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approximation results of Csorgo, Horvath, and Steinebach (1987) and a large-deviation theorem of Groeneboom, Oosternhoff, and Ruymgaart (1979). One consequence is that for non-Poisson claim-arrivals, the large-deviation probabilities of ruin are noticeably affected by the decision to model many parallel policy lines in place of one line with correspondingly faster claim-arrivals.
MODERATE- AND LARGE- DEVIATION PROBABILITIES IN ACTUARIAL RISK THEORY

by

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Abstract

A general model for the actuarial Risk Reserve Process as a superposition of compound delayed-renewal processes is introduced and related to previous models which have been used in Collective Risk Theory. It is observed that nonstationarity of the portfolio "age-structure" within this model can have a significant impact upon probabilities of ruin. When the portfolio size is constant and the policy age-distribution is stationary, the moderate- and large-deviation probabilities of ruin are bounded and calculated using the strong approximation results of Csörgő, Horváth and Steinebach (1987) and a large-deviation theorem of Groeneboom, Oosterhoff, and Ruymgaart (1979). One consequence is that for non-Poisson claim-arrivals, the large-deviation probabilities of ruin are noticeably affected by the decision to model many parallel policy lines in place of one line with correspondingly faster claim-arrivals.

Key words: risk-reserve process; compound delayed-renewal process; superposition; moderate- and large-deviations; strong approximation.
1. Modelling the risk-reserve process.

The problem of exact and asymptotic calculation of ruin probabilities for a (large) insurer has a long and well-documented history. Early work by Lundberg and Cramer [see Cramer 1955 for historical references] modelling portfolios of fixed size in which claim-arrivals follow a Poisson process, has led to voluminous contributions in a few main directions. In case claims are taken to arrive according to a renewal process, Thorin (1982) surveys the large literature on exact evaluation of ruin probabilities by Wiener-Hopf and complex-analytic methods. Cramer himself had a largely actuarial motivation for initiating the study of large-deviation probabilities for sums of independent random variables: a summary of results along this line for ruin probabilities can be found in the book of Beekman (1974). The point of view that rescaled risk-reserve processes should be well-approximated distributionally by Wiener processes with linear drift has led (Iglehart 1969; Harrison 1977) to asymptotic formulas for ruin probabilities as portfolios become large. Other recent work attempts to use martingale or Markov process structure to generalize the classes of risk-reserve process models for which formulas for ruin-probabilities can be written down (e.g. Dassios and Embrechts 1987). Martingale inequalities generalizing Kolmogorov’s classical Exponential Bounds to Martingales have been exploited by Slud (1989) to obtain universal upper-bounds on ruin probabilities in terms of means and variances of claim amounts and inter-occurrence times.

The following model for an insurer’s risk reserve is slightly more general than the authors have seen written down elsewhere. Indeed, it probably has too many parameters to yield useful general information. However, the model does incorporate the main features of actuarial risk for life insurance and annuities, as described by about eighty years of work by actuaries and
Let the index \( k = 1, 2, \ldots, m \) enumerate the independent "lines" of insurance/annuity policies of a large insurer; let \( e_k \geq 0 \) denote the chronological time at which policy number \( k \) is first in force, and assume for simplicity that the policy lines, once started, never terminate but are renewed instantaneously at the successive times \( t_{kj} \) of deaths, at which times the claims \( X_{kj} \) (positive for insurance, negative for annuity) are presented to the insurer. We assume throughout that the times between the \( j \)'th and \((j+1)\)'th claims for policy line \( k \) (for \( j \geq 1 \)) are independent and identically distributed positive random variables \( Y_{kj} \) with hazard rate function \( h_k(\cdot) \) and expectation \( \lambda_k < \infty \). The waiting-time \( Y_{k0} - y_k(e_k) \) from \( e_k \) until the first claim by the \( k \)'th policy line is assumed to be independent of the later inter-claim times \( (Y_{kj} : j \geq 1) \); here \( y_k(e_k) \) is interpreted as a nonrandom initial age at time \( e_k \) of the first individual insured under policy line \( k \), and \( Y_{k0} \) is taken to be distributed according to the conditional distribution of \( Y_{k1} \) given that \( Y_{k1} > y_k(e_k) \). Thus the chronological claim-times \( t_{kj} \) can be expressed by the formula

\[
(1.1) \quad t_{kj} = e_k + (Y_{k0} - y_k(e_k)) + Y_{k1} + \ldots + Y_{kj}, \quad j = 0, 1, \ldots .
\]

The assumptions about the stochastic behavior of the policies are completed by taking the claim amounts \( (X_{kj} : j \geq 1) \) for policy line \( k \) to be independent and identically distributed with finite positive mean \( \mu_k \), and by taking the inter-claim times and claim amounts for different policy lines to be mutually independent. Finally, policy line \( k \) is assumed to pay premiums continuously to the insurer at the nonrandom constant rate \((1+\gamma_k)\mu_k \sigma_k / \lambda_k \) for all chronological times greater than \( e_k \).

In this model, the parameters \( e_k, y_k(e_k), \lambda_k, \gamma_k \) and \( \mu_k \) as well as the
functions $h_k(\cdot)$ are taken to be nonrandom and fixed. The stochastic aspects of policies and claims arise solely from the arrays $X_{k,j} : j > 1$ and $Y_{k,j} : j \geq 0$ of independent random variables.

We next define notations related to policy ages and to the Risk Reserve Process of a life insurer. For each $k$, let $t_{kj}$ be as in (1.1) and put

$$N_k(t) = \begin{cases} \max(j : t \geq t_{kj}) & \text{if } t \geq t_{k0} \\ 0 & \text{if } t_{k0} > t \geq 0 \end{cases}$$

$$y_k(t) = \begin{cases} t \max(t_{kj} : j \geq 0 \text{ and } t > t_{kj}) & \text{if } t \geq t_{k0} \\ y_k(e_k) + t - e_k & \text{if } t_{k0} \geq t \geq e_k \end{cases}$$

Then $N_k(\cdot)$ is the delayed-renewal counting process for the occurrence of claims under the $k$'th policy, and $y_k(\cdot)$ is the corresponding current-age or current-life process. The process $y_k(\cdot)$ is left-continuous, and it is a standard fact that $N_k(t) - \int_0^t h_k(y_k(s))ds$ is a martingale with respect to $t$ for each $k$.

