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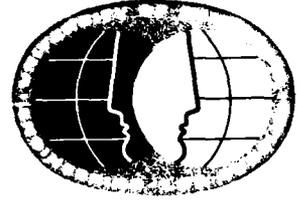
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STRONG AND WEAK STABILIZABILITY:

LYAPUNOV TYPE APPROACHES

W.M. MIYAJI

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This dissertation addresses the problem of determining stabilizing controls for distributed parameter systems. The focus is on controls which provide strong or weak stabilization to the system, which is appropriate for those situations in which exponential stability cannot be guaranteed. Stability is studied via Lyapunov-type functionals.

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STRONG AND WEAK STABILIZABILITY: LYAPUNOV TYPE APPROACHES

A dissertation submitted in partial satisfaction of the

requirements for the degree

Doctor of Philosophy in Electrical Engineering

By

Wendell Masanobu Miyaji

1988

The dissertation of Wendell Masanobu Miyaji is approved.

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1988

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Dedication

To Masanobu C. and Rose M. Miyaji, my parents
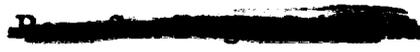
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STRONG AND WEAK STABILIZABILITY: LYAPUNOV TYPE APPROACHES

by

Wendell Masanobu Miyaji

Doctor of Philosophy in Electrical Engineering

University of California, Los Angeles, 1988

Professor Nhan Levan, Chair

This dissertation addresses the problem of determining stabilizing controls for distributed parameter systems. The focus is on controls which provide strong or weak stabilization to the system. Much of the prior work in this area has emphasized exponentially stabilizing a system. Compared to exponential stability, weak and strong stability are less desirable properties. However, there are situations, unlike finite dimensions systems, under which infinite dimensional systems can not be exponentially stabilized. In such casesl, we propose weak or strong stability.

One approach to the stabilization of finite dimensional systems and and exponential stabilization of infinite dimensional systems has been the use of Lyapunov type functionals. This is one technique which is developed and extended here, to provide new conditions for strong or weak stability. We present a new

functional, and if this functional is strictly positive, a certain semigroup will be strongly stable. This functional also suggests an inequality relation which, if satisfied guarantees the weak stablity of uniformly bounded semigroups.

We also examine the relationship between contraction semigroups on a Hilbert space and shift semigroups on a related Hilbert space. In particular, we find strongly stable semigroups to to be equivalent in a certain sense to a backward shift semigroup. This provides an alternative view point for strong stability.

Since stable semigroups are uniformly bounded and since this condition is important in verifying stability we examine this phenomena. Some new observations are presented to illustrate conditions under which perturbations of uniformly bounded semigroups remain uniformly bounded.

Chapter 1

INTRODUCTION

This dissertation presents research into the stabilization of distributed parameter systems using feedback controls. In particular the focus is on controls which provide strong or weak stabilization to the system. Most of the previous work in this area has emphasized providing a system with exponential stabilization. One approach to the study of exponential stability has been the use of Lyapunov type functionals. It is this technique which will be considered and applied to the investigation of strong and weak stabilization. Compared to exponential stability, weak and strong stability are less desirable properties. However, as we will see later, there are situations under which exponential stability is not possible. This fact emphasizes the need for feedback controls which will either strongly or weakly stabilize a system.

A vibrating beam or string are examples of distributed parameter systems. The state in each of these system might be represented as the position of the beam or string relative to its equilibrium and appropriate time derivatives. Then by stabilizing a system we are considered with the problem of given any

state of the system at an initial time, can a control $u(t)$ be selected, so that the state tends to zero in an appropriate sense. In the above examples, the control represents applying a force at the free end of the beam or in a distributed way on the string. Feedback means that the state is used instantaneously to determine the control at any given time. In particular we will focus on linear feedback which means that the control in selected to be a linear function of the state.

The norm of a Hilbert space in the system context can be thought of as a measure of the potential energy of these elastic systems. We will define precisely exponential, strong and weak stability. A system is exponentially stable if the norm tends to zero at some negative exponential rate as the time increases. For strongly stable system the norm tends to zero, but there is no exponential rate at which bounds this limit. The concept of weak stability is fundamentally different.

In chapter 2, a review of the essential mathematical definitions is made. In particular we look at linear systems on Hilbert or Banach spaces describe by abstract differential equations and the use of strongly continuous semigroups of linear bounded operators to represent the solutions of these equations. The concept of a semigroup is key to the entire dissertation, so we look at some of the critical properities of semigroups. In terms of these semigroups we precisely define the notions of exponential, strong and weak stability, and present simple examples of systems with each of these properties. The chapter concludes with a brief discussion of controllability.

Chapter 3 presents new conditions for a system to be weakly stable. The motivation for this approach is a well known theorem concerning the exponentially stability of system on a Hilbert space given by Datko [1] which we

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summarize. When a certain positive operator related to a uniformly bounded semigroup defines a functional with properties like those of Datko's Lyapunov functional we find that the semigroup can be shown to be weakly stable. We mention the relationship of this approach to weak stability implied by Nagy-Foais decomposition arguments.

Next, in the fourth chapter, strong stability characteristics are examined. First we present an integral which will serve as a Lyapunov functional. We show that the existence of this integral is equivalent the existence of an operator solution to a particular inner product equation. The interesting case is when the operator solution defines an equivalent norm. This is a sufficient condition for strong stability of the related semigroup. We then show sone related conditions for the existence of this equivalent norm and present two interesting examples. Exact controllability is shown to be associated to this criteria.

Also in the strong stability chapter, we discuss the particular case of strongly stable and exponentially stable contraction. One interesting observation is the fact that a strong stable contraction is unitarily equivalent to a certain backshift operator. This suggest that the backward shift is the archetype of a strongly stable semigroup on a Hilbert space.

A critical step to verifying strong or weak stability using the techniques we develop in chapter two and three is, checking, whether or not, the semigroup of interest is a uniformly bounded semigroup is uniformly bounded. In chapter 5, we first present conditions for a feedback system to generate a uniformly bounded semigroup when the uncontrolled system gives rise to a uniformly bounded semigroup. Then the concepts developed in chapters three and four are combined to present new weak and strong stabilization results.

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Chapter 2

PRELIMINARIES

In this chapter we will review the important concepts in the study of infinite dimensional systems theory and in particular, those ideas which are critical to the development of the results presented in this dissertation.

First we will discuss the class of systems to be considered. The semigroup theory, which is the indispensable basis of this approach is examined. The key concepts of stability and controllability will then be expatiated.

2.1 INFINITE DIMENSIONAL SYSTEMS

Many problems of interest in control theory can be described by the inhomogeneous equation

$$\dot{x}(t) = Ax + Bu \tag{1}$$

with initial state

$$x(0) = x_o \quad x_o \in \mathcal{D}(A)$$

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The A in this case is the infinitesimal generator of a C_o-semigroup and B is a linear bounded operator. In this section we will define these terms and related concepts.

The solution of such a differential equation then takes the form

$$x(t) = T(t)x(0) + \int_0^t T(t-s)Bu(s)\,ds$$

When the solutions $x(t)$, $t \geq 0$, are in a separable Hilbert Space $\mathcal{H}$, the state space need not be finite dimensional. In such cases we refer to equation (1) as representing an "infinite dimensional system".

The family of linear bounded operators $T(t)$, $t \geq 0$, forms a C_o-semigroup. This means that the following properties are satisfied for all $s, t \geq 0$.

- $||T(t)|| < \infty$

- $T(s+t) = T(s)T(t)$

- $T(0)x = x$ for all $x \in \mathcal{H}$

- the mapping $t \mapsto T(t)x$ is continuous in "t" for each $x \in \mathcal{H}$

These are the key properties satisfied by the matrix exponential, $exp\{At\}$, when A is a matrix, which are required to generalize the finite dimensional state space to an infinite dimensional space. We note that in the particular case where A is a linear bounded operator, the representation $exp\{At\}$ also holds. The norm above is the operator norm on the Hilbert space, $\mathcal{H}$

$$||T(t)|| = \sup_{||x||=1} ||T(t)x||$$

Generally, A is a closed linear operator from $\mathcal{D}(A)$, the domain of A, to the Hilbert Space, $\mathcal{H}$. An element, $x \in \mathcal{H}$, is in the domain of A if the limit

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$$\lim_{t^+ \to 0} \frac{T(t)x - x}{t}$$

is defined. In this case, the limit defines the value of Ax. We then have the representation

$$Ax = \frac{d}{dt}T(t)x|_{t=0}$$

and formally, $x(t) = T(t)x_o$, $t \geq 0$, solves the initial value problem

$$\dot{x}(t) = Ax, \; x(0) = x_o$$

The operator A is densely defined if the set $D(A)$ is a dense subspace of the Hilbert Space, $\mathcal{H}$. The Hille-Yosida [2] [3] and Generation Theorems provide us with conditions for an arbitrary linear operator A to generate a C_o-semigroup. The resolvent set, $\rho(A)$ of A, is the set of complex numbers

$$\{\lambda \; \text{complex} | \lambda I - A \; : \; D(A) \to \mathcal{H} \; \text{is one to one and onto and}$$

$$(\lambda I - A)^{-1} \; \text{is a linear bounded operator} \}$$

For these $\lambda \in \rho(A)$, the operator $(\lambda I - A)^{-1}$ is called the resolvent of A.

A C_o-semigroup is a contraction semigroup, if $||T(t)|| \leq 1$, $t \geq 0$, and is uniformly bounded when $||T(t)|| \leq M$, $t \geq 0$ for some positive $M \geq 1$. Contraction semigroups have two useful properties to note here. The operator A is said to be dissipative if for all $x \in D(A)$

$$Re[Ax, x] \leq 0$$

The infinitesimal generators of contraction semigroups are dissipative. When $T(t)$, $t \geq 0$ is a contraction semigroup, then its adjoint semigroup $T(t)$, $t \geq 0$ is also a contraction semigroup.

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Theorem 1 (Hille-Yosida) *A is the generator of a C_o-contraction semigroup if and only if A is a closed, densely defined and each positive λ is in the resolvent of set and satisfies*

$$||\lambda(\lambda I - A)^{-1}|| \leq 1$$

For general semigroups the following Generation theorem holds.

Theorem 2 *A is a generator of a C_o-semigroup $T(t)$, $t \geq 0$ if and only if A is closed, densely defined and there exist constants, $M \geq 1$ and $\omega \in R$ such that $\lambda \in \rho(A)$ for each $\lambda > \omega$ and satisfies*

$$||(\lambda - \omega)^n(\lambda I - A)^{-n}|| \leq M$$

for each $\lambda > \omega$ and when n is a positive integer. In which case we have $||T(t)|| \leq Me^{\omega t}$.

The last inequality is refered to as the exponential growth property.

In the original system (1) we will frequently be interested in the case where the control $u(t)$ is chosen to be a linear state feedback control, $u(t) = Fx(t)$. Here F is some other linear and preferably bounded operator. In this case we obtain the homogeneous system equation

$$\dot{x}(t) = (A + BF)x(t) \tag{2}$$

The solution to this differential equation then satisfies the integral equation

$$x(t) = T(t)x_o + \int_0^t T(t - s)BFx(s)\, ds$$

For certain classes of BF a better representation is possible.[4]

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Theorem 3 (Phillips) *Let A generate a C_o-semigroup $T(t)$, $t \geq 0$ on a Hilbert space $\mathcal{X}$ and $P : \mathcal{X} \to \mathcal{X}$ is a linear bounded operator. Then $A + P$ is also a generator of a C_o-semigroup.*

If B and F are both linear bounded operators then composition BF is also a linear bounded operator. So the sum $A + BF$ is once again the infinitesimal generator of some C_o-semigroup. This is one way to insure the existence of a solution to the feedback homogeneous system. If $S(t)$, $t \geq 0$, is the semigroup generate by $A + BF$, then the following relations are satisfied.

$$S(t)x_o = T(t)x_o + \int_0^t T(t-s)BFx(s)\,ds$$
$$= T(t)x_o + \int_0^t T(t-s)BFS(s)x_o\,ds$$

The feedback BF need not be bounded. If both A and BF are dissipative the the following theorem provides conditions under which $A + BF$ still generates a contraction semigroup

Theorem 4 *Let A be the infinitesimal generator of a C_o-contraction semigroup. Suppose that P is dissipative and $D(A) \supseteq D(P)$. If there are constants $0 \leq a \leq 1$, and $b \geq 0$ such that*

$$||Px|| \leq a||Ax|| + b||x||$$

for all $x \in D(A)$. Then $A + P$ generates a C_o-contraction semigroup.

