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SENSITIVITY ANALYSIS FOR

SHORTEST HAMILTONIAN PATH AND

TRAVELING SALESMAN PROBLEMS

by

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Abstract

Given the shortest Hamiltonian path (or tour) H^o in an undirected weighted graph, the sensitivity analysis problem consists in finding by how much we can perturb each edge weight individually without changing the optimality of H^o.

The maximum increment and decrement of the edge weight that preserve the optimality of H^o is called edge tolerance with respect to the solution H^o. A method of computing lower bounds of edge tolerances based on solving the sensitivity analysis problem for appropriate relaxations of the shortest Hamiltonian path (tour) problem is presented.

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1. Introduction

The problem considered in this paper belongs to so called *sensitivity analysis* in combinatorial optimization (see e.g. [2]). This term is used for a phase of solution procedure when an optimal solution of problem has been already found and additional calculations are performed in order to investigate, how this optimal solution depends on changes of problems parameters.

In this paper two well known (see e.g. [7]) combinatorial optimization problems are considered: the shortest Hamiltonian path problem in undirected weighted graph and the symmetric traveling salesman problem. It is assumed that an optimal solution of given problem is known. The goal of sensitivity analysis consists in finding by how much we can perturb each edge weight individually without changing the optimality of the solution. The maximum increment and decrement of the edge weight that preserve the optimality of solution are called the edge tolerances with respect to this solution.

In this paper, a method of computing lower bounds of the edge tolerances with respect to the optimal solution of the shortest Hamiltonian path problem and traveling salesman problem is described. The method is based on solving the sensitivity analysis problem for appropriate relaxation of the original optimization problem. A general idea of this approach was presented in [8]. In this paper we give a description of the approach and its microcomputer implementation and we report preliminary results of computational experiments.

This paper is organized as follows. In section 2 we introduce a notation and give some preliminary results concerning the relations between the sensitivity analysis for the original problem and its relaxation. In section 3 we describe algorithms for performing a sensitivity analysis for problem relaxations. A choice of appropriate relaxation of the original problem is discussed in sections 3 and 4. Section 4 contains also a description of implementation of the method and results of numerical experiments.

2. Notation and preliminary results

Let $G = (V,E,C)$ be an undirected weighted graph with a set of vertices $V = \{1,\dots,n\}$ and a set of edges $E = \{e_1,\dots,e_m\} \subset V \times V$. $C \in \bar{R}^{n \times n}$, where $\bar{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$, is a matrix of edge weights. (If $e = (i,j) \notin E$, then $c(i,j) = \infty$.) The subgraph (V,Q,C) of G will be identified with a set of its edges Q and by $l(Q) = \sum_{e \in Q} c(e)$ we will denote a weight of the subgraph.

Let $\mathcal{H}$ be the set of Hamiltonian paths in G with fixed ends in vertices 1, n and let $\mathcal{H}$ denote the set of Hamiltonian tours in G. Two well known combinatorial problems: the shortest Hamiltonian path problem (SHPP) and the traveling salesman problem (SHTP) are formulated as follows

$$\min\{l(H) : H \in \mathcal{H}\} \qquad \text{(SHPP)}$$

$$\min\{l(\bar{H}) : \bar{H} \in \bar{\mathcal{H}}\} \qquad \text{(SHTP)}$$

In this paper the shortest Hamiltonian path problem will be mainly considered. The approach for the traveling salesman problem is similar; the differences are pointed out if necessary.

Assume that H^o is a (known) optimal solution of the SHPP in the graph G, i.e.,

$$H^o = \arg\min\{l(H) : H \in \mathcal{H}\}$$

The *tolerance problem* is formulated as follows:

Given H^o, find for $e \in E$ values $c^+(e)$, $c^-(e)$, such that H^o is optimal for any perturbed graph $G' = (V,E,C')$, in which $c'(i,j) = c(i,j)$ if $(i,j) \neq e$ and $c(e) - c^-(e) \leq c'(e) \leq c(e) + c^+(e)$.

The values $c^+(e)$, $c^-(e)$ are called *upper and lower tolerances* of the edge e with respect to the optimal solution H^o. Edge tolerances with respect to optimal solution of the SHTP are defined in the same way.

Let

$$\mathcal{H}_e = \{H \in \mathcal{H} : e \in H\}$$

and

$$\mathcal{H}^e = \{H \in \mathcal{H} : e \notin H\}$$

The following proposition expresses the edge tolerances $c^+(e)$, $c^-(e)$, $e \in E$, by auxiliary optimization problems over sets $\mathcal{H}^e$, $\mathcal{H}_e$. (We will assume that if a minimization problem is infeasible, then its optimal value is equal to ∞).

