CMS Technical Summary Report #88-31

DERIVATION AND IMPLICIT SOLUTION OF THE HARRY DYM EQUATION, AND ITS CONNECTIONS WITH THE KORTEWEG-DE VRIES EQUATION

W. Hereman, P. P. Banerjee and
M. R. Chatterjee

UNIVERSITY OF WISCONSIN

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

April 1988

(Received April 8, 1988)

Approved for public release
Distribution unlimited

Sponsored by

Air Force Office of Scientific Research
Bolling Air Force Base
Washington, DC 20332

National Science Foundation
Washington, DC 20550

- / -

UNIVERSITY OF WISCONSIN-MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

DERIVATION AND IMPLICIT SOLUTION OF THE HARRY DYM EQUATION, AND ITS CONNECTIONS WITH THE KORTEWEG-DE VRIES EQUATION

W. Hereman[1,*], P. P. Banerjee[1,**] and M. R. Chatterjee[2]

Technical Summary Report #88-31
April 1988

ABSTRACT

The Harry Dym equation, which is related to the classical string problem, is derived in three different ways. An implicit cusp solitary wave solution is constructed via a simple direct method. The existing connections between the Harry Dym and the Korteweg-de Vries equations are uniformised and simplified, and transformations between their respective solutions are carried out explicitly. Whenever possible, physical insights are provided. (KF)

AMS (MOS) Subject Classification: 35G20

PACS Classification: 3.40K

Key Words: cusp soliton, Harry Dym, KdV, direct method, implicit solitary wave

[1]Also Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706.

[2]Department of Electrical Engineering, SUNY at Binghamton, Binghamton, NY 13901.

[*]Supported in part by the Air Force Office of Scientific Research under Grant No. 85-NM-0263.

[**]On leave of absence from Department of Electrical Engineering, Syracuse University, Syracuse, NY 13244. Supported in part by the National Science Foundation under Grant No. ECS-8603643, the National Science Foundation Presidential Young Investigator Award under Grant No. MIP 8657765 and a matching grant from Xerox Corporation, a grant from the Rome Air Force Development Center, and from the Air Force Office of Scientific Research under Grant No. 85-NM-0263.

Derivation and implicit solution of the Harry Dym equation, and its connections with the Korteweg-de Vries equation

W Hereman†*, P P Banerjee†+ and M R Chatterjee‡

†Mathematics Department and Center for the Mathematical Sciences, University of Wisconsin-Madison, Madison, WI 53706, USA

‡Department of Electrical Engineering, SUNY at Binghamton, Binghamton, NY 13901, USA

1. Introduction

In trying to discover possible nonlinear PDEs that could be solved using our real exponential approach (Hereman *et al* 1985, 1986), we stumbled upon the Harry Dym (HD) equation $r_t = r^3 r_{3x}$ that has nonlinearity and dispersion coupled together. This prototype of an evolution equation, that admits a cusp solitary wave solution (Wadati *et al* 1980), appears in many disguises, namely, $r_t = (1-r)^3 r_{3x}$, $r_t = (r^{-\frac{1}{2}})_{3x}$ and $(r^2)_t = (r^{-1})_{3x}$. The first one is occasionally referred to as the cusp-soliton equation (Kawamoto 1984a,b, 1985a). True, it failed to fit in our scheme but it certainly kindled our interest. According to some of the early references (Kruskal 1975, Magri 1978, Wadati *et al* 1979, 1980, Sabatier 1979a,b, Dijkhuis and Drohm 1979, Case 1982, Yi-Shen 1982, Calogero and Degasperis 1982) the origin of the HD lies in various private communications with Harry Dym. In search of truth, we contacted Dr. Harry Dym who replied, "In the spring of 1974, ... Martin Kruskal delivered a few lectures on the isospectral theory. ... Motivated by these lectures, I developed some analogues for the string equation. The HD equation, as Martin later termed it, was one of the outcomes." Their collaboration resulted in developing a draft of "a fairly complete theory" which is "gathering dust" in Dr. Kruskal's office.

From the *fact* files, the HD is a completely integrable nonlinear evolution equation (Wadati *et al* 1979, 1980) which can be solved by the Inverse Scattering Transform (IST). Dijkhuis and Drohm (1979) and Calogero and Degasperis (1982) discuss the HD equation as a special case of a new broad class of nonlinear PDEs tractable by IST. The HD has a bi-Hamiltonian structure (Magri 1978, Case 1982, Olver 1986); it possesses an infinite number of conservation laws and infinitely many symmetries (Magri 1978, Ibragimov 1985, Olver 1986); and reciprocal Bäcklund transformations (Rogers and Nucci 1986, Nucci 1988).

* Supported in part by the Air Force Office of Scientific Research under Grant No. 85-0263

+ On leave of absence from Department of Electrical Engineering, Syracuse University, Syracuse, NY 13244, USA. Supported in part by the National Science Foundation under Grant no. ECS-8603643, the National Science Foundation Presidential Young Investigator Award under Grant no. MIP 8657765 and a matching grant from Xerox Corporation, a grant from the Rome Air Force Development Center, and from the Air Force Office of Scientific Research under Grant no. 85-NM-0263.

However the HD equation does not possess the Painlevé property (Weiss 1983, 1986, Steeb and Louw 1987, Fokas 1987, Hereman and Van den Bulck 1988) indicating that the Painlevé property is at most sufficient, but not neccesary, for integrability.

Yi-Shen (1982) and Sabatier (1979a, b, 1980) recall the derivation of the HD equation. The first author starts from the isospectral equation modelling the classical string problem with a varying elastic constant. Sabatier independently rediscovers the HD and its generalisations within the Lax formalism (Lax 1968). In order to make this paper a self-contained tutorial of the HD equation, we shall first review and simplify these calculations, and also include a heuristic derivation of the HD to provide some physical motivation.