For example, if A is dissipative, we can choose the feedback $u(t) = -B^*x(t)$. Then if the domain of BB^* contains the domain of A, $A - BB^*$ is the generator of another C_o-contraction semigroup. B and consequently B^* need not be bounded operators.

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2.2 NOTIONS OF STABILITY

We would like to investigate conditions for the stability of the homogeneous system

$$\dot{x}(t) = Ax(t)$$

A system of this form can be obtain from the original equation (1) by either applying an appropriate state feedback, $u(t) = Fx(t)$, in which case we obtain equation (2), or simply by setting the control to zero. In this case a concern would be: When does the solution $x(t)$ tend to zero as t tends to infinity? And in would sense does this convergence occur. In the infinite dimensional case it is possible to define this notion from many different viewpoints. The strongest definition commonly considered is that of exponential stability.

Definition 1 *The system (2.2) is exponentially stable if there is an $M \geq 1$ and an $\omega_o > 0$ such that for $t \geq O$*

$$\|T(t)\| \leq Me^{-\omega_o t}$$

The key point in this definition is that the norm of the state decreases at a known exponential rate. This is a very desirable property, however in many real systems obtaining exponential stability is not possible, as we shall see later. The next best and a milder form is strong stability.

Definition 2 *The system (2.2) is strongly stable if there for every $x \in \mathcal{H}$*

$$\lim_{t \to \infty} \|T(t)x\| = 0$$

Here the norm still tends to zero however there is no fixed rate at which this convergence occurs.

Definition 3 *The system (2.2) is weakly stable if there for every $x, y \in \mathcal{H}$*

$$\lim_{t \to \infty} [T(t)x, y] = 0$$

For a weakly stable system the norm may not decrease at all. If a semigroup $T(t)$, $t \geq 0$ is exponentially or weakly stable, then the its adjoint semigroup $T(t)^*$, $t \geq 0$ is also respectively, exponentially or weakly stable. However, the adjoint semigroup of a strongly stable semigroup is only weakly stable.

The simplest example of an exponentially stable semigroup on an arbitrary Hilbert space is multiplication by the scalar factor, $e^{-\omega_o t}$, where $\omega_o > 0$,

$$y = T(t)x$$

$$y = e^{-\omega_o t}x$$

The backward shift is a good example of a strongly stable semigroup. Let us consider the Hilbert space $\mathcal{H} = L^2[R^+, \mathcal{H}_1]$, where $\mathcal{H}_1$ is another separable Hilbert space with norm $\| \cdot \|_{\mathcal{H}_1}$. In this case suppose that $f \in \mathcal{H}$. Let $U(\cdot)$ denote the unit step function,

$$U(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The backward shift semigroup on this space takes the form

$$T(t)f = g$$

$$g(x) = f(x + t)$$

Then

$$\|T(t)f\|_{L^2}^2 = \int_0^\infty \|f(x + t)\|_{\mathcal{H}_1}^2 \, dx$$
$$= \int_t^\infty \|f(x)\|_{\mathcal{H}_1}^2 \, dx$$

And since, by definition,

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$$\|f\|_{L^2}^2 = \int_0^\infty \|f(x)\|_{\mathcal{H}_1}^2 \, dx$$

it follows that

$$\lim_{t \to \infty} \|T(t)f\|_{L^2} = 0$$

However, if we consider the family of elements of $\mathcal{H}$ defined by $f_t \in \mathcal{H}$

$$f_t(x) = f(x - t)U(x - t)$$

Now, if we recognize that $\|f\| = \|f_t\|$, and that $T(t)f = f_t$, a.e., we see that

$$\|f_t\| = \|T(\tau)f_t\|$$

for $\tau \le t$. There does not exist a negative exponential growth rate to bound this semigroup.

Another example of a strongly stable can be constructed using a multiplication operator. This time consider the space $\mathcal{H} = L^2[R^+, R]$. Define a function

$$q(x) = \begin{cases} -x & \text{for } 0 \le x \le 1 \\ -1 & \text{for } x > 1 \end{cases}$$

Then take the infinitesimal generator of a semigroup to be given by $g = Af$, $g(x) = q(x)f(x)$. In this case the semigroup is given by $h = T(t)k$, $h(x) = e^{tq(x)}k(x)$ or

$$h(x) = \begin{cases} e^{-xt}k(x) & \text{for } 0 \le x \le 1 \\ e^{-t}k(x) & \text{for } x > 1 \end{cases}$$

First of all note that since $e^{tq(x)} \le 1$ this semigroup is a contraction. To see that this is at least strongly stable first consider a step function

$$k(x) = (1/\sqrt{b - a})U(t - a)U(b - t) \text{ for } b > a > 0$$

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Then $||T(t)k|| \leq ||e^{tq(a)}k||$ which tends to zero as $t \to \infty$. The more interesting case is $k(x) = (1/\sqrt{a})U(x)U(a-x)$ for $1 > a > 0$. Then

$$
\begin{aligned}
||T(t)k||^2 &= \int_0^a e^{-2tx} \frac{1}{a} dx \\
&= \frac{1}{2ta}(1 - e^{-ta}) \\
&< \frac{1}{2ta}
\end{aligned}
$$

And the last term converges to zero.

Next we consider whether this example is exponentially stable. It suffices to consider again $k(x) = (1/\sqrt{a})U(x)U(a-x)$ for $1 > a > 0$. Suppose there was an $0 < \omega_o < 1/2$ such that $||T(t)k|| \leq e^{-\omega_o t}$. However, we choose $a = \omega_o/2$ and then $t = 1/2a$ we see that

$$
||T(t)k||^2 = \frac{1}{2ta}(1 - e^{-2ta}) > e^{-\omega_o t}
$$

So this example is not exponentially stable.

For the same space, $\mathcal{H} = L^2(R^+, \mathcal{H}_1)$, the forward shift is a weakly stable isometric semigroup.

$$
\begin{aligned}
h &= F(t)k \\
h(x) &= k(x-t)U(x-t)
\end{aligned}
$$

Then to verify that this semigroup is weakly stable, first recall that finite linear combinations of step functions of the form $U(t-a)U(b-t)$ are dense in $\mathcal{H}$. Then for $t \geq b$,

$$
\int_0^\infty [h(x), U(x-a)U(b-x)] dx = 0
$$

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2.3 CONTROLLABILITY

In order to stabilize a system of the form (1), it must feasible to find control that will "steer" the system to the origin. The characteristic of a system which permits us to select a control to transfer the state of the system to another state is known as controllability. This property is also refered to as reachability. As is the situation for stability of infinite dimensional systems, there are in addition various definitions for controllability. See Dolecki [5] for many other definitions of controllability.

A very cogent notion is exact controllability. In this case, from the origin, for any arbitrary state, there is a control, that for some finite time will drive to this system to the state. As a consequence, starting at any state, there is a control to transfer any state to the origin in finite time. More precisely we have

Definition 4 *For the system (1), the reachable set for the time 't' is*

$$K(t) = \{x(t) = \int_0^t T(t - \tau)Bu(\tau)\,d\tau \ \text{ for every admissible } u(\cdot)\}$$

This system is then said to be exactly controllable if

$$\bigcup_{t \in [0,\infty)} K(t) = \mathcal{H}$$

Example 1 *Consider again the backward shift semigroup on $L^2[(0,\infty), \mathcal{H}_1] = \mathcal{H}$. Let $B : \mathcal{H} \to \mathcal{H}$ be the operator defined by*

$$Bf(\theta) = \begin{cases} 0 & \text{if } \theta \le t_o \\ f(\theta) & \text{if } \theta > t_o \end{cases}$$

To see that this system is exactly controllable, we must find the control u which satisfies

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$$y(t,\theta) = \int_0^t T(t-\tau)Bu(\tau,\theta)\,d\tau$$

Since the effect of the projection B is to annihilate $u(\theta)$ for $\theta < t_o$ we obtain

$$y(t,\theta) = \int_0^t T(t-\tau)u(\tau,\theta)U(\theta - t_o)\,d\tau$$

and applying the left-shift to the control we find

$$y(t,\theta) = \int_0^t u(\tau,\theta + t - \tau)U(\theta + t - \tau - t_o)\,d\tau$$

When we eliminate the step function we obtain the relation

$$y(t,\theta) = \int_{\inf\{t,t-t_o+\theta\}}^t u(\tau,\theta + t - \tau)\,d\tau \text{ for } \theta + t > t_o$$

and zero otherwise. We see here that it is necessary to have $t > t_o$. Let us assume that there is an $\epsilon > 0$ such that $t - t_o > \epsilon$. Let us now consider separately the two cases indicated by the upper limit of the previous integral.

CASE 1: When $\theta > t_o$ suppose that we choose the control to satisfy the relation

$$u(\tau,\theta + t - \tau) = \frac{1}{t}y(t,\theta)$$

With the change of variables $\sigma = \tau$, and $\psi = \theta + t - \tau$ we obtain $\theta = \psi - t + \sigma$, and then

$$u(\sigma,\psi) = \frac{1}{t}y(t,\psi + \sigma - t)$$

for $\psi + \sigma > t$ and zero otherwise. If $y(t,\cdot)$ is in L^2 then unequivocally so is $u(\sigma,\cdot)$.

CASE 2: If $\theta < t_o$ we choose the control to satisfy the relation

$$u(\tau,\theta + t - \tau) = \frac{1}{t + \theta - t_o}y(t,\theta)$$

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With the change of variables $\sigma = \tau$, and $\psi = \theta + t - \tau$ we obtain $\theta = \psi - t + \sigma$, and then

$$u(\sigma, \psi) = \frac{1}{\psi + \sigma - t_o} y(t, \psi + \sigma - t)$$

for $\psi + \sigma > t$ and zero otherwise. To see that $u(\sigma, \cdot) \in L^2$ note that $\psi + \sigma - t + t - t_o > \epsilon$ and consequently

$$\int_0^\infty \|u(\sigma, \psi)\|^2 \, d\psi = \int_0^\infty \frac{1}{(\psi + \sigma - t + t - t_o)^2} \|y(t, \psi - t + \sigma)\|^2 \, d\psi < \infty$$

Thus we see the the above system is exactly controllable. Next note that the operator B is self-adjoint.

In many practical applications the operator B is compact and even finite dimensional. As one would expect, it is difficult for a control to exert influence on the entire state space. In fact, it has been shown that it is not possible to exactly control such as system. Specifically, Triggiani [6] has shown that

Theorem 5 *If the semigroup $T(t)$, $t \geq 0$, or the control operator B is compact then the system (1) is not exactly controllable.*

A more practical notion is approximately controllability which has important ramifications in the weak stabilizability of the system (1). In this case, the subspace which is exactly controllable is a dense subspace.

Definition 5 *The system (1) is approximately controllable if*

$$\overline{\bigcup_{t \in [0,\infty)} K(t)} = \mathcal{H}$$

When the system is not approximately controllable we will refer to $M_C = \overline{\bigcup_{t \in [0,\infty)} K(t)}$ as the controllable subspace and define the uncontrollable subspace M_{UC} to be the orthogonal complement of M_C.

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Chapter 3

WEAK STABILITY

Although weak stability is not as desirable as strong or exponentially stability, since the conditions for obtaining weak stability are simpler, some practical results can be derived. We begin in this chapter by recounting a well known result for exponential stability and then show how this suggests an approach for studying the weak stability of infinite dimensional systems on Hilbert space. An alternative approach to weak stability and stabilizability may be found in [7].

Datko [8] proved the following,

Theorem 6 *Let* $T(t)$, $t \geq 0$ *be a strongly continuous semigroup with infinitesimal generator* A *on a Hilbert space* $\mathcal{H}$. *Then the following conditions are equivalent.*

1. $T(t)$, $t \geq 0$ is exponentially stable.

2. There exists a positive, self-adjoint operator, $P > 0$, satisfying

$$[PAx, x] + [x, PAx] = -[x, x] \tag{3}$$

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for x in the domain of A.

