/d\varsigma$ and $(1/s_0)\partial \mathbf{R}/\partial \nu_0$, i.e.,

$$\text{(i):} \quad \begin{bmatrix} \dfrac{\partial \mathbf{R}}{\partial \beta_{n0}} \\ \dfrac{\partial \mathbf{R}}{\partial \beta_{e0}} \end{bmatrix} = \begin{bmatrix} \cos\nu_0 & -\sin\nu_0 \\ \sin\nu_0 & \cos\nu_0 \end{bmatrix} \begin{bmatrix} (t - t_0)\,\dfrac{\partial \mathbf{R}}{\partial \varsigma} \\ \dfrac{1}{s_0}\,\dfrac{\partial \mathbf{R}}{\partial \nu_0} \end{bmatrix} . \tag{4.39}$$

From (4.4) and (4.5), we observe that every term of $\partial \mathbf{R}/\partial \nu_0$ contains one of the four expressions $\partial \sin\phi/\partial \nu_0$, $\partial \cos\phi/\partial \nu_0$, $\partial \sin\theta/\partial \nu_0$, or $\partial \cos\theta/\partial \nu_0$. From (4.46), (4.50), and (4.57), we also see that each of these contains a factor of $\sin\varsigma$. Thus, $(1/s_0)\, \partial \mathbf{R}/\partial \nu_0$ contains a factor of $\sin\varsigma/s_0$ which has the finite (and generally nonzero) limit of $(t - t_0)$ as s_0 goes to zero.

Another aspect which should be mentioned is the potential for negative ς. It is entirely possible, during a steepest descent procedure, to drive the state variable s_0 negative. This poses no problems provided we interpret it as motion in the opposite direction to the indicated course. More precisely, suppose we are faced with a negative $\dot{\varsigma}$, and, hence, negative ς. We may define primed variables by $\dot{\varsigma}' = -\dot{\varsigma}$; $\varsigma' = -\varsigma$; $\nu'_0 = \nu_0 + \pi$; and $\nu' = \nu + \pi$. Equations (4.40) to (4.43) of the next subsection remain invariant under this change of variables. Since ς' is positive, we may interpret the negative ς to be the same motion as $-\varsigma$ with the course reversed, i.e., with the course replaced by (course $+\ \pi$). More simply stated, we may work with equations (4.40) to (4.58) with impunity even if ς is negative. Also, if a particular set of computations results in negative ς, then the course ν_0 should be interpreted as the direction opposite to the physical course.

SUMMARY OF SPHERICAL TRIANGLE EQUATIONS

For convenience, we assemble here the equations from Appendix A, which are needed to compute $\sin\phi$, $\cos\phi$, $\sin\nu$, $\cos\nu$, $\sin\theta$, $\cos\theta$, and their derivatives. Note that when unsubscripted variables appear on the right-hand side, they must be computed via previous equations.

$$\sin\phi = \sin\phi_0 \cos\varsigma + \cos\phi_0 \sin\varsigma \cos\nu_0 \tag{4.40a}$$

$$\cos\phi = \sqrt{1 - \sin^2\phi} \qquad \phi \in (-\pi/2, \pi/2) \tag{4.40b}$$

$$\cos\Delta = \frac{\cos\varsigma - \sin\phi_0 \sin\phi}{\cos\phi_0 \cos\phi} \tag{4.41a}$$

$$\sin\Delta = \frac{\sin\varsigma \sin\nu_0}{\cos\phi} \tag{4.41b}$$

$$\cos\nu = \cos\nu_0 \cos\Delta - \sin\nu_0 \sin\Delta \sin\phi_0 \tag{4.42a}$$

$$\sin\nu = \cos\phi_0 \frac{\sin\nu_0}{\cos\phi} \tag{4.42b}$$

$$\sin\theta = \cos(\lambda_0 + \theta_g)\sin\Delta + \sin(\lambda_0 + \theta_g)\cos\Delta \tag{4.43a}$$

$$\cos\theta = \cos(\lambda_0 + \theta_g)\cos\Delta - \sin(\lambda_0 + \theta_g)\sin\Delta \tag{4.43b}$$

$$\frac{\partial \sin\phi}{\partial \phi_0} = \cos\varsigma\cos\phi_0 - \sin\varsigma\cos\nu_0\sin\phi_0 \tag{4.44a}$$

$$\frac{\partial \cos\phi}{\partial \phi_0} = -\frac{\sin\phi}{\cos\phi}\frac{\partial \sin\phi}{\partial \phi_0} \tag{4.44b}$$

$$\frac{\partial \sin\phi}{\partial \lambda_0} = \frac{\partial \cos\phi}{\partial \lambda_0} = 0 \tag{4.45}$$

$$\frac{\partial \sin\phi}{\partial \nu_0} = -\cos\phi_0\sin\varsigma\sin\nu_0 \tag{4.46a}$$

$$\frac{\partial \cos\phi}{\partial \nu_0} = -\frac{\sin\phi}{\cos\phi}\frac{\partial \sin\phi}{\partial \nu_0} \tag{4.46b}$$

$$\frac{\partial \sin\phi}{\partial \varsigma} = \cos\phi\cos\nu \tag{4.47a}$$

$$\frac{\partial \cos\phi}{\partial \varsigma} = -\sin\phi\cos\nu \tag{4.47b}$$

$$\frac{\partial \sin\Delta}{\partial \phi_0} = -\frac{\sin\Delta}{\cos\phi}\frac{\partial \cos\phi}{\partial \phi_0} \tag{4.48a}$$

$$\frac{\partial \cos\Delta}{\partial \phi_0} = -\frac{\sin\phi}{\cos\phi} + \frac{\cos\Delta}{\cos\phi_0}\sin\phi_0 - \frac{\sin\phi_0}{\cos\phi_0\cos\phi}\frac{\partial \sin\phi}{\partial \phi_0} - \frac{\cos\Delta}{\cos\phi}\frac{\partial \cos\phi}{\partial \phi_0} \tag{4.48b}$$

$$\frac{\partial \sin\Delta}{\partial \lambda_0} = \frac{\partial \cos\Delta}{\partial \lambda_0} = 0 \tag{4.49}$$

