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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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ly faster claim-arrivals.

MODERATE- AND LARGE- DEVIATION PROBABILITIES IN ACTUARIAL RISK THEORY

by

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Abstract

A general model for the actuarial Risk Reserve Process as a superposition of compound delayed-renewal processes is introduced and related to previous models which have been used in Collective Risk Theory. It is observed that nonstationarity of the portfolio "age-structure" within this model can have a significant impact upon probabilities of ruin. When the portfolio size is constant and the policy age-distribution is stationary, the moderate- and large- deviation probabilities of ruin are bounded and calculated using the strong approximation results of Csörgö, Horváth and Steinebach (1987) and a large-deviation theorem of Groeneboom, Oosterhoff, and Ruymgaart (1979). One consequence is that for non-Poisson claim-arrivals, the large-deviation probabilities of ruin are noticeably affected by the decision to model many parallel policy lines in place of one line with correspondingly faster claim-arrivals.

Keywords: asymptotics; mathematical models (KR)

1. Modelling the risk-reserve process.

The problem of exact and asymptotic calculation of ruin probabilities for a (large) insurer has a long and well-documented history. Early work by Lundberg and Cramer [see Cramer 1955 for historical references] modelling portfolios of fixed size in which claim-arrivals follow a Poisson process, has led to voluminous contributions in a few main directions. In case claims are taken to arrive according to a renewal process, Thorin (1982) surveys the large literature on exact evaluation of ruin probabilities by Wiener-Hopf and complex-analytic methods. Cramer himself had a largely actuarial motivation for initiating the study of large-deviation probabilities for sums of indepen-independent random variables: a summary of results along this line for ruin probabilities can be found in the book of Beekman (1974). The point of view that rescaled risk-reserve processes should be well-approximated distribution-ally by Wiener processes with linear drift has led (Iglehart 1969; Harrison 1977) to asymptotic formulas for ruin probabilities as portfolios become large. Other recent work attempts to use martingale or Markov process struc-ture to generlaize the classes of risk-reserve process models for which formu-las for ruin-probabilities can be written down (e.g. Dassios and Embrechts 1987). Martingale inequalities generalizing Kolmogorov's classical Exponential Bounds to Martingales have been exploited by Slud (1989) to obtain universal upper-bounds on ruin probabilities in terms of means and variances of claim amounts and inter-occurrence times.

The following model for an insurer's risk reserve is slightly more gener-al than the authors have seen written down elsewhere. Indeed, it probably has too many parameters to yield useful general information. However, the model does incorporate the main features of actuarial risk for life insurance and annuities, as described by about eighty years of work by actuaries and

probabilists.

Let the index $k = 1, 2, \ldots, m$ enumerate the independent "lines" of insurance/annuitity policies of a large insurer; let $e_k \geq 0$ denote the chronological time at which policy number k is first in force, and assume for simplicity that the policy lines, once started, never terminate but are renewed instantaneously at the successive times t_{kj} of deaths, at which times the claims X_{kj} (positive for insurance, negative for annuity) are presented to the insurer. We assume throughout that the times between the j'th and $(j+1)$'th claims for policy line k (for $j \geq 1$) are independent and identically distributed positive random variables Y_{kj} with hazard rate function $h_k(\cdot)$ and expectation $\lambda_k < \infty$. The waiting-time $Y_{k0} - y_k(e_k)$ from e_k until the first claim by the k'th policy line is assumed to be independent of the later inter-claim times $(Y_{kj} : j \geq 1)$; here $y_k(e_k)$ is interpreted as a nonrandom initial age at time e_k of the first individual insured under policy line k, and Y_{k0} is taken to be distributed according to the conditional distribution of Y_{k1} given that $Y_{k1} > y_k(e_k)$. Thus the chronological claim-times t_{kj} can be expressed by the formula

$$(1.1) \qquad t_{kj} = e_k + (Y_{k0} - y_k(e_k)) + Y_{k1} + \ldots + Y_{kj}, \quad j = 0, 1, \ldots .$$

The assumptions about the stochastic behavior of the policies are completed by taking the claim amounts $(X_{kj} : j \geq 1)$ for policy line k to be independent are identically distributed with finite positive mean μ_k, and by taking the inter-claim times and claim amounts for different policy lines to be mutually independent. Finally, policy line k is assumed to pay premiums continuously to the insurer at the nonrandom constant rate $(1+\gamma_k)\mu_k\sigma_k/\lambda_k$ for all chronological times greater than e_k.

In this model, the parameters $e_k, y_k(e_k), \lambda_k, \gamma_k$ and μ_k as well as the

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functions $h_k(\cdot)$ are taken to be nonrandom and fixed. The stochastic aspects of policies and claims arise solely from the arrays $X_{kj} : j > 1)$ and $Y_{kj} : j \geq 0)$ of independent random variables.

We next define notations related to policy ages and to the Risk Reserve Process of a life insurer. For each k, let t_{kj} be as in (1.1) and put

$$(1.2) \qquad N_k(t) = \begin{cases} 1+\max(j : t \geq t_{kj}) & \text{if } t \geq t_{k0} \\ 0 & \text{if } t_{k0} > t \geq 0 \end{cases}$$

$$(1.3) \qquad y_k(t) = \begin{cases} t-\max(t_{kj} : j \geq 0 \text{ and } t > t_{kj}) & \text{if } t \geq t_{k0} \\ y_k(e_k)+t-e_k & \text{if } t_{k0} \geq t \geq e_k. \end{cases}$$

Then $N_k(\cdot)$ is the delayed-renewal counting process for the occurrence of claims under the k'th policy, and $y_k(\cdot)$ is the corresponding current-age or current-life proces. The proces $y_k(\cdot)$ is left-continuous, and it is a standard fact that $N_k(t) - \int_0^t h_k(y_k(s))ds$ is a martingale with respect to t for each k.