3. For every x in $\mathcal{H}$:

$$\int_0^\infty \|T(t)x\|^2 \, dt < \infty$$

The unique operator P which satisfies (3) is defined by the expression

$$Px = \int_0^\infty T(t)^* T(t) x \, dt, \quad x \in \mathcal{H}$$

The functional (3) is

$$[Px, x] = \int_0^\infty \|T(t)x\|^2 \, dt$$

This theorem can also be generalized slightly in the following sense. Instead, if there is a solution P_1, positive and self-adjoint such that for some self-adjoint, strictly positive, $R \geq \gamma I$, $\gamma > 0$, to

$$[P_1 Ax, x] + [x, P_1 Ax] = -[Rx, x]$$

Then the system (2.2) would be exponentially stable. In this case we have

$$P_1 x = \int_0^\infty T(t)^* R T(t) x \, dt$$

This variation is verified by recognizing that R defines an equivalent norm $([Rx, x])^{1/2}$. By transforming back to the original space, the result of the previous theorem is obtained.

It is evident from (3) that

$$[PT(t)x, T(t)x] \leq [Px, x], \quad x \in \mathcal{H} \tag{4}$$

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We will investigate connections between operators P and semigroups $T(t), t \geq 0$ which satisfy this relationship and in particular the ramifications on the weak stability of the system (2.2).

When $P \geq 0$ and satisfies (4) there is a functional defined by another linear operator P and when applied to $T(t)x$ this functional converges to zero. We have

Proposition 1 *Let P be a self-adjoint, non-negative linear bounded operator and assume $T(t) \neq I$. Then there exists a $P \geq 0$ and $P \not\equiv 0$ such that*

$$\lim_{t \to \infty} [PT(t)x, T(t)x] = 0$$

proof For $0 \leq t_1 \leq t_2$,

$$[T(t_2)^* PT(t_2)x, x] \leq [T(t_1)^* PT(t_1)x, x]$$

$T(t)^* PT(t)$ is self-adjoint, non-negative, and non-increasing. Consequently, when we apply the uniform boundedness principle, we find, $T(t)^* PT(t)$ converges strongly to a non-negative, self-adjoint operator. We will denote by C^2 this limit.

$$\lim_{t \to \infty} T(t)^* PT(t)x = C^2 x$$

It is easy to see that $C^2 \leq P$ and that $C^2 = T(t)^* C^2 T(t)$. Now let us define $P = P - C^2$ and then

$$\lim_{t \to \infty} [PT(t)x, T(t)x] = \lim_{t \to \infty} [PT(t)x, T(t)x] - [T(t)^* C^2 T(t)x, x] = 0$$

This then defines the functional desired by this proposition.

The following simple example illustrates the key property of P.

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Example 2 *Consider the case where ϕ_n, $n = 1, \ldots$ is an orthonormal basis for*
H and

$$T(t)x = \sum_{n=1}^{\infty} e^{-\alpha_n t}[x, \phi_n]\phi_n$$

One example of P which meets the condition of Proposition (1) is

$$Px = \sum_{n=1} \frac{1 + sgn(Re(\alpha_n))}{n}[x, \phi_n]\phi_n$$

where the α_n are scalars.

By modifying slightly, the P that appears in the previous proposition we obtain conditions on for which the semigroup $T(t)$, $t \geq 0$ is weakly stable. The assumption that the semigroup be uniformly bounded is not restrictive since all weakly stable semigroups are uniformly bounded.

Proposition 2 *If $P > 0$ and $T(t), t \geq 0$ is uniformly bounded, $\|T(t)\| \leq M$, then $T(t)$, $t \geq 0$ is weakly stable.*

proof Define $Q^2 = P$. Then we have that

$$\lim_{t \to \infty} \|QT(t)x\| = 0$$

Since

$$[T(t)x, Qy] \leq \|QT(t)x\| \, \|y\|$$

we find

$$\lim_{t \to \infty}[T(t)x, Qy] = 0$$

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Since Q is positive, the range of Q is dense in $\mathcal{H}$. For any arbitrary any $z \in \mathcal{H}$, there is a sequence $\{y_n\}_{n=1}^{\infty}$ such that $Qy_n \to z$. Then we have the relation

$$[T(t)x, z] - [T(t)x, Qy_n] \leq \|T(t)x\| \, \|z - Qy_n\|$$
$$\leq M \, \|z - Qy_n\|$$

And then since Qy_n converges strongly to z we have

$$[T(t)x, z] = \lim_{n \to \infty} [T(t)x, y_n]$$

We then see that the semigroup is weakly stable since,

$$\lim_{t \to \infty} [T(t)x, z] = 0$$

for each $x, z \in \mathcal{H}$.

In the case of Proposition (1) we also have that $C = 0$ if and only if

$$\lim_{t \to \infty} [PT(t)x, T(t)x] = 0, \quad x \in \mathcal{H}$$

since $\|Cx\|^2 \leq [PT(t)x, T(t)x]$ for every $t \geq 0$ and each $x \in \mathcal{H}$.

Next let us examine the action of P on the Hilbert space $\mathcal{H}$. First denote by $\mathcal{M}$ the set

$$\mathcal{M} = \{x \in \mathcal{H} : [PT(t)x, T(t)x] = [Px, x] \text{ for every } t \geq 0\}$$

Now consider any arbitrary $y \in \mathcal{M}$, it satisfies the property

$$[PT(t)y, T(t)y] = [Py, y]$$

Since this equality holds for every $t \geq 0$, in the limit as $t \to \infty$ we also have

$$\|Cy\|^2 = [Py, y]$$

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or equivalently,

$$[(P - C^2)y, y] = 0$$

This means that $P - C^2 \geq 0$ as well as $Py = 0$ and $\mathcal{M} \subseteq \mathcal{N}(P)$. On the other hand if $z \in \mathcal{N}(P)$, $(P - C^2)z = 0$ and

$$[Px, x] = ||Cz||^2$$

From here we see that

$$[PT(t)z, T(t)z] \leq [Pz, z] = ||Cz||^2 \leq [PT(t)z, T(t)z]$$

Consequently, $[PT(t)z, T(t)z] = [Pz, z]$ for every $t \geq 0$ and every $z \in \mathcal{N}(P)$ and $\mathcal{N}(P) \subseteq \mathcal{M}$. To summarize we have that $\mathcal{N}(P) = \mathcal{M}$. This equality is a attribute of the system in the following example.

Example 3 *Suppose that ϕ_n, $n = 1, \ldots$ together with ψ_m, $m = 1, \ldots$ form an orthogonal basis for H and the semigroup $T(t)$, $t \geq 0$ is given by*

$$T(t)x = \sum_{n=1}^{\infty} e^{(-\alpha_n + i\beta_n)t}[x, \phi_n]\phi_n + \sum_{m=1}^{\infty} e^{(i\gamma_m)t}[x, \psi_m]\psi_m$$

where α_n, β_n and γ_m are real and the α_n are also positive. Then

$$T(t)^*x = \sum_{n=1}^{\infty} e^{(-\alpha_n - i\beta_n)t}[x, \phi_n]\phi_n + \sum_{m=1}^{\infty} e^{(-i\gamma_m)t}[x, \psi_m]\psi_m$$

Assume that P is defined by

$$Px = \sum_{n=1}^{\infty} \frac{1}{n}[x, \phi_n]\phi_n + \sum_{m=1}^{\infty} \frac{1}{m}[x, \psi_m]\psi_m$$

We can compute C^2 by

$$C^2 = \lim_{t \to \infty} T(t)^* P T(t) x$$

$$= \lim_{t \to \infty} \sum_{n=1}^{\infty} e^{(-\alpha_n - i\beta_n)t} \frac{1}{n} e^{(-\alpha_n + i\beta_n)t} [x, \phi_n] \phi_n$$

$$+ \lim_{t \to \infty} \sum_{m=1}^{\infty} e^{(-i\gamma_m)t} \frac{1}{m} e^{(i\gamma_m)t} [x, \psi_m] \psi_m$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} [x, \psi_m] \psi_m$$

Then for P,

$$Px = \sum_{n=1}^{\infty} \frac{1}{n} [x, \phi_n] \phi_n$$

Finally we recognize that $N(P) = \overline{span\{\phi_n\}}$.

In addition, when $[PT(t)x, T(t)x] \leq [Px, x]$ and $P \geq 0$, there is a subspace of H where a quasi-similar contraction semigroup can be found. First define by Q the linear bounded, self-adjoint, non-negative operator satisfying

$$P = Q^2$$

Another Hilbert space can be constructed by completing the range of Q. Denote this space by

$$\aleph = \overline{\mathcal{R}(Q)} \subseteq H$$

We can define a family of bounded linear operators $Z(t) : \aleph \to \aleph$, $t \geq 0$, to be quasisimilar to $T(t)$, $T \geq 0$

$$Z(t)Qx = QT(t)x \quad x \in H$$

$Z(t)$, $t \geq 0$, is easily seen to be a contraction semigroup [9]. The Nagy-Foias Decomposition [10] can now be applied to $Z(t)$, $t \geq 0$. The Hilbert space $\aleph$ can be decomposed into two orthogonal subspaces

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$$\mathcal{H} = \mathcal{H}_{cnu} \oplus \mathcal{H}_u$$

where the unitary subspace is defined as the set

$$\mathcal{H}_u = \{x \in \mathcal{H} \mid \|Z(t)x\| = \|x\| = \|Z(t)^*x\|\}$$

and the completely non-unitary subspace is its orthogonal complement.

$$\mathcal{H}_{cnu} = \mathcal{H}_u^{\perp}$$

More, importantly the contraction semigroup $Z(t)$ also can be decomposed according to it restriction to these two subspaces.

$$Z(t) = Z_{cnu}(t) \oplus Z_u(t)$$

The restriction of $Z(t)$ to the unitrary subspace is a semigroup

$$Z_u(t) = Z(t)|_{\mathcal{H}_u}$$

as well as the restriction to the completely non-unitary portion of $\mathcal{H}$

$$Z_{cnu}(t) = Z(t)|_{\mathcal{H}_{cnu}}$$

Since $\mathcal{H}_u$ and $\mathcal{H}_{cnu}$ are orthogonal subspaces there are projection operators for each subspace. Let us denote by P_u the self-adjoint orthogonal projection of $\mathcal{H}$ to $\mathcal{H}_u$, by P_{cnu} the self-adjoint orthogonal projection of $\mathcal{H}$ to $\mathcal{H}_{cnu}$. As a consequence of space decomposition we might observe that the projection operators commute with the semigroup $Z(t)$. For $x \in \mathcal{H}$

$$Z(t)P_u x = P_u Z(t)x$$

$$Z(t)P_{cnu} x = P_{cnu} Z(t)x$$

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When we apply these relations we find that

$$\begin{aligned}
[PT(t)x, T(t)x] &= \|Z(t)Qx\|^2 \\
&= \|Z(t)P_u Qx\|^2 + \|Z(t)P_{cnu}Qx\|^2 \\
&= \|P_u QT(t)x\|^2 + \|P_{cnu}QT(t)x\|^2 \\
&= [QP_u QT(t)x, T(t)x] + [QP_{cnu}QT(t)x, T(t)x]
\end{aligned}$$

Furthermore, for the unitary part of $Z(t)$

$$\|Z(t)P_u Qx\|^2 = \|P_u x\|^2$$

or

$$[QP_u QT(t)x, T(t)x] = [QP_u Qx, x]$$

In summary of the above, we have the ensuing proposition

Proposition 3 *If $T(t)$, $t \geq 0$ is a C_o semigroup and there exists a $P \geq 0$ such that*

$$[PT(t)x, T(t)x] \leq [Px, x]$$

then there exist $P_1, P_2 \geq 0$ such that

$$[PT(t)x, T(t)x] = [P_1 x, x] + [P_2 T(t)x, T(t)x]$$

Note that if in addition $P_u = 0$ then $T(t), t \geq 0$ is weakly stable.