Amongst the first researchers to study the connections between the HD and the KdV equations we mention Ibragimov (1981, 1985) Calogero and Degasperis (1982) and Weiss (1983, 1986). Later on Kawamoto (1985b), Rogers and Nucci (1986) and Steeb and Louw (1987) investigated the links to the mKdV equation. Underlying all these connections, there exists a rigorous though involved Lie-Bäcklund algebra. Excellent references on this approach are in books by Ibragimov (1985) and Olver (1986). In keeping with a tutorial point of view, we therefore recognise the need for uniformisation and simplification of these connections in order to provide a better insight. Also, as part of our simplifying efforts, we rederive the implicit solution to the HD equation in the simplest possible way, without the aid of heavy machinery like the IST. We obtain a particular exact solution which cannot, however, be expressed in closed form owing to the presence of a transcendental phase. Although the links between the equations have been extensively studied, limited attention has been paid to connections between the various solutions. In our opinion this is vital, since a customary technique in nonlinear science is to generate solutions to a nonlinear PDE, from known solutions of another.

The organisation of this paper is as follows. In Section 2, we first formally derive the HD from the string problem with a varying elastic constant and where the eigenvalue λ is constant w.r.t. a *parameter t*. We next employ the Lax operator technique for this isospectral eigenvalue problem. Lastly, we retrieve the HD by a heuristic method. This involves the derivation of the linear part of the PDE from the known dispersion relation, and suitable modification of the coefficients to account for nonlinear effects.

As stated earlier, the HD is conventionally solved using IST. We present, in Section 3, a novel direct integration method to construct the implicit cusp-type single solitary wave solution of the HD equation. Guided by an *a priori* knowledge of the final result, we assume a form of the solution which is inherently implicit. Mathematically speaking, this is achieved by a change of variable that depends on the solution of the equation itself. As will be explained later, the introduction of this type of variable is commensurate with the physical basis for implicit solutions of other nonlinear PDEs, viz., the nonlinear kinematic equation (for other examples see e.g. Wadati *et al* 1979, 1980 and Kawamoto 1984a, b. 1985a).

Now, the HD equation has intriguing links with other nonlinear evolution equations, viz., the ubiquitous KdV, and a Liouville-type equation (Hereman and Banerjee 1988).

In Section 4, we therefore summarise some of the existing links between the HD and the KdV by transforming one into the other and by providing the connections between the corresponding eigenvalue problems. Scattered work done by Ibragimov (1981, 1985), Weiss (1983, 1986) and Kawamoto (1985b) have contributed to the overall picture we draw in this section. The procedures we employ comprise the Băcklund transformation method, a technique involving the *Schwarzian* derivative, and the Cole-Hopf and Miura transformations. Every procedure above needs to be augmented by either an explicit-to-implicit or an implicit-to-explicit transformation. It is thus not surprising that the single solitary wave solution to the HD is an implicit one. For a better understanding we compare, wherever possible, different steps in every conversion procedure. For instance, it is realised that the Schwarzian derivative, which occurs here in the context of Painlevé analysis, may be conceived of as a combination of the Miura and the Cole-Hopf transformations.

We next turn to transforming the known solution(s) of one equation to that of the other. This is accomplished in Section 5. Of particular interest is the case where, by starting from the cusp-soliton solution of the HD, we are able to derive a new closed form, though singular, solution of the KdV equation.

2. Derivation of the Harry Dym equation

In this Section we shall derive the HD equation by (a) starting from the classical string problem, (b) employing the Lax operator technique and (c) using a heuristic approach.

Method (a):

Consider the ODE

$$\psi_{1,xx} = -\frac{\lambda}{r^2(x;t)}\psi_1 \tag{1}$$

which models the classical string problem where the string, for instance, has a varying elastic constant (Sabatier 1979a, b, 1980, Dijkhuis and Drohm 1979, Yi-Shen 1982, Calogero and Degasperis 1982). In (1), λ denotes the eigenvalue and $r(x;t)$ is a bounded, positive function of x. Furthermore, we assume that $r(x;t) \to 0$ as $|x| \to \infty$. For $r(x;t) = 1$, Eq. (1) reduces to the standard Schrödinger equation. If $r(x;t) \neq 1$, Eq. (1) can be transformed back to that standard equation by a suitable change of variables (see Section 4). Both r and ψ_1 depend on the parameter t. In order to find an integral representation for the solution of (1) that also incorporates the conditions, we first reduce (1) to an equivalent system of first-order equations. Defining $\psi_2 = \psi_{1,x}$, Eq. (1) may then be rewritten as

$$\Psi_x = M\Psi, \tag{2a}$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ -\frac{\lambda}{r^2} & 0 \end{pmatrix}. \tag{2b}$$

We remark here that the above decomposition is not unique in the sense that the *state variables* ψ_1 and ψ_2 could be chosen differently.

It was noted by Schlesinger (1912) that solvable nonlinear PDEs arise as the integrability conditions when a linear ODE as in (2) is deformed to, for instance,

$$\Psi_t = N\Psi, \quad N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

in such a way as to preserve some *characteristic* (for instance, the spectrum λ) of the equation. Thus, setting $\lambda_t = 0$ (isospectral case), the integrability conditions

$$\Psi_{xt} = \Psi_{tx} \tag{4}$$

yield the structure equation

$$M_t - N_x + [M, N] = 0, \tag{5}$$

where we define the commutator $[M, N] = MN - NM$. Incorporating the explicit forms for M and N (from (2b) and (3)) leads to four coupled equations, one of which describes the evolution of $r(x,t)$:

$$C + \frac{\lambda}{r^2}B = A_x, \tag{6a}$$

$$D - A = B_x, \tag{6b}$$

$$\lambda(2\frac{r_t}{r^3} - \frac{A}{r^2} + \frac{D}{r^2}) = C_x, \tag{6c}$$

$$-\frac{\lambda}{r^2}B - C = D_x. \tag{6d}$$

From (6a) and (6d), it follows that

$$D = -A, \tag{7}$$

where we have neglected a constant of integration. Using (6a,b) and (7) in (6c), the evolution of r may be rewritten as

$$\frac{r_t}{r^3} = -\frac{1}{4\lambda}B_{3x} + \frac{r_x}{r^3}B - \frac{B_x}{r^2}. \tag{8}$$