<u>Proposition 1.</u> If $e \in H^o$, then $c^-(e) = \infty$ and

$$c^+(e) = \min\{l(H) : H \in \mathcal{H}^e\} - l(H^o). \qquad (1)$$

If $e \notin H^o$, then $c^+(e) = \infty$ and

$$c^-(e) = \min\{l(H) : H \in \mathcal{H}_e\} - l(H^o). \qquad (2)$$

<u>Proof.</u> Consider an edge $e \in H^o$. It is obvious that any decrement of the weight $c(e)$ does not change the optimality of H^o, so $c^-(e) = \infty$. If the weight of e increases and the weights of all other edges remain unchanged, then weights of all Hamiltonian paths belonging to H_e also increase in the same way, but weights of paths in H^e are still the same. Therefore H^o remains optimal as long as the increase of the weight of e is not greater than the difference between the weight of the shortest Hamiltonian path in H^e and the value $l(H^o)$. The proof of the second part of Proposition 1 is analogous.

□

Similar fact may be proved for edge tolerances in the SHTP.

Proposition 1 suggests that a calculation of edge tolerances may be a difficult task, because in order to find the tolerances for a particular edge, one has to know the optimal value of an auxiliary optimization problem, which is in general as difficult as the original SHPP (unless this value is a by-product of solving the original problem). Another explanation of difficulty of this sensitivity analysis arises from the observation that the tolerance problem is closely connected to a problem of finding adjacent vertices in the SHPP or the SHTP polytope, which is known to be NP-hard [7].

The goal of this paper is to propose an approach which allows to compute in an efficient way lower bounds of edge tolerances, i.e., values $d^+(e)$, $d^-(e)$, $e \in E$, satisfying the conditions $d^+(e) \leq c^+(e)$, $d^-(e) \leq c^-(e)$, $e \in E$. Such lower bounds are also of practical value, because they imply that for particular edge e, the solution H^o remains still optimal if the weight of e belongs to an interval $[c(e) - d^-(e), c(e) + d^+(e)]$. Calculation of lower bounds seems to be much easier than calculation of edge tolerances, because in order to find $c^+(e)$, $c^-(e)$ one must, in fact, exploit necessary and sufficient conditions of the optimality of H^o. To calculate $d^+(e)$, $d^-(e)$ it is enough to have only some sufficient optimality conditions for H^o. The notion of necessary and sufficient optimality conditions in seldom the case in combinatorial optimization, whereas sufficient optimality conditions are provided by different relaxations of the original problem and related dual problems. The choice of appropriate relaxation is discussed in sections 4 and 5. In this paper as a relaxation of the SHPP, the shortest spanning tree problem (SSTP) is chosen, and to calculate bounds of edge tolerances for the SHTP, the shortest 1-tree problem (S1TP) is used (see e.g. [7]).

Let us consider a pair of problems - the SHPP and the SSTP - and let v(SHPP), v(SSTP) denote its optimal values, i.e.,

$$v(\mathrm{SHPP}) = \min\{l(H) : H \in \mathcal{H}\},$$

$$v(\mathrm{SSTP}) = \min\{l(T) : T \in \mathcal{T}\},$$

where $\mathcal{T}$ is a set of spanning trees in G. Usually, the SSTP is not a good relaxation of the SHPP (if we measure a quality of relaxation by the difference between the optimal values of both problems). But it is well known that this difference may be significantly reduced (see e.g. [7], Chapter 10) by appropriate modification of edge weights. This modification consists in replacement of the original edge weights $c(i,j)$, $(i,j) \in E$, by values $c^p(i,j)$ defined as follows:

$$c^p(i,j) = c(i,j) + p(i) + p(j) \tag{3}$$

where $p(i)$, $p(j)$, $i,j \in V$, are elements of so called *penalty vector* $p = (p(1),\dots,p(n))^T \in \mathbf{R}^n$. Denote by C^p modified edge weight matrix and let $G^p = (V,E,C^p)$. The weight of subgraph Q in G^p will be denoted by $l^p(Q)$. It is well known that the such modification of the graph does not change the set of optimal solutions of the SHPP. The following proposition states that this is also true for edge tolerances. The same facts hold also for the SHTP.

Proposition 2. Edge tolerances $c^+(e)$, $c^-(e)$, $e \in E$, are the same for any modified graph $G^p = (V,E,C^p)$, $p \in \mathbf{R}^n$.

Proof. This is a simple consequence of Proposition 1. It is easy to see that for $p \in \mathbf{R}^n$ the value $d(H',H") = l^p(H') - l^p(H")$ does not depend on p for any $H', H" \in \mathcal{H}$. But according to (1) and (2), if $c^+(e), c^-(e) < \infty$, then $c^+(e) = d(H^e,H^o)$, $e \in E$, and $c^-(e) = d(H_e,H^o)$, $e \in E\backslash H^o$, where $H^e = \arg\min\{l(H) : H \in \mathcal{H}^e\}$, $H_e = \arg\min\{l(H) : H \in \mathcal{H}_e\}$. If for some $e \in H^o$, $c^+(e) = \infty$ or for $e \in E\backslash H^o$, $c^-(e) = \infty$, this means that corresponding set $\mathcal{H}^e$ or $\mathcal{H}_e$ is empty; which, obviously does not depend on the vector p.