Example 4 *Suppose that ϕ_n, $n = 1, \ldots$, ψ_m, $m = 1, \ldots$ and ξ_p, $p = 1, \ldots$ jointly form an orthogonal basis for H and the semigroup $T(t)$, $t \geq 0$ is given by*

$$\begin{aligned}
T(t)x &= \sum_{n=1}^{\infty} e^{(-\alpha_n + i\beta_n)t}[x, \phi_n]\phi_n + \sum_{m=1}^{\infty} e^{(i\gamma_m)t}[x, \psi_m]\psi_m \\
&\quad + \sum_{p=1}^{\infty} e^{(\delta_p + i\epsilon_p)t}[x, \xi_p]\xi_p
\end{aligned}$$

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where α_n, β_n, γ_m, δ_p, and ϵ_p are real and the α_n and δ_n are also positive. Then define P by

$$Px = \sum_{n=1}^{\infty} \frac{1}{n}[x, \phi_n]\phi_n + \sum_{m=1}^{\infty} \frac{1}{m}[x, \psi_m]\psi_m$$

This P satisfies $[PT(t)x, T(t)x] \leq [Px, x]$. Corresponding to the above development we have the following relations.

$$Qx = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}[x, \phi_n]\phi_n + \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}}[x, \psi_m]\psi_m$$

$$\mathcal{H} = span\{\phi_n, \psi_n\}$$

$$Z(t)x = \sum_{n=1}^{\infty} e^{(-\alpha_n + i\beta_n)t}[x, \phi_n]\phi_n + \sum_{m=1}^{\infty} e^{(i\gamma_m)t}[x, \psi_m]\psi_m$$

$$\mathcal{H}_{cnu} = span\{\phi_n\}$$

$$\mathcal{H}_u = span\{\psi_m\}$$

$$Z_{cnu}(t)x = \sum_{n=1}^{\infty} e^{(-\alpha_n + i\beta_n)t}[x, \phi_n]\phi_n$$

$$Z_u(t)x = \sum_{m=1}^{\infty} e^{(i\gamma_m)t}[x, \psi_m]\psi_m$$

$$P_u x = \sum_{m=1}^{\infty} \frac{1}{m}[x, \psi_m]\psi_m$$

$$P_{cnu} x = \sum_{n=1}^{\infty} \frac{1}{n}[x, \phi_n]\phi_n$$

P will be positive in the case where the ξ_n are all zero. P_u is 0 if the ψ_n do not exist. We then have $[PT(t)x, T(t)x] \leq [Px, x]$ and the semigroup $T(t)$, $t \geq 0$ is weakly stable.

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Once we can verify the weak stability of a system in some cases, it is easy to show that the system is strongly stable. In particular when the resolvent operator of the infinitesimal generator is compact or if the semigroup itself is compact, showing that the semigroup is weakly stable is sufficient for strong stability. Moreover, if A has compact resolvent BF is bounded the $A + BF$ also has compact resolvent so that if BF weak stabilizes A it also strongly stabilizes A.

Chapter 4

STRONG STABILITY

We present in this chapter new conditions for verifying the strong stability of some systems. We start by developing a new Lyapunov functional. This is shown to be equivalent to the existence of an operator solution to a certain inner product equation. When this functional defines an equivalent norm we see that the associated system is strongly stable. Some interesting conditions related to this equivalent norm are then explored. These criteria will be applied to the strong stabilization problem.

4.1 A NEW LYAPUNOV TYPE CONDITION FOR STRONG STABILITY

The major thrust is based on the integral

$$\int_0^\infty \|B^* T(t)x\|^2 \, dt < \infty$$

We will use this functional to obtain a sufficient condition for the strong stability

of a C_o–semigroup. First we prove the existence of a solution to a certain Lyapunov type equation is equivalent to the finiteness of this integral.

Theorem 7 *A necessary and sufficient condition for the convergence of the integrals*

$$\int_0^\infty ||B^* T(t)x||^2 \, dt \tag{5}$$

for every x in H is the existence of a self-adjoint linear operator P on H such that P is non-negative and satisfies

$$[PAx, x] + [x, PAx] = -||B^*x||^2 \ \text{for} \ x \in D(A) \tag{6}$$

Moreover,

$$Px = \int_0^\infty T(t)^* BB^* T(t)x \, dt$$

satisfies (6). $T(t)^*, t \geq 0$ is the adjoint semigroup of $T(t), t \geq 0$ with the infinitesimal generator A^*, the adjoint of A.

proof [Sufficiency] Suppose there exists a self-adjoint operator $P : H \rightarrow H$, $P \geq 0$, such that for all $x \in D(A)$ equation (6) is satisfied. For each $x \in H$ define $V(x,t) = [PT(t)x, T(t)x]$. Since P is non-negative, $V(x,t)$ is non-negative for all $t \geq 0$.

Suppose $x \in D(A)$, then $V(x,t)$ is differentiable with respect to t and

$$\frac{d}{dt} V(x,t) = [PAT(t)x, T(t)x] + [T(t)x, PAT(t)x] = -||B^* T(t)x||^2$$

Integrating we obtain

$$V(x,t) - V(x,0) = \int_0^t -||B^* T(\tau)x||^2 \, d\tau$$

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Equivalently,

$$0 \leq V(x,t) = V(x,0) - \int_0^t \|B^*T(\tau)x\|^2 \, d\tau$$

And for all $t \geq 0$ and $x \in \mathcal{D}(A)$

$$V(x,0) \geq \int_0^t \|B^*T(\tau)x\|^2 \, d\tau \qquad (7)$$

The inequality

$$\|B^*T(t)(x_n - x)\| \leq \|B^*\| M e^{\omega t} \|x_n - x\|$$

shows that if $x_n \to x$ then $B^*T(t)x_n \to B^*T(t)x$ uniformly on compact intervals of $[0,\infty)$. Hence the inequality (7) holds for all $x \in H$ since $\mathcal{D}(A)$ is dense in H. So we have that for all $x \in H$

$$[Px, x] = V(x,0) \leq \int_0^\infty \|B^*T(t)x\|^2 \, dt$$

proving sufficiency.

[Necessity] Assume that for all $x \in H$ the integral (5) is finite. For each $t \geq 0$, define the self-adjoint non-negative operator $P(t)$ by

$$P(t)x = \int_0^t T(\tau)^* BB^*T(\tau) \, d\tau$$

Note that for each $x, y \in H$, $P(t)$ satisfies:

1. $[P(t)x, y] = [P(t)y, x]$

2. $0 \leq [P(t_1)x, x] \leq [P(t_2)x, x]$ for $0 \leq t_1 \leq t_2$

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3.

$$\begin{aligned}
|[P(t)x,y]|^2 &= \left| \int_0^t [B^*T(s)x, B^*T(s)y]\, ds \right|^2 \\
&\leq \left(\int_0^t \|B^*T(s)x\|\, \|B^*T(s)y\|\, ds \right)^2 \\
&\leq \left(\int_0^t \|B^*T(s)x\|^2\, ds \right) \left(\int_0^t \|B^*T(s)y\|^2\, ds \right) \\
&\leq \int_0^\infty \|B^*T(s)x\|^2\, ds \int_0^\infty \|B^*T(s)y\|^2\, ds \\
&< \infty
\end{aligned}$$

So we have that

$$\sup_{t \in [0,\infty)} |[P(t)x,y]| < \infty$$

Applying the uniform boundedness principle,

$$\sup_{t \in [0,\infty)} \|P(t)\| < \infty$$

Since $P(t)$ is increasing with respect to t, there is a $P \geq 0$ such that

$$\lim_{t \to \infty} \|P(t)x - Px\| = 0$$

Denote this P by

$$Px = \int_0^\infty T(t)^* B B^* T(t) x\, dt$$

$P \geq P(t)$ and $[Px, x] < \infty$ by assumption. Now consider

$$[PT(t)x, T(t)x] = \int_t^\infty \|B^*T(\tau)x\|^2\, d\tau$$

Differentiating with respect to t yields

$$[PAT(t)x, T(t)x] + [T(t)x, PAT(t)x] = -\|B^*T(t)x\|^2$$

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Setting $t = 0$ we have the desired relation

$$[PAx, x] + [x, PAx] = -||B^*x||^2$$

which is defined by $x \in \mathcal{D}(A)$. This completes the proof of Theorem 7.

Given the existence of the integral (5) we have the following condition for strong stability.

Theorem 8 *If there exists an $\alpha > 0$, satisfying*

$$\alpha||x||^2 \leq \int_0^\infty ||B^*T(t)x||^2 \, dt < \infty \tag{8}$$

Then the semigroup $T(t)$, $t \geq 0$ is strongly stable.

proof First define

$$P = \int_0^\infty ||B^*T(s)x||^2 \, ds \tag{9}$$

and

$$P(t) = \int_t^\infty ||B^*T(s)x||^2 \, ds \tag{10}$$

From (8) we find the sufficiency for strong stability.

$$
\begin{aligned}
\alpha||T(t)x||^2 &\leq \int_0^\infty ||B^*T(\tau)T(t)x||^2 \, d\tau \\
&= \int_t^\infty ||B^*T(\tau)x||^2 \, d\tau \\
&= [(P - P(t))x, x] \\
&\leq ||(P - P(t))x|| \, ||x||
\end{aligned}
$$

Since

$$\lim_{t \to \infty} ||Px - P(t)x|| = 0$$

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it follows that $\lim_{t \to \infty} ||T(t)x|| = 0$. Hence $T(t)$, $t \geq 0$ is strongly stable.

A solution to the Lyapunov equation, unfortunately, does not guarantee that P defines an equivalent norm. As indicated above verifiable conditions are known in the exponentially stable case, when $B^* = I$. Pazy [11] showed:

Theorem 9 $T(t), t \geq 0$ *is exponentially stable if and only if for* $1 \leq p < 0$

$$||x||_p = \left(\int_0^\infty ||T(t)x||^p \, dt \right)^{1/p} < \infty$$

Moreover, if there exist constants $t_o > 0$ *and* $c > 0$ *such that* $||T(t_o)x|| \geq c||x||$ *for every* $x \in H$ *then* $||x||$ *and* $||x||_p$ *define equivalent norms.*

This is necessary as we prove in the following proposition. We find however that a similar condition does not carry over to our case

Proposition 4 *Suppose that for some* $\alpha > 0$,

$$\alpha||x||^2 \leq \int_0^\infty ||B^*T(t)x||^2 \, dt < \infty$$

for every $x \in H$ *and* $T(t)$, $t \geq 0$ *is a uniformly bounded semigroup with bound* M. *Then there exist constants* $c > 0$ *and* $t_o > 0$ *such that*

$$c||x|| \leq ||T(t_o)x|| \tag{11}$$

for every $x \in H$.

proof Suppose that no $c > 0$ and t_o exist. Then for every $\epsilon_1 > 0$ and every $t > 0$ there is an $x \in H$ such that $||x|| = 1$ and $||T(t)x|| < \epsilon_1$. We can estimate integral (5), where $0 < \tau < \infty$, by

$$\int_0^\infty ||B^*T(t)x||^2 \, dt = \int_0^\tau ||B^*T(t)x||^2 \, dt$$

$$+ \int_\tau^\infty ||B^*T(t)x||^2 \, dt$$

$$\leq \tau ||B^*||^2 M^2 ||x||^2 + [PT(\tau)x, T(\tau)x]$$

$$\leq \tau ||B^*||^2 M^2 ||x||^2 + ||P|| \, ||T(\tau)x||^2$$

Since $||x|| = 1$, by selecting

1. $\tau < \frac{\epsilon}{2||B^*||^2 M^2}$

2. $\epsilon_1 < \sqrt{\frac{\epsilon}{2||P||}}$

we have the arbitrary bound

$$\int_0^\infty ||B^*T(t)x||^2 \, dt < \epsilon$$

Since for some $x \in H$ with $||x|| = 1$, the integral (5) can be made arbitrarily small, the integral is not bounded below. This contradicts the given conditions so the proposition is true.

Unfortunately, this condition is not sufficient. We also observe that for a similar construction, if $B^*T(t_o)$ has a bounded inverse then the semigroup $T(t)$, $t \geq 0$ is actually exponentially stable. We find then

Proposition 5 *Suppose that $\int_0^\infty ||B^*T(t)x||^2 \, dt < \infty$ and for some $c > 0$, $t_o > 0$ and for every $x \in H$*

$$||B^*T(t_o)x|| \geq c||x|| \tag{12}$$

Then

1. $T(t), t \geq 0$ is exponentially stable.

2. The following are equivalent norms

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(a) $||x||$

(b) $(\int_0^\infty ||T(t)x||^2 \, dt)^{1/2}$

(c) $(\int_0^\infty ||B^*T(t)x||^2 \, dt)^{1/2}$

proof

1. Substituting $T(t)x$ for x in equation (12) we obtain

$$c^2||T(t)x||^2 \leq ||B^*T(t_o + t)x||^2$$

Integrating from 0 to ∞ yields

$$\begin{aligned}
\int_0^\infty ||T(t)x||^2 &\leq \frac{1}{c^2}\int_0^\infty ||B^*T(t_o + t)x||^2 \, dt \\
&= \frac{1}{c^2}\int_{t_o}^\infty ||B^*T(t)x||^2 \, dt \\
&\leq \frac{1}{c^2}\int_0^\infty ||B^*T(t)x||^2 \, dt \\
&< \infty
\end{aligned}$$

Since $\int_0^\infty ||T(t)x||^2 \, dt$ is finite for all $x \in H$ the semigroup is exponentially stable by Theorem (6).

