PERIODIC SOLUTIONS OF PERTURBED SUPERQUADRATIC HAMILTONIAN SYSTEMS

Yiming Long

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

December 1987

(Received October 7, 1987)
PERIODIC SOLUTIONS OF PERTURBED SUPERQUADRATIC HAMILTONIAN SYSTEMS

Yiming Long

Technical Summary Report #88-23

December 1987

ABSTRACT

In this paper, we prove the existence of infinitely many distinct T-periodic solutions of the perturbed Hamiltonian system

\[ \dot{z} = J(H'(z) + f(t)) \]

under the conditions that \( H \) is of \( C^1 \), superquadratic, and possesses exponential or polynomial growth at infinity and that \( f \) is of \( W^{1,2} \) and T-periodic, via minimax methods.

AMS (MOS) Subject Classifications: 34C25, 58E05, 58F05

Key Words: Hamiltonian systems, superquadratic, perturbation, monotone truncations; a priori estimates; minimax methods; multiple periodic solutions

This research was supported in part by the United State Army under Contract No. DAAL03-87-K-0043, the Air Force Office of Scientific Research under Grant No. AFOSR-87-0202, the National Science Foundation under Grant No. MCS-8110556 and the Office of Naval Research under Grant No. N00014-88-K-0134.
PERIODIC SOLUTIONS OF PERTURBED SUPERQUADRATIC HAMILTONIAN SYSTEMS

Yiming Long

§1. Introduction and main results

We consider the existence of periodic solutions of a perturbed Hamiltonian system

\begin{equation}
\dot{z} = J(H'(z) + f(t))
\end{equation}

where \( z, f : \mathbb{R} \rightarrow \mathbb{R}^{2N}, \dot{z} = \frac{dz}{dt} = (0 \quad -I), I \) is the identity matrix on \( \mathbb{R}^N \), \( H : \mathbb{R}^{2N} \rightarrow \mathbb{R}, H' \) is its gradient. Let \( \cdot \cdot\cdot \cdot \) and \( \cdot \) denote the usual inner product and norm on \( \mathbb{R}^{2N} \). \( H \) will be required to satisfy the following conditions,

(H1) \( H \in C^1(\mathbb{R}^{2N}, \mathbb{R}) \).

(H2) There exist \( \mu > 2, r_0 > 0 \) such that \( 0 < \mu H(z) < H'(z) \cdot z, \\nabla \vert z \rvert > r_0 \).

(H3) There exist \( 0 < \frac{q_2}{2} < q_1 < q_2 < 2 \) and \( \alpha_i, \tau_i > 0, \beta_i > 0 \) for \( i = 1, 2 \) such that

\[
\alpha_1 |z|^{q_1} - \beta_1 < H(z) < \alpha_2 |z|^{q_2} + \beta_2 \quad \forall z \in \mathbb{R}^{2N}
\]

or

(H4) There exist \( 1 < p_1 < p_2 < 2p_1 + 1, \alpha_i > 0, \beta_i > 0 \) for \( i = 1, 2 \)

such that

\[
\alpha_1 |z|^{p_1 + 1} - \beta_1 < H(z) < \alpha_2 |z|^{p_2 + 1} + \beta_2 \quad \forall z \in \mathbb{R}^{2N}.
\]

Our main results are

This research was supported in part by the United States Army under Contract No. DAAL03-87-K-0043, the Air Force Office of Scientific Research under Grant No. AFOSR-87-0202, the National Science Foundation under Grant No. MCS-8110556 and the Office of Naval Research under Grant No. N00014-88-K-0134.
Theorem 1.2. Let $H$ satisfy conditions (H1)-(H3). Then for any given $T > 0$ and $T$-periodic function $f \in W^{1/2,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^{2N})$, (1.1) possesses an unbounded sequence of $T$-periodic solutions.

Theorem 1.3. The conclusion of Theorem 1.2 holds under conditions (H1), (H2) and (H4).

Such global existence problems have been studied extensively in recent years. For the autonomous case of (1.1) (i.e. $f \equiv 0$), the result was proved by Rabinowitz under the conditions (H1) and (H2) only ([17, 19]). His proof is based on a group symmetry possessed by the corresponding variational formulation. When one considers the forced vibration problem (1.1) ($f$ does depend on $t$), such a symmetry breaks down. Bahri and Berestycki ([2]) studied the perturbed problem (1.1) and proved the conclusion of Theorem 1.3 by assuming (H2), (H4) and (H4') : $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$. Our Theorem 1.3 weakens their condition on the smoothness of $H$ and Theorem 1.2 allows $H$ to increase faster at infinity.

Our proof extends Rabinowitz' basic ideas used in [18,19] and ideas used in [12,13]. In order to get the smoothness and compactness of corresponding functionals, we introduce a sequence of truncation functions $\{H_n\}$ of $H$ in $C^1$ and corresponding modified functionals $\{J_n\}$ and $J$ of $I$, where

\begin{equation}
I(z) = \int_0^{2\pi} \left( \frac{1}{2} \dot{z}^2 + Jz - H(z) - f*z \right) dt.
\end{equation}

We make $\{H_n\}$ be monotone increasing to $H$ and $\{J_n\}$ be monotone decreasing to $J$ as $n$ increases. These monotonicities also allow us to get $L^\infty$-estimates for the critical points of $J_n$ we found. We modify the treatment of the $S^1$-action on $W^{1,2}(S^1, \mathbb{R}^{2N})$ by introducing a simpler $S^1$-action on it to get upper estimates for certain minimax values. Combining with applications of Fadell-Rabinowitz cohomological index we get the multiple existence of periodic solutions of (1.1).
In §2 we define \( \{H_n\} \), \( \{J_n\} \) and \( J \). With the aid of an auxiliary space \( X \) we define sequences of minimax values \( \{a_k(n)\}, \{b_k(n)\} \) of \( J_n, \{a_k\}, \{b_k\} \) of \( J \) in §3, and discuss their properties in §4. §5 and §6 contain estimates from above and below for \( \{a_k\} \). We prove the existence of critical values of \( J_n \) in §7. Then in §8 by showing that the critical points of \( J_n \) for large \( n \) yields solutions of (1.1), we complete the proofs of our main theorems. Finally in §9 we discuss more general forced Hamiltonian systems.