$$\frac{\partial \sin\Delta}{\partial \nu_0} = \frac{\sin\varsigma\cos\nu_0}{\cos\phi} - \frac{\sin\Delta}{\cos\phi}\frac{\partial \cos\phi}{\partial \nu_0} \tag{4.50a}$$

$$\frac{\partial \cos\Delta}{\partial \nu_0} = -\frac{\sin\phi_0}{\cos\phi_0\cos\phi}\frac{\partial \sin\phi}{\partial \nu_0} - \frac{\cos\Delta}{\cos\phi}\frac{\partial \cos\phi}{\partial \nu_0} \tag{4.50b}$$

$$\frac{\partial \sin\nu}{\partial \phi_0} = -\frac{\sin\phi_0\sin\nu_0}{\cos\phi} - \frac{\sin\nu}{\cos\phi}\frac{\partial \cos\phi}{\partial \phi_0} \tag{4.51a}$$

$$\frac{\partial \cos\nu}{\partial \phi_0} = \cos\nu_0\frac{\partial \cos\Delta}{\partial \phi_0} - \sin\nu_0\frac{\partial \sin\Delta}{\partial \phi_0}\sin\phi_0 - \sin\nu_0\sin\Delta\cos\phi_0 \tag{4.51b}$$

$$\frac{\partial \sin\nu}{\partial \lambda_0} = \frac{\partial \cos\nu}{\partial \lambda_0} = 0 \tag{4.52}$$

$$\frac{\partial \sin\nu}{\partial \nu_0} = \cos\phi_0\frac{\cos\nu_0}{\cos\phi} - \frac{\sin\nu}{\cos\phi}\frac{\partial \cos\phi}{\partial \nu_0} \tag{4.53a}$$

$$\frac{\partial \cos\nu}{\partial \nu_0} = -\sin\nu_0 \cos\Delta + \cos\nu_0 \frac{\partial \cos\Delta}{\partial \nu_0} - \cos\nu_0 \sin\Delta \sin\phi_0 - \sin\nu_0 \sin\phi_0 \frac{\partial \sin\Delta}{\partial \nu_0} \tag{4.53b}$$

$$\frac{\partial \sin\nu}{\partial \varsigma} = \sin\nu \cos\nu \frac{\sin\phi}{\cos\phi} \tag{4.54a}$$

$$\frac{\partial \cos\nu}{\partial \varsigma} = - \sin^2\nu \frac{\sin\phi}{\cos\phi} \tag{4.54b}$$

$$\frac{\partial \sin\theta}{\partial \phi_0} = \cos(\lambda_0 + \theta_g) \frac{\partial \sin\Delta}{\partial \phi_0} + \sin(\lambda_0 + \theta_g) \frac{\partial \cos\Delta}{\partial \phi_0} \tag{4.55a}$$

$$\frac{\partial \cos\theta}{\partial \phi_0} = \cos(\lambda_0 + \theta_g) \frac{\partial \cos\Delta}{\partial \phi_0} - \sin(\lambda_0 + \theta_g) \frac{\partial \sin\Delta}{\partial \phi_0} \tag{4.55b}$$

$$\frac{\partial \sin\theta}{\partial \lambda_0} = -\sin(\lambda_0 + \theta_g) \sin\Delta + \cos(\lambda_0 + \theta_g) \cos\Delta \tag{4.56a}$$

$$\frac{\partial \cos\theta}{\partial \lambda_0} = -\sin(\lambda_0 + \theta_g) \cos\Delta - \cos(\lambda_0 + \theta_g) \sin\Delta \tag{4.56b}$$

$$\frac{\partial \sin\theta}{\partial \nu_0} = \cos(\lambda_0 + \theta_g) \frac{\partial \sin\Delta}{\partial \nu_0} + \sin(\lambda_0 + \theta_g) \frac{\partial \cos\Delta}{\partial \nu_0} \tag{4.57a}$$

$$\frac{\partial \cos\theta}{\partial \nu_0} = \cos(\lambda_0 + \theta_g) \frac{\partial \cos\Delta}{\partial \nu_0} - \sin(\lambda_0 + \theta_g) \frac{\partial \sin\Delta}{\partial \nu_0} \tag{4.57b}$$

$$\frac{\partial \sin\theta}{\partial \varsigma} = \cos\theta \frac{\sin\nu}{\cos\phi} \tag{4.58a}$$

$$\frac{\partial \cos\theta}{\partial \varsigma} = - \sin\theta \frac{\sin\nu}{\cos\phi} \tag{4.58b}$$

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Appendix A: GREAT CIRCLE MOTION AND ITS DERIVATIVES

A standard spherical triangle is pictured in Figure A.1. We are considering motion along the great circle arc ς from P_1 to P_2 where the latitude ϕ_1 and course ν_1 of the initial point as well as the distance ς (in radians) are known. (The longitude λ_1 is also known, but not relevant at the moment.) We wish to determine the new latitude ϕ_2, the change in longitude Δ, the new course ν_2, and their partial derivatives. Figure A.1, as well as standard spherical trigonometric formulae, assumes that all angles and sides of the triangle in question lie in the interval $[0, \pi]$. This is true of the colatitudes $\pi/2 - \phi_1$ and $\pi/2 - \phi_2$; however, ν_1, ς, Δ, and ν_2 will generally be in the interval 0 to 2π. To handle this problem, we transform to variables which satisfy the desired conditions and distinguish them by the addition of a prime. Four cases occur, as illustrated in Figure A.2 and summarized in Table A.1.

Table A.1. Transformations to standard spherical triangle.

case	transf of initial values	transf of new values
0: $\nu \leq \pi$, $\varsigma \leq \pi$		$\nu_2 = \pi - \nu'_2$
1: $\nu > \pi$, $\varsigma \leq \pi$	$\nu'_1 = 2\pi - \nu_1$	$\nu_2 = \pi + \nu'_2$ $\Delta = -\Delta'$
2: $\nu \leq \pi$, $\varsigma > \pi$	$\nu'_1 = \pi - \nu_1$ $\varsigma' = 2\pi - \varsigma$	$\Delta = -\Delta'$
3: $\nu > \pi$, $\varsigma > \pi$	$\nu'_1 = \nu_1 - \pi$ $\varsigma' = 2\pi - \varsigma$	$\nu_2 = 2\pi - \nu'_2$

For those transformations omitted, the primed value equals the unprimed value.

