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# EXACT LOWER CONFIDENCE LIMITS FOR THE RELIABILITY OF k OF n SYSTEMS FOR ZERO FAILURES AND NONCONSERVATIVENESS OF THE MAXIMUS METHOD

Andrew P. Soms



UNIVERSITY OF WISCONSIN

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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## ABSTRACT

The Buehler (1957) optimal $1-\alpha$ lower confidence limit on the reliability of $k$ of $n$ systems of independent components is derived for the case of zero failures and equal sample sizes. The limiting form of the lower confidence limit is obtained for $n-1$ of $n$ systems as $n$ goes to infinity. This result is used to show the nonconservativeness of the Maximus method given by Spencer and Easterling (1986).



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* Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201.

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## 1. The Lower Confidence Limit

We consider a $k$ of $n$ system of independent components, i.e., a system where the components function independently and the system works if and only if at least $k$ of the subsystems do. Let $p_i$ be the probability that the $i^{th}$ subsystem functions. Thus

$$h(\tilde{p}) = P\left(\sum_{i=1}^{n} Z_i \geq k\right) \quad , \tag{1}$$

where $\tilde{p} = (p_1,\ldots,p_n)$, $h(\tilde{p})$ is the system reliability corresponding to $\tilde{p}$ and $Z_i = 1$ if the $i^{th}$ subsystem functions, $0$ otherwise. The cases $k = 1$ and $k = n$ correspond to parallel and series systems, respectively. A practical matter of great importance is to place a good $1-\alpha$ lower confience limit on $h(\tilde{p})$ when binomial data is available on the subsystems, i.e., when the observed values $y_1, y_2,\ldots,y_n$ of $Y_1, Y_2,\ldots,Y_n$ are available, where the $Y_i$'s are independent binomial random variables with parameters $m_i$ and $p_i$. In other words, $Y_i$ is the number of successes in $m_i$ trials of the $i^{th}$ subsystem. Buehler (1957) gave a general solution to this problem which is generally computationally difficult. We will specialize this to the case when $\tilde{m} = (m_1,m_2,\ldots,m_n) = (m,m,\ldots,m)$ and $\tilde{y} = (y_1,y_2,\ldots,y_n) = \tilde{m} = (m,m,\ldots,m)$. In this case the $1-\alpha$ lower confidence limit $a_{\tilde{m}}$ on $h(\tilde{p})$ is

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$$a_{\tilde{m}} = \inf h(\tilde{p}) \;\Big|\; \prod_{i=1}^{n} p_i^m > \alpha \quad . \tag{2}$$

More generally, $a_{\tilde{y}}$ is given by

$$a_{\tilde{y}} = \inf h(\tilde{p}) \;\big|\; P\big(\tilde{z} \,|\, g(\tilde{z}) > g(\tilde{y})\big) > \alpha \quad ,$$

where $g(\tilde{y})$ is a reasonable point estimator of $h(\tilde{p})$. The only property of $g$ that we will assume in the sequel is that $g(\tilde{m})$ is the unique maximum of $g(\tilde{y})$. This insures that $a_{\tilde{m}} > a_{\tilde{y}}$, $\tilde{y} \neq \tilde{m}$, and gives the reasonable result that $a_{\tilde{m}}$ is the largest lower confidence limit. Since $h(\tilde{p})$ is a nondecreasing function in each $p_i$, without loss of generality (2) may be rewritten as

$$a_{\tilde{m}} = \inf h(\tilde{p}) \;\Big|\; \Big(\prod_{i=1}^{n} p_i\Big)^m = \alpha \quad . \tag{3}$$

One of the results of Pledger and Proschan (1971, p. 92) states that, subject to $\prod_{i=1}^{n} p_i = \alpha^{1/m}$, which is the condition in (3), $h(\tilde{p})$ is minimized by choosing the $p_i$ to be equal, i.e., $p_i = \alpha^{1/(nm)}$ (they prove that $h(\tilde{p})$ is a Schur-convex function of $R_i = -\ell n\, p_i$, which implies the above). So the $1-\alpha$ lower confidence limit $a_{\tilde{m}}$ is given by

$$a_{\tilde{m}} = \sum_{i=k}^{n} \binom{n}{i} \Big(\alpha^{\frac{1}{nm}}\Big)^{i} \Big(1-\alpha^{\frac{1}{nm}}\Big)^{n-i} \quad , \tag{4}$$

since for all the $p_i$ equal (1) implies that $\sum_{i=1}^{n} Z_i$ is a binomial random variable with parameters $n$ and $\alpha^{1/(nm)}$. For $k=1$ and $k=n$, (4) is already known (see, e.g., Spencer and Easterling (1986)). We summarize the above results in Theorem 1.1.