Since the proofs of Theorems 1.2 and 1.3 are similar, we shall carry out the details for the first one only and make some comments on the second in §8. For the details of the proof of Theorem 1.3, we refer to [13].

Acknowledgement. The author wishes to express his sincere thanks to Professor Paul H. Rabinowitz for his advise, help and encouragement, especially for his suggestions on truncation functions. He also thanks Professor Stephen Wainger and Dr. James Wright for interesting discussions on imbedding theorems.
§2. Modified functionals

By rescaling time if necessary we can assume \(T = 2\pi\). Let \(E = \mathbb{R}^{2N}\). The scalar product in \(L^2\) naturally extends as the duality pairing between \(E\) and \(E' = \mathbb{R}^{2N}\). Thus for \(z \in E\), the actional integral \(\frac{1}{2} A(z)\) is well defined, where

\[
A(z) = \int_0^{2\pi} z \cdot Jz \, dt.
\]

For \(k \in \mathbb{N}\), write \(\overline{k} = \left[\frac{k}{N}\right], \overline{k} = k - \left[\frac{k}{N}\right]\), where \(\left[\frac{k}{N}\right]\) is the integer part of \(\frac{k}{N}\). Let \(e_1, ..., e_{2N}\) denote the usual orthonormal basis in \(\mathbb{R}^{2N}\). We write \(i = \sqrt{-1}\), and denote for \(k \in \mathbb{N}\)

\[
\begin{align*}
\hat{\varphi}_k &= (\sin \overline{k}t)e_\overline{k} - (\cos \overline{k}t)e_{\overline{k}+N}, \\
i\hat{\varphi}_k &= (\cos \overline{k}t)e_{\overline{k}} + (\sin \overline{k}t)e_{\overline{k}+N}, \\
\hat{\psi}_k &= (\sin \overline{k}t)e_\overline{k} + (\cos \overline{k}t)e_{\overline{k}+N}, \\
i\hat{\psi}_k &= (\cos \overline{k}t)e_{\overline{k}} - (\sin \overline{k}t)e_{\overline{k}+N}.
\end{align*}
\]

Let \(E^+_{m,n} = \text{span}\{\hat{\varphi}_k, i\hat{\varphi}_k \mid m < k < n\}, E^-_{m,n} = \text{span}\{\hat{\psi}_k, i\hat{\psi}_k \mid m < k < n\}\) for \(m, n \in \mathbb{N}\). Let \(E^+ = \bigoplus_{N+1}^{\infty} E^+_{N+1, +\infty}, E^- = \bigoplus_{N+1}^{\infty} E^-_{N+1, +\infty}\) and \(E^0 = \text{span}\{\hat{\varphi}_k, i\hat{\varphi}_k \mid 1 < k < N\}\). Then \(E = E^+ \oplus E^- \oplus E^0\) and \(A(z)\) is positive definite, negative definite and null on \(E^+, E^-\) and \(E^0\) respectively. For \(z = z^+ + z^- + z^0 \in E^+ \oplus E^- \oplus E^0 = E\), we take as a norm for \(E\)

\[
|z|_E^2 = A(z^+) - A(z^-) + |z^0|^2.
\]

Under this norm, \(E\) becomes a Hilbert space and \(E^+, E^-, E^0\) are orthogonal subspaces of \(E\) with respect to the inner product associated with this norm, as well as with the \(L^2\) inner product. (2.2) gives a basis for \(E\).

A result of Brezis and Wainger [6] implies that \(E\) is compactly embedded into \(L^p(s^1, \mathbb{R}^{2N})\), \(1 < p < +\infty\) and the Orlicz space \(L^M_{\mathbb{R}^{2N}}\) with \(M(z) = e^\tau |z|^q - \sum_{k=0}^{n} |k!| |z|^k, \forall \tau > 0, 0 < q < 2\) and \(n \in \mathbb{N}, nq > 1\), and \(E \subset L^M_{\mathbb{R}^{2N}}\). Note that \(q = 2\) is the critical imbedding value (cf. [6,10]). We shall use the following version of their imbedding theorem,
Lemma 2.3. For $\tau > 0$, $0 < q < 2$, there exist constants $C_1, C_2 > 0$ depending only on $\tau$ and $q$ such that

$$\int_0^{2\pi} \exp(\sigma |z|^q) dt < C_1 \exp(C_2 (|\alpha z|_E)^{2-q}) \quad \forall \sigma > 0, z \in E.$$  

Proof. We use the notations in [6] and only prove the Lemma for $E = W^{1,2}(S^1, \mathbb{C})$.