Letting

$$N(t) = \sum_{k=1}^{\infty} N_k(t) \quad \text{and} \quad \pi(t) = \sum_{k=1}^{\infty} (1+\gamma_k) \frac{\mu_k}{\lambda_k} \max \{t-e_k, 0\}$$

respectively denote the total number of claims and total premiums paid up to time t on all policy lines, and denoting by $U = R(0)$ the insurer's cash risk-reserve at time 0, we define the *risk-reserve process*

$$(1.4) \qquad R(t) = U + \pi(t) - \sum_{k=1}^{\infty} \sum_{j=1}^{N_k(t)} X_{kj}.$$

This process consists of a deterministic upward drift minus a superposition of independent compound delayed-renewal counting-process. The primary task of

actuarial risk theory for a life insurer is to study the level-crossing proba-
bilities

(1.5) $\qquad P(t) = P\{R(t) \leq 0 \text{ for some } t \in [0,T]\}, \quad T \leq \infty.$

The model which we have just described contains far too many parameters
for a general analysis. Therefore, the simplification to a single class of
policies (i.e., to the case where $h_k(\cdot) = h(\cdot)$, the law of X_{kj} is the same
for all k, and $\gamma_k = \gamma$) has been common to all theoretical approaches to
this subject. A further simplification which has virtually always been
assumed is that the portfolio is of fixed size m at all times $t \geq 0$, i.e.,
that $e_k = 0$ for $k = 1,\ldots,m$. The reason for these simplifying assumptions
is not that the complexities modelled by parameters, $e_k, \gamma_k, \lambda_k, \mu_k$, etc., do
not exist in practice, but rather to limit difficulties of analysis and varie-
ties of phenomena. Under the simplified model of this paragraph,

(1.6) $\quad \{X_{kj}\}_{j \geq 1}$ and $\{Y_{kj}\}_{j \geq 1}$ are each i.i.d. arrays, $e_k \equiv 0$

$\qquad\qquad$ for all k, and $\gamma_k \equiv \gamma$ does not depend on k.

In the special case $h(\cdot) \equiv \lambda$, we have Cramer's (1955) famous Collective Risk
model, where the process $N(\cdot)$ is a superposition of m independent
Poisson (λ) processes, and the age-parameters $y_k(e_k) \equiv y_k(0)$ play no role.
Precisely the same model arises if $h(\cdot) \equiv \lambda$ is replaced by λm, with T
replaced by mT and m by 1. For this reason, when Iglehart (1969) wished
to analyze a more general model with nonexponential random times between claim
occurrences, he fixed $m = 1$ (and $y_1(0) = 0$). Our model for nonconstant
$h(\cdot)$ allows for a number m of independent policies or policy-classes which
in realistic cases will be much larger than the finite time-horizon T over
which ruin-probabilities should be calculated. [Iglehart in taking $m = 1$
naturally regarded the time-horizon mT as very large.] In this situation,

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the initial policy-"ages" $\{y_k(0) : k = 1,\ldots,m\}$ may be very nonstationary, in the sense that the empirical measure for $\{y_k(t) : k = 1,\ldots,m\}$ might be seriously nonconstant as t varies. Such a case would arise if the policy-holders at t = 0 were at much higher risk (through self-selection or some other selection mechanism) than the general population of potential "age-0" policyholders and if $h(\cdot)$ were a monotonically increasing function. It is not surprising that such nonstationarity could dominate other stochastic effects contributing to the values of ruin-probabilities (1.5). Since actuaries do not typically attempt to produce theoretical models of nonstationarity among their insured populations, we are naturally led to the further assumption, *which we adopt from now on*, that the policy-age processes $y_k(t)$ are for each k strictly stationary stochastic processes in t, i.e., (Karlin and Taylor 1975), that

$$y_k(0) \quad \text{has density} \quad \frac{1}{\lambda} \exp\left[-\int_0^{\cdot} h(x)dx\right].$$

In the strictly stationary fixed-portfolio-size setting just described, we set ourselves the task of describing the asymptotic behavior of the ruin-probability (1.5) as m gets large and (1.5) gets small, where U, T and γ are allowed to depend on m but where $h(\cdot)$, λ, and μ are not. We do not treat in detail the opposite case where U, T, and γ vary with m and behave in such a way that P(T) defined in (1.5) has a finite limit as $T \to \infty$. This case is adequately covered by the following Theorem, which is proved either by the same methods as Iglehart's (1969) main result, or alternatively by the discussion in Section 4 of Csörgö et al (1987a) in case m remains bounded.

<u>Theorem 1.1</u>. Suppose that $\underline{X} \equiv \{X_{k,i} : k \geq 1, i \geq 1\}$ and $\underline{Y} \equiv \{Y_{k,j} : k \geq 1, j \geq 1\}$ are independent arrays of i.i.d. random variables with

$$EX_{1,1} = \mu, \quad \text{Var}(X_{1,1}) = \alpha^2, \quad EY_{1,1} = \lambda, \quad \text{Var}(Y_{1,1}) = \beta^2$$

and $P\{Y_{1,1} > t\} \equiv \exp\{-\int_0^t h(x)dx\}$ for $0 \leq t < \infty$. In addition, let $\{(y_k(0), Y_{k,0})\}_{k=1}^\infty$ be an i.i.d. sequence of random pairs independent of $\underline{X}, \underline{Y}$, with joint density of $(y_1(0), Y_{1,0})$ at (s,t) given by $\lambda^{-1}h(t) \exp\{-\int_0^t h(x)dx\}$ for $0 < s < t$, and by 0 for other (s,t). For each k, let $N_k(t)$ be defined by

$$N_k(t) = \begin{cases} 0 & \text{if } Y_{k,0} - y_k(0) > t \\ 1 + \max\{j : Y_{k,0} + Y_{k,1} + \ldots + Y_{k,j} > y_k(0) + t\} & \text{if } Y_{k,0} - y_k(0) \leq t \end{cases}$$

(i.e., as above with $e_k \equiv 0$).

Now suppose that $\{T_n\}_{n=1}^\infty$, $\{m_n\}_{n=1}^\infty$, $\{\gamma(n)\}_{n=1}^\infty$, and $\{U_n\}_{n=1}^\infty$ are sequences of positive constants such that as $n \to \infty$, m_n and T_n are bounded below, with $m_n \cdot T_n \to \infty$ and $T_n \to T \leq \infty$ and

$$(m_n T_n)^{1/2} \gamma(n) \to a < \infty \quad \text{and} \quad (m_n T_n)^{-1/2} U_n \to u < \infty.$$

Let

$$(1.7) \qquad R_n(t) \equiv U_n + (1+\gamma(n))m_n \frac{\mu}{\lambda} t - \sum_{k=1}^{m_n} \sum_{j=1}^{N_k(t)} X_{kj}, \quad 0 \leq t \leq T_n.$$