□

Let $p \in \mathbf{R}^n$ be arbitrary penalty vector and define $\Delta(p)$ to be equal to the difference between the optimal values of the SHPP and the SSTP in G^p. Moreover, let T^p be the optimal solution of the SSTP for G^p and define $t_p^+(e,T^p)$ $(t_p^-(e,T^p))$, $e \in E$, be an upper (lower) tolerance of e with respect to T^p regarded as an optimal solution of the SSTP in G^p, i.e., $t_p^+(e,T^p)$ $(t_p^-(e,T^p))$ is equal to the maximum increment (decrement) of the weight of e, which does not change the optimality of T^p. Then the following fact hold:

Lemma 1: For $p \in R^n$ and $e \in H^o \cap T^p \cup (E \backslash H^o) \cap (E \backslash T^p)$

$$c^+(e) \geq t_p^+(e, T^p) - \Delta(p) \qquad (4)$$

and

$$c^-(e) \geq t_p^-(e, T^p) - \Delta(p) \qquad (5)$$

Proof. We will prove only (4); the proof of (5) is analogous. If $e \in E \backslash H^o$, then $c^+(e) = \infty$ and (4) holds. Assume then that $e \in H^o \cap T^p$ and let

$$t_p^e = \min\{l^p(T) : T \in \mathcal{T}^e\} \qquad (6)$$

where $\mathcal{T}^e = \{T \in \mathcal{T}: e \notin T\}$. Using the same arguments as in the proof of Proposition 1 it is easy to show that

$$t_p^+(e, T^p) = t_p^e - l^p(T^p) \qquad (7)$$

From Propositions 1 and 2

$$c^+(e) = l_p^e - l^p(H^o) \qquad (8)$$

where $l_p^e = \min\{l^p(H) : H \in \mathcal{H}^e\}$ (9)

The problem (6) is a relaxation of the problem (9), which implies that $l_p^e \geq t_p^e$ and now from (7) and (8) we have

$$c^+(e) \geq t_p^+(e,T^p) + l^p(T^p) - l^p(H^o) = t_p^+(e,T^p) - \Delta(p) \qquad \square$$

An analogue of Lemma 1 may be also proved for the SHTP and the S1TP as its relaxation.

Some comments concerning Lemma 1 are necessary. Two special cases have to be considered:

CASE 1^o - when there exists a penalty vector $p^* \in R^n$ such that $\Delta(p^*) = 0$;

CASE 2^o - when there is so called *duality gap* $\Delta > 0$, where $\Delta = \inf\{\Delta(p) : p \in R^n\}$.

In the Case 1^o, $H^o = \arg\min\{l^{p^*}(T) : T \in \mathcal{T}\}$ and from Lemma 1 we have the following inequalities for $e \in E$:

$$c^+(e) \geq t_{p^*}^+(e,H^o) \tag{10}$$

$$c^-(e) \geq t_{p^*}^-(e,H^o) \tag{11}$$

In section 3 we will show that bounds for $c^+(e)$, $c^-(e)$ provided by the inequalities (10), (11) may be slightly improved, because in the Case 1^o stronger inequalities hold:

$$c^+(e) \geq t_{p^*}^+(e,H^o) + \min\{t_{p^*}^+(u,H^o) : u \in H^o \setminus \{e\}\} \tag{12}$$

$$c^-(e) \geq t_{p^*}^-(e,H^o) + \min\{t_{p^*}^-(u,H^o) : u \in E \setminus H^o \setminus \{e\}\} \tag{13}$$

In order to use inequalities (10), (11) or (12) (13) to calculate lower bounds for the edge tolerances $c^+(e)$, $c^-(e)$, $e \in E$, in the Case 1^o two problems have to be solved:

(i) a penalty vector $p^* \in R^n$ satisfying $\Delta(p^*) = 0$ must be found;

(ii) edge tolerances $t^+_{p^*}(e,H^o)$, $t^-_{p^*}(e,H^o)$, $e \in E$, for the SSTP in G^{p^*} have to be calculated.

A solution of problem (ii) is described in section 3. A method of solving the problem (i) is discussed in section 4.

In the Case 2^o bounds for $c^+(e)$, $c^-(e)$ obtained from Lemma 1 are weaker, because $\Delta(p) > 0$ for any $p \in R^n$. Moreover, Lemma 1 does not provide bounds for edges belonging to $(H^o \backslash T^p) \cup (T^p \backslash H^o)$. In section 3 we will prove a theorem which specifies bounds for $c^+(e)$, $c^-(e)$ in this case, but they still may be weak. Thus, to calculate bounds for edge tolerances in the Case 2^o it is required to find a penalty vector $\bar{p}$, for which $\Delta(\bar{p})$ is possibly small and the cardinality of the set $H^o \backslash T^{\bar{p}}$ is small as well. This problem is discussed in section 4.