For $z \in E$, write $z(t) = C_0 + \sum \frac{C_n}{n \neq 0} \int \epsilon \int \epsilon \in L(2, \omega), g(t) = \sum C_n \epsilon \int \epsilon \in L(2,2)$. By (8) of [6],

$$\frac{2}{\sigma t |z|^q} < (\epsilon \sigma t)^q |z|^q + \frac{2}{2-q}$$

$$+ \left( \epsilon \sigma t \right)^q C(1 + |\log t|) \|k\|^2 \|L(2, \omega)\| \|g\|^2 \|L(2,2)\| + \epsilon \frac{2}{2-q} \text{ for } 0 < t < 1.$$  

Choose $\epsilon = \frac{1}{\sigma t} \left( \epsilon \sigma t \right)^q \|k\|^2 \|L(2, \omega)\| \|g\|^2 \|L(2,2)\|$ - $\frac{q}{2} \epsilon$, then from $\|g\| \|L(2,2)\| < C \|z\|_E$ and

$$\int_0^{2\pi} e^{\sigma t |z|^q} dt = \int_0^{2\pi} e^{\sigma t |z|^q} dt, \text{ we get the lemma.}$$

Note that $\text{Fix}(T_0) \equiv \{ z \in E \mid T_0 z = z, \forall \theta \in [0,2\pi] \} = E^0$.  

By Lemma 2.3 and (H3), $I(z)$ defined in (1.4) is a continuous functional on $E$ and formally the critical points of $I$ correspond to the solutions of (1.1).

Without loss of generality we assume $r_0 > 1$. Set $a_0 = \min_{|z| = r_0} \mathcal{H}(z), \quad -5-$
\[ \theta_0 = \beta_1 + \max_{z \leq r_0} |H(z)|, \text{ where } \beta_1 \text{ is given by } (H3). \text{ Conditions } (H1) \text{ and } (H2) \text{ imply that for some } \beta_3 > 0, \]
\[
\begin{cases}
\alpha_0 |z|^{\mu < \beta} \leq H(z), & \forall |z| > r_0 \\
\alpha_0 |z|^{\mu < \beta} \leq \beta_0 < \frac{1}{\mu} (H'(z) \cdot z + \beta_3), & \forall z \in \mathbb{R}^{2N}.
\end{cases}
\]
(2.4)

Choose \( \sigma \in (0, 1) \) such that \( \mu \sigma > 2 \). We have

**Proposition 2.5.** Assume conditions (H1) and (H2). Then there exists a sequence \( \{K_n\} \subset \mathbb{R} \) and a sequence of functions \( \{H_n\} \) such that

1. \( 0 < K_0 < K_n < K_{n+1}, \forall n \in \mathbb{N} \text{ and } K_n \to +\infty \text{ as } n \to +\infty \) where \( K_0 = \max \{1, r_0, \frac{\beta_0}{\alpha_0(1-\sigma)}\} \).
2. \( H_n \in C^1(\mathbb{R}^{2N}, \mathbb{R}) \forall n \in \mathbb{N} \).
3. \( H_n(z) = H(z) \forall n \in \mathbb{N} \text{ and } |z| < K_n \).
4. \( H_n(z) < H_{n+1}(z) < H(z) \forall n \in \mathbb{N} \text{ and } z \in \mathbb{R}^{2N} \).
5. \( 0 < \mu \sigma H_n(z) < H_n'(z) \cdot z, \forall n \in \mathbb{N} \text{ and } |z| > r_0 \).
6. For \( n \in \mathbb{N} \), there exists a constant \( \lambda_0 > 1 \) independent of \( n \) and \( C(n) > 0 \) such that

\[ |H_n'(z)|^{\lambda_0} < C(n)(H_n'(z) \cdot z + 1) \forall z \in \mathbb{R}^{2N} \]

Since the proof of this proposition is rather technical and lengthy, we put it in the Appendix.

Similar to (2.4), there is a constant \( \beta_4 > 0 \) independent of \( n \) such that

\[
\begin{cases}
\alpha_0 |z|^{\mu \sigma < \beta} < H_n(z), & \forall n \in \mathbb{N} \text{ and } |z| > r_0 \\
\alpha_0 |z|^{\mu \sigma < \beta} < \beta_0 < \frac{1}{\mu \sigma} (H_n'(z) \cdot z + \beta_4), & \forall n \in \mathbb{N} \text{ and } z \in \mathbb{R}^{2N}.
\end{cases}
\]
(2.6)

For \( n \in \mathbb{N} \), we define

\[ I_n(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H_n(z) \text{dt} - \int_0^{2\pi} f \cdot z \text{dt} \forall z \in \mathbb{E} \]
Lemma 2.7.

1°. \( I_n \in C^1(E,\mathbb{R}), \forall n \in \mathbb{N} \).

2°. \( I(z) \prec I_{n+1}(z) \prec I_n(z), \forall n \in \mathbb{N} \) and \( z \in E \).

Proof: For 1° we refer to [5,17]. 2° follows from 4° of Proposition 2.5.

From now on in this section, we define \( \tau = \frac{1}{4}\sqrt{3\mu_0+10} \).