Then, as $n \to \infty$,

(1.8)

$$P\{R_n(T) \leq 0 \text{ for some } t \in [0, T_n]\} \to P\{\sup(W(\theta_s^2) - a\frac{\mu}{\lambda}s - u : 0 \leq s \leq 1) > 0\}$$

where $W(\cdot)$ is a standard Wiener process and

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$$\theta_s^2 = \begin{cases} \lambda^{-3}s(\lambda^2\alpha^2+\mu^2\beta^2) & \text{if } T = \infty, \\ (\lambda^{-1}\alpha^2 s+\mu^2 T^{-1}\text{Var}(N_1(sT))) & \text{if } T < \infty. \end{cases}$$

In the remainder of this paper, we retain the setting of Theorem 1.1 (i.e., of (1.6) – (1.7))) except that the parameter sequence $\{m_n\}$, $\{T_n\}$, $\{\gamma(n)\}$, and $\{U_n\}$ will be assumed to behave in such a way that $P\{R_n(t) \leq 0$ for some $t \in [0,T_n]\} \to 0$ as $n \to \infty$. This restriction corresponds to the actuarial requirement which motivated Cramer's development of Large Deviations theory, namely that the probability of ruin should become small in a definite way as a function of parameters when the scale of an insurer becomes large.

2. Moderation-Deviation Ruin Probabilities.

Throughout the rest of the paper, we assume

(A.1) The random variables X_{kj} and Y_{kj} have finite moment generating functions $E \exp(sX_{kj})$ and $E \exp(sY_{kj})$ for $0 < s \leq s_0$; also Y_{kj} has a finite density $h(0)$ at 0.

(A.2) As $n \to \infty$, $U_n/(m_nT_n) + \gamma(n) = \mathcal{O}(1)$ and $(m_nT_n)^{-1/2} = o\left[U_n/(m_nT_n) + \gamma(n)\right]$.

Since our primary goal is to find asymptotic expressions for $P_n(T_n) \equiv P\{R_n(t) \leq 0$ for some $t \in [0,T_n]\}$, we will repeatedly apply the following technical result on tail-probabilities for parameterized sums of i.i.d. random variables.

Lemma 2.0 [essentially proved as Theorem 10 of Chapter VIII in Petrov 1972]. For a fixed subset Θ of $\mathbb{R}$, let $\{\zeta(\theta)\}_\theta$ be a family of mean-0 random

variables which satisfies for some $r > 0$

$$\sup_{\theta \in \Theta} E\, e^{r\zeta(\theta)} < \infty, \quad \inf_{\theta \in \Theta} E\{\zeta(\theta)\}^2 > 0.$$

For each θ, let $\{\zeta_k(\theta)\}_{k=1}^{\infty}$ be an independent identically distributed sequence of random variables with the same law as $\zeta(\theta)$. Then there exist constants $d, \eta > 0$ not depending upon θ, and a family of analytic functions $\psi(\cdot, \theta)$, such that for all $x \in [0, \eta]$, $\theta \in \Theta$, and $n \geq 1$,

$$\left| \log\left[P\left\{ \sum_{k=1}^{m} \zeta_k(\theta) > mx(E\zeta^2(\theta))^{1/2} \right\} / (1 - \Phi(xm^{1/2})) \right] - mx^3 \psi(x, \theta) \right| \leq d \cdot (m^{-1/2} + x).$$

Here $\psi(0, \theta) = E\zeta^3(\theta)/[6(E\zeta^2(\theta))^{3/2}]$ is $1/6$ times the "skewness" of $\zeta(\theta)$.

The problem we have set ourselves is to understand the asymptotics of ruin probabilities on the time-interval $[0, T_n]$. In realistic settings, there will be a minimum time $t_0 > 0$ (not depending upon n) before the first claims can be filed, and this will make certain results easier to state on $[t_0, T_n]$. It turns out that when $\gamma(n) = \mathcal{O}(U_n/m_n)$ the same results are true with t_0 replaced by 0, while if $U_n/m_n = o(\gamma(n))$ the ruin-probabilities on $[0, t_0]$ for sufficiently small t_0 can be shown to have the same behavior as if claim-arrivals were Poisson — a situation covered by Theorem 2.3 below.

Lemma 2.1. Suppose that $\{X_{kj}\}$ and $\{Y_{kj}\}$ satisfying (A.1) are as in Theorem 1.1, and suppose that $R_n(t)$ is as defined in that theorem except that the nonrandom sequences $\{m_n\}$, $\{U_n\}$, $\{\gamma(n)\}$ satisfy (A.2). Then if $\delta \equiv \delta_n$ is any sequence of positive numbers such that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, there exists $c_0 > 0$ such that for all $t_0 \geq 0$

$$\sup_{t_0 \leq t \leq T_n} P\{R_n(t) \leq 0\} \leq P\{ \inf_{t_0 \leq t \leq T_n} R_n(t) \leq 0\}$$

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$$\leq \delta_n^{-1} T_n e^{-c_0 m_n \delta_n^{1/3}} + \sum_{j:\, t_0 \leq j\delta \leq T_n} P\{R_n(j\delta) \leq \delta^{1/2} m_n\}.$$

<u>Proof</u>. The first inequality is obvious. To prove the second, observe that

$$P\{\inf_{t_0 \leq t \leq T_n} R_n(t) \leq 0\} \leq \sum_{j:\, t_0 \leq j\delta \leq T_n} P\{\inf_{j\delta \leq t \leq (j+1)\delta \wedge T_n} R_n(t) \leq 0\}$$

where $x \wedge y$ denotes $\min\{x,y\}$, while if $0 \leq a < b = a + \delta \leq T_n$

$$P\{\sup_{a \leq t \leq b} \sum_{k=1}^{m_n} \left[\sum_{j=N_k(a)+1}^{N_k(t)} X_{kj} - \frac{(t-a)\mu}{\lambda} \right] \geq \delta^{1/2} m(n)\}$$

$$\leq P\left\{ \sum_{k=1}^{m_n} \sum_{j=N_k(a)+1}^{N_k(t)} X_{kj}^+ + \frac{\delta|\mu|}{\lambda} m_n \geq \delta^{1/2} m(n) \right\}$$

$$= P\left\{ \sum_{k=1}^{m_n} \sum_{j=1}^{N_k(\delta)} X_{kj}^+ \geq (\delta^{1/2} - \frac{\delta|\mu|}{\lambda})m_n \right\}$$

$$\leq P\left\{ \sum_{k=1}^{m_n} N_k(\delta) \geq m_n \delta^{2/3} \quad \text{or} \quad \sum_{j \leq \delta^{2/3} m_n} X_{1j}^+ \geq (\delta^{1/2} - \frac{\delta|\mu|}{\lambda})m_n \right\}.$$