<u>Theorem 1.1</u>. The lower $1-\alpha$ confidence limit $a_{\tilde{m}}$ on $h(\tilde{p})$, the reliability of a $k$ of $n$ system, when all the subsystem test results have equal sample sizes $m$ and zero failures are observed, is given by (4).

For purposes of the next section, we shall let $k = n-1$ and consider the limit of (4) as $n \to \infty$. When $k = n-1$, (4) becomes

$$a_{\underset{\sim}{m}} = n\left(\alpha^{\frac{1}{nm}}\right)^{n-1}\left(1-\alpha^{\frac{1}{nm}}\right) + \alpha^{\frac{1}{m}} \tag{5}$$

and, as $n \to \infty$, this tends to

$$b_{\underset{\sim}{m}} = \alpha^{1/m}\frac{(-\ell n\ \alpha)}{m} + \alpha^{1/m} = e^{\ell n\ \alpha/m}\left(1 + \frac{-\ell n\ \alpha}{m}\right) \tag{6}$$

(in fact, it is readily verified that the limit for a $n-k$ of $n$ system is $e^{\ell n\ \alpha/m}\sum_{i=0}^{k}\left(\frac{-\ell n\ \alpha}{m}\right)^i/i!$, but this is of no interest here).

## 2. Nonconservativeness of the Maximus Method.

We first consider the limiting behavior of the Maximus method under the same assumptions as above, i.e., an $n-1$ of $n$ system as $n \to \infty$ with zero observed failures. Using the description of Spencer and Easterling (1986), we have $Q_0 = 0$ and $Q_1 = 1/(m/(n-1) + 1)^n$, where we have used the unpooling on the $n$ $(n-1)$ component series sytems in parallel to give independent subsystem data with $m/(n-1)$ trials and $0$ failures. This yields the effective binomial sample size $N_S$,

$$N_S = (1-Q_1)/Q_1 = \frac{1-1/(m/(n-1)+1)^n}{1/(m/(n-1)+1)^n} \tag{7}$$

and observed failures $F_S = 0$. As $n \to \infty$, the limit in (7) is

$$N_S = e^m - 1 \tag{8}$$

and so the limiting Maximus lower confidence limit $a_M$ is

$$a_M = \alpha^{1/(e^m-1)} \approx 1 + \ell n\ \alpha/e^m \quad . \tag{9}$$

For the purpose of comparison, we take $\alpha = .01$ and $m = 20$. Then the exact lower confidence limit $b_{\underset{\sim}{m}}$ is .9772. The corresponding Maximus figure

$b_M$, obtained from (9) is $b_M = 1 - 9.5 \times 10^{-9}$. The true $\alpha$, $\alpha_t$, corresponding to a lower confidence limit of $1 - 9.5 \times 10^{-9}$ is $\alpha_t = .998$ (we emphasize this is $\alpha$, not $1-\alpha$) and is obtained by solving (6) in reverse, i.e., letting $b_{\tilde{m}} = 1 - 9.5 \times 10^{-9}$ and solving for the corresponding $\alpha$. So there are points $\tilde{p}$ in the parameter space whose coverage probability comes arbitrarily close to .002 from above when the nominal or desired minimal coverage probability is .99. These are n-dimensional parameter points $\tilde{p}$ such that $\tilde{p} = (p,\ldots,p)$ and $p^n < (.998)^{1/m}$ but is arbitrarily close to it.

We also consider a finite case. Let $n=5$, $\alpha = .01$, $m = 20$. Then $a_{\tilde{m}}$ is .9815 and Maximus gives .9994, which corresponds to a true $\alpha$, $\alpha_t$, of $\alpha_t = .448$. So there are ponts $\tilde{p} = (p,p,p,p,p)$ in the parameter space with $p^5 < (.448)^{1/20}$, but arbitrarily close to it, for which the actual coverage is arbitrarily close to .552 from above as contrasted with a nominal or desired minimal coverage probability of .99.

## 3. Conclusions.

We have shown in this paper that there can be large differences in nominal and actual coverages when the Maximus method is used. This method is used extensively by government agencies. The use of such an apparently nonconservative method should be carefully considered.

## REFERENCES

Buehler, R. J. (1957), "Confidence Intervals for the Product of Two Binomial Parameters," Journal of the American Statistical Association, 52, 482-493.

Pledger, G. and Proschan, F. (1971), "Comparisons of Order Statistics and of Spacings from Heterogeneous Distributions," in Optimizing Methods in Statistics, ed. J. S. Rustagi, New York: Academic Press, pp. 89-113.

Spencer, F. W. and Easterling, R. G. (1986), "Lower Confidence Bounds on System Reliability Using Component Data: The Maximus Methodology," in Reliability and Quality Control, ed. A. P. Basu, Elsevier Science Publishers, pp. 353-367.