Letting

$$N(t) = \sum_{k=1}^{\infty} N_k(t) \quad \text{and} \quad \pi(t) = \sum_{k=1}^{\infty} (1+y_k) \frac{\mu_k}{\lambda_k} \max\{t-e_k,0\}$$

respectively denote the total number of claims and total premiums paid up to time $t$ on all policy lines, and denoting by $U = R(0)$ the insurer's cash risk-reserve at time 0, we define the risk-reserve process

$$R(t) = U + \pi(t) - \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} X_{k,j}$$

This process consists of a deterministic upward drift minus a superposition of independent compound delayed-renewal counting-process. The primary task of
actuarial risk theory for a life insurer is to study the level-crossing probabilities

\[ P(t) = P(R(t) \leq 0 \text{ for some } t \in [0,T]), \quad T \leq \infty. \]

The model which we have just described contains far too many parameters for a general analysis. Therefore, the simplification to a single class of policies (i.e., to the case where \( h_k(\cdot) = h(\cdot) \), the law of \( X_{kj} \) is the same for all \( k \), and \( \gamma_k = \gamma \)) has been common to all theoretical approaches to this subject. A further simplification which has virtually always been assumed is that the portfolio is of fixed size \( m \) at all times \( t \geq 0 \), i.e., that \( e_k = 0 \) for \( k = 1, \ldots, m \). The reason for these simplifying assumptions is not that the complexities modelled by parameters, \( e_k, \gamma_k, \lambda_k, \mu_k \), etc., do not exist in practice, but rather to limit difficulties of analysis and varieties of phenomena. Under the simplified model of this paragraph,

\[ (1.6) \quad \{X_{kj}\}_{j \geq 1} \text{ and } \{Y_{kj}\}_{j \geq 1} \text{ are each i.i.d. arrays, } e_k = 0 \text{ for all } k, \text{ and } \gamma_k = \gamma \text{ does not depend on } k. \]

In the special case \( h(\cdot) = \lambda \), we have Cramer's (1955) famous Collective Risk model, where the process \( N(\cdot) \) is a superposition of \( m \) independent Poisson \( \lambda \) processes, and the age-parameters \( y_k(e_k) = y_k(0) \) play no role. Precisely the same model arises if \( h(\cdot) = \lambda \) is replaced by \( \lambda m \), with \( T \) replaced by \( mT \) and \( m \) by 1. For this reason, when Iglehart (1969) wished to analyze a more general model with nonexponential random times between claim occurrences, he fixed \( m = 1 \) (and \( y_1(0) = 0 \)). Our model for nonconstant \( h(\cdot) \) allows for a number \( m \) of independent policies or policy-classes which in realistic cases will be much larger than the finite time-horizon \( T \) over which ruin-probabilities should be calculated. [Iglehart in taking \( m = 1 \) naturally regarded the time-horizon \( mT \) as very large.] In this situation,
the initial policy-"ages" \( \{y_k(0) : k = 1, \ldots, m\} \) may be very nonstationary, in the sense that the empirical measure for \( \{y_k(t) : k = 1, \ldots, m\} \) might be seriously nonconstant as \( t \) varies. Such a case would arise if the policyholders at \( t = 0 \) were at much higher risk (through self-selection or some other selection mechanism) than the general population of potential "age-0" policyholders and if \( h(\cdot) \) were a monotonically increasing function. It is not surprising that such nonstationarity could dominate other stochastic effects contributing to the values of ruin-probabilities (1.5). Since actuaries do not typically attempt to produce theoretical models of nonstationarity among their insured populations, we are naturally led to the further assumption, which we adopt from now on, that the policy-age processes \( y_k(t) \) are for each \( k \) strictly stationary stochastic processes in \( t \), i.e., (Karlin and Taylor 1975), that

\[
y_k(0) \text{ has density } \frac{1}{\lambda} \exp \left[ -\int_0^\cdot h(x)dx \right].
\]

In the strictly stationary fixed-portfolio-size setting just described, we set ourselves the task of describing the asymptotic behavior of the ruin-probability (1.5) as \( m \) gets large and (1.5) gets small, where \( U, T \) and \( \gamma \) are allowed to depend on \( m \) but where \( h(\cdot), \lambda, \) and \( \mu \) are not. We do not treat in detail the opposite case where \( U, T, \) and \( \gamma \) vary with \( m \) and behave in such a way that \( P(T) \) defined in (1.5) has a finite limit as \( T \to \infty \). This case is adequately covered by the following Theorem, which is proved either by the same methods as Iglehart's (1969) main result, or alternatively by the discussion in Section 4 of Csörgö et al (1987a) in case \( m \) remains bounded.
Theorem 1.1. Suppose that \( X = \{X_{k,1} : k \geq 1, 1 \geq 1 \} \) and \( Y = \{ Y_{k,j} : k \geq 1, j \geq 1 \} \) are independent arrays of i.i.d. random variables with
\[
EX_{1,1} = \mu, \ Var (X_{1,1}) = \sigma^2, \ EY_{1,1} = \lambda, \ Var (Y_{1,1}) = \beta^2
\]
and \( P(Y_{1,1} > t) = \exp(-\int_{0}^{t} h(x)dx) \) for \( 0 \leq t < \infty \). In addition, let
\[
\{(y_{k}(0),Y_{k,0})\}_{k=1}^{\infty}
\]
be an i.i.d. sequence of random pairs independent of \( X, Y \), with joint density of \( (y_{1}(0),Y_{1,0}) \) at \((s,t)\) given by
\[
\lambda^{-1} h(t) \exp(-\int_{0}^{t} h(x)dx) \quad \text{for} \quad 0 < s < t, \quad \text{and by} \quad 0 \quad \text{for other} \quad (s,t).
\]
For each \( k \), let \( N_{k}(t) \) be defined by
\[
N_{k}(t) = \begin{cases}
0 & \text{if} \quad Y_{k,0} - y_{k}(0) > t \\
1 + \max\{j : Y_{k,0} + Y_{k,1} + \ldots + Y_{k,j} > y_{k}(0) + t\} & \text{if} \quad Y_{k,0} - y_{k}(0) \leq t
\end{cases}
\]
(i.e., as above with \( e_k = 0 \)).