The choice $B = -4\lambda r$ (Yi-Shen 1982) leads straightforwardly to the HD equation

$$r_t = r^3 r_{3x}. \tag{9}$$

Method (b):

Following Lax (1968), we can derive an eigenvalue problem of the type $L\tilde{\psi} = \lambda\tilde{\psi}$ with $\lambda_t = 0$ for a nonlinear evolution equation expressible in the form

$$L_t = [B, L], \tag{10}$$

where L and B are linear spatial operators (respectively symmetric self-adjoint, and antisymmetric) which may depend on r and its spatial derivatives. Now the ODE (1) may be modified to

$$[\frac{1}{r^{m-2}}\frac{d^2}{dx^2}r^m]\tilde{\psi} + \lambda\tilde{\psi} = 0, \tag{11}$$

where $\tilde{\psi}$ is a new wave function defined as $\tilde{\psi} = \psi_1/r^m$, and where m is an integer. From the constraint of self-adjointness, it readily follows that $m = 1$, yielding

$$L = r\frac{d^2}{dx^2}r, \tag{12}$$

so that (10) becomes

$$\frac{r_t}{r}L + L\frac{r_t}{r} = [B, L]. \tag{13}$$

To construct the antisymmetric operator B, we set, following Sabatier (1979b),

$$B = AL + LA \tag{14}$$

where A is antisymmetric; thus $[B, L] = [A, L]L + L[A, L]$. Comparing with (13), it follows that

$$[A, L] = \frac{r_t}{r}, \tag{15}$$

provided the LHS of the above equation reduces to a multiplicative operator and (15) yields the required nonlinear PDE. Noting that

$$L = (r\frac{d}{dx})(\frac{d}{dx}r) \stackrel{\text{def}}{=} ST, \tag{16}$$

we set $A = S + T$, so that

$$\begin{aligned}[A, L] &= S[S,T] - [S,T]T \\ &= [S,[S,T]] + [S,T](S - T) \\ &= r(rr_{2x})_x - rr_x r_{2x} \\ &= r^2 r_{3x},\end{aligned} \tag{17}$$

where we have used the observations $T - S = r$ and $[S,T] = rr_{2x}$. From (15) and (17), the HD equation (9) follows.

Method (c):

The heuristic approach to constructing nonlinear evolution and wave equations involves deriving the linear PDE from a (known) dispersion relation and, thereafter, incorporating nonlinear corrections to the (linear) phase velocity c_0 (Korpel and Banerjee 1984). Consider the dispersion relation

$$\omega = c_0 k - \gamma k^3. \tag{18}$$

where ω denotes the angular frequency and k the propagation constant. The linear PDE is obtained through replacing ω and k by their respective operators $\omega \to -i\partial/\partial\tilde{t}$, $k \to i\partial/\partial\tilde{x}$ in (18) :

$$\tilde{r}_{\tilde{t}} + c_0\tilde{r}_{\tilde{x}} + \gamma\tilde{r}_{3\tilde{x}} = 0. \tag{19}$$

The nonlinear extension is now done by assuming that γ, instead of c_0, is a nonlinear function of $\tilde{r}$

$$\gamma = \gamma_3\tilde{r}^3, \tag{20}$$

where γ_3 is constant. Incorporating this in (19) together with the scalings

$$x = \tilde{x} - c_0\tilde{t}, \;\; t = \tilde{t}, \;\; r(x,t) = -(\gamma_3)^{\frac{1}{3}}\tilde{r}(\tilde{x},\tilde{t}), \tag{21}$$

we readily retrieve the HD equation as in (9).

3. Derivation of the implicit single solitary wave solution

In this section, we employ direct integration to construct the (implicit) one solitary wave solution of the HD equation previously found by Wadati *et al* (1980) using the IST. We are guided by the fact that the solitary wave solution has a singularity in its derivative at its (bounded) peak value owing to the presence of a transcendental phase $\epsilon(x,t)$ (see figure 1). We therefore seek for a solution to (9) in the form

$$r(x,t) = F(f) \tag{22a}$$

where

$$f(x,t) = K(x - x_0 + Vt) + K\epsilon(x,t) \tag{22b}$$

with

$$K\epsilon(x,t) = G(f). \tag{22c}$$

In the above scheme, x_0 is a constant, K is the wavenumber and V is the anticipated velocity of the solitary wave.

At this point, we wish to speculate on the physical basis of implicit solutions of nonlinear PDEs in general. Readers are reminded here of the real exponential approach to solving nonlinear evolution and wave equations (Korpel 1978, Hereman *et al* 1985, 1986) where the final solution is assumed to be built up from the nonlinear mixings of real exponential solutions to the *linear dispersive* part of the PDE. Alternatively, we may think of constructing a particular solution from the solution to the *nonlinear nondispersive* part of the PDE. This is a valid conjecture, since the nonlinear nondispersive part of, say, the KdV equation in $u(X,T)$, i.e., the nonlinear kinematic equation (Whitham 1974),

$$u_T + \alpha u u_X = 0, \tag{23}$$

possesses shock wave solutions that are intrinsically implicit :

$$u(X,T) = g(X - \alpha u(X,T)T). \tag{24}$$

In a more general sense, we may therefore think of solutions to the entire nonlinear dispersive PDE to be of the form (Banerjee and Hereman 1988a)

$$u(X,T) = F(f) \tag{25a}$$

with

$$f(X,T) = H_1(f)X - H_2(f)T + H_3(f). \tag{25b}$$

Examining (22), we note that it fits the scheme described above.

Returning to the construction of the implicit solution to (9), we first note from (22) that

$$\frac{\partial}{\partial t} = (\frac{KV}{1-G_f})\frac{d}{df}, \quad \frac{\partial}{\partial x} = (\frac{K}{1-G_f})\frac{d}{df}. \tag{26}$$

Eq. (9) then transforms to

$$\frac{V}{K^2}\frac{F_f}{F^3} - \frac{d}{df}[(\frac{1}{1-G_f})\frac{d}{df}(\frac{F_f}{1-G_f})] = 0. \tag{27}$$

A first integration, followed by multiplication by F_f and a subsequent integration yields

$$F_f{}^2 = c_1(1-G_f)^2F + c_2(1-G_f)^2 + \frac{V}{K^2}\frac{(1-G_f)^2}{F}, \tag{28}$$

where c_1, c_2 are integration constants.