Lemma 2.8. There is \( \beta_5 > 0 \) independent of \( n \) such that

\[
\frac{\tau(3\mu_0+2)}{2(\tau-1)} \frac{1}{L^2} \left\| I \right\|_{L^2}^2 < \frac{3\mu_0+2}{8} \int_0^{2\pi} (H_n(z) + \beta_0) dt + \frac{\mu_0-2}{2(\tau-1)} \frac{1}{L^2} \left\| I \right\|_{L^\mu_0}^{\mu_0}.
\]

Proof. By (2.6)

\[
\frac{3\mu_0+2}{8} \int_0^{2\pi} (H_n(z) + \beta_0) dt - \frac{\tau(3\mu_0+2)}{2(\tau-1)} \frac{1}{L^2} \left\| I \right\|_{L^2}^2 > \frac{\mu_0-2}{8} \left( A(z) \right)^2 + 1 + \beta_5 \quad \forall n \in \mathbb{N} \) and \( z \in E \).

Since \( \mu_0 > 2 \), (2.9) holds.

Lemma 2.10. There is \( \beta_6 > 0 \) independent of \( n \) such that for any \( n \in \mathbb{N}, z \in E \) if \( I_n^1(z) = 0 \) then

\[
(2.11) \quad \frac{3\mu_0+2}{8} \int_0^{2\pi} (H_n(z) + \beta_0) dt + \beta_5 \leq \left( \left( \frac{1}{2} A(z) \right)^2 + 1 \right)^{1/2} + \beta_6.
\]

Proof. From \( \langle I_n^1(z), z \rangle = 0 \), we get that

\[
(2.12) \quad \frac{\mu_0}{2} \int_0^{2\pi} (H_n(z) + \beta_0) dt + \frac{1}{2} \int_0^{2\pi} f \cdot z dt + \pi \beta_4 \quad \text{(by (2.6))}
\]

\[
> \frac{3\mu_0+2}{8} \int_0^{2\pi} (H_n(z) + \beta_0) dt + \frac{\mu_0-2}{16} a_0 \left\| I \right\|_{L^\mu_0}^{\mu_0} \quad \text{if} \quad \left\| I \right\|_{L^2} \left\| I \right\|_{L^2} - \pi \beta_4 - 1.
\]
Since $\mu \sigma > 2$, (2.11) holds.

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s) = 1$ if $s < 1$, $\chi(s) = 0$ if $s > \tau$, and $-\frac{2}{\tau - 1} < \chi''(s) < 0$ if $1 < s < \tau$, where $\tau$ is defined before Lemma 2.8.

For $n \in \mathbb{N}$ and $z \in E$ we define

$$\varphi_0(z) = \left(\frac{1}{2} A(z)\right)^2 + 1)^{1/2} + \beta_6,$$

$$\varphi(z) = \frac{3\mu \sigma + 2}{8} \int_0^{2\pi} (H(z) + \beta_0) dt + \beta_5,$$  

$$\psi(z) = \chi\left(\frac{\varphi(z)}{\tau \varphi_0(z)}\right), \quad \psi_n(z) = \chi\left(\frac{\psi_n(z)}{\tau \varphi_0(z)}\right),$$

$$J(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H(z) dt - \psi(z) \int_0^{2\pi} f \cdot z dt,$$

$$J_n(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H_n(z) dt - \psi_n(z) \int_0^{2\pi} f \cdot z dt.$$

For these functionals on $E$ we have

**Lemma 2.13.**

1°. $\psi_n \in C^1(E, \mathbb{R})$, $J_n \in C^1(E, \mathbb{R})$, $\forall n \in \mathbb{N}$.

2°. $\psi \in C(E, \mathbb{R})$, $J \in C(E, \mathbb{R})$.

**Proof.** For 1° we refer to [5,17], 2° follows from (H3) and Lemma 2.3.

**Lemma 2.14.** For any $m, n \in \mathbb{N}$, $m > n$ and $z \in E$,

1°. $0 < \frac{1}{2} \int_0^{2\pi} (H_m(z) - H_n(z)) dt < J_n(z) - J_m(z) < \frac{3}{2} \int_0^{2\pi} (H_m(z) - H_n(z)) dt$,

2°. $0 < \frac{1}{2} \int_0^{2\pi} (H(z) - H_n(z)) dt < J_n(z) - J(z) < \frac{3}{2} \int_0^{2\pi} (H(z) - H_n(z)) dt$.

**Proof.** We only prove 1°. The proof of 2° is similar. Let $\text{supp } \psi_n$ be the closure of $\{z \in E \mid \psi_n(z) \neq 0\}$ in $E$.

If $z \notin \text{supp } \psi_n \cup \text{supp } \psi_m$, then by 4° of Proposition 2.5

$$J_n(z) - J_m(z) = \int_0^{2\pi} (H_m(z) - H_n(z)) dt > 0.$$
If \( z \in \text{supp } \psi_n \cup \text{supp } \psi_m \), then \( \varphi_j(z) < \tau^2 \varphi_0(z) \) for \( j = n \) or \( j = m \).

By Lemma 2.8,

\[
\varphi_j(z) > \frac{\tau(3\mu + 2)}{2(\tau - 1)} \frac{1}{L^2} \int_{L^2} z^j.
\]

So

\[
\frac{2}{\tau - 1} \cdot \frac{3\mu + 2}{8} \frac{1}{L^2} \frac{1}{L^2} \frac{\varphi_j(z)}{\tau \varphi_0(z)} < \frac{1}{2}.
\]

(2.15)

On the other hand

\[
J_n(z) - J_m(z) = \int_0^{2\pi} (H_m(z) - H_n(z)) dt + \frac{\varphi_m(z) - \varphi_n(z)}{\tau \varphi_0(z)} \int_0^{2\pi} f(z) dt,
\]

where we have used the mean value theorem with a number \( \zeta \) between \( \frac{\varphi_m(z)}{\tau \varphi_0(z)} \) and \( \frac{\varphi_n(z)}{\tau \varphi_0(z)} \). By the definition of \( \varphi_m \) and \( \varphi_n \), we get,