Now suppose that \( \{T_{n}\}_{n=1}^{\infty}, \{m_{n}\}_{n=1}^{\infty}, \{\gamma(n)\}_{n=1}^{\infty}, \) and \( \{U_{n}\}_{n=1}^{\infty} \) are sequences of positive constants such that as \( n \rightarrow \infty \), \( m_{n} \) and \( T_{n} \) are bounded below, with \( m_{n} \cdot T_{n} \rightarrow \infty \) and \( T_{n} \rightarrow T \leq \infty \) and
\[
(m_{n} T_{n})^{1/2} \gamma(n) \rightarrow a < \infty \quad \text{and} \quad (m_{n} T_{n})^{-1/2} U_{n} \rightarrow u < \infty.
\]
Let
\[
R_{n}(t) = U_{n} + (1+\gamma(n))m_{n} \frac{\mu_{k}}{\lambda} - \sum_{k=1}^{m_{n}} \sum_{j=1}^{N_{k}(t)} X_{k,j}, \quad 0 \leq t \leq T_{n}.
\]
Then, as \( n \rightarrow \infty \),
\[
P(R_{n}(T) \leq 0 \text{ for some } t \in [0,T_{n}]) \rightarrow P(\sup(W^{2}_{s}) - a_{s}^{\mu_{k}} - u : 0 \leq s \leq 1) > 0
\]
where \( W(\cdot) \) is a standard Wiener process and
\[ \theta^2_s = \begin{cases} \lambda^{-3} s (\lambda^2 \alpha^2 + \mu^2 \beta^2) & \text{if } T = \infty, \\ (\lambda^{-1} \alpha^2 s + \mu^2 T^{-1} \text{Var}(N_1(sT))) & \text{if } T < \infty. \end{cases} \]

In the remainder of this paper, we retain the setting of Theorem 1.1 (i.e., of (1.6) - (1.7))) except that the parameter sequence \( \{m_n\}, \{T_n\}, \{\gamma(n)\}, \) and \( \{U_n\} \) will be assumed to behave in such a way that \( \mathbb{P}(R_n(t) \leq 0) \text{ for some } t \in [0, T_n] \rightarrow 0 \) as \( n \rightarrow \infty \). This restriction corresponds to the actuarial requirement which motivated Cramer's development of Large Deviations theory, namely that the probability of ruin should become small in a definite way as a function of parameters when the scale of an insurer becomes large.


Throughout the rest of the paper, we assume

(A.1) The random variables \( X_{kj} \) and \( Y_{kj} \) have finite moment generating functions \( \mathbb{E} \exp(sX_{kj}) \) and \( \mathbb{E} \exp(sY_{kj}) \) for \( 0 < s \leq s_0 \); also \( Y_{kj} \) has a finite density \( h(0) \) at \( 0 \).

(A.2) As \( n \rightarrow \infty \), \( U_n/(m_n T_n) + \gamma(n) = O(1) \) and \( (m_n T_n)^{-1/2} = o\left[U_n/(m_n T_n) + \gamma(n)\right] \).

Since our primary goal is to find asymptotic expressions for \( \mathbb{P}_n(T_n) = \mathbb{P}(R_n(t) \leq 0 \text{ for some } t \in [0, T_n]) \), we will repeatedly apply the following technical result on tail-probabilities for parameterized sums of i.i.d. random variables.

**Lemma 2.0 [essentially proved as Theorem 10 of Chapter VIII in Petrov 1972].**

For a fixed subset \( \Theta \) of \( \mathbb{R} \), let \( \{\zeta(\Theta)\}_\Theta \) be a family of mean-0 random
variables which satisfies for some $r > 0$

$$\sup_{\theta \in \Theta} E e^{r \zeta(\theta)} < \infty, \quad \inf_{\theta \in \Theta} E(\zeta(\theta))^2 > 0.$$ 

For each $\theta$, let $\{\zeta_k(\theta)\}_{k=1}^\infty$ be an independent identically distributed sequence of random variables with the same law as $\zeta(\theta)$. Then there exist constants $d, n > 0$ not depending upon $\theta$, and a family of analytic functions $\psi(\cdot, \theta)$, such that for all $x \in [0, n]$, $\theta \in \Theta$, and $n \geq 1$,

$$\log \left[ P \left( \sum_{k=1}^m \zeta_k(\theta) > mx(\zeta(\theta))^{1/2}/(1-\Phi(xm^{1/2})) \right) - mx^3 \psi(x, \theta) \right] \leq d \cdot (m^{-1/2} + x).$$

Here $\psi(0, \theta) = E \zeta^3(\theta)/[6(E \zeta^2(\theta))^{3/2}]$ is $1/6$ times the "skewness" of $\zeta(\theta)$.

The problem we have set ourselves is to understand the asymptotics of ruin probabilities on the time-interval $[0, T]$. In realistic settings, there will be a minimum time $t_0 > 0$ (not depending upon $n$) before the first claims can be filed, and this will make certain results easier to state on $[t_0, T]$. It turns out that when $\gamma(n) = O(U_n/m_n)$ the same results are true with $t_0$ replaced by 0, while if $U_n/m_n = o(\gamma(n))$ the ruin-probabilities on $[0, t_0]$ for sufficiently small $t_0$ can be shown to have the same behavior as if claim-arrivals were Poisson — a situation covered by Theorem 2.3 below.

**Lemma 2.1.** Suppose that $\{X_{k,j}\}$ and $\{Y_{k,j}\}$ satisfying (A.1) are as in Theorem 1.1, and suppose that $R_n(t)$ is as defined in that theorem except that the nonrandom sequence $\{m_n\}, \{U_n\}, \{\gamma(n)\}$ satisfy (A.2). Then if $\delta = \delta_n$ is any sequence of positive numbers such that $\delta_n \to 0$ as $n \to \infty$, there exists $c_0 > 0$ such that for all $t_0 \geq 0$

$$\sup_{t_0 \leq s \leq T_n} P(R_n(t) \leq 0) \leq P(\inf_{t_0 \leq s \leq T_n} R_n(t) \leq 0).$$
\[ \leq \delta^{-1} t_n e^{-c_0 n \delta^{1/3}} + \sum_{j: t_0 \leq j \leq T_n} P(R_n(j) \leq \delta^{1/2} m_n). \]