Here x^+ denotes $\max\{x,0\}$. By two applications of Lemma 2.0, using $\text{Var } N_1(\delta_n) = \delta_n/\lambda + \mathcal{O}(\delta_n^2)$ as $n \to \infty$, we find

$$P\left\{ \sum_{k=1}^{m_n} N_k(\delta) \geq m_n \delta^{2/3} \right\} = P\left\{ \frac{\sum_{k=1}^{m_n} N_k(\delta) - \delta m_n/\lambda}{[m_n \text{ Var } N_1(\delta)]^{1/2}} \geq \frac{m_n(\delta^{2/3} - \delta/\lambda)}{[m_n \text{ Var } N_1(\delta)]^{1/2}} \right\}$$

$$= \{1 - \Phi((\delta^{2/3} - \frac{\delta}{\lambda})[m_n/\text{Var } N_1(\delta)]^{1/2}) \cdot W(1 + \mathcal{O}(m_n^{-1/2}))$$

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$$\leq \frac{1}{2} \exp\{-c_0 m_n \delta_n^{1/3}\}$$

and

$$P\left\{ \sum_{j=1}^{[\delta^{2/3} m_n]} X_{1j}^+ \geq \delta^{1/2} - \frac{\delta|\mu|}{\lambda})m_n \right\}$$

$$= \left\{ 1 - \Phi\left[\frac{\delta^{1/2} - \frac{\delta|\mu|}{\lambda} - \delta^{2/3} E X_{11}^+}{\delta^{1/3}} \left[\frac{m_n}{\text{Var } X_{11}^+}\right]^{1/2} \right] \right\} \cdot (1 + O(m_n^{-1/2} \delta_n^{1/2}))$$

$$\leq \frac{1}{2} \exp(-c_0 m_n \delta_n^{1/3})$$

for some $c_0 > 0$ and all n. The Lemma follows immediately from these estimates. $\qquad\qquad\square$

The message of Lemma 2.1 is that asymptotically as $n \to \infty$, in very general circumstances the ruin-probability $P\{ \inf_{t_0 \leq t \leq T_n} R_n(t) < 0\}$ differs from $\sup_{t_0 \leq t \leq T_n} P\{R_n(t) < 0\}$ by a factor which is at least 1 and in most examples satisfying (A.1) - (A.2) can be bounded by a constant not depending on n.

The assumption which makes $P_n(T_n)$ a *moderate-deviation* probability is

(A.3) $$U_n/(m_n T_n) + \gamma(n) = o(1) \quad \text{as} \quad n \to \infty.$$

For the present, assume also that $T_n \equiv T < \infty$. The main result of this section is

<u>Theorem 2.2</u>. Assume (1.6) together with (A.1) - (A.3), where $T_n \equiv T < \infty$, and define θ_s as in Theorem 1.1. Fix $t_0 > 0$, and put

$$z_n(t) \equiv \frac{U_n + m_n \gamma(n) \mu t / \lambda}{m_n^{1/2} \theta_t}, \quad \xi \equiv \xi_n = \min_{t_0 \leq t \leq T} z_n(t).$$

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Then for some positive constants c_0 (as in Lemma 2.1), c_1 and c_2,

$$(1-\Phi(\xi_n))e^{-c_1 m_n^{-1/2}\xi_n^3} \le P\{R_n(t) \le 0 \text{ for some } t \in [t_0,T]\}$$

$$\le \sum_{j:\, t_0 \le j\delta_n \le T} \left\{1-\Phi(z_n(j\delta_n) - \frac{(\delta_n m_n)^{1/2}}{\theta_{j\delta_n}})\right\} e^{c_1 m_n^{-1/2}\xi_n^3} + \frac{T}{\delta_n}e^{-c_0 m_n \delta_n^{1/3}}$$

where $\{\delta_n\}$ is arbitrary subject to $\delta_n \to 0$ and $m_n \delta_n \to \infty$ as $n \to \infty$. In particular, taking $d_n \equiv m_n^{-2}\xi_n^4$ we find

$$\left| \ln\left[P\{R_n(t) \le 0 \text{ for some } t \in [t_0,T]\}/(1-\Phi(\xi_n)) \right] \right| \le c_3 m_n^{-1/2}\xi_n^3$$

where c_3 is another positive constant not depending on n.

<u>Proof</u>. By Lemma 2.0 with $t = \theta \in \Theta \equiv [t_0,T]$ and $\zeta_k(\theta) = N_k(t) - \mu t/\lambda$ for $k = 1,\ldots,m_n$, there is a finite constant $c_1 > 0$ not depending on n or t such that

$$\left| \ln\left[P\{R_n(t) \le 0\}/\{1-\Phi(z_n(t))\} \right] \right| \le c_1 m_n^{-1/2} z_n^3(t).$$

The rest follows immediately from Lemma 2.1. If $\delta_n = m_n^{-2}\xi_n^4$, then it is easy to check that uniformly for $j\delta_n \in [t_0,T]$

$$(\delta_n m_n)^{1/2}/\theta_{j\delta_n} = o(z_n(j\delta_n))$$

and

$$|\ln \delta_n| \ll \left(\ln\left[1-\Phi(z_n(j\delta_n) - \frac{\sqrt{\delta_n m_n}}{\theta_{j\delta_n}}) \right] \right) \ll m_n \delta_n^{1/3}$$

as $n \to \infty$, where $A_n \ll B_n$ means the same thing as $A_n = o(B_n)$. $\quad\square$

While Lemmas 2.0 and 2.1 can be made to yield more delicate estimates of

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$\ln[P_n(T)/\{1-\Phi(\xi_n)\}]$ than the one in Theorem 2.2, the main point of the theorem is that under (A.1) - (A.3), with $T_n \equiv T < \infty$, the asymptotically dominant term in $\ln P\{\inf\limits_{[t_0,T_n]} R_n(t) < 0\}$ is $-\frac{1}{2}\xi_n^2 = -\frac{1}{2} \inf\limits_{t_0 \le t \le T_n} \left[\dfrac{U_n + m_n \gamma(n)\mu t/\lambda}{m_n^{1/2}\theta_t}\right]^2$.

Another setting in which this same statement holds is the standard one where T_n becomes large but m_n does not.