To solve for F, we must first eliminate G from (28) by assuming a relation between G_f and F. The obvious choice,

$$1 - G_f = F, \tag{29}$$

reduces (28) to an ODE which can be readily solved either by employing the real exponential approach (Hereman *et al* 1985, 1986) or by direct integration. Using the latter, we get

$$f = \int (\frac{V}{K^2}F + c_2F^2 + c_1F^3)^{-\frac{1}{2}} dF, \tag{30}$$

where the integration constant can be absorbed in x_0. For the choice

$$c_1 = \frac{V}{K^2}, \quad c_2 = -\frac{2V}{K^2}, \tag{31}$$

Eq. (30) may be readily evaluated as

$$F(f) = \tanh^2(\frac{\sqrt{V}}{2K}f). \tag{32a}$$

From (29) we then obtain

$$G(f) = \frac{2K}{\sqrt{V}}\tanh(\frac{\sqrt{V}}{2K}f). \tag{32b}$$

For later use, we write down the implicit solution to the HD equation in the original variables x and t :

$$r(x,t) = \tanh^2[\frac{\sqrt{V}}{2}(x - x_0 + Vt + \epsilon(x,t))] \tag{33a}$$

with

$$\epsilon(x,t) = \frac{2}{\sqrt{V}}\tanh[\frac{\sqrt{V}}{2}(x - x_0 + Vt + \epsilon(x,t))]. \tag{33b}$$

This type of cusp-soliton solution (which is plotted in figure 1) has also been obtained for coupled systems of evolutions equations. Kawamoto (1984a,b, 1985a) discusses two examples : an Ito-type system and a normalised Boussinesq equation.

4. Connections between the HD and the KdV equations

Having derived the cusp-solitary wave solution to the HD, we would like to investigate how this solution maps to particular (hopefully new) solutions of the KdV. Toward this goal, we first reconstruct the transformations between the HD and the KdV equations according to the following scheme :

(1) by deriving the HD from the KdV equation using
   (i) the Bäcklund transformation obtained by Weiss (1983, 1986),
   (ii) the *Schwarzian* transformation proposed by Ibragimov (1981, 1985);
(2) by retracting the KdV from the HD using the Cole-Hopf and the Miura transformations as suggested by Kawamoto (1985b);
(3) by transforming the spectral problem associated with the HD equation to the one for the KdV equation (Calogero and Degasperis 1982).

We remark that in addition to every transformation above, we need an *explicit (implicit)* to *implicit (explicit)* transformation in going from the KdV (HD) to the HD (KdV). This explains why the solitary wave solution of the HD equation is implicit in nature. Detailed calculations on the conversion of the cusp-soliton solution of the HD to particular solutions of KdV (and vice versa) will be presented in Section 5.

4.1. Transformation from the KdV to HD

(i) Bäcklund transformation method.

This conversion process from the KdV to the HD can be summarized in the following three steps :

1. We first use the auto-Bäcklund transformation

$$u(X,T) = \frac{12}{\alpha}[\ln\phi(X,T)]_{2X} + u_2(X,T) \tag{34}$$

derived by Weiss (1983, 1986) in the context of the weak Painlevé analysis of the KdV equation

$$u_T + \alpha u u_X + u_{3X} = 0. \tag{35}$$

The function $\phi(X,T)$ plays a crucial role as a *singular* or *pole* manifold (Newell *et al* 1987) in the Painlevé formalism and in Hirota's bilinear method as a new dependent variable (Matsuno 1984, Gibbon *et al* 1985). We remark that $u_2(X,T)$, like $u(X,T)$, must also be a solution to (35). Hence, from subtracting the respective equations, we obtain

$$(u-u_2)_t + (u-u_2)_{3X} + \frac{\alpha}{2}[(u-u_2)^2 + 2u_2(u-u_2)]_X = 0. \tag{36}$$

We now substitute for $u - u_2$ from (34) into (36) and integrate once w.r.t. x. After a little algebra and equating the coefficients of ϕ^{-1} and ϕ^{-2} we obtain the two nontrivial relations :

$$\phi_{XT} + \alpha\phi_{2X}u_2 + \phi_{4X} = 0. \tag{37a}$$

$$\phi_X\phi_T + \alpha{\phi_X}^2 u_2 + 4\phi_X\phi_{3X} - 3{\phi_{2X}}^2 = 0, \tag{37b}$$

The coefficients of ϕ^{-3} and ϕ^{-4} identically vanish. Upon elimination of u_2 from (37) and one integration w.r.t. X, we get

$$\frac{\phi_T}{\phi_X} + \{\phi; X\} = \mu, \tag{38}$$

where the Schwarzian derivative (Hille 1976) is defined by

$$\{\phi; X\} = \frac{\phi_{3X}}{\phi_X} - \frac{3}{2}\left(\frac{\phi_{2X}}{\phi_X}\right)^2. \tag{39}$$

In (38) μ may depend on T but is constant in X. We recall the property of *Galilean invariance* of (38) : If $\phi(X,T;\mu)$ denotes the solution to (38), then

$$\phi(X,T;0) = \phi(X-\mu T, T; \mu). \tag{40}$$

2. This step involves the explicit to implicit transformation by interchanging the dependent variable ϕ and the independent variable X. This is effected by first recalling a property of the Schwarzian derivative (Hille 1976), viz.,

$$\{\phi; X\} = -{\phi_X}^2\{X; \phi\}. \tag{41}$$

Together with the observations

$$\phi_X = \frac{1}{X_\phi}, \quad T \text{ constant}, \tag{42}$$

and

$$\phi_T = \phi_X X_T = \frac{X_T}{X_\phi}, \tag{43}$$

(38) may be recast into the form

$$X_T = \mu + \frac{1}{{\phi_X}^2}\{X;\phi\}, \tag{44}$$

with $X = X(\phi, T)$.