3. Edge tolerances for shortest spanning tree and 1-tree

The problem of calculating edge tolerances of the shortest spanning tree has been addressed in several papers (see e.g. [1, 3, 12]). In this section we review at first some fundamental facts on which sensitivity analysis for the shortest spanning tree is based. Next we discuss in detail implementations of algorithms for finding edge tolerance with respect to special spanning tree which is also a Hamiltonian path. We close this section by proving some useful result concerning relations between edge tolerances and lengths of spanning trees.

Let T^0 be the shortest spanning tree in $G = (V,E,C)$. The following well known proposition (see e.g. [12]) formulates necessary and sufficient optimality conditions for T^0:

<u>Proposition 3.</u> T^0 is the shortest spanning tree in G if and only if for any $e \in E \setminus T^0$

$$c(e) \geq c(w) \text{ for } w \in U(e) \tag{14}$$

where U(e) is a subset of edges belonging to the unique path in T^0 joining the ends of e. □

Denote for $e \in T^0$

$$W(e) = \{w \in E \setminus T^0 : e \in U(w)\} \tag{15}$$

The following fact is a straightforward consequence of Proposition 3:

<u>Proposition 4.</u> If $e \in T^0$, then $t^-(e) = \infty$ and

$$t^+(e) = \min\{c(w) : w \in W(e)\} - c(e) \tag{16}$$

If $e \in E \setminus T^0$, then $t^+(e) = \infty$ and

$$t^-(e) = c(e) - \max\{c(u) : u \in U(e)\} \tag{17}$$

□

Proposition 4 provides a method of computing edge tolerances with respect to T^0 by finding minimum weight edge belonging to W(e) for any $e \in T^0$ and maximum weight edge in U(e) for any $e \notin T^0$. This may be done simultaneously by using an auxiliary graph ([11]) called *transmuter*. A transmuter is a directed acyclic graph which contains one vertex v(e') of in-degree zero for any $e' \in T^0$, one vertex v(e") of out-degree zero for any $e" \in E \setminus T^0$ and arbitrary number of additional vertices. Moreover, in a transmuter there exists a path from vertex v(e') to vertex v(e") if and only if $e" \in W(e')$. It was shown in [10] that for given spanning tree T in G a transmuter containing $O(m\ \alpha(m,n))$ vertices can be constructed in $O(m\ \alpha(m,n))$ time (where $\alpha(m,n)$ is a functional inverse of Ackerman's function [10]). Given a transmuter, a labeling procedure was described in [12] to compute all edge tolerances in $O(m\ \alpha(m,n))$ time using O(m) space. It is the best known complexity of algorithm for finding all edge tolerances with respect to general shortest spanning tree, although there is some doubt, whether complicated data structures used may lead to a computationally efficient procedure (see [10]).

In [3] simpler data structures were proposed to compute all edge tolerances for (general) shortest spanning tree in O(m log n) time using O(m) space.

In [9] two methods which may be used to compute edge tolerances were described: the first has time and space complexity $O(n^2)$, the second has running time O(mn) and requires O(m) space.

Any of the methods mentioned above may be used to calculate edge tolerances with respect to the shortest spanning tree in the Case 2° (see section 2). But in the Case 1°, T^0 is a particular spanning tree which is also a path and more efficient algorithms may be proposed.

Let T^0 be the shortest spanning tree in G = (V,E,C). Assume that T^0 is also a Hamiltonian path in G and, moreover, the vertices of G are numbered in such a way, that $T^0 = \{(1,2), (2,3), \ldots, (n-1,n)\}$.

Then sets U(e), W(e), $e \in E$, appearing in Proposition 4 are defined as follows:

For $e \in E \backslash T^0$, i.e., $e = (k,l)$, $k=1,\dots,n-2$, $l=k+2,\dots,n$,

$$U(e) = \{(i,i+1) : k \leq i \leq l-1\} \tag{18}$$

For $e \in T^0$, i.e., $e = (i,i+1)$, $i=1,\dots,n-1$,

$$W(e) = \{(k,l) : 1 \leq k \leq i,\ i+1 \leq l \leq n,\ (k,l) \neq (i,i+1)\} \tag{19}$$

Figure 1 illustrates subsets of elements of edge weight matrix C for which appropriate minima and maxima must be calculated according to formulae (16), (17) and (18), (19).