<u>Theorem 2.3</u>. Assume that (A.1) - (A.3) hold but that $T_n \to \infty$ while $m_n \equiv m$ remains bounded. Then with the same notations ω_t, $Z_n(t)$, and ξ_n as in Theorem 2.2, and θ_t^2 replaced by $t\,\theta^2$, where $\theta^2 \equiv \lambda^{-1}\alpha^2 + \lambda^{-3}\mu^2\beta^2$,

$$P\{R_n(t) \le 0 \text{ for some } t \in [t_0, T_n]\} =$$

$$= P\left(\max\left\{W(t) - \dfrac{U_n + m_n \gamma(n)\mu t/\lambda + d_1(\log T_n + x_n)}{\theta\sqrt{m_n}} : t_0 \le t \le T_n\right\} \ge 0\right) + d_2 e^{-d_3 x_n}$$

for an arbitrary sequence $\{x_n\}$, where $|d_i|$ for $i = 1,2$ are bounded by finite constants not depending on n, and where $d_3 > 0$. When $x_n = o((m_n T_n)^{1/2})$, $U_n^2/(m_n T_n) + \gamma^2(n) = o(x_n)$, and $t_0 = 0$, as $n \to \infty$

$$(2.1) \quad \ln P\{R_n(t) < 0 \text{ for some } t \le T_n\} \sim -\frac{1}{2} \min\limits_{0 \le t \le T_n} \dfrac{(U_n + m_n\gamma(n)\mu t/\lambda)^2}{\theta^2 m_n t}.$$

<u>Proof</u>. Apply Theorem 1.1 of Csörgö et al (1987b) to each of the independent compound renewal processes $\sum\limits_{i=1}^{N_k(t)} X_{ki}$, for $k = 1, \ldots, m$, to conclude that on a possibly larger probability space there exist independent Wiener processes $W_1(\cdot), \ldots, W_m(\cdot)$ such that

$$P\left\{\sup\limits_{0 \le t \le T_n} \left|\sum\limits_{i=1}^{N_k(t)} X_{ki} - \mu t/\lambda - \theta W_k(t)\right| \ge A \log T_n + x_n\right\} \le B e^{-C x_n}$$

for some positive constants A, B, C and arbitrary $\{x_n\}$. Therefore, defining the new Wiener process $\bar{W}(t) \equiv m^{-1/2}(W_1(t) + \ldots + W_m(t))$, we have

$$P\{ \sup_{0 \leq t \leq T_n} |\sum_{k=1}^{m} \sum_{l=1}^{N_k(t)} X_{kl} - m\mu t/\lambda - \theta m^{1/2}\bar{W}(t)| \geq m(A \log T_n + x_n)\}$$

$$\leq Bme^{-Cx_n}.$$

The first part of the Theorem now follows immediately from the definition of $R_n(t)$. Now by (A.3) it is possible to choose x_n so that as $n \to \infty$, $\xi_n^2 = o(x_n)$ while $x_n = o((m_n T_n)^{1/2})$, and we do so. Then (2.1) with this choice of x_n yields $P\{R_n(t) < 0$ for some $t \in [0, T_n]\}$

$$\sim P\{ \max_{t \leq T_n} (\bar{W}(t) - \frac{U_n + m_n \gamma(n)\mu t/\lambda + d_1(\log T_n + x_n)}{m_n^{1/2}\theta}) \geq 0\}$$

$$= P\{ \max_{0 \leq s \leq 1} (W(s) - \frac{U_n + m_n \gamma(n)\mu s T_n/\lambda + d_1(\log T_n + x_n)}{(m_n T_n)^{1/2}\theta}) \geq 0\},$$

for the Wiener process $W(s) \equiv \bar{W}(sT_n)/T_n^{1/2}$. Now the exact expression for the last probability, as given by Bartlett (1946), is

$$1 - \Phi(a_n + b_n) + e^{-2a_n b_n}(1 - \Phi(a_n - b_n))$$

where $a = a_n = (U_n + d_1 \log T_n + d_1 x_n)\theta^{-1}(m_n T_n)^{-1/2}$ and $b = b_n = \mu\gamma(n)(m_n T_n)^{1/2}/(\lambda\theta)$. Thus, since (A.2) implies that $a_n + b_n \to \infty$ as $n \to \infty$, we find that $\ln P\{R_n(t) < 0$ for some $t \leq T_n\}$ is asymptotic to $-\frac{1}{2}(a_n + b_n)^2$ if $a_n \geq b_n$ and to $-2a_n b_n$ if $a_n < b_n$.

In the present setting, where m_n remains bounded and $T_n \to \infty$ as $n \to \infty$ and (A.2) holds, we have $(\gamma_n T_n + U_n)/T_n^{1/2} \to \infty$. Since x_n has been chosen to be $o(T_n^{1/2})$, we have $\log T_n + x_n = o(U_n + \gamma_n T_n)$, so that as $n \to \infty$

$$a_n \sim a_n' \equiv \theta^{-1} U_n (m_n T_n)^{-1/2}.$$

Thus we have proved

$$(2.2) \quad \ln P\{R_n(t) < 0 \text{ for some } t \in [0, T_n]\} \sim \begin{cases} \frac{1}{2}(a_n' + b_n)^2 & \text{if } a_n' \geq b_n \\ -2a_n' b_n & \text{if } a_n' < b_n. \end{cases}$$

It remains only to check that the right hand side of (2.2) coincides with

$$-\frac{1}{2} \min_{0 \leq t \leq T_n} \frac{(U_n + m_n \gamma(n)\mu t/\lambda)^2}{\theta^2 m_n t} = -\frac{1}{2}\left[\min_{0 \leq s \leq 1} \frac{a_n' + b_n s}{\sqrt{s}}\right]^2,$$

which follows easily by elementary calculus. □

Remarks. (i) The theorems of this section cover the cases where T_n is fixed and m_n gets large or where m_n is fixed and T_n becomes large as $n \to \infty$. Of course, with small changes the proofs of these theorems remain valid under some other assumptions concerning the limiting behavior of sequences $\{m_n\}$ and $\{T_n\}$. For example, Theorem 2.2 remains valid as stated if T is replaced by a sequence $\{T_n\}$ which becomes large in such a way that $\ln(T_n/\delta_n) = o(\xi_n^2)$ as $n \to \infty$. Similarly, minor changes in the proof of Theorem 2.3 yield the same result if the constant m is replaced by a sequence $\{m_n\}$ converging to ∞ in such a way that

$$m_n \log T_n + U_n^2/T_n + m_n \gamma^2(n) = o(U_n + \gamma(n) m_n T_n).$$

(ii) When the random variables Y_{kj} are exponentially distributed with mean λ^{-1}, i.e., when the claim-arrival processes $N_k(t)$ are Poisson with rate λ, the "memoryless" property of claim arrivals immediately implies that the probabilities $P\{R_n(t) \leq 0 \text{ for some } t \leq T_n\}$ depend on $\{m_n\}$, $\{T_n\}$ only

through the products $m_n \cdot T_n$. Writing these probabilities with m_n and $T_n \equiv t_0$ replaced by $T'_n \equiv t_0 > 0$ and $m'_n \equiv m_n T_n$, we find from Theorem 2.3 that as $n \to \infty$

$$\ln P\{T_n(t) < 0 \text{ for some } t \in [0, t_0]\} \sim -\frac{1}{2} \min_{0 \le t \le t_0} \frac{(U_n + m_n \gamma(n) \mu t/\lambda)^2}{\theta^2 m_n t}.$$

Just as in (2.2), the expression on the right can be written explicitly as a continuous function of t_0, λ, U_n, m_n, and $\gamma(n)$.