3. Finally, we define a new dependent variable

$$w = \frac{1}{X_\phi}, \tag{45}$$

and study the evolution of $w(\phi, T)$. We note that

$$w_T = -\frac{X_{\phi T}}{{X_\phi}^2} = -w^2 X_{\phi T} \tag{46}$$

which can be reexpressed, using (44), as

$$\begin{aligned} w_T &= -\mu_\phi w^2 - w^2[w w_{2\phi} + \frac{1}{2}{w_\phi}^2]_\phi \\ &= w^3 w_{3\phi} - \mu_\phi w^2. \end{aligned} \tag{47}$$

Consider the special case where $\mu_\phi = 0$. Upon defining new variables

$$x = \phi, \quad t = T, \quad r(x,t) = w(\phi, T), \tag{48}$$

we readily obtain the HD equation (9).

(ii) The Schwarzian transformation method.

The next conversion method from the KdV to the HD is partly motivated by the inverse transformation due to Kawamoto (1985b). In essence, Kawamoto's procedure involves an implicit to explicit transformation to reduce the HD as in (9) to an intermediate equation similar to (38), and then using the Cole-Hopf transformation on ϕ_X to yield the mKdV equation. The Miura transformation finally links the mKdV to the KdV.

To go from the KdV to the HD we will apply the above transformations in reverse order to yield the intermediate equation (38). In that sense, this method differs from (i) in the first step. We first use the Miura transformation to $v(X,T)$:

$$u = v^2 + i\sqrt{\frac{6}{\alpha}}v_X, \tag{49a}$$

followed by the Cole-Hopf transformation on the potential $\phi_X(X,T)$:

$$v = -i\sqrt{\frac{3}{2\alpha}}\frac{\phi_{2X}}{\phi_X}. \tag{49b}$$

After some tedious algebra, which has been performed using MACSYMA, we can identify the relation

$$L_1(\phi_T + \{\phi; X\}\phi_X) = 0 \tag{50}$$

where the operator L_1 is given by

$$L_1 = \frac{3}{\alpha}\left(\frac{1}{\phi_X}\frac{\partial^3}{\partial X^3} - 3\frac{\phi_{2X}}{{\phi_X}^2}\frac{\partial^2}{\partial X^2} - \left(\frac{\phi_{3X}}{{\phi_X}^2} - 3\frac{{\phi_{2X}}^2}{{\phi_X}^3}\right)\frac{\partial}{\partial X}\right), \tag{51}$$

from which we recover the intermediate equation (38). With MACSYMA, we have obtained yet another but equivalent representation of (50) :

$$L_2\left(\frac{\phi_T}{\phi_X} + \{\phi; X\}\right) = 0, \tag{52}$$

with

$$L_2 = \frac{3}{\alpha}\left(\frac{\partial^3}{\partial X^3} + 2\{\phi; X\}\frac{\partial}{\partial X} + \{\phi; X\}_X\right). \tag{53}$$

The latter operator is similar to the anti-symmetric Lax operator for the KdV, wherein we have formally replaced $u(X,T)$ by a multiple of $\{\phi; X\}$.

Note that the successive transformations defined in (49) may indeed be combined to give

$$u = \frac{3}{\alpha}\{\phi; X\}, \tag{54}$$

which is the *Schwarzian* transformation proposed by Ibragimov (1981, 1985). The argument presented in Ibragimov's book may be summarised as follows : Suppose there exists a transformation from (38) to the form

$$\Psi_T + \Psi_{3X} + \Phi(\Psi, \Psi_X) = 0 \tag{55}$$

having a nontrivial Lie-Băcklund algebra. Then (38) is equivalent to either a linear equation with constant coefficients or the KdV equation. In our case, (49b) provides the relevant transformation to the mKdV equation where $\Psi = v$; $\Phi = \beta v^2 v_X$, which is equivalent to the KdV through the Miura transformation (49a).

The subsequent steps for the transformation are identical to steps 2 and 3 in (i).

4.2. Transformation from the HD to KdV

Along the lines of Kawamoto (1985b), we convert the HD equation to the KdV through the following three steps :

1. We first employ an implicit to explicit transformation by defining

$$X = \int_{-\infty}^{x} \frac{ds}{r(s,t)}, \quad T = -t, \tag{56}$$

with $R(X,T) = r(x(X,T), t(X,T))$ representing the new transformed dependent variable. We observe that

$$\begin{aligned} \frac{\partial}{\partial t} &= -\frac{\partial}{\partial T} + (-r r_{2x} + \tfrac{1}{2} r_x^2)\frac{\partial}{\partial X} \\ &= -\frac{\partial}{\partial T} - \left(\frac{R_{2X}R - \frac{3}{2}R_X^2}{R^2}\right)\frac{\partial}{\partial X}, \end{aligned} \tag{57a}$$

where use has been made of the HD equation (9) to replace all time derivatives of r in terms of spatial derivatives, and that

$$\frac{\partial}{\partial x} = \frac{1}{R}\frac{\partial}{\partial X}. \tag{57b}$$

After a little algebra, Eq. (9) can be expressed as

$$R_T + \frac{R_{3X}R^2 - 3R_{2X}R_X R + \frac{3}{2}R_x^3}{R^2} = 0. \tag{58}$$

In the above derivation we have used the fact that $r(x,t)$ and its spatial derivatives tend to zero as $|x| \to \infty$.

Before proceeding any further, we make the following remarks :

Remark 1 : The substitutions in (56) are equivalent to

$$X = \int_x^\infty \frac{ds}{r(s,t)}, \quad T = t. \tag{59}$$

Remark 2 : Comparing (58) with (38), it may be readily recognized that

$$R = \phi_X. \tag{60}$$

2. We now write

$$v = \frac{R_X}{R} \tag{61}$$

and find $v_T(X,T)$ with the use of (58). It is easy to check that v_T can be expressed entirely in terms of v and its spatial derivatives through the mKdV equation

$$v_T - \frac{3}{2}v^2 v_X + v_{3X} = 0. \tag{62}$$

3. The Miura transformation

$$u = -\frac{3}{2\alpha}(v^2 + 2v_X) \tag{63}$$

now relates (62) to the KdV equation (35). This may be readily verified by computing u_T from (63), using (62) to replace the time derivative(s) of v in terms of v and its spatial derivatives, and finally reexpressing in terms of u.