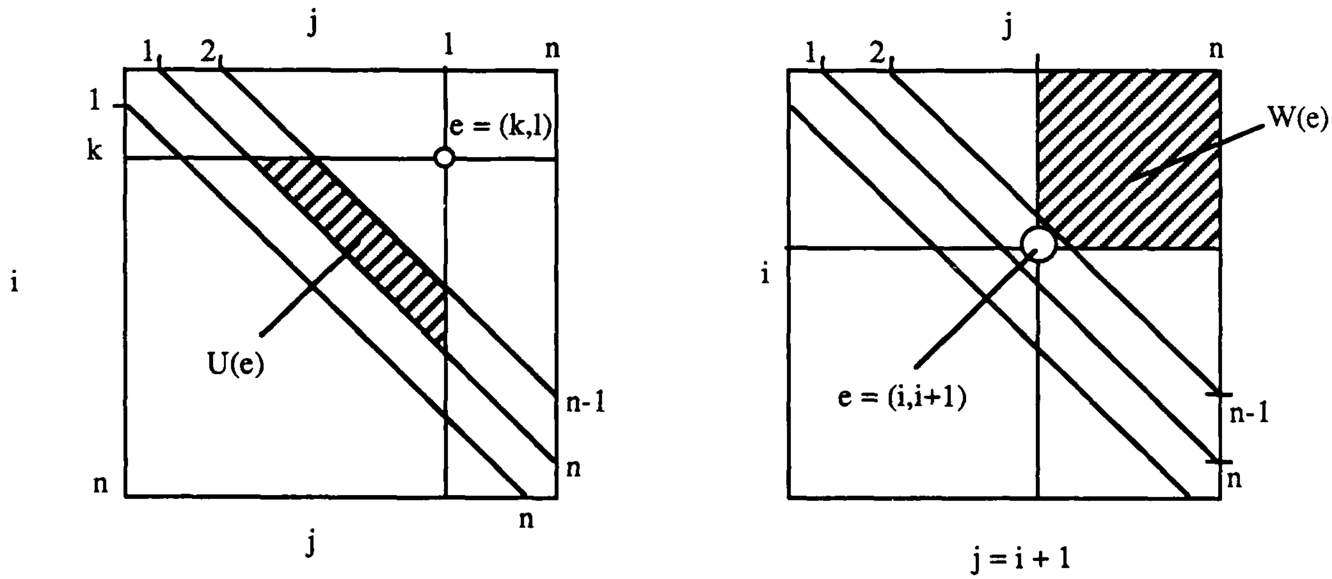

Fig. 1

If the graph G is dense, i.e., $m = \Theta(n^2)$, then the following simple labeling algorithms may be used to calculate edge tolerances $t^+(e), t^-(e)$, $e \in E$, in $O(n^2)$ time using $O(n^2)$ space. Let $w(i,j) \in \bar{R}$ be labels defined for $i = 0,1,\ldots,n$, $j = 1,\ldots,n,n+1$.

Algorithm for calculating $t^+(e)$, $e \in T^0$

Step 1 *(Initialization)* for i=1 to n-1 do w(i,n+1): = ∞;

for j:=2 to n do w(0,j):=∞;

Step 2 *(Labeling)* for i:=1 to n-2 do

for j:=n downto i+2 do w(i,j):= min{w(i-1,j), c(i,j), w(i,j+1)};

Step 3 *(Calculation of tolerances)* for i:=1 to n-1 do

$t^+(i,i+1) := \min\{w(i-1,i+1), w(i,i+2)\} - c(i,i+1)$.

Algorithm for calculating $t^-(e)$, $e \in E \setminus T^0$

Step 1 *(Initialization)* for i:=1 to n-1 do w(i,i+1):=c(i,i+1);

Step 2 *(Labeling and calculation of tolerances)*

```
for i:=2 to n-1 do
  for j:=1 to n-i do
    begin w(j,j+i):= min{w(j,j+i-1), w(j+1,j+i)};
          t⁻(j,j+i):=w(j,j+i)-c(j,j+i)
    end
```

If the graph is sparse, then more efficient algorithms may be used to calculate edge tolerances with respect to T^0.

Consider at first the problem of calculating

$$\max\{c(u) : u \in U(k,l)\} \quad (20)$$

where $U(k,l)$ is given by (18). In order to solve (20) for all $(k,l) \in E\setminus T^0$ efficiently let us store elements of set $U = \{c(i,i+1),\ i=1,\ldots,n-1\}$ using a data structure called a symmetric heap (see [6]). A *symmetric heap* $SH(U)$ is a directed binary tree containing one vertex for any element of the set U. The vertex $v(i)$, $i=1,\ldots,n-1$, of $SH(U)$ has a label $c(i,i+1)$ and the following properties are satisfied for any $k,l=1,\ldots,n-1$, $k \neq l$:

If $c(k,k+1) \leq c(l,l+1)$, then there is a path in $SH(U)$ from the vertex $v(l)$ to the vertex $v(k)$ and, moreover, if $k < l$, then $v(k)$ belongs to the left subtree of $v(l)$, otherwise $v(k)$ belongs to the right subtree of $v(l)$.

A symmetric heap $SH(U)$ may be constructed in $O(n)$ steps and, as it was observed in [6], any particular problem (20) for given k,l is equivalent to calculating the nearest common ancestor of vertices $v(k)$, $v(l-1)$ in $SH(U)$. But this problem may be solved in $O(1)$ time (see [4]) if a preprocessing requiring $O(n)$ time has been performed. This means, that all lower tolerances $t^-(e)$, $e \in E\setminus T^0$, may be calculated in $O(m)$ time and $O(m)$ space.