Now, if $\{Y_{kj}\}$ are i.i.d. with a nonexponential distribution with mean λ, then it is easy to show that for each $t_0 > 0$ there are numbers $\lambda_*(t_0) < \lambda < \lambda^*(t_0)$ such that $(N_k(t) : 0 \le t \le t_0)$ is (stochastically) larger than a Poisson process $N_{k*}(t)$ with rate $(\lambda_*(t_0))^{-1}$ and is (stochastically) smaller than a Poisson process $N_k^*(t)$ with rate $(\lambda^*(t_0))^{-1}$, and such that $\lambda^*(t_0) - \lambda_*(t_0) \to 0$ as $t_0 \to 0$. Then for each $t_0 > 0$, $\ln P\{R_n(t) < 0 \text{ for some } t \in [0, t_0]\}$ lies asymptotically for large n between the values

$$-\frac{1}{2} \min_{0 \le t \le t_0} (U_n + m_n \gamma(n) \frac{\mu t}{\lambda_*})^2 / (m_n t \frac{\alpha^2 + \mu^2}{\lambda^*})$$

and

$$-\frac{1}{2} \min_{0 \le t \le t_0} (U_n + m_n \gamma(n) \frac{\mu t}{\lambda^*})^2 / (m_n t \frac{\alpha^2 + \mu^2}{\lambda_*}).$$

By taking t_0 arbitrarily small but still positive, we conclude that under the hypotheses of Theorem 2.2, the conclusion of that theorem holds if t_0 is allowed to take the value 0.

(iii) A result like Theorem 2.3, but somewhat weaker, can be proved by appealing to Lemma 2.1. As can be seen from Remark (1) above, even Theorem 2.3 cannot be made to cover all interesting cases where T_n is of larger

order of magnitude than m_n. What is lacking is a strong approximation result with optimal rate for *superpositions* of large numbers of renewal processes over large times.

3. **Large-Deviation Ruin Probabilities**.

We continue our study of the asymptotics of $P_n(T_n) = P\{R_n(t) \le 0$ for some $t \in [0, T_n]\}$ under hypothesis (A.1), but now (A.2) - (A.3) is replaced by the following condition characterizing the "large-deviation" setting.

(A.2′) As $n \to \infty$, $\dfrac{U_n}{m_n T_n} \to \omega$ and $\gamma(n)\dfrac{\mu}{\lambda} \to \rho$, where $\omega, \rho \ge 0$ are such that

$\omega + \rho > 0$.

Note that (A.2′) implies (A.2), so that Lemma 2.1 still applies. As in Section 2, we treat first the case where $T_n \equiv T$ remains bounded.

Theorem 3.1. Assume that $\{X_{kj}\}$ and $\{Y_{kj}\}$ are as in Theorem 1.1 and satisfy (A.1), and that $\{m_n\}$, $\{U_n\}$, $\{\gamma(n)\}$ satisfy (A.2′) where $T_n \equiv T \in (0, \infty)$. Then, as $n \to \infty$,

$$m_n^{-1} \log P\{R_n(t) \le 0 \text{ for some } t \in [0,T]\} \to \sup_{0 \le t \le T} \inf_{\xi} \{-(\omega T + (\rho + \tfrac{\mu}{\lambda})t)\xi$$

(3.1)
$$+ \log p_{N(t)}(\varphi_x(\xi))\}$$

where $p_{N(t)}(s)$ for each t denotes the probability generating function $\sum_{j=0}^{\infty} p\{N_1(t) = j\}s^j$ of $N_k(t)$, and where φ_x is the moment generating function of the X_{kj} random variables.

Proof. For each fixed $t > 0$, Chernoff's (1952) Theorem, as given for example by Bahadur (1971, Section 3) says that as $n \to \infty$,

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$$m_n^{-1} \log P\{R_n(t) \le 0\} \rightarrow \inf_\xi \{-(\omega T + (\rho + \frac{\mu}{\lambda})t)\xi + \log E \exp[\xi \sum_{k=1}^{N_1(t)} X_{11}]\}$$

$$= \inf_\xi \{-(\omega T + (\rho + \frac{\mu}{\lambda})t)\xi + \log p_{N(t)}(\varphi_X(\xi))\}$$

where the right hand side is strictly negative and continuous in t, and is

also the limit of $m_n^{-1} \log P\{R_n(t) \le \delta_n^{1/2} m_n\}$ if δ_n is any sequence of con-

stants tending to 0. Apply Lemma 2.1 with $\delta_n \equiv m_n^{-1/2}$ to deduce (3.1). □

The preceding result takes a simple form because it gives information

only about the logarithmic order of magnitude of $P_n(T)$. Nevertheless, the

large-deviation "rate" given as the limit in (3.1) has played some historical

role in the collective-risk literature under the name "adjustment coefficient"

in cases where claim arrivals are Poisson. (See Moriconi 1985 for literature

references and extensions to other claim-arrival processes.) Moreover, as

with the adjustment-coefficient ("the Lundberg-de Finetti inequality"), the

large-deviation rate in (3.1) can be used to provide exact upper bounds for

the left hand side of (3.1) for finite n. Indeed, since the limit in

Chernoff's (1952) theorem is actually an upper bound, the method of proof of

Theorem 3.1 yields:

Corollary 3.2. Under assumption (A.1), for each finite n

$$\log P\{R_n(t) \le 0 \text{ for some } t \in [0,T]\} \le m_n T e^{-c_0 m_n^{2/3}}$$

$$+ m_n T \exp[\sup_{0 \le t \le T} \inf_\xi \{-(U_n - m_n^{1/2} + m_n(1+\gamma(n))\frac{\mu}{\lambda}t)\xi + \log p_{N(t)}(\varphi_X(\xi))\}]$$

where c_0 is as in Lemma 2.1 and can be written explicitly in terms of large-

deviation rates in Chernoff's Theorem for random variables X_{1j}^+ and Y_{1j}.