**Proof.** The first inequality is obvious. To prove the second, observe that

\[ P( \inf_{t \leq T_n} R_n(t) \leq 0) \leq \sum_{j: t_0 \leq j \leq T_n} P( \inf_{j \leq t \leq (j+1) \delta T_n} R_n(t) \leq 0) \]

where \( x \wedge y \) denotes \( \min(x, y) \), while if \( 0 \leq a < b = a + \delta \leq T_n \)

\[ P\left( \sup_{a \leq t \leq b} \sum_{k=1}^{m_n} \left( \sum_{j=N_k(a)+1}^{N_k(t)} X_{kj} - \frac{(t-a)\mu}{\lambda} \right) \geq \delta^{1/2} m(n) \right) \]

\[ \leq P\left\{ \sum_{k=1}^{m_n} \sum_{j=N_k(a)+1}^{N_k(t)} X_{kj}^+ + \frac{\delta |\mu|}{\lambda} m_n \geq \delta^{1/2} m(n) \right\} \]

\[ = P\left\{ \sum_{k=1}^{m_n} \sum_{j=1}^{N_k(\delta)} X_{kj}^+ \geq (\delta^{1/2} - \frac{\delta |\mu|}{\lambda}) m_n \right\} \]

\[ \leq P\left\{ \sum_{k=1}^{m_n} N_k(\delta) \geq \frac{m_n^{2/3}}{\delta^{2/3}} \text{ or } \sum_{j \leq \delta^{2/3} m_n} X_{kj}^+ \geq (\delta^{1/2} - \frac{\delta |\mu|}{\lambda}) m_n \right\}. \]

Here \( x^+ \) denotes \( \max(x, 0) \). By two applications of Lemma 2.0, using

\[ \text{Var } N_1(\delta_n) = \frac{\delta \lambda}{m_n} + O(\delta_n^2) \text{ as } n \to \infty, \]

we find

\[ P\left\{ \sum_{k=1}^{m_n} N_k(\delta) \geq \frac{m_n^{2/3}}{\delta^{2/3}} \right\} \leq \frac{\sum_{k=1}^{m_n} N_k(\delta) - \delta m_n / \lambda}{[m_n \text{ Var } N_1(\delta)]^{1/2}} \geq \frac{m_n^{(\delta^{2/3} - \delta/\lambda)}}{[m_n \text{ Var } N_1(\delta)]^{1/2}} \]

\[ = (1 - \Phi((\delta^{2/3} - \delta)^{1/2} [m_n / \text{ Var } N_1(\delta)]^{1/2}) \cdot W(1 + O(m_n^{-1/2})) \]

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\[ \leq \frac{1}{2} \exp(-c_0 n^d_{1/3}) \]

and

\[
P\left\{ \sum_{j=1}^{[d^{2/3}m_n]} x_{1j}^{+} \geq d^{1/2} - \frac{d|\mu|}{\lambda} n \right\}
= \left\{ 1 - \Phi \left( \frac{d^{1/2} - \frac{d|\mu|}{\lambda} - \frac{d^{2/3} \epsilon_{11}}{\delta^{1/3}}}{\delta^{1/3} \epsilon_{11}} \right) \right\} \left( 1 + o(\frac{1}{n^{1/2} d^{1/2}}) \right)
\leq \frac{1}{2} \exp(-c_0 n^d_{1/3})
\]

for some \( c_0 > 0 \) and all \( n \). The Lemma follows immediately from these estimates.

The message of Lemma 2.1 is that asymptotically as \( n \to \infty \), in very general circumstances the ruin-probability \( P\{ \inf_{t_0 \leq t \leq T_n} R(t) < 0 \} \) differs from \( \sup_{t_0 \leq t \leq T_n} P\{R(t) < 0\} \) by a factor which is at least 1 and in most examples satisfying (A.1) - (A.2) can be bounded by a constant not depending on \( n \).

The assumption which makes \( P_n(T_n) \) a moderate-deviation probability is

\[ (A.3) \quad U_n/(m_n T_n) + \gamma(n) = o(1) \quad \text{as} \quad n \to \infty. \]

For the present, assume also that \( T_n = T < \infty \). The main result of this section is

**Theorem 2.2.** Assume (1.6) together with (A.1) - (A.3), where \( T_n = T < \infty \), and define \( \theta_s \) as in Theorem 1.1. Fix \( t_0 > 0 \), and put

\[ z_n(t) = \frac{U_n + m_n \gamma(n) \mu t / \lambda}{m_n^{1/2} \theta_s t_0}, \quad \xi = \xi_n = \min_{t_0 \leq t \leq T_n} z_n(t). \]
Then for some positive constants $c_0$, $c_1$, and $c_2$,
\[
\left(1 - \Phi(x_n)\right)e^{-c_1m_n^{-1/2}x_n^3} 
\leq P(R_n(t) \leq 0 \text{ for some } t \in [t_0, T])
\]
\[
\leq \sum_{j: t_0 \leq j \delta_n \leq T} \left\{1 - \Phi(z_n(j \delta_n) - \frac{0.5m_n}{\theta_j \delta_n})\right\} e^{-c_1m_n^{-1/2}x_n^3} + \frac{T}{\delta_n} e^{-c_0m_n^{-1/3}}
\]
where $\{\delta_n\}$ is arbitrary subject to $\delta_n \to 0$ and $m_n \delta_n \to \infty$ as $n \to \infty$. In particular, taking $d_n = m_n^{-2} \epsilon_n^4$ we find
\[
|\ln \left[P(R_n(t) \leq 0 \text{ for some } t \in [t_0, T])/(1 - \Phi(z_n(t)))\right]| \leq c_3m_n^{-1/2}x_n^3
\]
where $c_3$ is another positive constant not depending on $n$.

**Proof.** By Lemma 2.0 with $t = \theta \in \theta \in [t_0, T]$ and $\zeta_k(\theta) = \frac{N_k(t) - \mu t}{\lambda}$ for $k = 1, \ldots, m_n$, there is a finite constant $c_1 > 0$ not depending on $n$ or $t$ such that
\[
|\ln \left[P(R_n(t) \leq 0) / (1 - \Phi(z_n(t)))\right]| \leq c_1m_n^{-1/2}x_n^3(t).
\]

The rest follows immediately from Lemma 2.1. If $\delta_n = m_n^{-2} \epsilon_n^4$, then it is easy to check that uniformly for $j \delta_n \in [t_0, T]$
\[
(\delta_n m_n)^{1/2} \theta_j \delta_n = a(z_n(j \delta_n))
\]
and
\[
|\ln \delta_n| \leq (\ln \left[1 - \Phi(z_n(j \delta_n) - \frac{\sqrt{m_n}}{\theta_j \delta_n})\right]) \leq m_n \delta_n^{1/3}
\]
as $n \to \infty$, where $A_n \ll B_n$ means the same thing as $A_n = o(B_n)$. \(\square\)

While Lemmas 2.0 and 2.1 can be made to yield more delicate estimates of
\[ \ln \left[ \frac{P_n(T)}{1 - \Phi(\xi_n)} \right] \] than the one in Theorem 2.2, the main point of the theorem is that under (A.1)-(A.3), with \( T_n = T < \infty \), the asymptotically dominant term in \( \ln P \left( \inf \frac{R(t)}{n} < 0 \right) \) is

\[ \frac{1}{2} \sum_{n} \inf_{0 \leq t \leq T_n} \left[ U_n + m \gamma(n) \mu / \lambda \right] \]

Another setting in which this same statement holds is the standard one where \( T_n \) becomes large but \( m_n \) does not.