We could perform all our derivations assuming case (0) and then justify the results for all cases by an appeal to analyticity of the individual variables (which implies a unique extension). However, with a bit more labor, we achieve the same end directly, using the following sign relationships (easily verified from Table A.1):

$$\cos\Delta = \cos\Delta' \tag{A.1}$$

$$\cos\varsigma = \cos\varsigma' \tag{A.2}$$

$$\frac{\sin\varsigma}{\sin\varsigma'} = \frac{\cos\nu_1}{\cos\nu'_1} = -\frac{\cos\nu_2}{\cos\nu'_2} \tag{A.3}$$

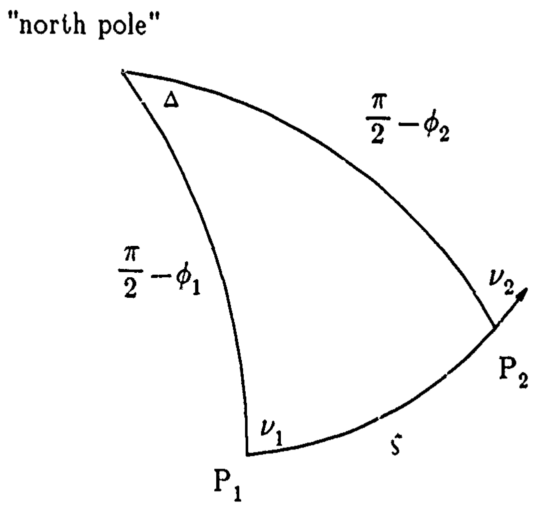

Figure A.1. Spherical triangle on unit sphere.

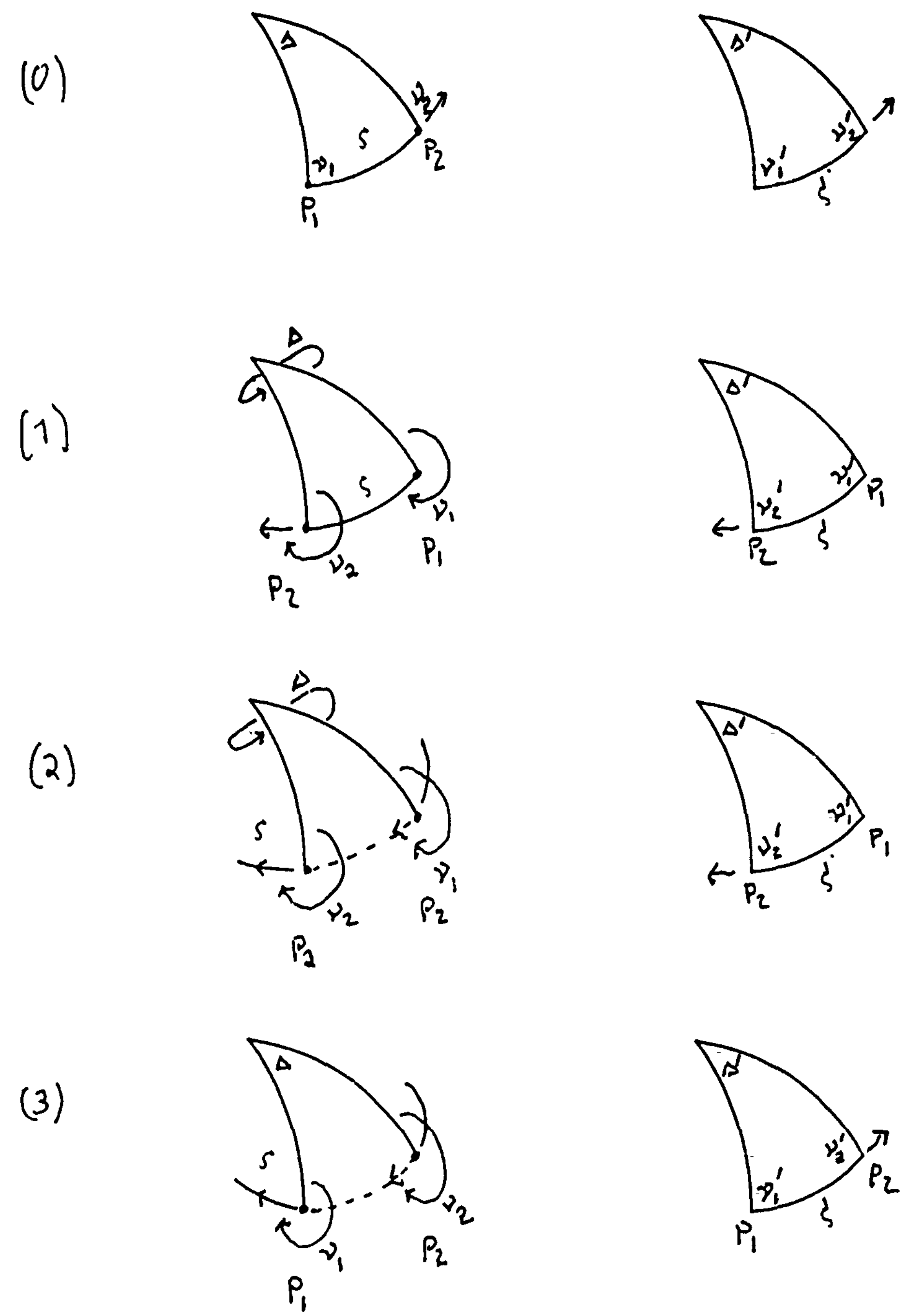

Figure A.2. The various cases for great circle motion from P_1 to P_2 and their relation to standard spherical triangle parameters.

$$\frac{\sin\nu_1}{\sin\nu'_1} = \frac{\sin\nu_2}{\sin\nu'_2} \tag{A.4}$$

$$\frac{\sin\Delta}{\sin\Delta'} = \frac{\sin\varsigma \sin\nu_1}{\sin\varsigma' \sin\nu'_1} \tag{A.5}$$

Equations (A.3) and (A.5) imply

$$\frac{\cos\nu_2}{\cos\nu'_2} = -\frac{\sin\nu_1 \sin\Delta}{\sin\nu'_1 \sin\Delta'} \tag{A.6}$$

Table A.2 contains a summary of the locations of the various equations.