(2.16) \[
J_n(z) - J_m(z) = \left( 1 + \frac{\chi'((\zeta) \frac{3\mu + 2}{8} \frac{1}{L^2} \frac{1}{L^2} \int_{L^2} f(z) dt} {\tau \varphi_0(z)} \right) \int_0^{2\pi} (H_m(z) - H_n(z)) dt.
\]

Since \( |\chi'((\zeta))| < \frac{2}{\tau - 1}, \forall \xi \in \mathbb{R} \), by (2.15) we get that

\[
\frac{1}{2} < 1 + \frac{\chi'((\zeta) \frac{3\mu + 2}{8} \frac{1}{L^2} \frac{1}{L^2} \int_{L^2} f(z) dt} {\tau \varphi_0(z)} \leq \frac{3}{2}.
\]

Combining with 4° of Proposition 2.5 and (2.16) we get the proof of 1°.

Corollary 2.17. For any \( n \in \mathbb{N} \) and \( z \in E \)

\[
J_n(z) > J_{n+1}(z) > J(z).
\]

Lemma 2.18. There exists \( \beta_7 > 0 \) independent of \( n \) such that for any \( n \in \mathbb{N}, z \in E \) and \( M > \beta_8 \), if \( J_n(z) > M \) then \( \varphi_0(z) > \frac{M}{2} \).

Proof. Since \( \mu \sigma > 2 \) and

\[
J_n(z) < \frac{1}{2} A(z) - \alpha_0 |z|^{\mu \sigma} + \frac{1}{L^2} |z|^{2\mu \sigma} + 2\pi \beta_0,
\]

-9-
there exists $C > 0$ independent of $n$ such that
\[ \varphi_0(z) = ((\frac{1}{2} A(z))^2 + 1)^{1/2} + \beta_6 > J_n(z) + \beta_6 - C \]
and this yields the Lemma.

\begin{proof}

Lemma 2.19. There exists a constant $\beta_8 > 0$ independent of $n$ such that for any $n \in \mathbb{N}$ and $z \in E$, if $J_n(z) > \beta_8$ and $<J_n'(z), z> = 0$, then $J_n(z) = I_n(z)$ and $J_n'(z) = I_n'(z)$.

For any $z, \zeta \in E$, we have that
\[ <J_n'(z), \zeta> = (1 + T_{n,1}(z)) \overline{A}(z, \zeta) \]

(2.20)
\[ - (1 + T_{n,2}(z)) \int_0^{2\pi} H_n'(z) \cdot \zeta \ dt - \psi_n(z) \int_0^{2\pi} f \cdot \zeta \ dt \]

where
\[ \overline{A}(z, \zeta) = \frac{1}{2} \int_0^{2\pi} (z \cdot \zeta + \zeta \cdot Jz) dt , \]
\[ T_{n,1}(z) = \chi'(\frac{\varphi_0(z)}{\tau \varphi_0(z)}) \frac{A(z)\varphi_0(z)}{2\tau^2 \varphi_0(z)(\varphi(z) - \beta_6)} \int_0^{2\pi} f \cdot z \ dt \]
\[ T_{n,2}(z) = \chi'(\frac{\varphi_0(z)}{\tau \varphi_0(z)}) \frac{3 \mu \sigma + 2}{8 \tau \varphi_0(z)} \int_0^{2\pi} f \cdot z \ dt . \]

If for large enough $\beta_8$, we have
\[ |T_{n,1}(z)| < \frac{\mu \sigma - 2}{16 \mu \sigma} , \quad |T_{n,2}(z)| < \frac{\mu \sigma - 2}{16 \mu \sigma} . \]

Then from (2.20) with $\zeta = z$, we get
\[ \frac{1}{2} A(z) > \frac{3 \mu \sigma + 2}{8} \int_0^{2\pi} (H_n(z) + \beta_8) dt + \frac{\mu \sigma - 2}{16} a_0 \frac{\mu \sigma - 2}{L} \int_0^{2\pi} L \frac{\mu \sigma}{L} - \frac{L^2}{L^2} \frac{L^2}{L^2} = \beta_4 - 1 . \]

This is (2.12). So $\varphi_0(z) > \varphi_n(z)$, thus $\psi_n(z) = 1$ and $\psi_n'(z) = 0$. This yields the lemma. Therefore we reduce to the proof of (2.21).

If $z \notin \text{supp} \psi_n$, $T_{n,1}(z) = T_{n,2}(z) = 0$. If $z \in \text{supp} \psi_n$, then
\[ \tau^2 \varphi_0(z) > \varphi_n(z) > \frac{3 \mu \sigma + 2}{8} a_0 \frac{\mu \sigma - 2}{L} . \]
Thus

\[ |T_{n,1}(z)| < \frac{1}{1 - \frac{1}{2} \left( \frac{\mu - 2}{2} \right) L_2 |z|^{\mu} \left( \frac{1}{\varphi_0(z)} \right)^{\frac{1}{\mu}} - 1}, \]

Similarly

\[ |T_{n,2}(z)| < M(\varphi_0(z))^{-\mu} - 1 \]

for some constant \( M > 0 \) independent of \( n \) and \( z \), by Lemma 2.18, this implies (2.21) and completes the proof of the Lemma.