**Theorem 2.3.** Assume that (A.1)-(A.3) hold but that \( T_n \rightarrow \infty \) while \( m_n = m \) remains bounded. Then with the same notations \( \omega_t, Z_n(t), \) and \( \xi_n \) as in Theorem 2.2, and \( \theta_t^2 \) replaced by \( t \theta_t^2 \), where \( \theta_t^2 = \lambda - \frac{1}{2} \alpha + \lambda - \frac{3}{2} \beta^2 \).

\[ P \{ R_n(t) \leq 0 \text{ for some } t \in [t_0, T_n] \} = \]

\[ \mathbb{P} \left( \max_{0 \leq t \leq T_n} \frac{U_n + m \gamma(n) \mu / \lambda + d_1 \left( \log T_n + x_n \right)}{\theta \sqrt{n}_{n}} \right) = \mathbb{P} \left( \max_{0 \leq t \leq T_n} \frac{U_n + m \gamma(n) \mu / \lambda + d_1 \left( \log T_n + x_n \right)}{\theta \sqrt{n}_{n}} \right) + d_2 e^{-d_3 x_n} \]

for an arbitrary sequence \( \{x_n\} \), where \( |d_1| \) for \( 1 = 1, 2 \) are bounded by finite constants not depending on \( n \), and where \( d_3 > 0 \). When \( x_n = \alpha((m T_n)^{1/2}) \), \( U_n/(m T_n) + \gamma^2(n) = \alpha(x_n) \), and \( t_0 = 0 \), as \( n \rightarrow \infty \)

\[ (2.1) \ln P \{ R_n(t) < 0 \text{ for some } t \leq T_n \} \sim -\frac{1}{2} \min_{0 \leq t \leq T_n} \frac{(U_n + m \gamma(n) \mu / \lambda)^2}{\theta^2 m_t} \]

**Proof.** Apply Theorem 1.1 of Csörgő et al (1987b) to each of the independent compound renewal processes \( \sum_{i=1}^{k} X_{ki} \) for \( k = 1, \ldots, m \), to conclude that on a possibly larger probability space there exist independent Wiener processes \( W_1(\cdot), \ldots, W_m(\cdot) \) such that

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T_n} \left| \sum_{i=1}^{k} X_{ki} - \mu t / \lambda - \theta W_k(t) \right| \geq A \log T_n + x_n \right) \leq Be^{-C x_n} \]
for some positive constants $A, B, C$ and arbitrary $\{x_n\}$. Therefore, defining

the new Wiener process $\bar{W}(t) = m^{-1/2}(W_1(t) + \ldots + W_m(t))$, we have

$$P\left( \sup_{0 \leq t \leq T_n} \sum_{k=1}^{m} \sum_{i=1}^{N_k(t)} X_{ki} - m\mu t / \lambda - \Theta m^{1/2} \bar{W}(t) \right) \geq m(\Delta \log T_n + x_n) \right) \right) \leq Cx_n.$$ 

The first part of the Theorem now follows immediately from the definition of $R_n(t)$. Now by (A.3) it is possible to choose $x_n$ so that as $n \to \infty$, $\xi_n^2 = 0(x_n)$ while $x_n = o((mT_n)^{1/2})$, and we do so. Then (2.1) with this choice of $x_n$ yields

$$P\left( R_n(t) < 0 \text{ for some } t \in [0, T_n] \right) \right) \sim \exp \left( \frac{U + m \gamma(n) \mu T_n / \lambda + d_1 (\log T_n + x_n)}{m \gamma(n) \mu T_n / \lambda + d_1 (\log T_n + x_n)} \right) \geq 0 \right) \right),$$

for the Wiener process $W(s) = \bar{W}(sT_n)/T_n^{1/2}$. Now the exact expression for the last probability, as given by Bartlett (1946), is

$$1 - \Phi(a_n + b_n) + e^{-2a_n b_n} n(1 - \Phi(a_n - b_n))$$

where $a_n = a_n = (U + d_1 \log T_n + d_1 x_n)^{-1/2} (mT_n)^{-1/2}$ and $b_n = \mu T_n (mT_n)^{1/2}/(\lambda \theta)$. Thus, since (A.2) implies that $a_n + b_n \to \infty$ as $n \to \infty$, we find that

$$\ln P\left( R_n(t) < 0 \text{ for some } t \leq T_n \right)$$

is asymptotic to $-\frac{1}{2}(a_n + b_n)^2$ if $a_n \geq b_n$ and to $-2a_n b_n$ if $a_n < b_n$.

In the present setting, where $m$ remains bounded and $T_n \to \infty$ as $n \to \infty$ and (A.2) holds, we have $(\gamma T_n + U) / T_n^{1/2} \to \infty$. Since $x_n$ has been chosen to be $o(T_n^{1/2})$, we have $\log T_n + x_n = o(U + \gamma T_n)$, so that as $n \to \infty$.
Thus we have proved

\[
(a'_n + b_n)^2 \quad \text{if} \quad a'_n \geq b_n
\]

\[
-2a'_n b_n \quad \text{if} \quad a'_n < b_n.
\]

It remains only to check that the right hand side of (2.2) coincides with

\[
\frac{1}{2} \min_{0 \leq s \leq t} \frac{(U + m_n \gamma(n) \mu t/\lambda)^2}{\theta^2 m_n t} = - \frac{1}{2} \left[ \min_{0 \leq s \leq 1} \frac{a'_n + b_n}{\sqrt{s}} \right]^2
\]

which follows easily by elementary calculus.

Remarks. (1) The theorems of this section cover the cases where \( T_n \) is fixed and \( m_n \) gets large or where \( m_n \) is fixed and \( T_n \) becomes large as \( n \to \infty \).