4.3. Connection between the respective eigenvalue problems

We will complete the connection between the HD and the KdV equations by pointing out the link between their respective eigenvalue problems. To this end, we first employ the implicit to explicit transformation described by (56), which is similar to the one employed by Calogero and Degasperis (1982). The eigenvalue problem (1) for the HD equation is then equivalent to

$$\frac{\theta_{2X}R - \theta_X R_X}{R} = -\lambda\theta; \tag{64}$$

where $\theta(X;T) = \psi_1(x(X,T);t(T))$. Now defining

$$\Omega(X;T) = R^{-\frac{1}{2}}\theta(X;T) \tag{65}$$

and eliminating θ from (64), we obtain

$$\Omega_{2X} + [\frac{1}{2}\frac{R_{2X}}{R} - \frac{3}{4}\frac{R_X^2}{R^2} + \lambda]\Omega = 0, \tag{66a}$$

or, using (60),

$$\Omega_{2X} + [\frac{1}{2}\{\phi;X\} + \lambda]\Omega = 0. \tag{66b}$$

Note that if we adopt the Cole-Hopf transformation (61), then (66a) may be also written as

$$\Omega_{2X} + [\frac{1}{4}(v^2 - 2v_X) - \lambda]\Omega = 0 \tag{67}$$

and thence as

$$\Omega_{2X} + [\tilde{u} - \lambda]\Omega = 0, \tag{68}$$

provided $\tilde{u}(X,T)$ is a solution to the KdV equation

$$\tilde{u}_T - 6\tilde{u}\tilde{u}_X + \tilde{u}_{3X} = 0. \tag{69}$$

Thus, we have transformed the eigenvalue problem for the HD to that for the KdV, where the corresponding eigenfunctions $\psi_1(=\theta)$ and Ω are related through (65). The explicit form for $R(X,T)$ will be discussed in the following section.

5. Connections between the solutions of the HD and KdV equations

Using the knowledge of the connections between the HD and the KdV equations discussed in the previous section, we shall proceed to transform the known solution(s) of each equation to the solution(s) of the other. The transformation of the implicit solution of the HD to explicit solution(s) of the KdV is more interesting and will be taken up first.

We shall first apply the implicit to explicit transformation prescribed by (56) to the cusp-solitary wave solution of the HD as in (33). From the latter equation

$$\frac{1}{r} = 1 + \epsilon_x , \tag{70}$$

hence, from (56),

$$X = x + \epsilon(x,t) \tag{71}$$

since $x + Vt + \epsilon(x,t)$, which is the argument of the $\tanh^2$ function in (33a), tends to zero as $|x + Vt| \to \infty$ (see figure 1). Consequently, from (33a) and the definition of R as in (56),

$$R(X,T) = \tanh^2[\frac{\sqrt{V}}{2}(X - X_0 - VT)], \tag{72}$$

where $X_0(= x_0)$ is a constant.

For later use, we will write down the expression for ϕ from (60). Using (72), and one integration w.r.t. X, we get

$$\phi = X - X_1(T) - \frac{2}{\sqrt{V}}\tanh[\frac{\sqrt{V}}{2}(X - X_0 - VT)], \tag{73}$$

where $X_1(T)$ is an integration constant.

From (61) and (72), we arrive at the singular solution to the mKdV equation (62) :

$$v(X,T) = 2\sqrt{V}\operatorname{cosech}[\sqrt{V}(X - X_0 - VT)]. \tag{74}$$

The solution to the KdV equation (35) may now be calculated using the Miura transformation (63). Straightforward algebra yields the familiar soliton solution

$$u(X,T) = \frac{3V}{\alpha}\operatorname{sech}^2[\frac{\sqrt{V}}{2}(X - X_0 - VT)]. \tag{75}$$

An alternate solution of the KdV may be derived by starting from (73) and employing the auto-Bäcklund representation of u in terms of ϕ, as in (34). Indeed, since ϕ should also satisfy (38), it is readily checked upon substituting (73), that $X_1(T) = \text{constant} = X_1$ and $\mu = 0$. Now, u_2 can be computed using either of the relations (37a,b). This gives again the one-solitary wave solution for u_2, as in (75). Returning to (34), a new, but singular, solution to the KdV equation reads

$$u(X,T) = -\frac{3V}{\alpha}[\frac{(\frac{V}{4}(X - X_1)^2 + 1)\tanh^2[\frac{\sqrt{V}}{2}(X - X_0 - VT)] - \frac{V}{4}(X - X_1)^2}{(\tanh[\frac{\sqrt{V}}{2}(X - X_0 - VT)] - \frac{\sqrt{V}}{2}(X - X_1))^2}]. \tag{76}$$

We remark that the inverse process of transforming the familiar sech^2 solution of the KdV to that of the HD leads to, for instance, a rational solution $r(x,t) = \frac{\sqrt{V}}{2}(x - 1)$.

Other possible solutions to the HD and the KdV equations are presently under investigation (Banerjee and Hereman 1988b).

6. Conclusions

In retrospect, we have provided some straightforward derivations of the HD equation. A direct integration method was used to derive a simple particular implicit solution. We have consolidated the various links between the HD, the KdV and the mKdV equations, and used these to provide connections between their solutions.

Although at this stage, the HD equation is more of theoretical significance than of applicative relevance, we hope that this paper contributes not only to provide a better insight into cusp solitary waves, but also serves as a review on the HD equation. The straightforward mathematical techniques employed throughout this paper should prove helpful in deriving implicit solutions to other nonlinear PDEs and in (as a spin-off) discovering new solutions to the KdV equation. Work on this is currently in progress.