We say \( J_n \) satisfies the Palais-Smale condition (PS) if whenever a sequence \( \{z_j\} \) in \( E \) satisfies that \( \{J_n(z_j)\} \) is bounded and \( J_n'(z_j) \to 0 \) as \( j \to +\infty \), then \( \{z_j\} \) possesses a convergent subsequence.

**Lemma 2.22.** For any \( n \in \mathbb{N} \), \( J_n \) satisfies (PS) on \( [J_n]_{\beta_8} = \{z \in E \mid J_n(z) \geq \beta_8\} \), where the constant \( \beta_8 > 0 \) is defined in Lemma 2.19 which is independent of \( n \).

**Proof.** Since \( J_n'(z_j) \to 0 \), we may assume \( |\langle J_n'(z_j), z \rangle| < |z|_E \forall z \in E \). By the assumption \( \beta_8 < J_n(z_j) < M \), from (2.20) and (2.21) we get

\[
M + |\langle J_n'(z_j), z \rangle| \leq J_n(z_j) - \frac{1}{2(1+T_{n,1}(z_j))} \langle J_n'(z_j), z_j \rangle
\]

\[
= \frac{1+T_{n,2}(z_j)}{2(1+T_{n,1}(z_j))} \int_0^{2\pi} H_n(z_j) \cdot z_j dt - \int_0^{2\pi} H_n(z_j) dt
\]

\[
- \left( 1 - \frac{1}{2(1+T_{n,1}(z_j))} \right) \psi_n(z_j) \int_0^{2\pi} f \cdot z_j dt
\]

\[
> \frac{\mu - 2}{16\mu} \int_0^{2\pi} H_n(z_j) \cdot z_j dt + \frac{1}{4} \left( \mu - 2 \right) \int_0^{2\pi} (H_n(z_j) + \beta_0) dt
\]

\[
- \frac{2}{3} \left( \frac{1}{L_2} \right) L_2^2 - 2\pi (\beta_4 - \beta_0).
\]
Therefore there is $M_1 > 0$ independent of $n$ and $j$ such that

\begin{equation}
L_{\mu^g}^{j} + \int_0^{2\pi} (H_n'(z_j) \cdot z_j + \beta_d) dt < M_1 (L_{\mu^g}^{j} + 1).
\end{equation}

Write $z_j = z_j^+ + z_j^- + z_j^0$. Since $z_j^0 = \frac{1}{2\pi} \int_0^{2\pi} z_j dt$, there is $M_2 > 0$ independent of $n$, $j$ such that

\begin{equation}
|z_j^0| < L_{\mu^g}^{j} < M_2 (L_{\mu^g}^{j} + 1).
\end{equation}

From (2.20) for $\langle J_n'(z_j), z_j^+ \rangle$, we get

\begin{equation}
\frac{1}{2} L_{\mu^g}^{j} + \frac{1}{2} \sum |z_j^+|^2 < (1 + T_n, 1(z_j)) A(z_j) < \frac{3}{2} \int_0^{2\pi} |H_n'(z_j)| |z_j^+| dt + \int L_{\mu^g}^{j} L_{\mu^g}^{j} + L_{\mu^g}^{j} + 1.
\end{equation}

By 6° of Proposition 2.5 and (2.23) we get

\begin{equation}
\int_0^{2\pi} |H_n'(z_j)| |z_j^+| dt < \left( \int_0^{2\pi} |H_n'(z_j)| \lambda_0 \right)^{1/\lambda_0} L_{\mu^g}^{j} \lambda_0/(\lambda_0 - 1) < M_3(n) (L_{\mu^g}^{j} + 1)
\end{equation}

for some constant $M_3(n) > 0$ independent of $j$. Thus there is $M_4(n) > 0$ independent of $j$ such that

\begin{equation}
L_{\mu^g}^{j} < M_4(n) (L_{\mu^g}^{j} + 1).
\end{equation}

Similarly

\begin{equation}
L_{\mu^g}^{j} < M_5(n) (L_{\mu^g}^{j} + 1)
\end{equation}

for some $M_5(n) > 0$ independent of $j$. Combining with (2.24), we get a constant $M_6(n) > 0$ independent of $j$ such that

\begin{equation}
L_{\mu^g}^{j} < M_6(n).
\end{equation}

Let $P^\perp : E \to E^\perp$ be the orthogonal projections. From (2.20)

\begin{equation}
P^\perp J_n'(z_j) = t(1 + T_n, 1(z_j)) z_j^+ + P_n(z_j)
\end{equation}

where $P_n$ is a compact operator by (H1), (H2) and Proposition 2.5. Since
\[ |T_{n,1}(z_j)| < \frac{1}{16}, \]

\[ z_j^+ = (1 + T_{n,1}(z_j))^{-1} f_j \left( (1 + T_{n,1}(z_j))^{-1} p_j n(z_j) \right). \]

By (2.25) this shows that \{z_j^+\} and \{z_j^-\} are precompact in \( E \). By (2.25) \{z_j^0\} is also precompact, therefore \{z_j^0\} is precompact in \( E \), and the proof is complete.

\[ \square \]

**Lemma 2.26:** There exists a constant \( \beta_0 > 0 \) such that

\[ |J(z) - J(T_\theta z)| < \beta_0 (\log (|J(z)| + 1) + 1) \quad \forall z \in E. \tag{2.27} \]

where \( q_1 \) is defined in (H3).