Of course, with small changes the proofs of these theorems remain valid under some other assumptions concerning the limiting behavior of sequences \( \{m_n\} \) and \( \{T_n\} \). For example, Theorem 2.2 remains valid as stated if \( T \) is replaced by a sequence \( \{T_n\} \) which becomes large in such a way that

\[
\ln(T_n/\delta) = o(\xi_n^2) \quad \text{as} \quad n \to \infty.
\]

Similarly, minor changes in the proof of Theorem 2.3 yield the same result if the constant \( m \) is replaced by a sequence \( \{m_n\} \) converging to \( \infty \) in such a way that

\[
m_n \log T_n + U^2/n + m_n \gamma^2(n) = o(U + \gamma(n)m T_n).
\]

(11) When the random variables \( Y_{k,j} \) are exponentially distributed with mean \( \lambda^{-1} \), i.e., when the claim-arrival processes \( N_k(t) \) are Poisson with rate \( \lambda \), the "memoryless" property of claim arrivals immediately implies that the probabilities \( P(R_n(t) \leq 0 \quad \text{for some} \quad t \leq T_n) \) depend on \( \{m_n\}, \{T_n\} \) only.
through the products $m_n T_n$. Writing these probabilities with $m_n$ and $T_n = t_0$ replaced by $T'_n = t_0 > 0$ and $m'_n = m_n T_n$, we find from Theorem 2.3 that as $n \to \infty$

$$\ln P(T_n(t) < 0 \text{ for some } t \in [0,t_0]) \sim -\frac{1}{2} \min_{0 \leq t \leq t_0} \frac{(U_n + m_n \gamma(n) \mu t/\lambda)^2}{\theta^2 n^2 m_n t}.$$ 

Just as in (2.2), the expression on the right can be written explicitly as a continuous function of $t_0, \lambda, U_n, m_n$, and $\gamma(n)$.

Now, if $\{Y_{jk}\}$ are i.i.d. with a nonexponential distribution with mean $\lambda$, then it is easy to show that for each $t_0 > 0$ there are numbers $\lambda_*(t_0) < \lambda < \lambda^*(t_0)$ such that $\{N_k(t) : 0 \leq t \leq t_0\}$ is (stochastically) larger than a Poisson process $N_k*(t)$ with rate $(\lambda^*(t_0))^{-1}$ and is (stochastically) smaller than a Poisson process $N_k^*(t)$ with rate $(\lambda_*(t_0))^{-1}$, and such that $\lambda^*(t_0) - \lambda_*(t_0) \to 0$ as $t_0 \to 0$. Then for each $t_0 > 0$, $\ln P(R_n(t) < 0 \text{ for some } t \in [0,t_0])$ lies asymptotically for large $n$ between the values

$$-\frac{1}{2} \min_{0 \leq t \leq t_0} \frac{(U_n + m_n \gamma(n) \mu t/\lambda)^2}{\theta^2 n^2 m_n t}$$

and

$$-\frac{1}{2} \min_{0 \leq t \leq t_0} \frac{(U_n + m_n \gamma(n) \mu t/\lambda)^2}{\theta^2 n^2 m_n t}.$$ 

By taking $t_0$ arbitrarily small but still positive, we conclude that under the hypotheses of Theorem 2.2, the conclusion of that theorem holds if $t_0$ is allowed to take the value 0.

(iii) A result like Theorem 2.3, but somewhat weaker, can be proved by appealing to Lemma 2.1. As can be seen from Remark (1) above, even Theorem 2.3 cannot be made to cover all interesting cases where $T_n$ is of larger

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order of magnitude than $m_n$. What is lacking is a strong approximation result with optimal rate for superpositions of large numbers of renewal processes over large times.

3. Large-Deviation Ruin Probabilities.

We continue our study of the asymptotics of $P_n(T_n) = P(R_n(t) \leq 0$ for some $t \in [0,T_n]$) under hypothesis (A.1), but now (A.2) - (A.3) is replaced by the following condition characterizing the "large-deviation" setting.

\[(A.2') \text{ As } n \to \infty, \frac{U_n}{m_nT_n} \to \omega \text{ and } \gamma(n)^{\frac{\mu}{\lambda}} \to \rho, \text{ where } \omega, \rho \geq 0 \text{ are such that } \omega + \rho > 0.\]

Note that (A.2') implies (A.2), so that Lemma 2.1 still applies. As in Section 2, we treat first the case where $T_n = T$ remains bounded.

**Theorem 3.1.** Assume that $\{X_{kj}\}$ and $\{Y_{kj}\}$ are as in Theorem 1.1 and satisfy (A.1), and that $\{m_n\}, \{U_n\}, \{\gamma(n)\}$ satisfy (A.2') where $T_n = T \in (0,\infty)$. Then, as $n \to \infty$,

$$m_n^{-1} \log P(R_n(t) \leq 0 \text{ for some } t \in [0,T]) \to \sup_{0 \leq s \leq T} \inf_{\xi} \left\{ -(\omega T + (\rho + \frac{\mu}{\lambda})t) \xi - \log p_N(t)(\varphi_x(\xi)) \right\}$$

(3.1)

where $p_N(t)(s)$ for each $t$ denotes the probability generating function $\sum p(N(t) = j)s^j$ of $N_k(t)$, and where $\varphi_x$ is the moment generating function of the $X_{kj}$ random variables.

**Proof.** For each fixed $t > 0$, Chernoff's (1952) Theorem, as given for example by Bahadur (1971, Section 3) says that as $n \to \infty$,
\[
\frac{1}{n} \log P\{R_n(t) \leq 0\} \rightarrow \inf_n \left\{ -\left( \omega T + (\rho + \frac{\mu}{\lambda}) t \right) \xi + \log E \exp \left[ \xi \sum_{k=1}^{N_1(t)} X_{1k} \right] \right\}
\]

\[
= \inf_n \left\{ -\left( \omega T + (\rho + \frac{\mu}{\lambda}) t \right) \xi + \log P_N(t) (\varphi_x(\xi)) \right\}
\]

where the right hand side is strictly negative and continuous in \( t \), and is also the limit of \( \frac{1}{n} \log P\{R_n(t) \leq \delta_n^{1/2} m_n \} \) if \( \delta_n \) is any sequence of constants tending to 0. Apply Lemma 2.1 with \( \delta_n = m_n^{-1/2} \) to deduce (3.1). \( \square \)