It remains to find analogues for Theorem 3.1 in the case where $m_n \equiv m$ is bounded and $T_n \to \infty$. Again the result will follow from Lemma 2.1 together with known results in Large Deviations Theory. It should be noted that the method of Strong Approximation by a Wiener process with drift cannot possibly yield a Large-Deviation rate under (A.2') because the error-term $e^{-d_3 x_n}$ in Theorem 2.3 would become the dominant term

Theorem 3.3. Assume (A.1) and (A.2') in the setting of Theorem 1.1, where $m_n \equiv 1 < \infty$ and $T_n \to \infty$ as $n \to \infty$. Then the limit as $n \to \infty$ of

$$T_n^{-1} \log P\{R_n(t) \leq 0 \text{ for some } t \in [0, T_n]\}$$

exists and is given by the formula

$$\sup_{r > 0} \inf\{ r \log \phi_X(t_0) + t_1 - \omega t_0 + r \log \phi_Y(-(\rho + \tfrac{\mu}{\lambda})t_0 - t_1) : t_1, t_0 \geq 0\}.$$

Proof. Since it makes no difference in the result, we assume for convenience in this proof that the "age" y_0 at time 0 is 0, so that the new policy-lifetime $Y_{1,0}$ begins at time 0. If $R_n(t) \leq 0$ for some $t \in [0, T_n]$, then for some integer M, $\sum_{j=0}^{M-1} Y_{1j} \leq T_n$ and also $U_n + (1+\gamma(n))\tfrac{\mu}{\lambda} \sum_{j=0}^{M-1} Y_{1j} - \sum_{j=1}^{M} X_{1j} \leq 0$. For each $a, b, T > 0$ and each integer M, let

$$p(a, b, M, T) \equiv P\{\sum_{j=0}^{M-1} Y_{1j} \leq T, \quad aT + b \sum_{j=0}^{M-1} Y_{1j} - \sum_{j=1}^{M} X_{1j} \leq 0\}.$$

Then for each n,

(3.2) $\max_{M \geq 1} p(U_n/T_n, (1+\gamma(n))\tfrac{\mu}{\lambda}, M, T_n) \leq P\{R_n(t) \leq 0 \text{ for some } t \in [0, T_n]\} \leq$

$$\leq P\{\sum_{j=0}^{T_n^2-1} Y_{1j} \leq T_n\} + \sum_{M=1}^{T_n^2} p(U_n/T_n, (1+\gamma(n))\tfrac{\mu}{\lambda}, M, T_n).$$

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Since $P\{\sum_{j=0}^{T_n^2-1} Y_{1j} \leq T_n\} = a(e^{-\varepsilon T_n^2})$ for some $\varepsilon > 0$, the idea of our proof is to find $\lim_{T\to\infty} T^{-1} \ln p(a,b,[rT],T)$ for each a,b,r and to observe that this limit varies continuously with respect to a and b.

As in Chernoff's (1952) Theorem, a simple upper bound for $p(a,b,M,T)$ suggests the form for its logarithmic order of magnitude when $M = [rT] =$ greatest integer less than or equal to rT:

$$p(a,b,M,T) \leq E\left\{ I_{[\sum_{j=0}^{M-1} Y_{1j} \leq T_n, \ aT+b\sum_{j=0}^{M-1} Y_{1j} - \sum_{j=0}^{M} X_{1j} \leq 0]} \right.$$

$$\left. e^{t_0\left(\sum_{j=1}^{M} X_{1j} - b\sum_{j=0}^{M-1} Y_{1j} - aT\right) + t_1\left(T - \sum_{j=0}^{M-1} Y_{1j}\right)} \right\}$$

$$\leq E\left\{ e^{-t_0 aT + t_1 T} \ e^{(-bt_0 - t_1)\sum_{j=0}^{M-1} Y_{1j}} \ e^{t_0 \sum_{j=1}^{M} X_{1j}} \right\}.$$

Therefore

(3.3)

$$T^{-1} \log p(a,b,M,T) \leq \inf_{t_0, t_1 \geq 0}\left\{ -at_0 + t_1 + \frac{1}{M}\log \phi_Y(-bt_0 - t_1) + \frac{M}{T}\log \phi_X(t_0) \right\}.$$

Moreover, Theorem 5.1 of Groeneboom, Oosterhoff, and Ruymgaart (1979) implies for each $a,b,r > 0$ that

(3.4) $\lim_{T\to\infty} T^{-1} \log p(a,b,[rT],T) = p_*(a,b,r) =$

$$= \inf_{t_0, t_1 \geq 0} \{ r\log\phi_X(t_0) + r\log\phi_Y(-bt_0 - t_1) + t_1 - at_0 \}$$

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and it is easy to see that the expression (3.4) varies monotonically and con-
tinuously with respect to a and b, and that its maximum over $r \geq 0$ is
achieved. It now follows immediately from (3.2) - (3.4) and (A.2′) that for
arbitrarily small $\delta > 0$ and all large n,

$$\sup_{r \geq 0} \log p_*(\omega+\delta, \tfrac{\mu}{\lambda}+\rho+\delta, r) - \delta \leq T_n^{-1} \log P\{R_n(t) \leq 0 \text{ for some } t \leq T_n\}$$

$$\leq T_n^{-1} \log (T_n^2) + T_n^{-1} \sup_{r \geq 0} \log p_*(\omega-\delta, \tfrac{\mu}{\lambda}+\rho-\delta, r) + \delta.$$

Taking limits as first $n \rightarrow \infty$ and then $\delta \rightarrow 0$ completes the proof. □

Remarks. (i) If claim arrivals are Poisson, i.e., if $\phi_Y(s) = (1-\lambda s)^{-1}$, then
it is an instructive exercise to verify that the formula just proved agrees
with (3.1) with $T \equiv 1$, $m_n \equiv T_n$.

(ii) In the setting of Theorem 3.3 if $m_n \equiv m \geq 2$, then a slightly more
complicated application of Groeneboom, Oosterhoff and Ruymgaart's (1979)
Theorem 5.1 yields the large-deviation rate. Thus a result analogous to
Theorem 3.3 holds for general m.