2. Since B^* is bounded, we see that

$$\frac{c}{||B^*||}||x|| \leq ||T(t_o)x||$$

for every $x \in H$. Thus $T(t_o)$ is bounded below and Pazy's theorem [11] applies. $||x||$ and $(\int_0^\infty ||T(t)x||^2 \, dt)^{1/2}$ are equivalent norms. Moreover,

$$\begin{aligned}
c^2\int_0^\infty ||T(t)x||^2 \, dt &\leq \int_0^\infty ||B^*T(t)x||^2 \, dt \\
&\leq ||B^*||^2 \int_0^\infty ||T(t)x||^2 \, dt
\end{aligned}$$

So all three are equivalent norms. $\square$

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In the following example, we find that integral (5) is precisely equal to $||x||^2$.

Example 5 *Suppose that $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$. Consider the semigroup*

$$T(t)x = \sum_{n=1}^{\infty} e^{-t/n} [x, \phi_n] \phi_n$$

Note that this semigroup is strongly but not exponentially stable. Take B^* to be the compact operator defined by

$$B^* x = \sum_{n=1}^{\infty} \sqrt{2/n} \, [x, \phi_n] \phi_n$$

Then

$$B^* T(t)x = \sum_{n=1}^{\infty} e^{-t/n} \sqrt{2/n} \, [x, \phi_n] \phi_n$$

$$||B^* x||^2 = \sum_{n=1}^{\infty} \frac{2}{n} e^{-2t/n} |[x, \phi_n]|^2$$

So we have that

$$\begin{aligned} \int_0^{\infty} ||B^* T(t)x||^2 \, dt &= \sum_{n=1}^{\infty} \frac{2}{n} \int_0^{\infty} e^{-2t/n} \, dt |[x, \phi_n]|^2 \\ &= \sum_{n=1}^{\infty} |[x, \phi_n]|^2 \\ &= ||x||^2 \end{aligned}$$

For this example $P = I$ and

$$Ax = \sum_{n=1}^{\infty} -\frac{1}{n} [x, \phi_n] \phi_n$$

So we see the "Lyapunov" equation holds.

We might also note that the boundedness of B^* is not necessary in the use of integral (5) to verify strong stability. In the following example, (5) is an equivalent norm when restricted to the domain of A and C^* is unbounded.

Example 6 *Consider the backward shift semigroup defined by $T(t)f = f(\theta + t)$, where $f \in L_2[(0, \infty), \mathcal{H}_1] = \mathcal{H}$, and $\mathcal{H}_1$ is another Hilbert space with norm $||\cdot||_{\mathcal{H}_1}$. The domain of A is the space of absolutely continuous functions contained in $\mathcal{H}$. On this subspace, we can define the operator $C^* : \mathcal{H} \to \mathcal{H}_1$ by $C^*f = f(0)$. Then we have $C^*T(t)f = f(t)$ and $\mathcal{D}(A) = \mathcal{D}(C^*)$. In which case,*

$$\int_0^\infty ||C^*T(t)f||_{\mathcal{H}_1}^2 \, dt = \int_0^\infty ||f(t)||_{\mathcal{H}_1}^2 \, dt = ||f||_{\mathcal{H}}^2$$

The fact that $\int_0^\infty ||C^*T(t)f||_{\mathcal{H}_1}^2 \, dt = ||f||_{\mathcal{H}}^2$ implies that $T(t), t \geq 0$ is strongly stable on $\mathcal{D}(A)$ as suggested by Theorem 4. Since the semigroup is also uniformly bounded, strong stability on the dense subspace $\mathcal{D}(A)$ extends to strong stability on all of $\mathcal{H}$.

In the above development we require that the semigroup be uniformly bounded. An important question is then when is the feedback semigroup also uniformly bounded. We have one case where this can be verified.

Proposition 6 *Let P be a linear bounded, non-negative operator satisfying*

$$[Px, x] = \int_0^\infty ||B^*T(t)x||^2 \, dt$$

Assume that P defines an equivalent norm. Then the feedback semigroup $S(t)$, $t \geq 0$ generated by $A - BB^*P$ is uniformly bounded.

proof Substituting $T(t)x$ for x we obtain

$$[PT(t)x, T(t)x] = \int_0^\infty ||B^*T(t + \tau)x||^2 \, d\tau$$

If this is differentiated with respect to t we get

$$\frac{d}{dt}[PT(t)x, T(t)x] = 2Re[PAT(t)x, T(t)x] = -||B^*T(t)x||^2$$

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Then set $t = 0$ and subtract $2Re[P(a - BB^*P)x, x]$ to obtain

$$2Re[P(A - BB^*P)x, x] = -||B^*x||^2 - 2||B^*PS(t)x||^2$$

Substituting back $S(t)x$ for x, we see that

$$\frac{d}{dt}[PS(t)x, S(t)x] = -||B^*S(t)x||^2 - ||B^*PS(t)x||^2 \leq 0$$

So $[PS(t)x, S(t)x] \leq [Px, x]$. Since $[Px, x]$ defines an equivalent norm, there exist constants α_1 and α_2 such that $0 < \alpha_1 < \alpha_2$ and

$$\alpha_1||x||^2 \leq [Px, x] \leq \alpha_2||x||^2$$

We find that

$$\alpha_1||S(t)x||^2 \leq [PS(t)x, S(t)x] \leq [Px, x] \leq \alpha_2||x||^2$$

And

$$||S(t)x|| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}||x|| \quad \square$$

We should note that since P defines an equivalent norm $T(t)$, $t \geq 0$, is already a strong stable semigroup. If we define a self-adjoint non-negative D by $D^2 = BB^* + 2PBB^*P$, we also have

$$2Re[P(A - BB^*P)x, x] = -||Dx||^2$$

and applying Theorem 7

$$[Px, x] = \int_0^\infty ||DS(t)x||^2 \, dt$$

Consequently, $S(t)$, $t \geq$ is also strongly stable.

The following theorem of Triggiani suggests alternate criteria for the applicability of the preceding theory.

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Theorem 10 *Let U be a Hilbert space, and let the space of admissible controls be $L^2(U)$. The system (1) is exactly controllable if and only if there exist $t > 0$ and $\gamma > 0$ such that*

$$\int_0^t \|B^*T(\tau)^*x\|^2 \, d\tau > \gamma \|x\|^2$$

To obtain conditions like those we considered previously in this chapter, a sufficient condition would then be the exact controllablity of the system (A^*, B). In this case the above theorem indicates the existence of a $t > 0$ and a $\gamma > 0$ such that

$$\int_0^t \|B^*T(\tau)x\|^2 \, d\tau > \gamma \|x\|^2$$

In this case if the above integral is finite as $t \to \infty$ we can apply Proposition 4.

We can state that

Corollary 1 *Suppose U is a Hilbert space, and let the space of admissible controls be $L^2(U)$. If the system (A^*, B) is exactly controllable and there is a $P > 0$ satisfying $[PAx, x] + [x, PAx] = -\|B^*x\|^2$, then the semigroup $T(t)$, $t \geq 0$ is strongly stable.*

4.2 STRONGLY STABLE CONTRACTIONS

Previously, we saw that the backward shift semigroup on the half-infinite interval was strongly stable. Another interesting approach to investigating the strong stability of a contraction semigroup is to consider similarity to the backward shift. In this section we investigate the relationship between backward shift semigroups on various spaces and the original semigroup.

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Since an exponentially stable contraction semigroup is an extreme example of a strongly stable contraction semigroup let us examine briefly this special case first. As demonstrated earlier, if $T(t)$, $t \geq 0$ is an exponentially stable semigroup then there is a positive P such that

$$[Px, x] = \int_0^\infty ||T(t)x||^2 \, dt$$

In this case, we can ask what are the ramifications of the assumption that P defines an equivalent norm, that means that P is strictly positive. Consider an linear operator $V : \mathcal{X} \to L^2(R^+, \mathcal{X})$ defined by $y = Vx$ and $y(t) = T(t)x$. Moreover consider the inner product defined by P, namely $[x, y]_P = [Px, y]$. Then we can write

$$||x||_P^2 = [Px, x] = \int_0^\infty ||T(t)x||^2 \, dt$$

The range space of V, $\mathcal{R}(V)$ is a subspace of $L^2(R^+, \mathcal{X})$ and if V is viewed as a map $V : (\mathcal{X}, ||\cdot||_P) \to L^2(R^+, \mathcal{X})$ then V is an isometry. We can define a backward shift on the space $L^2(R^+, \mathcal{X})$, $B(t)$, $t \geq 0$. For any $x \in (\mathcal{X}, ||\cdot||_P)$ we have the relation

$$B(t)Vx = VT(t)x \tag{13}$$

Relation (13) shows that an exponentially stable contraction semigroup is "unitarily equivalent" to the backward shift restricted a subspace of the associated space $L^2(R^+, \mathcal{X})$. We will see that similar constructions exist for strongly stable semigroups on Hilbert spaces. On Banach spaces the relationship between strongly stable semigroups and backward shift semigroups on related spaces is even simpler.

The following two theorems are of interest to us.

Theorem 11 *[12] Any strongly continuous contractive semigroup can be extended to a strongly continuous coisometric semigroup.*

Coisometric means that the semigroup has an isometric adjoint semigroup.

Theorem 12 *[12] Let $V(t)$, $t \geq 0$ be a strongly continuous isometric semigroup on K. Then there are Hilbert spaces K and $\mathcal{L}$ and a strongly continuous unitary semigroup $U(t)$, $t \geq 0$ on $\mathcal{L}$, such that $V(t)$ is unitarily equivalent to $B^*(t) \oplus U(t)$, $t \geq 0$, where $B^*(t)$ is the forward translation semigroup on $L^2(R^+, K)$.*

Fillmore then suggests the following two problems which we resolve here.

Theorem 13 *A strongly continuous contractive semigroup is unitarily equivalent to a part of a backward translation semigroup if and only if $T(t) \to 0$ strongly as $t \to 0$.*

proof: As in the proof of the first Fillmore theorem the following development holds. The backward shift semigroup on an L^2 space is strongly stable. The adjoint of the backward shift is a forward shift which is an isometric semigroup, so the backward shift is coisometric. Conversely, let us consider a semigroup which is strongly stable. Define the bilinear form

$$[x, y]_1 = -[Ax, y] - [x, Ay]$$

on $\mathcal{D}(A)$. Since the semigroup is contractive this is non-negative. Let $\mathcal{N} = \{x \in \mathcal{D}(A) : [x, x]_1 = 0\}$, then $\mathcal{D}(A)/\mathcal{N}$ is a pre-Hilbert space and define the completion to be K. Take $W : \mathcal{D}(A) \to L^2(R^+, K)$ as $Wx(t) = T(t)x$. And we have

$$\|Wx\|^2_{L^2(R^+, K)} = \lim_{n \to \infty} \int_0^n \|T(t)x\|^2_1 \, dt$$

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$$= \lim_{n \to \infty} -\int_0^n 2Re[AT(t)x, T(t)x]\, dt$$

$$= \lim_{n \to \infty} -\int_0^n \frac{d}{dt}||T(t)x||^2\, dt$$

$$= ||x||^2 - \lim_{n \to \infty} ||T(n)x||^2$$

$$= ||x||^2$$

In this case W is an isometry on $\mathcal{D}(A)$ and consequently W can be extended to $\mathcal{H}$. From the fact that W is an isometry and hence bounded from below, the inverse, W^{-1} is defined on the range of W. Then in this case we must have for $y(\cdot) \in \mathcal{R} = W\mathcal{H} \subset L^2(\mathcal{R}^+, \mathcal{H})$, an $x \in K$ such that $y(t) = T(t)x$ and $W^{-1}y = x$.

Moreover, for $x \in \mathcal{D}(A)$, we have

$$(WT(s)x)(t) = T(t)T(s)x = T(t+s)x = (B(s)W)(t)$$

where $B(t)$, $t \geq 0$ is the backward shift on $L^2(R^+, K)$. We see that $\mathcal{R}$ is an invariant subspace of $B(t)$, $t \geq 0$. $T(t)$, $t \geq 0$ and $B(t)|_\mathcal{R}$, $t \geq 0$ are unitarily equivalent.