Acknowledgements

The authors would like to acknowledge the support of the National Science Foundation under Grant no. ECS-8603643, the National Science Foundation Presidential Young Investigator Award under Grant no. MIP 8657765 and a matching grant from Xerox Corporation, a grant from the Rome Air Force Development Center, and from the Air Force Office of Scientific Research under Grant no. 85-NM-0263. PPB also wishes to thank the Departments of Mathematics and Electrical Engineering at the University of Wisconsin-Madison for their hospitality.

References

Banerjee P P and Hereman W 1988a *On the construction of solitary wave solutions starting from implicit solutions to the nonlinear nondispersive system* in preparation

Banerjee P P and Hereman W 1988b *Some particular solutions to the Korteweg-de Vries equation constructed via the Painlevé analysis* in preparation

Calogero F and Degasperis A 1982 *Spectral Transform and Solitons 1* (Amsterdam: North Holland)

Case K M 1982 *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems* AIP Conference Proceedings, eds Tabor M and Treve Y M (New York: Am. Inst. of Physics) pp 186-7

Dijkhuis H and Drohm J K 1979 *Internal Report 148 Institute of Theoretical Physics* (Groningen) pp 1-15

Fokas A S 1987 *Stud. Appl. Math.* 77 253-99

Gibbon J D, Radmore P, Tabor M and Wood D 1985 *Stud. Appl. Math.* **72** 39-63

Hereman W and Banerjee P P 1988 *private communication*

Hereman W, Banerjee P P, Korpel A, Assanto G, Van Immerzeele A and Meerpoel A 1986 *J. Phys. A: Math. Gen.* **19** 607-28

Hereman W, Korpel A and Banerjee P P 1985 *Wave Motion* **7** 283-90

Hereman W and Van den Bulck E 1988 *Proc. of Workshop on Finite Dimensional Integrable Nonlinear Dynamical Systems* eds Leach P G L and Steeb W-H (Singapore: World Science) pp 112-24

Hille E 1976 *Ordinary Differential Equations in the Complex Domain* (New York : Wiley)

Ibragimov N H 1981 *C. R. Acad. Sci. Paris* Sér I **293** 657-60

Ibragimov N H 1985 *Transformation Groups Applied to Mathematical Physics* (Boston: Reidel)

Kawamoto S 1984a *J. Phys. Soc. Japan* **53** 1203-5

Kawamoto S 1984b *J. Phys. Soc. Japan* **54** 2465-71

Kawamoto S 1985a *J. Phys. Soc. Japan* **54** 1701-9

Kawamoto S 1985b *J. Phys. Soc. Japan* **54** 2055-6

Korpel A 1978 *Phys. Lett.* **68A** 179-81

Korpel A and Banerjee P P 1984 *Proc. IEEE* **72** pp 1109-30

Kruskal M D 1975 *Lecture Notes in Physics* vol 38, ed Moser J (Berlin: Springer) pp 310-54

Lax P D 1968 *Comm. Pure Appl. Math.* **21** 467-90

Magri F 1978 *J. Math. Phys.* **19** 1156-62

Matsuno Y 1984 *Bilinear Transform Method* (New York: Academic)

Newell A C, Tabor M and Zeng Y B 1987 *Physica D* **29** 1-68

Nucci M C 1988 *J. Phys. A: Math. Gen.* **21** 73-9

Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

Rogers C and Nucci M C 1986 *Physica Scripta* **33** 289-92

Sabatier P C 1979a *Lett. Nuovo Cimento* **26** 477-82

Sabatier P C 1979b *Lett. Nuovo Cimento* **26** 483-6

Sabatier P C 1980 *Lecture Notes in Physics* vol 120, eds Boiti M, Pempinelli F and Soliani G (Berlin: Springer) pp 85-102

Schlesinger L 1912 *J. Reine Angew. Math.* **141** 96-145

Steeb W-H and Louw J A 1987 *Physica Scripta* **36** 11-4

Yi-Shen L 1982 *Lett. Nuovo Cimento* **70** 1-12

Wadati M, Ichikawa Y H and Shimizu T 1980 *Prog. Theor. Phys.* **64** 1959-67

Wadati M, Konno K and Ichikawa Y H 1979 *J. Phys. Soc. Japan* **47** 1698-700

Weiss J 1983 *J. Math. Phys.* **24** 1405-13

Weiss J 1986 *J. Math. Phys.* **27** 1293-305

Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)