To calculate all upper tolerances $t^+(e)$, $e \in T^0$, a simple algorithm requiring a sorting of values $c(e)$, $e \subset E\setminus T^0$, may be constructed and this problem may be solved in $O(n \log m)$ time and $O(m)$ space. But it is not known, whether there is of linear complexity ($O(m)$ time and space) algorithm to calculate upper tolerances of edges in the Case 1^o.

Edge tolerances with respect to the shortest 1-tree $\bar{T}^0$ can be computed in a similar way. An approach is based on simple Proposition 5 which is an analog of Proposition 4 and which we will state without proof. Let $\bar{T}^0$ be an optimal solution of the S1TP in $G = (V,E,C)$, i.e., $\bar{T}^0 = T_1 \cup \{(1,k), (1,l)\}$, where T_1 is the shortest spanning tree in the graph $G_1 = (V\backslash\{1\}, E_1, C)$ obtained from G by removing the vertex 1, and (1,k), (1,l) are two shortest edges incident to the vertex 1. By $W_1(e)$, $U_1(e)$ we denote subsets of edges of G_1 defined for T_1 in the same way as the sets W(e), U(e) for T^0.

<u>Proposition 5:</u> If $e \in T_1$, then $t^-(e) = \infty$ and

$t^+(e) = \min\{c(w): w \in W_1(e)\} - c(e)$.

If $e \in E_1\backslash T_1$, then $t^+(e) = \infty$ and $t^-(e) = c(e) - \max\{c(u): u \in U_1(e)\}$.

If $e \in E_1' = E\backslash E_1\backslash\{(1,k), (1,l)\}$, then $t^+(e) = \infty$ and

$t^-(e) = c(e) - \max\{c(1,k), c(1,l)\}$.

Furthermore, $t^-(1,k) = t^-(1,l) = \infty$ and $t^+(1,k) = c(1,k) - \min\{c(e) : e \in E_1'\}$,

$t^+(1,l) = c(1,l) - \min\{c(e) : e \in E_1'\}$.

□

We will close this section by proving a result which establishes a relation between the edge tolerances with respect to the shortest spanning tree and the value of difference between the weights of the shortest spanning tree and an arbitrary spanning tree.

<u>Theorem 1.</u> Let T^0 be the shortest spanning tree in G and T be an arbitrary spanning tree in G. Then

$$l(T) - l(T^0) \geq \max\{ \sum_{r \in T^0\backslash T} t^+(r), \sum_{q \in T\backslash T^0} t^-(q)\} \tag{21}$$

Proof. Consider two subsets of edges: $R = T^0 \setminus T$ and $Q = T \setminus T^0$. It is known (see [5], Theorem 1), that there exists a bijection ψ *from R into Q, such* that for every edge $r \in R$, $T_r = T^0 \setminus \{r\} \cup \{\psi(r)\}$ is a spanning tree in G and $c(\psi(r)) - c(r) \geq 0$. From the fact that T_r is a spanning tree it follows, that $\psi(r) \in W(r)$ and from (16) we have the inequality $t^+(r) \leq c(\psi(r)) - c(r)$ and further $l(T) - l(T^0) = \sum_{r \in R} [c(\psi(r)) - c(r)] \geq \sum_{r \in R} t^+(r)$. Similarily, for every edge $q \in Q$, $T^0 \cup \{q\} \setminus \{\psi^{-1}(q)\}$ is also a spanning tree and this implies that $\psi^{-1}(q) \in U(q)$. Now from (17) we have $t^-(q) \leq c(q) - c(\psi^{-1}(q))$ and finally $l(T) - l(T^0) = \sum_{q \in Q} [c(q) - c(\psi^{-1}(q))] \geq \sum_{q \in Q} t^-(q)$.

□

As corollaries of Theorem 1 we obtain some properties of edge tolerances with respect to the shortest Hamiltonian path, which were stated without proof in section 2.

Let for some $p \in R^n$, H^o and T^p be optimal solutions of the SHPP and the SSTP in $G^p = (V,E,C^p)$. As before, $t_p^+(e,T^p)$, $t_p^-(e,T^p)$, $e \in E$, are edge tolerances with respect to T^o and $c^+(e)$, $c^-(e)$, $e \in E$, are edge tolerances with respect to H^o.