Table A.2. Equation numbers for various computations .

indep variable	sin/cos	derivatives of sin/cos $\partial/\partial\phi_1$	$\partial/\partial\lambda$	$\partial/\partial\nu_1$	$\partial/\partial\varsigma$
ϕ_2	A.7	A.10	A.11	A.13	A.15
Δ	A.8	A.20	A.21	A.22	
ν_2	A.9	A.25	A.26	A.27	A.30
θ_2	A.32	A.33	A.34	A.35	A.38

FUNCTIONS OF P_2

In the equations which follow, we shall assume that no point of the arc P_1P_2 passes through one of the poles. To solve for ϕ_2, we use the law of cosines[†] applied to the sides of the spherical triangles in Figure A.2. Thus,

$$\sin\phi_2 = \sin\phi_1 \cos\varsigma' + \cos\phi_1 \sin\varsigma' \cos\nu'_1$$

$$= \sin\phi_1 \cos\varsigma + \cos\phi_1 \sin\varsigma \cos\nu_1, \tag{A.7a}$$

and

† Our choice of equations in what follows is dictated by a desire to avoid indeterminate forms and simplify sign determination. Thus, only cosines of the latitude appear in denominators.

$$\cos\phi_2 = \sqrt{1 - \sin^2\phi_2} \qquad \phi_2 \in (-\pi/2, \pi/2). \tag{A.7b}$$

Note that we have used (A.2) and (A.3) in the derivation of (A.7a).

Similarly for Δ',

$$\cos\Delta' = \frac{\cos\varsigma' - \sin\phi_1 \sin\phi_2}{\cos\phi_1 \cos\phi_2}$$

which, by (A.1) and (A.2), implies

$$\cos\Delta = \frac{\cos\varsigma - \sin\phi_1 \sin\phi_2}{\cos\phi_1 \cos\phi_2}. \tag{A.8a}$$

From the law of sines,

$$\sin\Delta' = \frac{\sin\varsigma' \sin\nu'_1}{\cos\phi_2}$$

so that (c.f., (A.5))

$$\sin\Delta = \frac{\sin\varsigma \sin\nu_1}{\cos\phi_2}. \tag{A.8b}$$

The law of cosines for angles yields

$$\cos\nu'_2 = -\cos\nu'_1 \cos\Delta' + \sin\nu'_1 \sin\Delta' \sin\phi_1;$$

that is, by (A.1), (A.3), and (A.6),

$$\cos\nu_2 = \cos\nu_1 \cos\Delta - \sin\nu_1 \sin\Delta \sin\phi_1. \tag{A.9a}$$

Again, from the law of sines,

$$\sin\nu'_2 = \cos\phi_1 \frac{\sin\nu'_1}{\cos\phi_2},$$

or, using (A.4),

$$\sin\nu_2 = \cos\phi_1 \frac{\sin\nu_1}{\cos\phi_2}, \tag{A.9b}$$

Note that in this section, as in those which follow, some expressions may contain quantities which must be computed from previous equations. For example, (A.9b) depends on $\cos\phi_2$, which may be computed using (A.7).

DERIVATIVES OF $\sin\phi_2$ and $\cos\phi_2$

From equations (A.7a),

$$\frac{\partial \sin\phi_2}{\partial \phi_1} = \cos\varsigma \cos\phi_1 - \sin\varsigma \cos\nu_1 \sin\phi_1. \tag{A.10a}$$

Also,

$$\frac{\partial \cos\phi_2}{\partial \phi_1} = -\frac{\sin\phi_2}{\cos\phi_2} \frac{\partial \sin\phi_2}{\partial \phi_1} \tag{A.10b}$$

$$\frac{\partial \sin\phi_2}{\partial \lambda_1} = \frac{\partial \cos\phi_2}{\partial \lambda_1} = 0 \ , \tag{A.11}$$

where we have used

$$\frac{\partial \cos\phi_2}{\partial w} = -\frac{\sin\phi_2}{\cos\phi_2} \frac{\partial \sin\phi_2}{\partial w} \ , \tag{A.12}$$

which follows from (A.7b) for arbitrary w.

The partials with respect to ν_1 are, from (A.7a) and (A.12),

$$\frac{\partial \sin\phi_2}{\partial \nu_1} = -\cos\phi_1 \sin\varsigma \sin\nu_1 \tag{A.13a}$$

$$\frac{\partial \cos\phi_2}{\partial \nu_1} = -\frac{\sin\phi_2}{\cos\phi_2} \frac{\partial \sin\phi_2}{\partial \nu_1}. \tag{A.13b}$$

Before computing the derivatives with respect to ς, we note, from Figure A.1, that taking the derivative with respect to ς at P_2 is the same as taking the derivative at $P_2 = P_1$ by setting ς equal to zero, and then replacing the subscript 2 with the subscript 1. That is, the rate of change of $\sin\phi$ or $\cos\phi$ with respect to ς at an arbitrary point depends only on that point. As we shall see, this provides us with a quick method for simplifying the resulting expressions. Following the indicated procedure, we have from (A.7a)

$$\left.\frac{\partial \sin\phi_2}{\partial \varsigma}\right|_{\varsigma=0} = \left(-\sin\phi_1 \sin\varsigma + \cos\phi_1 \cos\varsigma \cos\nu_1\right)\Big|_{\varsigma=0}$$

$$= \cos\phi_1 \cos\nu_1. \tag{A.14}$$

Then, since (A.14) is only a function of P_1 and since P_1 in this context is arbitrary, we may replace it by P_2 to get

$$\frac{\partial \sin\phi_2}{\partial \varsigma} = \cos\phi_2 \cos\nu_2. \tag{A.15a}$$

Using (A.12), we find

$$\frac{\partial \cos\phi_2}{\partial \varsigma} = -\sin\phi_2 \cos\nu_2 \ . \tag{A.15b}$$