(iii) There is another way of proving that the limit in Theorem 3.3
exists. If the variables Y_{11} are assumed to be essentially bounded, then

$$\left(t + y_0 - \sum_{j=0}^{N_1(t)} Y_{1j}, \; X_{1,N_1(t)+1} \right)$$

defines a Markov process in t which satisfies the hypotheses of Theorems 6.9
and Corollary 7.21 of Stroock (1984). The large-deviation rate calculated in
(3.4) of Theorem 3.3 above therefore coincides with a more complicated general
expression given by Stroock.

(iv) If the claim-counting processes $N_k(t)$ are allowed to be non-

Poisson with independent increments, then Lynch and Sethuraman (1987) provide abstract large-deviation expressions analogous to Theorem 3.3.

The large-deviation rates calculated in Theorems 3.1 and 3.3 give two different possible meanings to the logarithmic order of magnitude of the ruin probability under (A.1) and (A.2'). This rate is interesting as the proper generalization of the classical "adjustment coefficient" mentioned just after Theorem 3.1 above. It should therefore also be interesting to observe that the rate-numbers arising in these Theorems do differ in general! That is, for non-Poisson claim arrivals the asymptotics of the logarithm of the ruin-probability as a multiple of $m_n \cdot T_n$ depends on whether m_n is large or T_n is. For example, if the variables X_{ij} are independent and exponentially distributed with mean μ, and if the Y_{ij} are Gamma$(2, \frac{1}{2}\lambda)$, then for various combinations of parameter-values μ, λ, ω, and γ we display in the following table the large-deviation rates obtained in Theorems 3.1 and 3.3.

Table 1. Large-deviation rate numbers from Theorems 3.1 and 3.3 for various combinations of parameters λ, γ, and ω, where the claim-interoccurrence times Y are $\Gamma(2,\frac{1}{2}\lambda)$ and the claim amounts X are exponential with mean μ standardized to 1 unit.

Parameters			Rate for case m=n, T=1	Rate for case m=1, T=n
λ	γ	ω		
0.1	0.05	2.0	-0.1821	-0.1835
		1.0	-0.0688	-0.0693
		0.5	-0.0313	-0.0316
	0.02	0.5	-0.0156	-0.0157
		0.25	-0.00653	-0.00659
		0.12	-0.00322	-0.00326
		0.1	-0.00256	-0.00261
		0.06	-0.00152	-0.00157
		0.03	-0.00074	-0.00078
0.25	0.05	1.0	-0.1014	-0.1033
	0.02	0.5	-0.0255	-0.0260
		0.25	-0.00851	-0.00868
		0.1	-0.00258	-0.00263
0.5	0.05	1.0	-0.1511	-0.1562
	0.02	0.5	-0.0409	-0.0424
		0.25	-0.0125	-0.0130
		0.1	-0.00302	-0.00315
		0.05	-0.00127	-0.00131
		0.01	-0.000226	-0.000261
1.0	0.05	1.0	-0.2257	-0.2390
		0.5	-0.732	-0.0781
		0.25	-0.241	-0.0258
		0.1	-0.00644	-0.00693
1.0	0.02	0.5	-0.06617	-0.0707
		0.25	-0.0197	-0.0212
		0.1	-0.00418	-0.00450

The Table shows that the superposed stationary renewal model (m = n, T = 1, as in Theorem 3.1) generally yields a larger large-deviation rate than the more usual single renewal process model (m = 1, T = n, as in Theorem 3.3). The difference does not ever seem very large for the Gamma$(2,\frac{1}{2}\lambda)$ interoccurrence distribution used in calculating Table 1: the differences range from less than 1% to as much as about 7% for the parameters shown. As would be expected, the most pronounced differences arise when the expected

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interoccurrence time λ is relatively large compared to the time-horizon (which was taken in 1 in the case of fixed T), when the reserve $U = \omega n$ and loading γ are relatively small.

We next argue that the parameter values chosen in Table 1 are reasonable, and that the differences shown for the large-deviation rates could make some practical difference. Imagine a large life-insurer with, say 1,000 policy lines of average face amount $50,000, and that we are interested in its solvency over a time-horizon of 50 years. This means our $\mu = 1$ is measured in units of $50,000, and that the unit T of time is 50 years. For these n = 1000 policy lines, we assume the loading γ to be either 2% or 5%, and the initial risk-reserve for such a company might be of the order of 5 to 20 million dollars, which in our units would mean that $\omega = U/n$ would lie somewhere in the range from 0.1 to 2.0. Finally, the expected time until a claim for a randomly selected policy might be from 10 to 30 years, which in our units makes λ lie in the range of 0.2 to 0.6. Now consider a particular combination of parameters, say $\lambda = 0.25$, $\gamma = 0.02$, and $\omega = 0.25$. The upper bound on the probability of ruin provided by the usual "coefficient of adjustment" (which is how we refer to the large-deviation rat in Theorem 3.3) would be $e^{-1000*0.00868} = 0.00017$, while the corresponding upper bound using the correct model of superposed renewal processes in Theorem 3.1 would be $e^{-1000*0.00851} = 0.00020$. While the difference between these upper bounds is not large, we conclude that with $\text{Gamma}(2, \frac{1}{2}\lambda)$ interoccurrence times (which have not been chosen to be as different from exponential as might be reasonable), ruin probabilities should be inflated as much as 15% above what one would calculate using ordinary collective-risk models. Note that because of the results of section 2 above, the relative difference between the two ways of calculating probability of ruin can be noticeable only when the probabili-

ties themselves are extremely small. For this reason, we might expect the difference to have practical importance only if the claim *interoccurence-time* distribution is dramatically different from exponential.

4. Conclusion.

We have in this paper surveyed the asymptotics of actuarial ruin-probabilities under a family of superposed-compound-renewal-process models. When the ruin-probabilities are moderate-deviation probabilities (i.e., satisfy (A.2) - (A.3)), we found that the top-order asymptotic term in the ruin-probability is the same under any of our models as long as the product $m_n T_n$ of portfolio-size by time-horizon is the same. However, when the ruin-probabilities are *large*-deviation probabilities, the logarithmic order of magnitude of the ruin-probability can depend noticeably on which of the superposed-compound-renewal process models is used. Actuaries should be concerned whether the usual model of Collective Risk Theory (the model with $m_n \equiv 1$, $T_n \to \infty$) is as appropriate in modelling a large insurance portfolio as the model with many independent lines of insurance ($m_n \to \infty$, with T_n either fixed or $o(m_n)$). We think it is not.

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