**Proof:** A direct application of Hölder inequality shows that there exists \( c_1 > 0 \) such that

\[ \frac{1}{2\pi} \int_{0}^{2\pi} \frac{|z|}{|dt|^q_1} < \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{|z|^q_1} dt + c_1 \quad \forall z \in E. \]

If \( z \in \text{supp} \psi, |J(z)| > (\frac{3\mu^2}{8\tau^2} - 1) \int_{0}^{2\pi} (H(z) + \beta_0) dt - c_2 \int_{0}^{2\pi} |z| dt - c_3 \]

\[ > c_4 \int_{0}^{2\pi} \frac{1}{|z|^q_1} dt - c_5. \]

Here we used (H3), and \( c_j \)'s denote positive constants. Therefore

\[ |J(z) - J(T_\theta z)| < 2 \psi(z) \int_{0}^{2\pi} |f_\theta z| dt < c_6 \psi(z) \int_{0}^{2\pi} |z| \]

\[ < \frac{1}{q_1} (|J(z)| + 1) + 1 \]

for some \( \beta_0 > 0 \) and the proof is complete.

\[ \square \]
§3. A minimax structure

For $k \in \mathbb{N}$, $k > N+1$, we define $V_k(E) = E_{N+1}^k \oplus E^{-} \oplus E^0$. By (2.6) and Corollary 2.17, there exists $R_k$ for $k > N+1$ such that $1 < R_k < R_{k+1}$ and $J(z) < J_m(z) < 0 \forall n \in \mathbb{N}$, $z \in V_k(E)$ with $\|z\|_E > R_k$.

Let $D_k(E) = V_k(E) \cap B_k(E)$, $B_k(E) = \{ z \in E \mid \|z\|_E < R_k \}$.

For $z = z^0 + z^+ + z^- \in E$ we write

$$z^+ = \sum_{k > N+1} \rho_k^e \varphi_k^e, \quad z^- = \sum_{k > N+1} \sigma_k^e \psi_k^e,$$

where $\rho_k^e, \sigma_k^e > 0$, $\varphi_k^e, \psi_k^e \in [0,2\pi]$.

Then we have

$$\|z\|_E^2 = |z^0|^2 + 2\pi \sum_{k > N+1} k(\rho_k^2 + \sigma_k^2),$$

and

$$T_\theta z = z^0 + \sum_{k > N+1} \psi_k^e T_\theta \varphi_k^e, \quad \psi_k = i(\beta_k + \kappa \theta) \varphi_k + i(\alpha_k + \kappa \theta) \psi_k \quad \forall \theta \in [0,2\pi].$$

We define a new $S^1$-action on $E$ by

$$T_\theta^* z = z^0 + \sum_{k > N+1} \varphi_k^e T_\theta \psi_k^e, \quad \varphi_k = i(\alpha_k + \kappa \theta) \varphi_k + i(\beta_k + \kappa \theta) \psi_k \quad \forall \theta \in [0,2\pi]$$

for $z$ with expression (3.1) and denote $E$ with $T_\theta^*$ by $X$. We define an $S^1$-action on $X \times E$ by

$$T_\theta^* (x,z) = (T_\theta^* x, T_\theta^* z) \quad \forall (x,z) \in X \times E \quad \text{and} \quad \theta \in [0,2\pi].$$

We also define $S^1$-invariant sets, equivariant maps, invariant functionals for $X$, $E$, $X \times E$ in a usual way (cf. [9]). Let $E$ (or $X,F$) denote the family of closed (in $E$) (or $X,X \times E$) $S^1$-invariant subsets in $E \setminus \{0\}$ (or $X \setminus \{0\}, X \times E \setminus \{0\}$). Then $F$ contains $\times \{0\}$ and $\{0\} \times E$. For $B \in X$, we say a map $h : B \to E$ is $S^1$-equivariant if $h(T_\theta^* x) = T_\theta h(x) \quad \forall x \in B$ and $\theta \in [0,2\pi]$. We also denote by $V_k(X), D_k(X), B_k(X)$ the sets in $X$ corresponding to $V_k(E), D_k(E), B_k(E)$, etc. We introduce the Fadell-
Rabinowitz cohomological index theory on $F$ (cf. [9]).

**Lemma 3.2:** There is an index theory on $F$ i.e. a mapping
$$
\gamma : F \to \{0\} \cup N \cup \{\infty\}
$$
such that if $A, B \in F$

1°. $\gamma(A) < \gamma(B)$. if there exists $h \in C(A, B)$ with $h$ being $S^1$-equivariant.

2°. $\gamma(A \cup B) < \gamma(A) + \gamma(B)$.

3°. If $B \subset (X \times E) \setminus (X^0 \times E^0)$ and $B$ is compact, then $\gamma(B) < \infty$ and there is a constant $\delta > 0$ such that $\gamma(N_{\delta}(B, X \times E)) = \gamma(B)$ where
$$
N_{\delta}(B, X \times E) = \{z \in X \times E \mid Lz - B_{X \times E} \subset \delta\}.
$$

4°. If $S \subset (X \times E) \setminus (X^0 \times E^0)$ is a 2n-1 dimensional invariant sphere, then $\gamma(S) = n$.

By identifying $X$ with $X \times \{0\}$, and $E$ with $\{0\} \times E$, we may view that the index theory $\gamma$ is defined on both $X$ and $E$.

For $x \in X, z \in E$, we write $z \sim x$ if $z^0 = x^0, \rho_k(z) = \rho_k(x), \sigma_k(z) = \sigma_k(x), \hat{\sigma}_k(z) = \hat{\sigma}_k(x)$ and $\beta_k(z) = \beta_k(x) \forall k > N+1$.