For completeness, let us show that this is equivalent to just taking the derivative of (A.7) with respect to ς. We have from (A.8a)

$$\cos\phi_2 = \frac{\cos\varsigma - \sin\phi_1 \sin\phi_2}{\cos\Delta \cos\phi_1}, \tag{A.16}$$

so that

$$\cos\nu_1 \cos\Delta \cos\phi_2 = \cos\nu_1 \frac{\cos\varsigma - \sin\phi_1 \sin\phi_2}{\cos\phi_1}$$

$$= \frac{\cos\nu_1}{\cos\phi_1} (\cos\varsigma - \sin^2\phi_1 \cos\varsigma - \sin\phi_1 \cos\phi_1 \sin\varsigma \cos\nu_1)$$

$$= \frac{\cos\nu_1 \cos\varsigma}{\cos\phi_1} \cos^2\phi_1 - \cos^2\nu_1 \sin\phi_1 \sin\varsigma \ , \tag{A.17}$$

where the second last line follows from substitution of (A.7a). Also, from the law of sines for Figure A.1 (equation (A.8b)),

$$\cos\phi_2 \sin\Delta \sin\nu_1 \sin\phi_1 = \sin\varsigma \sin^2\nu_1 \sin\phi_1. \tag{A.18}$$

Multiplying (A.9a) by $\cos\phi_2$ and substituting (A.17) and (A.18) in the resulting expression, we get

$$\cos\phi_2 \cos\nu_2 = \cos\nu_1 \cos\phi_1 \cos\varsigma - \sin\phi_1 \sin\varsigma \tag{A.19}$$

Then (A.15a) equals the left-hand side of (A.19) while the right-hand side is, indeed, the derivative of (A.7a) with respect to ς.

DERIVATIVES OF $\sin\Delta$ **AND** $\cos\Delta$

Differentiating (A.8b), we have

$$\frac{\partial \sin\Delta}{\partial \phi_1} = -\frac{\sin\Delta}{\cos\phi_2}\,\frac{\partial \cos\phi_2}{\partial \phi_1}. \tag{A.20a}$$

From (A.8a)

$$\frac{\partial \cos\Delta}{\partial \phi_1} = -\frac{\sin\phi_2}{\cos\phi_2} + \frac{\cos\Delta}{\cos\phi_1}\sin\phi_1 - \frac{\sin\phi_1}{\cos\phi_1\cos\phi_2}\,\frac{\partial \sin\phi_2}{\partial \phi_1} - \frac{\cos\Delta}{\cos\phi_2}\,\frac{\partial \cos\phi_2}{\partial \phi_1}. \tag{A.20b}$$

Finally,

$$\frac{\partial \sin\Delta}{\partial \lambda_1} = \frac{\partial \cos\Delta}{\partial \lambda_1} = 0\ . \tag{A.21}$$

The partials with respect to ν_1 are also found by differentiating equations (A.8). From (A.8b)

$$\frac{\partial \sin\Delta}{\partial \nu_1} = \frac{\sin\varsigma\cos\nu_1}{\cos\phi_2} - \frac{\sin\Delta}{\cos\phi_2}\,\frac{\partial \cos\phi_2}{\partial \nu_1}. \tag{A.22a}$$

Similarly, from (A.8a),

$$\frac{\partial \cos\Delta}{\partial \nu_1} = -\frac{\sin\phi_1}{\cos\phi_1\cos\phi_2}\,\frac{\partial \sin\phi_2}{\partial \nu_1} - \frac{\cos\Delta}{\cos\phi_2}\,\frac{\partial \cos\phi_2}{\partial \nu_1}; \tag{A.22b}$$

For ς we find that we shall only need the derivatives at $\varsigma = 0$, so we proceed as in the derivation of (A.14). Equations (A.8a) and (A.15) imply

$$\left.\frac{\partial \cos\Delta}{\partial \varsigma}\right|_0 = \left.\left(-\frac{\sin\varsigma}{\cos\phi_1\cos\phi_2} - \frac{\cos\Delta}{\cos\phi_2}\,\frac{\partial \cos\phi_2}{\partial \varsigma} - \frac{\sin\phi_1}{\cos\phi_1\cos\phi_2}\,\frac{\partial \sin\phi_2}{\partial \varsigma}\right)\right|_0$$

$$= 0 - \frac{1}{\cos\phi_1}(-\sin\phi_1\cos\nu_1) - \frac{\sin\phi_1}{\cos^2\phi_1}(\cos\phi_1\cos\nu_1)$$

$$= 0. \tag{A.23}$$

From (A.8b)

$$\left.\frac{\partial \sin\Delta}{\partial \varsigma}\right|_0 = \left.\left(\frac{\cos\varsigma \sin\nu_1}{\cos\phi_2} - \frac{\sin\Delta}{\cos\phi_2}\,\frac{\partial \cos\phi_2}{\partial\,\varsigma}\right)\right|_0$$

$$= \frac{\sin\nu_1}{\cos\phi_1} \tag{A.24}$$

DERIVATIVES OF $\sin\nu_2$ AND $\cos\nu_2$

From (A.9b)

$$\frac{\partial \sin\nu_2}{\partial \phi_1} = -\frac{\sin\phi_1 \sin\nu_1}{\cos\phi_2} - \frac{\sin\nu_2}{\cos\phi_2}\,\frac{\partial \cos\phi_2}{\partial \phi_1}, \tag{A.25a}$$

and from (A.9a)

$$\frac{\partial \cos\nu_2}{\partial \phi_1} = \cos\nu_1\,\frac{\partial \cos\Delta}{\partial \phi_1} - \sin\nu_1\,\frac{\partial \sin\Delta}{\partial \phi_1}\,\sin\phi_1 - \sin\nu_1 \sin\Delta \cos\phi_1 \ . \tag{A.25b}$$

Of course,

$$\frac{\partial \sin\nu_2}{\partial \lambda_1} = \frac{\partial \cos\nu_2}{\partial \lambda_1} = 0 \ . \tag{A.26}$$

Next, taking derivatives of (A.9) with respect to ν_1, we get

$$\frac{\partial \sin\nu_2}{\partial \nu_1} = \cos\phi_1\,\frac{\cos\nu_1}{\cos\phi_2} - \frac{\sin\nu_2}{\cos\phi_2}\,\frac{\partial \cos\phi_2}{\partial \nu_1}, \tag{A.27a}$$

and

$$\frac{\partial \cos\nu_2}{\partial \nu_1} = -\sin\nu_1 \cos\Delta + \cos\nu_1\,\frac{\partial \cos\Delta}{\partial \nu_1} - \cos\nu_1 \sin\Delta \sin\phi_1$$

$$- \sin\nu_1 \sin\phi_1\,\frac{\partial \sin\Delta}{\partial \nu_1} \tag{A.27b}$$