Then next question is then to obtain a representation for W^*. Consider

$$[Wx, y]_{L^2(\mathcal{R}^+, K)} = \int_0^\infty [T(t)x, y(t)]_1\, dt$$

$$= -\int_0^\infty ([AT(t)x, y(t)] + [T(t)x, Ay(t)])\, dt$$

$$= -\int_0^\infty [x, (A^*T(t)^* + T(t)^*A)y(t)]\, dt$$

$$= \left[x, -\int_0^\infty (A^*T(t)^* + T(t)^*A)y(t)\, dt\right]$$

We see that for appropriate $y(\cdot) \in L^2(\mathcal{R}^+, K)$ and W^* is given by $W^*y = v$,

$$v = -\int_0^\infty (A^*T(t)^* + T(t)^*A)y(t)\, dt$$

Since $y(t) \in K$, for all $t \geq 0$ we have, first of all, that $y(t) \in \mathcal{D}(A)$ and secondly, for $t > 0$, $T(t)^*y(t) \in \mathcal{D}(A^*)$. Thus v is defined if we can identify those $y(\cdot)$ for which this integral is finite.

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When $y(\cdot) \in \mathcal{R}(W)$ then there is a $z \in \mathcal{H}$ such that $y(t) = T(t)z$ and in this case we have

$$
\begin{aligned}
[W^*x, y] &= -\int_0^\infty \left([AT(t)x, T(t)z] + [T(t)x, AT(t)z]\right) dt \\
&= -\lim_{n\to\infty} \int_0^n \frac{d}{dt}[T(t)x, T(t)z] \, dt \\
&= [x, z]
\end{aligned}
$$

Thus for $y(t) \in \mathcal{R}(W)$, at least, W^* is defined. In addition, in this case,

$$
\|y\|_{L^2(\mathcal{R}^+, K)} = \|z\|_{\mathcal{H}}
$$

so that

$$
\|W^*y\|_{\mathcal{H}} = \|y\|_{L^2(\mathcal{R}^+, K)}
$$

This same analysis applies when $y(\cdot)$ is a forward shift of $T(t)z$. Suppose that $y(t) = T(t - t_o)z\mu(t - t_o)$, for $t_o > 0$, where $\mu(t)$ is the unit step function. In this case we have

$$
\begin{aligned}
[Wx, y] &= -\int_0^\infty \left([AT(t)x, \mu(t - t_o)T(t - t_o)z] \right. \\
&\qquad\qquad \left. + [T(t)x, \mu(t - t_o)AT(t - t_o)z]\right) dt \\
&= -\int_{t_o}^\infty \left([AT(t)x, T(t - t_o)z] + [T(t)x, AT(t - t_o)z]\right) dt \\
&= -\lim_{n\to\infty} \int_{t_o}^n \frac{d}{dt}[T(t)x, T(t - t_o)z] \, dt \\
&= [x, T(t_o)^*z]
\end{aligned}
$$

Then for this y we have $w = W^*y$, $w = T(t_o)^*z$. So we have $\|W^*y\|_{\mathcal{H}} \leq \|y\|_{L^2(\mathcal{R}^+, K)}$.

In the following lemma we see that terms of the form Wx, together with the forward shifts, generate $L^2(\mathcal{R}^+, K)$.

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Lemma 1 *Consider the set*

$$M = \{y \in L^2(\mathcal{R}^+, K) : y = Wx \text{ or } y = F(t)Wx\}$$

where $F(t)$ is the forward shift on $L^2(\mathcal{R}^+, K)$. Then the closure of the span of M is $L^2(\mathcal{R}^+, K)$.

proof: [12] We show that we can construct step functions of $L^2(\mathcal{R}^+, K)$ from the elements of M. Take $\psi_{[0,a]}$ to be the indicator function on the interval $[0, a]$. Then $f = \psi_{[0,a]}x \in L^2(\mathcal{R}^+, K)$ if $x \in K$. Now define a set functions

$$g_\tau(t) = T(t + \tau)x\mu(\tau - t)$$

Since $T(t)x$ converges to x in the $\|\cdot\|_1$-norm as $t \to 0^+$, for any $\epsilon_o > 0$ there is an N such that $\|x - T(t)x\|^2 < \epsilon_o$ for $t \le 2/N$. $\square$

Finally then, since $M = L^2(\mathcal{R}^+, K)$, we recognize that W^* is defined on all of $L^2(\mathcal{R}^+, K)$ and that for $x \in K$, $x = W^*Wx$. Moreover, W is an isometry and W^* is an isometry on $\mathcal{R}$. On $L^2(\mathcal{R}^+, K)$, W^* is a contraction.

A semigroup $T(t)$, $t \ge 0$ is equivalent to a backward shift semigroup [left-shift] if and only if it is strongly continuous, coisometric and $T(t) \to 0$ strongly as $t \to 0$.

In this case we must show that the added condition, that the semigroup is coisometric implies that the subspace $\mathcal{R}$ is in fact the whole space $L^2(R^+, K)$. First of all, note that $\mathcal{R}$ is invariant under $B(t)$, $t \ge 0$ and $B(t)$ is coisometric. Consequently, $\mathcal{R}$ reduces $B(t)$.

A contractive exponentially stable semigroup is an interesting example of a contractive strongly stable semigroup. Let us add a few additional observations.

Proposition 7 *Suppose $T(t)$, $t \ge 0$ is a C_o-semigroup, exponentially stable, and contractive. Then there is a constant $\omega_o < 0$ such that*

$$\|T(t)\| \leq e^{\omega_o t}$$

proof: We know that for an exponentially stable semigroup

$$\|T(t)\| \leq M e^{-\alpha t}$$

for $\alpha > 0$, and $M \geq 1$. What we would like to point out is the fact that for contraction semigroups the bound M is in fact 1. In this proof we utilize the notion of the characteristic growth property [13]

Let $\omega(t) = \log \|T(t)\|$ for $t \geq 0$. Define $\omega_o = \inf_{t \geq 0} \omega(t)/t$. Since ω_o is finite we choose an "a" such that $\omega(a)/a \leq \omega_o + \epsilon$ for some arbitrarily small ϵ. Then set $t = ka + r$, with k, a non-negative integer, and $0 \leq r < a$.

Then

$$\begin{aligned}
\frac{\omega(t)}{t} &= \frac{\omega(ka+r)}{ka+r} \leq \frac{k\omega(a)}{ka+r} + \frac{\omega(r)}{ka+r} \\
&\leq \frac{\omega(a)}{a+r/k} + \frac{\omega(r)}{t}
\end{aligned}$$

$$\frac{\omega(t)}{t} \leq \omega_o + \epsilon + \frac{\omega(r)}{t} \tag{14}$$

For contraction semigroups

$$\omega(r) = \log \|T(r)\| \leq 0 \tag{15}$$

since $\|T(r)\| \leq 1$. Consequently (14) and (15) imply

$$\frac{\omega(t)}{t} \leq \omega_o + \epsilon$$

$$\|T(t)\| \leq e^{(\omega_o + \epsilon)t}$$

Finally, since ϵ is arbitrary, for an exponentially stable contraction semigroup,

$$\|T(t)\| \leq e^{\omega_o t}$$

for some $\omega_o < 0$. $\square$

Chapter 5

STABILIZATION

As was evinced in the proposition (2) and will be shown in this chapter, uniform boundedness is a key property in verifying the stability of a C_o-semigroup. In this section we will review the germane material and present two suitable conditions for a feedback semigroup to be uniformly bounded. Then we will combine these conditions for a uniformly bounded semigroup and the conditions for weak and strong stability. This will enable us to present new results on the stabilization problem.

It is more convenient in this case to focus on C_o-semigroups defined on Banach spaces. In this framework we lose the inner product, which is replaced by the notion of duality. We will denote by X, a Banach space with norm $|| \cdot ||$ and the dual space X^*. The dual space consists of the continuous linear functionals on the Banach space X.

Definition 6 *[14] For a Banach space X and the associated dual space X^* we define the multivalued duality map J by*

$$J(f) = \{\phi \in X^* : ||\phi||^2 = ||f||^2 = [f, \phi]\}$$

Define a duality section J of J by $J : X \rightarrow X^*$ and $J(f) \in \mathcal{J}(f)$ for every $f \in X$. Then the operator A is called dissipative with respect to the duality section J if for every $f \in \mathcal{D}(A)$, $Re[Af, J(f)] \leq 0$. A on X is dissipative if A is dissipative with respect to some duality section.

A is said to m-dissipative if it is dissipative and $\{\rho(A) \cap (0, \infty)\} = \emptyset$.

Based on these definitions we have the Lumer-Phillips form of the Hille-Yosida Theorem. [15]

Theorem 14 *Suppose A generates a contraction C_o-semigroup on X. Then*

1. $\overline{\mathcal{D}(A)} = X$

2. A is dissipative with respect to any duality section

3. $(0, \infty) \subset \rho(A)$

Conversely, if

1. $\overline{\mathcal{D}(A)} = X$

2. A is dissipative with respect to some duality section

3. $\{(0, \infty) \cap \rho(A)\} \neq \emptyset$

then A generates a contraction C_o-semigroup on X.

To verify the dissipativity of an operator the following condition may be useful.

Proposition 8 *[16] A is dissipative if and only if for each $\lambda > 0$*

$$\|(\lambda I - A)^{-1}\| \leq 1/\lambda$$

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We considered in the previous chapter a Hilbert space $\mathcal{H}$ and the feedback $-B^*$ to obtain the infinitesimal generator $A - BB^*$. When A is dissipative $A - BB^*$ also is dissipative so the semigroup generated by $A - BB^*$ is a contraction. However, when A is not a contraction, it has not been determined whether the resulting semigroup is a contraction or even uniformly bounded. We will present conditions under which feedback semigroups are uniformly bounded.

Suppose that $T(t)$, $t \geq 0$ was a uniformly bounded semigroup. We can make the following change of norm to obtain a space where "$T(t)$" is a contraction. Let us denote this new norm by $||\cdot||_n$ and denote by

$$I \quad : \quad (\mathcal{X}, ||\cdot||) \to (\mathcal{X}, ||\cdot||_n)$$

$$I^{-1} \quad : \quad (\mathcal{X}, ||\cdot||_n) \to (\mathcal{X}, ||\cdot||)$$

the identity maps relating the same element in each space. Suppose then that $y \in (\mathcal{X}, ||\cdot||)$, $z \in (\mathcal{X}, ||\cdot||_n)$ and $z = Iy$. Now define the new norm by

$$||z||_n = \sup_{t \in [0,\infty)} ||T(t)y||$$

First since $T(t)$, $t \geq 0$ is uniformly bounded,

$$||y|| \leq \sup_{t \in [0,\infty)} ||T(t)y|| \leq M||y||$$

and so

$$||y|| \leq \sup_{t \in [0,\infty)} ||z||_n \leq M||y||$$

We have an equivalent norm. The semigroup on the new space is then $T_n(t) = IT(t)I^{-1}$, $t \geq 0$. This is a contraction since

$$
\begin{aligned}
||T_n(t)z||_n &= \sup_{\tau \in [0,\infty)} ||T(t+\tau)y|| \\
&\leq \sup_{\tau \in [0,\infty)} ||T(\tau)y|| \\
&= ||z||_n
\end{aligned}
$$

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In theorem (4) we saw conditions under which a feedback semigroup might be a contraction semigroup. This theorem also holds in the Banach space case. This variation can be used in the Banach space framework to obtain the following theorem.

Theorem 15 *Let A be the infinitesimal generator of a uniformly bounded semigroup $T(t)$, $t \geq 0$ on a Banach space $(X, \|\cdot\|)$. Suppose that C is a bounded dissipative operator on $(X, \|\cdot\|_n)$. Then $A + I^{-1}CI$ also generates a uniformly bounded semigroup $S(t)$, $t \geq 0$.*

proof: On $(X, \|\cdot\|_n)$ then operators IAI^{-1} and C are dissipative and generate contraction semigroups. Since C is bounded we find a new semigroup $S_n(t)$, $t \geq 0$ generated by the sum $IAI^{-1} + C$. In the original space $I^{-1}CI$ is also bounded and the sum $A + I^{-1}CI$ also generates a semigroup $S(t) = I^{-1}S_n(t)I$. Then

$$
\begin{aligned}
\|S(t)y\| &\leq \|S_n(t)z\| \\
&\leq \|z\|_n \\
&\leq M\|y\|
\end{aligned}
$$

We see that the semigroup $S(t)$, $t \geq 0$ is uniformly bounded $\square$.

In the following example it is possible to verify that dissipative feedback actually decreases the bound of a uniformly non-contractive semigroup.