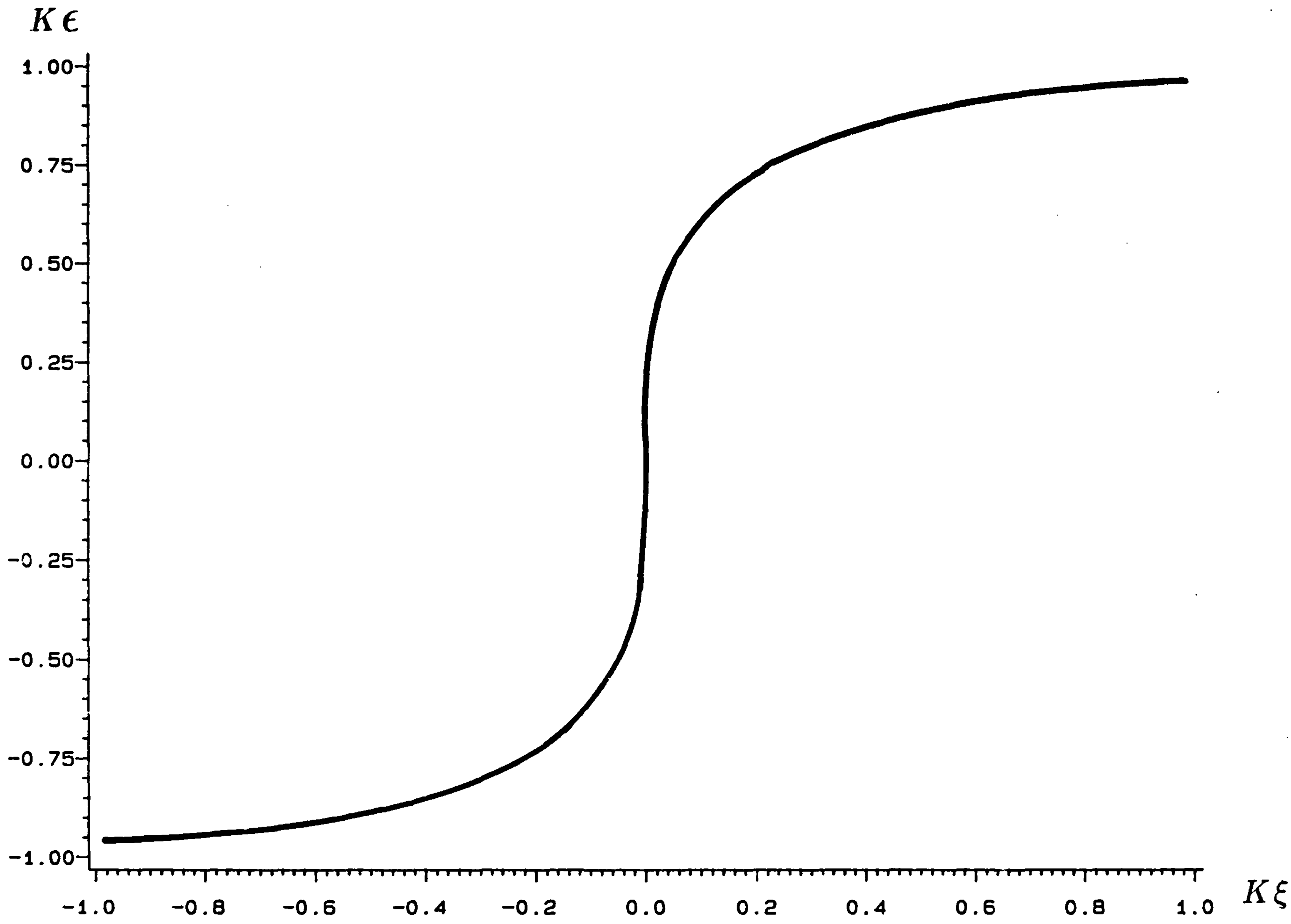

Figure 1a: Plot of $K\epsilon$ vs $K\xi$ where $\xi = x - x_0 + Vt$.

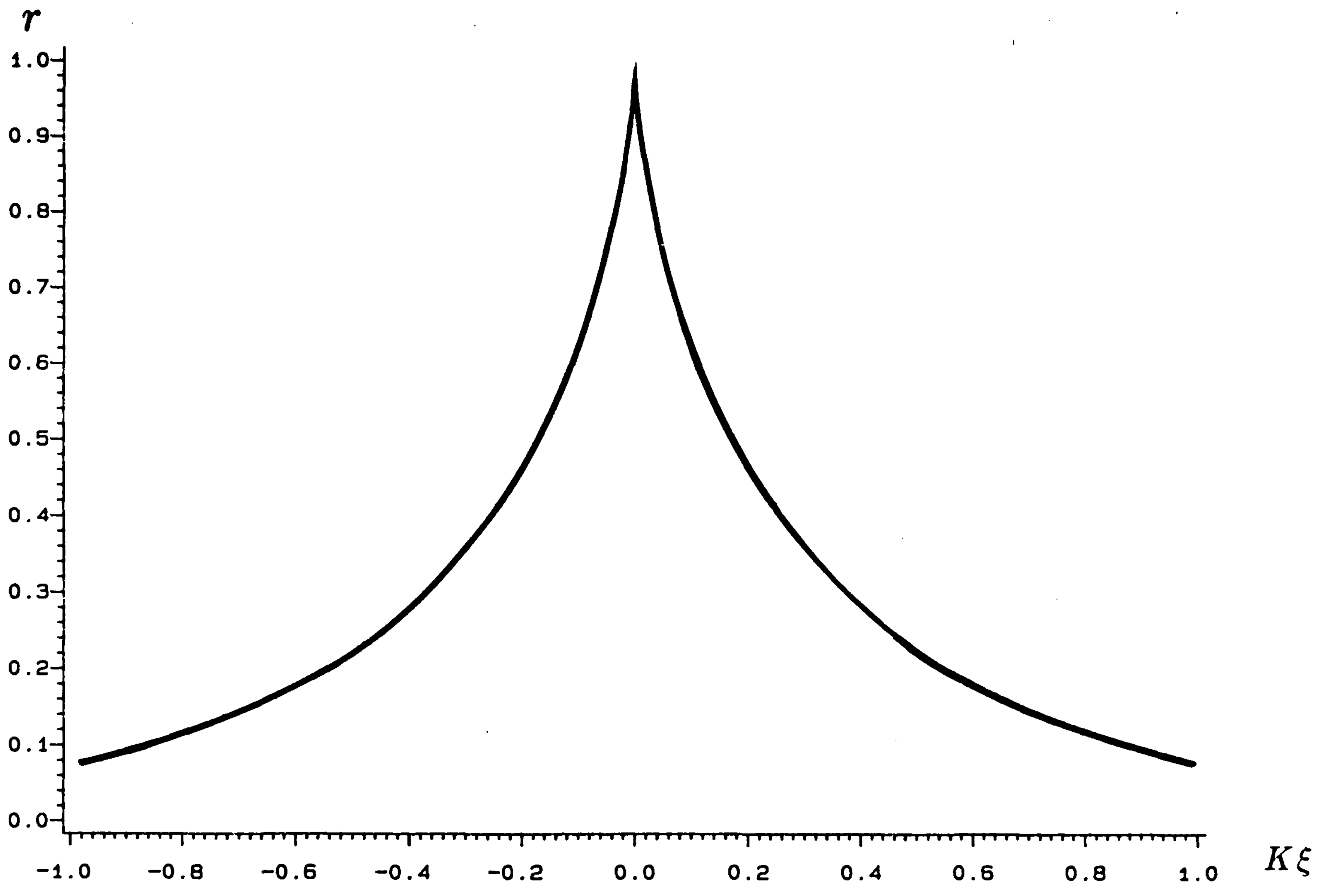

Figure 1b: Plot of r vs $K\xi$ where $\xi = x - x_0 + Vt$.