We employ the usual trick to obtain the partials with respect to ς. From (A.9b) and (A.15b):

$$\left.\frac{\partial \sin\nu_2}{\partial\varsigma}\right|_0 = -\left.\frac{\sin\nu_2}{\cos\phi_2}\,\frac{\partial\cos\phi_2}{\partial\varsigma}\right|_0$$

$$= \frac{\sin\nu_1}{\cos\phi_1}\sin\phi_1\cos\nu_1. \tag{A.28}$$

Similarly, from (A.9a) and (A.24),

$$\left.\frac{\partial\cos\nu_2}{\partial\varsigma}\right|_0 = \left.\left(\cos\nu_1\,\frac{\partial\cos\Delta}{\partial\varsigma} - \sin\nu_1\sin\phi_1\,\frac{\partial\sin\Delta}{\partial\varsigma}\right)\right|_0$$

$$= -\sin\nu_1\sin\phi_1\,\frac{\sin\nu_1}{\cos\phi_1} \tag{A.29}$$

Thus,

$$\frac{\partial\sin\nu_2}{\partial\varsigma} = \sin\nu_2\cos\nu_2\,\frac{\sin\phi_2}{\cos\phi_2} \tag{A.30a}$$

$$\frac{\partial\cos\nu_2}{\partial\varsigma} = -\sin^2\nu_2\,\frac{\sin\phi_2}{\cos\phi_2} \tag{A.30b}$$

DERIVATIVES OF $\sin\theta_2$ AND $\cos\theta_2$

Instead of functions of the longitude, we are generally concerned with those of θ, the sidereal time,

$$\theta_2 = \theta_{g_2} + \lambda_1 + \Delta\ , \tag{A.31}$$

where $\lambda_1 + \Delta$ is the longitude at P_2, and θ_{g_2} is the Greenwich sidereal time at point P_2, i.e., when the station is at P_2. Equation (A.31) implies

$$\sin\theta_2 = \cos(\lambda_1 + \theta_{g_2})\sin\Delta + \sin(\lambda_1 + \theta_{g_2})\cos\Delta \tag{A.32a}$$

$$\cos\theta_2 = \cos(\lambda_1 + \theta_{g_2}) \cos\Delta - \sin(\lambda_1 + \theta_{g_2}) \sin\Delta \tag{A.32b}$$

Thus,

$$\frac{\partial \sin\theta_2}{\partial \phi_1} = \cos(\lambda_1 + \theta_{g_2}) \frac{\partial \sin\Delta}{\partial \phi_1} + \sin(\lambda_1 + \theta_{g_2}) \frac{\partial \cos\Delta}{\partial \phi_1} \tag{A.33a}$$

$$\frac{\partial \cos\theta_2}{\partial \phi_1} = \cos(\lambda_1 + \theta_{g_2}) \frac{\partial \cos\Delta}{\partial \phi_1} - \sin(\lambda_1 + \theta_{g_2}) \frac{\partial \sin\Delta}{\partial \phi_1} \tag{A.33b}$$

$$\frac{\partial \sin\theta_2}{\partial \lambda_1} = -\sin(\lambda_1 + \theta_{g_2}) \sin\Delta + \cos(\lambda_1 + \theta_{g_2}) \cos\Delta \tag{A.34a}$$

$$\frac{\partial \cos\theta_2}{\partial \lambda_1} = -\sin(\lambda_1 + \theta_{g_2}) \cos\Delta - \cos(\lambda_1 + \theta_{g_2}) \sin\Delta \tag{A.34b}$$

$$\frac{\partial \sin\theta_2}{\partial \nu_1} = \cos(\lambda_1 + \theta_{g_2}) \frac{\partial \sin\Delta}{\partial \nu_1} + \sin(\lambda_1 + \theta_{g_2}) \frac{\partial \cos\Delta}{\partial \nu_1} \tag{A.35a}$$

$$\frac{\partial \cos\theta_2}{\partial \nu_1} = \cos(\lambda_1 + \theta_{g_2}) \frac{\partial \cos\Delta}{\partial \nu_1} - \sin(\lambda_1 + \theta_{g_2}) \frac{\partial \sin\Delta}{\partial \nu_1} \ . \tag{A.35b}$$

From (A.32), (A.23), and (A.24), we have

$$\left.\frac{\partial \sin\theta_2}{\partial \varsigma}\right|_{\Delta=0} = \cos(\theta_2 - \Delta) \left.\frac{\partial \sin\Delta}{\partial \varsigma}\right|_{\Delta=0} + 0$$

$$= \cos\theta_1 \frac{\sin\nu_1}{\cos\phi_1}, \tag{A.36}$$

and similarly

$$\frac{\partial \cos\theta_1}{\partial \varsigma} = -\sin\theta_1 \frac{\sin\nu_1}{\cos\phi_1} \ . \tag{A.37}$$

Again, since this must hold for all P_1 along the path[†], we have

$$\frac{\partial \sin\theta_2}{\partial \varsigma} = \cos\theta_2 \frac{\sin\nu_2}{\cos\phi_2} \tag{A.38a}$$

$$\frac{\partial \cos\theta_2}{\partial \varsigma} = - \sin\theta_2 \frac{\sin\nu_2}{\cos\phi_2}. \tag{A.38b}$$

† Note that the definition of θ as a function of P is independent of the point P_1, which permits replacing the subscript 1 with 2 in (A.36) and (A.37). This was not possible in (A.23) or (A.24) since Δ is defined in terms of P_1. Of course ς in (A.36) also depends on P_1, but if we define ς_b as the arc length relative to another point P_b, then we have $\varsigma_b = \varsigma$ - arclength $P_1P_b = \varsigma$ - constant, so that $\partial/\partial\varsigma = \partial/\partial\varsigma_b$, which is independent of the choice of the point P_b.

Appendix B: KEPLER MODEL AND DERIVATIVES

The Kepler model is specified by six elements and an epoch:

a = orbit semimajor axis
e = orbit eccentricity
M_0 = mean anomaly at epoch
i = inclination
Ω = longitude of ascending node
ω = argument of perigee
t_0 = epoch of elements.