Example 7 *On the Hilbert space, $\mathcal{H} = L_2[(0,\infty), R] \oplus L_2[(0,\infty), R]$ consider the projection operator given by $Q : L_2[(0,\infty), R] \rightarrow L_2[(0,\infty), R]$ onto the set of intervals $I = \bigcup_{n=1}^{\infty} [m_k - 1, m_k]$, where the m_k satisfy the relation*

$$
m_{k_1} < m_{k_2} \Rightarrow 4m_{k_1} \leq m_{k_2} - 1
$$

Then Q can be represented by

$$(Qf)(x) = \sum_{n=1}^{\infty} U(x + 1 - m_k)U(m_k - x)f(x)$$

First take the infinitesimal generator

$$A = \begin{bmatrix} \partial/\partial x & Q \\ 0 & -\partial/\partial x \end{bmatrix}$$

Then the semigroup generated by A is

$$T(t) = \begin{bmatrix} S^{\cdot}(t) & Q(t) \\ 0 & S(t) \end{bmatrix}$$

$S^{\cdot}(t)$ *and* $S(t)$ *are the left and right shifts.* $Q(t)$ *is represented by*

$$(Q(t)f) = \int_0^t S^{\cdot}(t - \tau)QS(\tau)f \, d\tau$$

Now we can apply a dissipative feedback given by

$$F = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & -\partial/\partial x \end{bmatrix}$$

F is also the generator of a contraction semigroup and so it is dissipative. Let us then investigate the boundedness of the semigroup $V(t)$, $t \geq 0$ *generated by* $A + F$, *noting that*

$$A + F = \begin{bmatrix} 2\partial/\partial x & Q \\ 0 & -2\partial/\partial x \end{bmatrix}$$

First note the effect the coefficients "2" on $\partial/\partial x$ *and* $\partial/\partial x$. *These generate the shift semigroups, respectively,*

$$L_2(t)f(x) = U(x - 2t)f(x - 2t)$$
$$R_2(t)f(x) = f(x + 2t)$$

Also define

$$\chi(x) = \sum_{n=1}^{\infty} U(x + 1 - m_k)U(m_k - x)$$

And for $g = Q(t)f$ we have

$$\begin{aligned} g(x) &= \int_0^t R_2(t - s)QL_2(s)f \, ds \\ &= \int_0^t \chi(x + 2t - 2s)U(x + 2t - 4s)f(x + 2t - 4s) \, ds \end{aligned}$$

Then the semigroup generated by $A + F$ can be expressed as

$$V(t) = \begin{bmatrix} L_2(t) & Q(t) \\ 0 & R_2(t) \end{bmatrix}$$

A meticulous calculation in the Appendix A shows that

$$\|T(t)\| \leq \sqrt{3}$$

while

$$\|V(t)\| \leq \sqrt{2}$$

Of course the point of the previous development is to provide another tool for stabilization. One would expect that if a bounded dissipative feedback is applied to a system where A generates a uniformly bounded semigroup, stability should be improved. At the very least, this should not destabilize an already stable system.

Recall that in the chapter on weak stability conditions we found that if for some uniformly bounded semigroup $T(t)$, $[PT(t)x, T(t)x] \leq [Px, x]$ and $P > 0$ then the semigroup is at least weakly stable. It would be helpful here to note the relationship between controllability of (A, B) and $(A + BF, B)$. The following result is applicable.

Theorem 16 *Let F be a bounded operator. Suppose A generates the semigroup $T(t)$, $t \geq 0$ and $A + BF$ generates the semigroup $S(t)$, $t \geq 0$. Then the approximate controllability of the systems (A, B) and $(A + BF, B)$ are equivalent.*

proof: First we need to derive two identities. Consider the system

$$\dot{x}(t) = A^* x(t) + F^* B^* x(t)$$

then

$$x(t) = T(t)^* x(0) + \int_0^t T(t-s)^* F^* B^* x(s) \, ds$$

or

$$x(t) = S(t)^* x(0)$$

Combining these two we have, after setting $x(0) = x$,

$$S(t)x = T(t)^* x + \int_0^t T(t-s)^* F^* B^* S(s)^* x \, ds \tag{16}$$

In a similar manner we use

$$\dot{x}(t) = (A + BF)^* x(t) - F^* B^* x(t)$$

to obtain

$$x(t) = S(t)^* x(0) - \int_0^t S(t-s)^* F^* B^* x(s) \, ds$$

as well as

$$x(t) = T(t)^* x(0)$$

These yield

$$T(t)^* x = S(t)^* tx + \int_0^t S(t-s)^* F^* B^* T(s)^* x \, ds \tag{17}$$

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To show that (A, B) controllable implies $(A + BF, B)$ controllable we assume that $(A + BF, B)$ is not controllable. Hence there is an $x \neq 0$ such that for all $t \geq 0$, $B^*S(t)^*x = 0$. From the identity (16) derived above we also have that $B^*T(t)^*x = 0$ for all $t \geq 0$. This contradicts the controllability of (A, B). So it must be true that

(A, B) controllable $\Rightarrow$ $(A + BF, B)$ controllable

In a similar way, to verify $(A + BF, B)$ controllable implies (A, B) controllable, we assume that (A, B) is not controllable. Then there is an $x \neq 0$ such that for all $t \geq 0$, $B^*T(t)^*x = 0$. Applying the identity (17) we again find $B^*S(t)^*x = 0$ for all $t \geq 0$. This contradicts the controllability of $(A + BF, B)$. So it must also be true that

$(A + BF, B)$ controllable $\Rightarrow$ (A, B) controllable

$\square$.

We can state the following proposition,

Proposition 9 *Let $T(t)$, $t \geq 0$ be a uniformly bounded C_o- semigroup with infinitesimal generator A. Assume $A + BF$ is the infinitesimal generator of the semigroup $S(t)$, $t \geq 0$. Suppose the system (A, B) is controllable, $BF = I^{-1}CI$ where is a C is a dissipative bounded operator as described above. and*

$$\int_0^\infty ||B^*S(t)^*x||^2 \, dt < \infty \tag{18}$$

then the semigroup $S(t)$, $t \geq 0$ is not only uniformly bounded, but also weakly stable.

proof: From the previous proposition we see that $(A+BF, B)$ is also controllable and then the integral (18) defines a positive operator P, where

$$Px = \int_0^\infty S(t)BB^*S(t)^*x\,dt$$

Moreover, $[PS(t)^*x, S(t)^*x] \leq [Px, x]$, so $S(t)^*$ is a weakly stable semigroup. Then the adjoint semigroup $S(t)$ is also weakly stable. $\square$.

We can make the usual extension to strong stability when the appropriate resolvent condition is satisfied. If the resolvent of A is compact then so is the resolvent of $A+BF$. In the case where the conditions of the previous proposition are satisfied then the weak stability of $S(t)$, $t \geq 0$ implies that the feedback semigroup is also strongly stable.

If the operator P is strictly positive, as would be implied by exact controllability of $(A + BF, B)$ then we would also find that the feedback semigroup is strongly stable.

There is one generalization of theorem (15) that we would like to point out before we leave this topic. Rather than a bounded perturbation, we consider the so called Kato perturbation [14].

Definition 7 C *is a Kato Perturbation of A if $D(C) \supset D(A)$ and if for every* $a > 0$ *there is a $c > 0$ such that*

$$||Cx|| \leq a||Ax|| + c||x||$$

For such perturbations we have the following perturbation theorem [3] for the generation of a contraction semigroup.

Theorem 17 *If A generates a C_o contraction semigroup and if C is a dissipative Kato perturbation of A, then $A + C$ generates a C_o contraction semigroup.*

Which leads us to the following theorem

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Theorem 18 *Let A be the infinitesimal generator of a uniformly bounded semi-group $T(t)$, $t \geq 0$ on a Banach space $\mathcal{X}$. Suppose that C is a contraction on $(\mathcal{X}, \| \cdot \|_n)$ and a Kato perturbation of the operator A. Then $A + I^{-1}CI$ also generates a uniformly bounded semigroup $S(t)$, $t \geq 0$.*

proof: We must verify that if

$$\|Cz\|_n \leq a\|IAI^{-1}z\| + b\|z\|$$

then $I^{-1}CI$ is also Kato bounded, in which case $A + I^{-1}CI$ also generates a uniformly bounded semigroup. First of all, since

$$\|y\| \leq \|z\|_n \leq M\|y\|$$

we have $\|I\| = M$ and $\|I^{-1}\|_n = 1$ and it follows then that

$$
\begin{aligned}
\|I^{-1}CIy\| &\leq \|I^{-1}\|_n\|CIy\| \\
&\leq aM\|Ay\| + bM\|y\|
\end{aligned}
$$

$\square$

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Appendix A

EXAMPLES OF UNIFORM BOUNDEDNESS

Since so much emphasis has been placed on contraction semigroups by other authors, one might question the significance of uniformly bounded semigroups. An interesting result of Packel [17] provides conditions under which a uniformly bounded semigroup is not similar to a contraction semigroup. He shows that

Theorem 19 *If $T(t)$, $t \geq 0$ is similar to a contraction semigroup and A is the infinitesimal generator of A, then if $W(A)$ is the weakly stable subspace of A,*

$$W(A) \bigcap W(A^*)^\perp = 0$$

Packel [17] also presented the first example of a uniformly bounded semigroup which is not similar to a contraction.

Example 8 *Consider the space $\mathcal{H} = L_2[(0,\infty), R] \oplus L_2[(0,\infty), R]$. Let $S^*(t)$, $t \geq 0$ denote the left shift semigroup and $S(t)$, $t \geq 0$ denote the right shift. Let*

$P(t)$, $t \geq 0$ be another family of linear bounded operators. Suppose then that

$$(S(t)f)(x) = f(x-t)U(x-t)$$
$$(S^{\cdot}f)(x) = f(x+t)$$
$$-(P(t)f)(x) = f(2 \cdot 4^k - t - x)$$

when $x \in [0, 2 \cdot 4^l - t]$ or $x \in (4^k - t, 4^k]$ or $k \geq l$ and $P(t)f(x) = 0$, otherwise. Here l is the unique integer such that $4^l < t \leq 4^{l+1}$. And the semigroup given by

$$T(t) = \begin{bmatrix} S^{\cdot}(t) & P(t) \\ 0 & S(t) \end{bmatrix}$$

is uniformly bounded, not similar to a contraction and $\|T(t)\| \leq 2$.

Another interesting uniformly bounded semigroup was constructed by Benchimol [18].

Example 9 *First define a projection operator*

$$P : L_2[(0, \infty), R] \to L_2[(0, \infty), R]$$

onto the set of intervals $I = \bigcup_{n=1}^{\infty} [3^k - 1, 3^k]$. *Then P can be represented by*

$$(Pf)(x) = \sum_{n=1}^{\infty} U(x + 1 - 3^k)U(3^k - x)f(x)$$

Next the infinitesimal generator is taken to be

$$A = \begin{bmatrix} \partial/\partial x & P \\ 0 & -\partial/\partial x \end{bmatrix}$$

In this case the semigroup generated by A is shown to be

$$T(t) = \begin{bmatrix} S^\cdot(t) & P(t) \\ 0 & S(t) \end{bmatrix}$$

Again, $S^\cdot(t)$ and $S(t)$ are the left and right shifts and $P(t)$ can be written as

$$(P(t)f) = \int_0^t S^\cdot(t - \tau)PS(\tau)f \, d\tau$$

In this case Benchimol computes the bound of $T(t)$ and finds that

$$\|T(t)\| \leq \sqrt{3}$$

Benchimol also notes that if the intervals are defined by $[m_k - 1, m_k]$ where these integers satisfy the relation

$$m_{k_1} < m_{k_2} \Rightarrow 2m_{k_1} \leq m_{k_2} - 1 \tag{19}$$

Then if $P : L_2[(0,\infty), R] \to L_2[(0,\infty), R]$ is once again the projection operator onto the set of intervals $I = \bigcup_{k=1}^\infty [m_k - 1, m_k]$, the semigroup $T(t)$, $t \geq 0$, as developed above, is also uniformly bounded semigroup which is not similar to a contraction.

It is possible to modify this generalization and illustrate our theorem on the application of a dissipative feedback to uniformly bounded semigroups.