Theorem 2. If $\Delta(p) = 0$ and $H^o = T^p$, then for $e \in E$

$$c^+(e) \geq t_p^+(e,H^o) + \min \{t_p^+(u,H^o) : u \in H^o \setminus \{e\}\}$$

and

$$c^-(e) \geq t_p^-(e,H^o) + \min\{t_p^-(u,H^o) : u \in E \setminus H^o \setminus \{e\}\}$$

Proof. If $c^+(e) < \infty$, then according to (1) we have $c^+(e) = l^p(H^e) - l^p(H^o)$, where $H^e = \arg\min \{l^p(H) : H \in \mathcal{H}^e\}$. It is easy to see that $|H^o \backslash H^e| \geq 2$. Obviously, $H^o, H^e \in \mathcal{T}$ and now because H^o is the shortest spanning tree in G^p and $e \in H^o \backslash H^e$, from (21) we have $c^+(e) = l^p(H^e) - l^p(H^o) \geq t_p^+(e,H^o) + t_p^+(u,H^o)$ for some $u \in H^o \backslash \{e\}$. The proof of second part of theorem is analogous. □

Theorem 3. If $H^o \nsubseteq T^p$, then for $e \in H^o \backslash T^p$

$$c^+(e) \geq \min\{t_p^-(q,T^p) : q \in E \backslash T^p \backslash \{e\}\} - \Delta(p) \tag{22}$$

and for $e \in T^p \backslash H^o$

$$c^-(e) \geq \min\{t_p^+(r,T^p) : r \in T^p \backslash \{e\}\} - \Delta(p) \tag{23}$$

Proof. We will prove only (22), because a proof of (23) is analogous. Consider $e \in H^o \backslash T^p$. If $c^+(e) < \infty$, then $c^+(e) = l(H^e) - l(H^o)$ and there exists a spanning tree T_2^e which is the second shortest spanning tree not containing e. Moreover, $l(T_2^e) \leq l(H^e)$ and $c^+(e) \geq l(T_2^e) - l(T^p) + (l(T^p) - l(H^o)) = l(T_2^e) - l(T^p) - \Delta(p)$. But $T_2^e \backslash T^p$ must contain some edge $q \in E \backslash T^p \backslash \{e\}$. Now from (21) we have $l(T_2^e) - l(T^p) \geq t^-(q)$ which implies (22). □

Bounds for edge tolerances provided by Theorem 3 (and Lemma 1) may be weak. In particular cases, values of right-hand sides of inequalities (22), (23) and (4), (5) may even be negative, which means that trivial bounds are obtained. Thus, although any penalty vector may be used to calculate $c^+(e)$,

$c^-(e)$, it is desired to have a vector p which gives small (if possible - equal to zero) values of $\Delta(p)$ and $|H^o \backslash T^p|$. This problem is discussed in the next section.

4. Computing of penalties

To calculate lower bounds of edge tolerances with respect to H^o, a penalty vector p is needed, for which $\Delta(p)$ and $|H^o \backslash T_p|$ are as small as possible. If the duality gap Δ is equal to zero, then such vector may be found as a solution of equation $\Delta(p) = 0$ and this guarantees also that $|H^o \backslash T^p| = 0$. Otherwise, one may try to solve this bicriteria problem by choosing as a vector p such a feasible solution of equation $\Delta(p) = \Delta$, for which $|H^o \backslash T^p|$ is minimal.

To solve $\Delta(p) = 0$ two attempts may be considered:

(i) The problem $\min\{\Delta(p) : p \in R^n\}$ may be solved exploiting properties of the function $\Delta(p)$ ($\Delta(p)$ is convex, piece-wise linear function on R^n) by some subgradient type procedure.

(ii) A feasible solution of $\Delta(p) = 0$ (if exists) may be calculated by finding a solution of the system of linear inequalities (24).

The later approach was used in computer implementation and it will be described in this section.

Define for a given graph $G = (V,E,C)$, $P(C) = \{p \in R^n$:

$$p(i) + p(j) - p(k) - p(k+1) \geq c(k,k+1) - c(i,j)$$

$$\text{for } (i,j) \in E,\ i = 1,\ldots,n-2 \quad j = i+2,\ldots,n \quad k = i,\ldots j-1\} \tag{24}$$

Theorem 4. Let $H^o = \{(1,2), (2,3),\ldots,(n-1,n)\}$ be an optimal solution of tne SHPP in $G^p = (V,E,C^p)$. Then $\Delta(p) = 0$ if and only if $p \in P(C)$.

Proof. $\Delta(p) = 0$ if and only if H^o is also an optimal solution of the SSTP in G^p, i.e., if necessary and sufficient optimality conditions formulated in Proposition 3 are satisfied. This means that for H^o inequalities (14) must hold for the graph G^p. But for the spanning tree H^o the sets $U(e)$, $W(e)$ are given by (18) and (19), and now it is easy to check, that if the inequalities (14) are formulated for H^o and the graph G^p, then we obtain a system of conditions defining $P(C)$. □

The number $S(G)$ of inequalities defining $P(C)$ is of order $O(mn)$. If $G = K_n$ (complete graph with n vertices), then

$$S(K_n) = \frac{1}{3}(2k-1)[k(2k+1)-3] \qquad \text{if } n = 2k,\ k=1,2,\ldots$$

and

$$S(K_n) = \frac{2}{3}k[(k+1)(2k+1)-3] \qquad \text{if } n = 2k+1,\ k=1,2,\ldots$$

Any vector p belonging to P(C) may be used as a penalty vector to compute lower bounds of edge tolerances with respect to H^o, although different vectors lead, in general, to different values of these bounds. If $P(C) = \Phi$, then it means that there is a positive duality gap Δ.