The first three quantities determine the position $\mathbf{r}^{(PQW)}$ and velocity $\dot{\mathbf{r}}^{(PQW)}$ for all time with respect to the orbital plane and relative to perigee, i.e., in the so-called perifocal coordinate system (**P** is the semimajor axis to the periapsis, **Q** rotated 90° in the direction of motion, and **W** determined by the right-hand rule [1]). The remaining three elements determine the orientation of the orbital plane. We then have, with IJK representing geocentric coordinates at epoch t_0,

$$\mathbf{r}^{(IJK)} = \mathbf{U}\,\mathbf{r}^{(PQW)}$$
$$\dot{\mathbf{r}}^{(IJK)} = \mathbf{U}\,\dot{\mathbf{r}}^{(PQW)}, \tag{B.1}$$

where

$\mathbf{U} =$

$$\begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix} \tag{B.2}$$

For notational convenience in the following, vectors with no superscript will be assumed to be in the PQW coordinate system. Those vectors which are written with only two components have their W-component equal to zero; that is, they lie in the orbital plane.

Two derived quantities, used in computing the orbit, are the mean anomaly at time t†

$$M \triangleq M_0 + \sqrt{\mu/a^3}\,(t - t_0) + 2\pi k\,, \tag{B.3}$$

† From a developmental viewpoint, the basic quantity is ν, the angle subtended at the focus of the ellipse. The mean anomaly M, as well as the eccentric anomaly E, are derived from ν.

where $0 \leq M < 2\pi$, and the eccentric anomaly E is determined implicitly by the equation

$$E - e \sin E = M . \tag{B.4}$$

The constant μ is the gravitational parameter (gravitational constant times the mass of the earth, [1]), and the quantity $\sqrt{\mu/a^3}$ is termed the mean motion since it is the rate of change of the mean anomaly M.

The position of the satellite at time t is given by

$$\mathbf{r} = (r \cos \nu, r \sin \nu) \ , \tag{B.5}$$

where r, the magnitude of $\mathbf{r}$, is determined from ([1], equation (4-2-14))

$$r \triangleq \| \mathbf{r} \| = a\,(1 - e \cos E) \, , \tag{B.6a}$$

and ν, the angle subtended by $\mathbf{r}$ with the P-axis at the focus of the ellipse, satisfies ([1], equations (4-2-12) and (4-2-13))

$$\cos \nu = \frac{a}{r}\,(\cos E - e) \tag{B.6b}$$

$$\sin \nu = \frac{a}{r}\,\sqrt{1 - e^2} \sin E \, . \tag{B.6c}$$

Thus,

$$\mathbf{r} = a\,(\cos E - e, \ \sqrt{1 - e^2} \sin E) \tag{B.7}$$

Then,

$$\dot{\mathbf{r}} = (-\dot{E}\, a \sin E, \ \dot{E}\, a \sqrt{1 - e^2} \cos E) \, . \tag{B.8}$$

But, differentiating (B.4) with respect to t, using $\dot{M} = \sqrt{\mu/a^3}$ from (B.3), and relationship (B.6a), we have

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}} \ , \tag{B.9}$$

so that

$$\dot{\mathbf{r}} = \frac{\sqrt{\mu a}}{r}(-\sin E,\ \sqrt{1-e^2}\cos E) \tag{B.10}$$

or

$$\dot{\mathbf{r}} = \sqrt{\frac{\mu}{a}}\left(\frac{-\sin E}{1-e\cos E},\ \sqrt{1-e^2}\,\frac{\cos E}{1-e\cos E}\right) . \tag{B.11}$$

Next we compute the partial derivatives of $\mathbf{r}$ and $\dot{\mathbf{r}}$ with respect to the elements. Equation (B.4) implies

$$dE\,(1-e\cos E) - \sin E\, de = dM\,, \tag{B.12}$$

i.e.,

$$\frac{\partial E}{\partial M} = \frac{1}{1-e\cos E} = \frac{a}{r} \tag{B.13a}$$

$$\frac{\partial E}{\partial e} = \frac{\sin E}{1-e\cos E} = \frac{a\sin E}{r} . \tag{B.13b}$$

For convenience, let

$$m \triangleq \sqrt{\frac{\mu}{a^3}}\,(t-t_0) \tag{B.14}$$

be the elapsed mean motion. Considering M a function of a and t, we have

$$\frac{\partial M}{\partial M_0} = 1\,, \tag{B.15}$$

$$\frac{\partial M}{\partial a} = -\frac{3}{2}\frac{1}{a}\sqrt{\frac{\mu}{a^3}}\,(t-t_0) = -\frac{3}{2}\frac{m}{a}\,, \tag{B.16}$$

and

$$\frac{\partial M}{\partial t} = \sqrt{\frac{\mu}{a^3}} . \tag{B.17}$$

From (B.13a)

$$\frac{\partial E}{\partial a} = \frac{\partial E}{\partial M}\frac{\partial M}{\partial a} = \frac{a}{r}\left(-\frac{3}{2}\frac{m}{a}\right) = -\frac{3}{2}\frac{m}{r} \tag{B.18}$$

We may now compute the partials of $\mathbf{r}$ with respect to the elements a, e, M_0.

$$\frac{\partial \mathbf{r}}{\partial a} = (\cos E - e,\ \sqrt{1-e^2}\sin E) + a\,(-\sin E,\ \sqrt{1-e^2}\cos E)\ \frac{\partial E}{\partial a}$$

$$= (\cos E - e,\ \sqrt{1-e^2}\sin E) - \frac{3}{2}m\left(\frac{-\sin E}{1 - e\cos E},\ \sqrt{1-e^2}\ \frac{\cos E}{1 - e\cos E}\right)$$

$$= \left(\cos E + \frac{3}{2}m\frac{\sin E}{1 - e\cos E},\ \sqrt{1-e^2}\sin E - \frac{3}{2}m\sqrt{1-e^2}\ \frac{\cos E}{1 - e\cos E}\right). \quad \text{(B.19)}$$

$$\frac{\partial \mathbf{r}}{\partial e} = a\left(-1,\ \frac{-e\sin E}{\sqrt{1-e^2}}\right) + a\,(-\sin E,\ \sqrt{1-e^2}\cos E)\frac{\partial E}{\partial e}$$

$$= a\left(-1 - \sin E\ \frac{\sin E}{1 - e\cos E},\ -\frac{e\sin E}{\sqrt{1-e^2}} + \sqrt{1-e^2}\sin E\ \frac{\cos E}{1 - e\cos E}\right). \quad \text{(B.20)}$$

$$\frac{\partial \mathbf{r}}{\partial M_0} = a\,(-\sin E,\ \sqrt{1-e^2}\cos E)\frac{\partial E}{\partial M_0}$$

$$= a\left(\frac{-\sin E}{1 - e\cos E},\ \sqrt{1-e^2}\ \frac{\cos E}{1 - e\cos E}\right), \quad \text{(B.21)}$$

since $\partial M/\partial M_0 = 1$.