The preceding result takes a simple form because it gives information only about the logarithmic order of magnitude of \( P_n(T) \). Nevertheless, the large-deviation "rate" given as the limit in (3.1) has played some historical role in the collective-risk literature under the name "adjustment coefficient" in cases where claim arrivals are Poisson. (See Moriconi 1985 for literature references and extensions to other claim-arrival processes.) Moreover, as with the adjustment-coefficient ("the Lundberg-de Finetti inequality"), the large-deviation rate in (3.1) can be used to provide exact upper bounds for the left hand side of (3.1) for finite \( n \). Indeed, since the limit in Chernoff’s (1952) theorem is actually an upper bound, the method of proof of Theorem 3.1 yields:

**Corollary 3.2.** Under assumption (A.1), for each finite \( n \)

\[
\log P\{R_n(t) \leq 0 \text{ for some } t \in [0,T] \} \leq m_n T e^{-c_0 m_n^{2/3}}
\]

\[
+ m_n T \exp \left[ \sup_{0 \leq t \leq T} \inf \left\{ -\left( U_n - m_n^{1/2} \right) \right. \right. \left. \left. \left( 1+\gamma(n) \right) \right. \right. \left. \left. \left( \frac{\mu}{\lambda} t \right) \xi + \log P_N(t) (\varphi_x(\xi)) \right\} \right]
\]

where \( c_0 \) is as in Lemma 2.1 and can be written explicitly in terms of large-deviation rates in Chernoff’s Theorem for random variables \( X_{1j}^+ \) and \( Y_{1j}^+ \).
It remains to find analogues for Theorem 3.1 in the case where $m_n = m$ is bounded and $T_n \to \infty$. Again the result will follow from Lemma 2.1 together with known results in Large Deviations Theory. It should be noted that the method of Strong Approximation by a Wiener process with drift cannot possibly yield a Large-Deviation rate under (A.2') because the error-term $e^{-d_3X_n}$ in Theorem 2.3 would become the dominant term.

**Theorem 3.3.** Assume (A.1) and (A.2') in the setting of Theorem 1.1, where $m_n = 1 < \infty$ and $T_n \to \infty$ as $n \to \infty$. Then the limit as $n \to \infty$ of

$$T_n^{-1} \log P(R_n(t) \leq 0 \text{ for some } t \in [0, T_n])$$

exists and is given by the formula

$$\sup_{r > 0} \inf \left\{ r \log \phi_X(t_0) + t_1 - \omega t_0 + r \log \phi_Y(-\frac{\mu}{\lambda} t_0 - t_1) : t_1, t_0 \geq 0 \right\}.$$

**Proof.** Since it makes no difference in the result, we assume for convenience in this proof that the "age" $y_0$ at time 0 is 0, so that the new policy-lifetime $Y_{1,0}$ begins at time 0. If $R_n(t) \leq 0$ for some $t \in [0, T_n]$, then for some integer $M$, $\sum_{j=0}^{M-1} Y_j \leq T_n$ and also $\sum_{j=0}^{M-1} \frac{\mu}{\lambda} Y_j - \sum_{j=1}^{M-1} X_j \leq 0$. For each $a, b, T > 0$ and each integer $M$, let

$$p(a, b, M, T) = P\left\{ \sum_{j=0}^{M-1} Y_j \leq T, a + b \sum_{j=0}^{M-1} Y_j - \sum_{j=1}^{M-1} X_j \leq 0 \right\}.$$

Then for each $n$,

$$\max_{M \geq 1} p(U_n/T_n, (1+\gamma(n))\frac{\mu}{\lambda}, M, T_n) \leq P(R_n(t) \leq 0 \text{ for some } t \in [0, T_n]) \leq \sum_{M=1}^{T_n^{2}} p(U_n/T_n, (1+\gamma(n))\frac{\mu}{\lambda}, M, T_n).$$

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Since \( P\{ \sum_{j=0}^{n} Y_{1j} \leq T_n \} = c(e^{-cT_n}) \) for some \( c > 0 \), the idea of our proof is to find \( \lim_{T \to \infty} \ln p(a,b,[rT],T) \) for each \( a, b, r \) and to observe that this limit varies continuously with respect to \( a \) and \( b \).

As in Chernoff's (1952) Theorem, a simple upper bound for \( p(a,b,M,T) \) suggests the form for its logarithmic order of magnitude when \( M = [rT] = \) greatest integer less than or equal to \( rT \):

\[
p(a,b,M,T) \leq E \left\{ \prod_{j=1}^{M-1} \left[ \sum_{j=0}^{M-1} Y_{1j} \leq Y_{1j} - \sum_{j=0}^{M-1} X_{1j} \geq 0 \right] \right\} e^{t_0 \left( \sum_{j=1}^{M-1} X_{1j} - b \sum_{j=0}^{M-1} Y_{1j} - aT \right) + t_1 \left( \sum_{j=0}^{M-1} Y_{1j} \right) - t_0 \left( \sum_{j=0}^{M-1} X_{1j} \right)}
\]

Therefore

\[(3.3) \quad T^{-1} \log p(a,b,M,T) \leq \inf_{t_0, t_1 \geq 0} \left\{ -at_0 + t_1 + \frac{1}{M} \log \phi_y(-bt_0 - t_1) + \frac{M}{T} \log \phi_x(t_0) \right\}.
\]

Moreover, Theorem 5.1 of Groeneboom, Oosterhoff, and Ruymgaart (1979) implies for each \( a, b, r > 0 \) that

\[(3.4) \quad \lim_{T \to \infty} T^{-1} \log p(a,b,[rT],T) = p_*(a,b,r) = \inf_{t_0, t_1 \geq 0} \left\{ r \log \phi_x(t_0) + r \log \phi_y(-bt_0 - t_1) + t_1 - at_0 \right\}.
\]
and it is easy to see that the expression (3.4) varies monotonically and continuously with respect to \( a \) and \( b \), and that its maximum over \( r \geq 0 \) is achieved. It now follows immediately from (3.2) - (3.4) and (A.2') that for arbitrarily small \( \delta > 0 \) and all large \( n \),

\[
\sup_{r \geq 0} \log p_p(\omega+\delta, \frac{\mu}{\lambda}+\rho+\delta, r) - \delta \leq T_n^{-1} \log P\{R_n(t) \leq 0 \text{ for some } t \leq T_n\},
\]

\[
\leq T_n^{-1} \log (T_n^2) + T_n^{-1} \sup_{r \geq 0} \log p_p(\omega-\delta, \frac{\mu}{\lambda}+\rho-\delta, r) + \delta.
\]

Taking limits as first \( n \to \infty \) and then \( \delta \to 0 \) completes the proof. \( \square \)

Remarks. (I) If claim arrivals are Poisson, i.e., if \( \phi_p(s) = (1-\lambda s)^{-1} \), then it is an instructive exercise to verify that the formula just proved agrees with (3.1) with \( T = 1, m_n = T_n \).