As a simple consequence of Theorem 4 we obtain the following fact:

Corollary 1. If for a given graph G there is zero duality gap Δ, then the optimality of arbitrary Hamiltonian path may be verified in a polynominal time.

Proof. It is an immediate consequence of the fact that P(C) is defined by polynominal number of inequalities and its consistency may be checked in a polynominal time by linear programming. □

Similar facts (which we will give without proof) hold for the SHTP. Define for $G = (V,E,C)$, $\bar{P}(C) = \{p \in R^n$:

$$p(k) - p(2) \geq c(1,2) - c(1,k) \text{ for } k=3,\ldots,n-1, \ (1,k) \in E,$$

$$p(k) - p(n) \geq c(1,n) - c(1,k) \text{ for } k=3,\ldots,n-1, \ (k,n) \in E,$$

$$p(i) + p(j) - p(k) - p(k+1) \geq c(k,k+1) - c(i,j)$$

$$\text{for } (i,j) \in E, \ i=2,\ldots,n-2,$$

$$j = i+2,\ldots,n, \ k=i,\ldots,j-1\}$$

Theorem 5. Let $\bar{H}^o = \{(1,2), (2,3), \ldots, (n-1,n), (n,1)\}$ be the shortest Hamiltonian tour in $G^p = (V,E,C^p)$. For $\bar{H}^o$ to be the shortest 1-tree in G^p it is necessary and sufficient that $p \in \bar{P}(C)$. □

5. Implementation of the method and conclusions

The method of calculating lower bounds of edge tolerances for the SHPP and the SHTP described in previous sections was implemented for IBM PC in Turbo Pascal 3.0. In the step of computing of edge tolerances for spanning trees and 1-trees simple $O(n^2)$ labeling procedures mentioned in section 3 are used. To calculate appropriate penalties an approach provided by Theorems 4 and 5 is used. Penalities are computed by solving linear programming problems

$$\min\{a^T p: p \in P(C)\}$$

$$\min\{a^T p: p \in \bar{P}(C)\} \tag{25}$$

Different objective vectors $a \in \mathbf{R}^n$ may be chosen and, usually, different penalties as well as different lower bounds for edge tolerances are obtained. In computational experiments $a = (1,\ldots,1)^T$ or $a = (0,\ldots,0)^T$ was mainly used. In the later case by solving (25) an existence of feasible solution of equation $\Delta(p) = 0$ is checked.

To solve (25) a simple specialized version of the revised simplex algorithm was implemented. As problem (25) has only n variables and large number of constraints (for example, for the SHPP in K_n, $n = 40$, the number of constraints exceeds 10000), the dual problem for (25) is solved and column generation technique is used. The computational experience is limited to rather small sizes of problems. In Table 1 computation times in seconds for IBM PC/XT with math-processor are reported. These times do not include input and output of data. All test problems were randomly generated as planar Euclidean SHTP.

In Table 1 n denotes the number of vertices, δ is a density of graph, τ^p_a, τ^p_{min}, τ^p_{max} are respectively - average, minimal and maximal times of computing penalties (for 5 problems), τ^t is a time of computing edge tolerances.

TABLE 1

n	δ	τ^p_a	τ^p_{min}	τ^p_{max}	τ^t
10	1	2.6	0.5	4.3	0.1
10	0.3	1.2	0.3	1.8	0.1
25	1	121.6	58.8	177.5	0.7
25	0.3	48.9	40.6	49.8	0.7
40	1	1572	877	2523	1.9
40	0.3	421	316	747	1.9

An approach described in section 2 may be used with different relaxations of original problem.

Let (P): $\min\{f(x) : x \in X\}$ denote the original (primal) programming problem and let x^o be its optimal solution. Denote by (R_q) a relaxation of (P) parameterized by some element q belonging to a specified set Q:

$$(R_q) \quad v(q) = \min\{f_q(x) : x \in X_q\}.$$

(For example, in the approach described in this paper a role of parameter q is played by the penalty vector p and $Q = R^n$).

As a dual problem for (P) the following problem may be considered:

$$\text{(D)} \qquad q^* = \arg \max_{q \in Q} v(q)$$

The relaxation (R_{q^*}) seems to be a good candidate to provide a sensitivity analysis for x^o by similar approach as used in this paper. In order to apply this approach one must be able to answer the following two auxiliary questions:

(i) How to perform a sensitivity analysis for the problem (R_{q^*})?

(ii) How to find in an efficient way q^*, if x^o is given, i.e., how to solve the dual problem when the solution of primal is known?

In some cases an answer for the later question is obtained as an inexpensive by-product of solving the original problem. This may be an important argument for the choice of relaxation, because as numerical results reported in Table 1 show, in this approach almost all computing time may be spent on solving problem (ii).

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