Next, we compute the derivatives of the velocity $\dot{\mathbf{r}}$. First, from (B.10),

$$\frac{\partial \dot{\mathbf{r}}}{\partial E} = -\frac{\dot{\mathbf{r}}}{r}\frac{\partial r}{\partial E} + \frac{\sqrt{\mu a}}{r}(-\cos E,\ -\sqrt{1-e^2}\sin E) \quad \text{(B.22)}$$

or, by (B.6a),

$$\frac{\partial \dot{\mathbf{r}}}{\partial E} = -\dot{\mathbf{r}}\ e\ \frac{\sin E}{1 - e\cos E} - \sqrt{\frac{\mu}{a}}\left(\frac{\cos E}{1 - e\cos E},\ \sqrt{1-e^2}\ \frac{\sin E}{1 - e\cos E}\right). \quad \text{(B.23)}$$

Note that equations (B.22) and (B.23) were computed by considering $\dot{\mathbf{r}}$ as a function of a, e and E. That shall be taken to be the meaning of $\partial\dot{\mathbf{r}}/\partial E$, i.e., that quantity is given by (B.22) or (B.23). However, we wish to consider a, e, M_0 as the independent variables. Thus, below, considering E as a function of a and M_0, we make use of $\partial E/\partial a$ as given by (B.18). We must be careful to be consistent, however, since $\dot{\mathbf{r}}$ is not expressed uniquely in terms of a, e, E, and M_0. The route used for the chain rule will be equation (B.10) with r from equation (B.6a) followed ' · substitution of E as determined from (B.4) and

(B.3).

From (B.10), (B.6a), and (B.18)

$$\frac{\partial \dot{\mathbf{r}}}{\partial a} = \frac{\dot{\mathbf{r}}}{2a} - \frac{\dot{\mathbf{r}}}{r}(1 - e\cos E) + \frac{\partial \dot{\mathbf{r}}}{\partial E}\frac{\partial E}{\partial a}$$

$$= -\frac{\dot{\mathbf{r}}}{2a} - \frac{3}{2}\frac{m}{r}\frac{\partial \dot{\mathbf{r}}}{\partial E}$$

$$= -\frac{1}{a}\left(\frac{\dot{\mathbf{r}}}{2} + \frac{3}{2}\frac{m}{1 - e\cos E}\frac{\partial \dot{\mathbf{r}}}{\partial E}\right). \tag{B.24}$$

Also,

$$\frac{\partial \dot{\mathbf{r}}}{\partial e} = -\frac{\dot{\mathbf{r}}}{r}(-a\cos E) + \frac{\sqrt{\mu a}}{r}\left(0,\ \frac{-\cos E}{\sqrt{1 - e^2}}e\right) + \frac{\partial \dot{\mathbf{r}}}{\partial E}\frac{\partial E}{\partial e}$$

$$= \frac{\cos E}{1 - e\cos E}\dot{\mathbf{r}} - \sqrt{\frac{\mu}{a}}\left(0,\ \frac{e}{\sqrt{1 - e^2}}\frac{\cos E}{1 - e\cos E}\right) + \frac{\sin E}{1 - e\cos E}\frac{\partial \dot{\mathbf{r}}}{\partial E}. \tag{B.25}$$

Lastly,

$$\frac{\partial \dot{\mathbf{r}}}{\partial M_0} = \frac{\partial \dot{\mathbf{r}}}{\partial E}\frac{\partial E}{\partial M_0} = \frac{1}{1 - e\cos E}\frac{\partial \dot{\mathbf{r}}}{\partial E}. \tag{B.26}$$

To compute the derivatives of $\mathbf{r}$ and $\dot{\mathbf{r}}$ in IJK coordinates with respect to an arbitrary element q, we use (B.1):

$$\frac{\partial \mathbf{r}}{\partial q^{(IJK)}} = \mathbf{U}\frac{\partial \mathbf{r}}{\partial q^{(PQW)}} \qquad \text{for } q = a \text{ or } e \text{ or } M_0 \tag{B.27}$$

$$\frac{\partial \mathbf{r}}{\partial q^{(IJK)}} = \frac{\partial \mathbf{U}}{\partial q}\mathbf{r}^{(PQW)} \qquad \text{for } q = i \text{ or } \Omega \tag{B.28}$$

The same equations hold for $\dot{\mathbf{r}}$. It remains to compute $\frac{\partial \mathbf{U}}{\partial q}$. We have

$$\frac{\partial \mathbf{U}}{\partial i} = \begin{bmatrix} \sin\Omega\sin\omega\sin i & \sin\Omega\cos\omega\sin i & \sin\Omega\cos i \\ -\cos\Omega\sin\omega\sin i & -\cos\Omega\cos\omega\sin i & -\cos\omega\cos i \\ \sin\omega\cos i & \cos\omega\cos i & -\sin i \end{bmatrix} \tag{B.29}$$

$$\frac{\partial \mathbf{U}}{\partial \Omega} = \begin{bmatrix} -U_{21} & -U_{22} & -U_{23} \\ U_{11} & U_{12} & U_{13} \\ 0 & 0 & 0 \end{bmatrix} \tag{B.30}$$

$$\frac{\partial \mathbf{U}}{\partial \omega} = \begin{bmatrix} U_{12} & -U_{11} & 0 \\ U_{22} & -U_{21} & 0 \\ U_{32} & -U_{31} & -\sin i \end{bmatrix} \tag{B.31}$$