Example 10 *On the same Hilbert space,* $\mathcal{X} = L_2[(0,\infty), R] \oplus L_2[(0,\infty), R]$ *Consider the projection operator given by* $Q : L_2[(0,\infty), R] \to L_2[(0,\infty), R]$ *onto the set of intervals* $I = \bigcup_{n=1}^\infty [m_k - 1, m_k]$, *where the* m_k *satisfy the relation*

$$m_{k_1} < m_{k_2} \Rightarrow 4m_{k_1} \leq m_{k_2} - 1$$

Then Q can be represented by

$$(Qf)(x) = \sum_{n=1}^{\infty} U(x + 1 - m_k) U(m_k - x) f(x)$$

Again we have the infinitesimal generator

$$A = \begin{bmatrix} \partial/\partial x & Q \\ 0 & -\partial/\partial x \end{bmatrix}$$

And the semigroup generated by A is

$$T(t) = \begin{bmatrix} S^{\cdot}(t) & Q(t) \\ 0 & S(t) \end{bmatrix}$$

$S^{\cdot}(t)$ *and* $S(t)$ *are the left and right shifts.* $Q(t)$ *is represented by*

$$(Q(t)f) = \int_0^t S^{\cdot}(t - \tau) Q S(\tau) f \, d\tau$$

In this case the choice of m_k *also satisfies Benchimol's condition (19). Benchimol's calculations are still valid and the bound for* $T(t)$ *is given by* $\|T(t)\| \leq \sqrt{3}$.

Now let us modify things a little bit. We know that on $L_2[(0, \infty), R]$, $\partial/\partial x$ *and* $-\partial/\partial x$ *generate contraction semigroups, namely, the shifts,* $S^{\cdot}(t)$ *and* $S(t)$.

On $\mathcal{H} = L_2[(0, \infty), R] \oplus L_2[(0, \infty), R]$ *the operator* F

$$F = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & -\partial/\partial x \end{bmatrix}$$

is also the generator of a contraction semigroup and so it is dissipative. Let us then investigate the boundedness of the semigroup $V(t)$, $t \geq 0$ *generated by* $A + F$, *noting that*

$$A + F = \begin{bmatrix} 2\partial/\partial x & Q \\ 0 & 2 - \partial/\partial x \end{bmatrix}$$

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First note the effect the coefficients "2" on $\partial/\partial x$ and $\partial/\partial x$. These generate the shift semigroups semigroups, respectively,

$$L_2(t)f(x) = U(x - 2t)f(x - 2t)$$

$$R_2(t)f(x) = f(x + 2t)$$

Also define

$$\chi(x) = \sum_{n=1}^{\infty} U(x + 1 - m_k)U(m_k - x)$$

And for $g = Q(t)f$ we have

$$
\begin{aligned}
g(x) &= \int_0^t R_2(t - s)QL_2(s)f\,ds \\
&= \int_0^t \chi(x + 2t - 2s)U(x + 2t - 4s)f(x + 2t - 4s)\,ds
\end{aligned}
$$

Then the semigroup generated by $A + F$ can be expressed as

$$V(t) = \left[\begin{array}{cc} L_2(t) & Q(t) \\ 0 & R_2(t) \end{array}\right]$$

Using similar techniques to Benchimol we show that this new semigroup is also uniformly bounded. First of all, for any pair of f and g we have

$$
\begin{aligned}
g(x) &= \frac{1}{2}\int_x^{x+2t} \chi(v)U(2v - x - 2t)f(2v - x - 2t)\,dv \\
&= \int_{\sup\{x,(x+2t)/2\}}^{2+2t} \chi(v)f(2v - x - 2t)\,dv
\end{aligned}
$$

Then if, $\chi_k(v) = U(v - m_k + 1)U(m_k - v)$ we have

$$g_k(x) = \int_{\sup\{x,(x+2t)/2\}}^{2+2t} \chi_k(v)f(2v - x - 2t)\,dv$$

and

$$g(x) = \sum_{k=1}^{\infty} g_k(x)$$

Eliminating the χ_k by applying the definition of the step functions we obtain for g_k

$$g_k(x) = \frac{1}{2} \int_{\sup\{x,(x+2t)/2,m_k-1\}}^{\inf\{x+2t,m_k\}} f(2v - x - 2t)\, dv$$

If we make the change of variables $u = 2v - x - 2t$ we obtain

$$g_k(x) = \frac{1}{4} \int_{\sup\{x-2t,0,2m_k-2-x-2t\}}^{\inf\{x+2t,2m_k-x-2t\}} f(2v - x - 2t)\, dv$$

A careful examination of the limits of this integral reveal that $g_k(x)$ equals zero if:

1. $x + 2t \leq 2mk - 2 - x - 2t \iff x \leq m_k - 2t - 1$

2. $2m_k - x - 2t \leq x - 2t \iff x \geq m_k$

3. $2m_k - x - 2t \leq 0 \iff x \geq 2m_k - 2t$

4. $x < 0$

We then see that

$$\text{support}\{g_k\} \subseteq [\sup\{0, m_k - 2t - 1\}, \inf\{m_k, 2m_k - 2t\}] \tag{20}$$

Breaking down the remaining calculations into three propositions adds to the comprehension of the following.

Proposition 10 *For $t > 0$ there is at most one $k > 0$ such that*

$$m_k \leq 2t < 4m_k \tag{21}$$

proof: Suppose that there are integers $k_1 < k_2$ such that

$$m_{k_1} \leq 2t < 4m_{k_1}$$

and

$$m_{k_2} \leq 2t < 4m_{k_2}$$

so

$$m_{k_1} \leq 2t < 4m_{k_2} \tag{22}$$

Since $k_1 < k_2$, $m_{k_1} < m_{k_2}$, and for every k, $4m_k \leq m_k - 1$ we have

$$4m_{k_1} \leq m_{k_2} - 1 \tag{23}$$

When equations (22) and (23) are combined we obtain

$$4m_{k_1} \leq m_{k_2} - 1 \leq 2t - 1 \leq 4m_{k_1} - 1$$

So the assumption that $k_1 < k_2$ must be false. $\square$

Proposition 11 *The functions $g_k(x)$ have disjoint supports.*

proof: In the last proposition it was shown that there was at most one t to satisfy (21). Given t, suppose there is a k_o such that (21) holds. In this case

$$\text{support}\{g_{k_o}(x)\} \subseteq [0, 2m_k - 2t]$$

Then let k_1 and k_2 be integers satisfying the inequalities $k_o < k_1 < k_2$. In which case

$$m_{k_1} > 2t \text{ and } m_{k_2} > 2t \tag{24}$$

(i) To show that the supports of g_{k_o} and g_{k_1} are disjoint, first note from (24) that

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$$2m_{k_1} - 2t > m_{k_1} \tag{25}$$

in addition, since $4m_{k_o} \leq m_{k_1} - 1$ we have the relation

$$2t < 4m_{k_o} \leq m_{k_1} - 1 \Rightarrow 0 < 4m_{k_o} - 2t \leq m_{k_1} - 2t - 1 \tag{26}$$

Then since the support of g_k is given by (20) we see that the support of g_k also satisfies

$$\text{support}\{g_{k_1}\} \subseteq [m_{k_1} - 2t - 1, m_{k_1}] \tag{27}$$

When one combines (25), (26), and (27), we see that the supports of g_{k_o} and g_{k_1} are disjoint.

(ii) To see that the supports of g_{k_1} and g_{k_2} are disjoint, it will be sufficient to deduce that $M_{k_1} \leq m_{k_2} - 2t - 1$. We note that

$$4m_{k_1} \leq m_{k_2} - 1 \Rightarrow 4m_{k_1} - 2t \leq m_{k_2} - 2t - 1 \tag{28}$$

and that

$$m_{k_1} < rm_{k_1} - 2t \leq m_{k2} - 2t - 1 \tag{29}$$

$\square$ Once we compute the bound for $\|Q(t)\|$ we will have completed the hard work.

Proposition 12 $\|Q(t)\| \leq \frac{1}{2}$

proof: Let's start with a couple of new definitions. Let

$$J(x) = \sup\{0, x - 2t, 2(m_k - 1) - x - 2t\}$$

$$K(x) = \inf\{x + 2t, 2m_k - x - 2t\}$$

Since

$$J(x) \leq 2(m_k - 1) - x - 2t$$

$$K(x) \geq 2m_k - x - 2t$$

It follows that $K(x) - J(x) \leq 2$. With this new notation we can rewrite g_k as

$$g_k(x) = \frac{1}{4} \int_{J(x)}^{K(x)} f(u)\, du$$

Suppose that $2t \leq m_k \leq 4t$, then applying (20) we see that

$$\text{support}\{g_k(x)\} \subseteq [0, 2m_k - 2t]$$

On the other hand, if $m_k > 2t$

$$\text{support}\{g_k\} \subseteq [2m_k - 2t - 1, m_k]$$

Now denote by $[a(k), b(k)]$ be the largest of these two intervals. Then we find that

$$
\begin{aligned}
\|g_k\|^2 &= \int_0^\infty |g_k(x)|^2\, dx \\
&= \frac{1}{16} \int_{a(k)}^{b(k)} \left(\int_{J(x)^{K(x)}} f(s)\, dx \right)^2 dx \\
&\leq \frac{1}{16} \left(\int_{J(x)}^{K(x)} 1^2\, ds \right) \left(\int_{J(x)}^{K(x)} f(s)^2\, ds \right)\, dx \\
&\leq \frac{1}{8} \left(\int_{J(x)}^{K(x)} f(s)^2\, ds \right)\, dx
\end{aligned}
$$

As a result of the definition of $[a(k), b(k)]$ we have the inclusion

$$[J(x), K(x)] \subseteq [a(k), b(k)]$$

and the above integral can be bounded as

$$\|g_k\|^2 \;\leq\; \frac{1}{8}\int_{a(k)}^{b(k)}\int_{a(k)}^{b(k)} U(K(x)-s)U(s-J(x))f^2(s)\,dx\,ds$$

$$\leq\; \frac{1}{8}\int_{a(k)}^{b(k)} f^2(s)\int_{a(k)}^{b(k)} U(K(x)-s)U(s-J(x))\,dx\,ds$$

If we replace $K(x)$ we see that

$$K(x)-s = \inf\{x+2t, 2m_k - x - 2t\} - s \leq 2m_k - x - 2t$$

$$U(K(x)-s) \leq U(2m_k - x - 2t - s)$$

If a similar manner we can replace $J(x)$ to obtain

$$s-J(x) \;=\; s-\sup(0, x-2t, 2(m_k-1)-x-2t)$$

$$=\; s + \inf(0, 2t-x, x+2t-2(m_k-1))$$

$$\leq\; s + x + t - 2(m_k-1))$$

And as a result,

$$U(s-J(x)) \leq U(s+x+t-2(m_k-1))$$

When these step functions are substituted back in

$$\int_{a(k)}^{b(k)} U(K(x)-s)U(s-J(x))\,ds$$

$$\leq\; \int_{a(k)}^{b(k)} U(2m_k - x - 2t - s)U(s+x+2t-2(m_k-1))\,dx$$

$$=\; \int_{\sup\{a(k),2(m_k-1)-s-2t\}}^{\inf\{2m_k-s-2t,b(k)\}} dx$$

$$\leq\; 2$$

From here we see that for each g_k

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$$\|g_k\|^2 \le \frac{1}{8} \int_{a(k)}^{b(k)} f(s)^2 \, ds \cdot 2$$

Since the supports of the individual g_k's are disjoint it is easy to compute the norm of g. We have for g

$$\begin{aligned}
\|g\|^2 &= \sum_{k=1}^{\infty} \|g_k\|^2 \\
&\le \sum_{k=1}^{\infty} \frac{1}{4} \int_{a(k)}^{b(k)} f(s)^2 \, ds \\
&\le \frac{1}{4} \|f\|^2
\end{aligned}$$

Finally we find that $\|g\| \le (1/2)\|f\|$ and

$$\|Q(t)\| \le \frac{1}{2}$$

Computing the norm of $T(t)f$ we see that

$$\begin{aligned}
\|T(t)f\|^2 &= \|S^*(t)f_1 + Q(t)f_2\|^2 + \|S(t)f_2\|^2 \\
&\le \left(\|f_1\| + \frac{1}{2}\|f_2\| \right)^2 + \|f_2\|^2 \\
&\le 2\left(\|f_1\|^2 + \frac{1}{4}\|f_2\|^2 \right) + \|f_2\|^2 \\
&\le 2\|f_1\|^2 + \frac{3}{2}\|f_2\|^2 \\
&\le 2\|f\|^2
\end{aligned}$$

Therefore $\|T(t)\| \le \sqrt{2}$ and the application of the dissipative feedback, in fact, decreases the bound for this particular example.

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