(ii) In the setting of Theorem 3.3 if \( m_n = m \geq 2 \), then a slightly more complicated application of Groeneboom, Oosterhoff and Ruymgaart's (1979) Theorem 5.1 yields the large-deviation rate. Thus a result analogous to Theorem 3.3 holds for general \( m \).

(iii) There is another way of proving that the limit in Theorem 3.3 exists. If the variables \( Y_{1j} \) are assumed to be essentially bounded, then

\[
N_1(t) \left[ t + y_{0} - \sum_{j=0}^{Y_{1j}} X_{1j} N_1(t)+1 \right]
\]

defines a Markov process in \( t \) which satisfies the hypotheses of Theorems 6.9 and Corollary 7.21 of Stroock (1984). The large-deviation rate calculated in (3.4) of Theorem 3.3 above therefore coincides with a more complicated general expression given by Stroock.

(iv) If the claim-counting processes \( N_k(t) \) are allowed to be non-
Poisson with independent increments, then Lynch and Sethuraman (1987) provide abstract large-deviation expressions analogous to Theorem 3.3.

The large-deviation rates calculated in Theorems 3.1 and 3.3 give two different possible meanings to the logarithmic order of magnitude of the ruin probability under (A.1) and (A.2'). This rate is interesting as the proper generalization of the classical "adjustment coefficient" mentioned just after Theorem 3.1 above. It should therefore also be interesting to observe that the rate-numbers arising in these Theorems do differ in general! That is, for non-Poisson claim arrivals the asymptotics of the logarithm of the ruin-probability as a multiple of \( m_n T_n \) depends on whether \( m_n \) is large or \( T_n \) is. For example, if the variables \( X_{i,j} \) are independent and exponentially distributed with mean \( \mu \), and if the \( Y_{i,j} \) are Gamma\((2, \frac{1}{2\lambda})\), then for various combinations of parameter-values \( \mu, \lambda, \omega, \gamma \) we display in the following table the large-deviation rates obtained in Theorems 3.1 and 3.3.
Table 1. Large-deviation rate numbers from Theorems 3.1 and 3.3 for various combinations of parameters $\lambda$, $\gamma$, and $\omega$, where the claim-interoccurrence times $Y$ are $\Gamma(2, \frac{1}{2}\lambda)$ and the claim amounts $X$ are exponential with mean $\mu$ standardized to 1 unit.

<table>
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<th>Parameters</th>
<th>Rate for case $m=n$, $T=1$</th>
<th>Rate for case $m=1$, $T=n$</th>
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The Table shows that the superposed stationary renewal model ($m = n$, $T = 1$, as in Theorem 3.1) generally yields a larger large-deviation rate than the more usual single renewal process model ($m = 1$, $T = n$, as in Theorem 3.3). The difference does not ever seem very large for the $\text{Gamma}(2, \frac{1}{2}\lambda)$ interoccurrence distribution used in calculating Table 1: the differences range from less than $1\%$ to as much as about $7\%$ for the parameters shown. As would be expected, the most pronounced differences arise when the expected
interoccurrence time $\lambda$ is relatively large compared to the time-horizon (which was taken in $T$ in the case of fixed $T$), when the reserve $U = \omega n$ and loading $\gamma$ are relatively small.

We next argue that the parameter values chosen in Table 1 are reasonable, and that the differences shown for the large-deviation rates could make some practical difference. Imagine a large life-insurer with, say 1,000 policy lines of average face amount $\$50,000$, and that we are interested in its solvency over a time-horizon of 50 years. This means our $\mu = 1$ is measured in units of $\$50,000$, and that the unit $T$ of time is 50 years. For these $n = 1000$ policy lines, we assume the loading $\gamma$ to be either 2% or 5%, and the initial risk-reserve for such a company might be of the order of 5 to 20 million dollars, which in our units would mean that $\omega = U/n$ would lie somewhere in the range from 0.1 to 2.0. Finally, the expected time until a claim for a randomly selected policy might be from 10 to 30 years, which in our units makes $\lambda$ lie in the range of 0.2 to 0.6. Now consider a particular combination of parameters, say $\lambda = 0.25$, $\gamma = 0.02$, and $\omega = 0.25$. The upper bound on the probability of ruin provided by the usual "coefficient of adjustment" (which is how we refer to the large-deviation rate in Theorem 3.3) would be $e^{-1000 \cdot 0.00862} = 0.00017$, while the corresponding upper bound using the correct model of superposed renewal processes in Theorem 3.1 would be $e^{-1000 \cdot 0.00851} = 0.00020$. While the difference between these upper bounds is not large, we conclude that with Gamma$(2, \frac{1}{2} \lambda)$ interoccurrence times (which have not been chosen to be as different from exponential as might be reasonable), ruin probabilities should be inflated as much as 15% above what one would calculate using ordinary collective-risk models. Note that because of the results of section 2 above, the relative difference between the two ways of calculating probability of ruin can be noticeable only when the probabili-
ties themselves are extremely small. For this reason, we might expect the
difference to have practical importance only if the claim interoccurrence-time
distribution is dramatically different from exponential.

4. Conclusion.

We have in this paper surveyed the asymptotics of actuarial ruin-
probabilities under a family of superposed-compound-renewal-process models.
When the ruin-probabilities are moderate-deviation probabilities (i.e., satisfy \(A.2\) - \(A.3\)), we found that the top-order asymptotic term in the ruin-
probability is the same under any of our models as long as the product \(m_n T_n\)
of portfolio-size by time-horizon is the same. However, when the ruin-
probabilities are large-deviation probabilities, the logarithmic order of
magnitude of the ruin-probability can depend noticeably on which of the
superposed-compound-renewal process models is used. Actuaries should be
concerned whether the usual model of Collective Risk Theory (the model with
\(m_n = 1, T_n \to \infty\)) is as appropriate in modelling a large insurance portfolio as
the model with many independent lines of insurance \((m_n \to \infty, \text{ with } T_n \text{ either}
fixed or } a(m_n)\). We think it is not.

References

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