For $x = x^0 + \sum_{k > N+1} (\rho_k e^{i\sigma_k} + \beta_k e^{i\beta_k}) \in X$, we define
$$
h(x) = x^0 + \sum_{k > N+1} (i k \rho_k e^{i\sigma_k} + i k \beta_k e^{i\beta_k})
$$
About this map $h$, we have

**Lemma 3.3:**

1°. $h \in C(X, E)$ and is surjective.

2°. $h$ is $S^1$-equivariant.

3°. $h(\partial B^\rho(X) \cap V_k(X)) = \partial B^\rho(E) \cap V_k(E) \forall \rho > 0, k > N+1$.

4°. $h(x) = z$ if $x \in X^0$ and $x \sim z$.

5°. If $(h(x_n))$ is convergent in $E$, $\{x_n\}$ is precompact in $X$. 

-15-
Proof: $1^\circ \sim 4^\circ$ are direct consequences of the definition of $h$. Suppose 

$x_n \in X$ and $h(x_n) + z$ in $E$ as $n \to \infty$. Write $x_n = (x_n^0, \rho_k(n), \alpha_k(n); \sigma_k(n); \beta_k(n))$. Since $2^n(\rho_k(n) - \rho_k(m))^2 < \|h(x_n) - h(x_m)\|_E^2$, we get that 

$(\rho_k(n))$, similarly $(\sigma_k(n))$, is convergent for each $k \in \mathbb{N}$. Since $(x_n^0)$ is precompact, and $\alpha_k(n), \beta_k(n) \in [0, 2\pi]$, we can choose a sequence $(n_j)$ in $\mathbb{N}$ such that $(x_{n_j}^0), (\alpha_k(n_j))$ and $(\beta_k(n_j))$ are convergent for every fixed $k \in \mathbb{N}$. Then $(x_{n_j})$ is convergent in $X$. This proves $5^\circ$.

We name the above map $h : X + E$ by "id". With the aid of "id" we can define minimax structures now.

**Definition 3.4.** For $j \in \mathbb{N}, j > N+1$, define $\Gamma_j$ to be the family of such maps $h$, which satisfy the following conditions:

1. $h \in C(D_j(X), E)$ and is $S^1$-equivariant.

2. $h = \text{id}$ on $(\partial B_j(X) \cap V_j(X)) \cup (X^0 \cap D_j(X)) = F_j(X)$.

3. $P^{-}h(x) = \alpha(x)P^{-}\text{id}(x) + \beta(x)$, $\forall x \in D_j(X)$, where $\alpha \in C(D_j(X), [1, \overline{\alpha}])$ with $1 < \overline{\alpha} < +\infty$ depending on $h$ and is $S^1$-invariant, $\beta \in C(D_j(X), E^-)$ is compact and $S^1$-equivariant, and $\beta = 0$ on $F_j(X)$.

4. $h(D_j(X))$ is bounded in $E$.

**Definition 3.5.** For $j \in \mathbb{N}, j > N+1$, define $\Lambda_j$ to be the family of maps $h$, which satisfy

1. $h \in C(D_{j+1}(X), E)$ and $h|_{D_j(X)} \in \Gamma_j$,

2. $h = \text{id}$ on $(\partial B_{j+1}(X) \cap V_{j+1}(X)) \cup ((B_{j+1}(X) \setminus B_j(X)) \cap V_j(X)) = G_j(X)$.

3. $P^{-}h(x) = \alpha(x)P^{-}\text{id}(x) + \beta(x)$, for any $x \in D_{j+1}(X)$ where $\alpha \in C(D_{j+1}(X), [1, \tilde{\alpha}])$ with $1 < \tilde{\alpha} < +\infty$ depending on $h$, $\beta \in C(D_{j+1}(X), E^-)$ is compact and $\beta = 0$ on $G_j(X)$, and $\alpha, \beta$
are extensions of the corresponding maps defined in 1° via the
definition of $\Gamma_2$.

4°. $h(D_{j+1}(X))$ is bounded in $E$.

Remark. $id \in \Gamma_j \cap A_j$ for any $j > N+1$.

Lemma 3.6. For $j > N+1$, any $h \in \Gamma_j$ can be extended to a map in $A_j$.

Proof. For $h \in \Gamma_j$, define $h = id$ on $G_j(X)$. This also extends $\alpha$ and $\beta$ in 3° of Definition 3.4 of $h$ by $\alpha = 1$ and $\beta = 0$ on $G_j(X)$. Now we use Dugundji extension theorem [7] to extend $\alpha, \beta$ to the whole $D_{j+1}(X)$. Since by this theorem the image of the extension mapping is contained in the closed convex hull of the original image, $\alpha \in C(D_{j+1}(X), [1,0])$ and $\beta \in C(D_{j+1}(X), E^-)$ is compact and 4° of Definition 3.5 holds. We use this theorem again to extend $p^+h, p^0h$ to the whole $D_{j+1}(X)$. ($p^0 : E \to E^0$ is the orthogonal projector.) Then define $h = p^+h + p^-h + p^0h$. It is easy to check that $h \in A_j$.

Now for $k \in \mathbb{N}, k > N+1$ we define

\[
A_k = \frac{h(D_j(X) \setminus Y)}{j > k, h \in \Gamma_j, Y \in X \text{ with } \gamma(Y) < j-k},
\]

\[
B_k = \frac{(h(D_{j+1}(X) \setminus Y)}{j > k, h \in A_j, Y \in X \text{ with } \gamma(Y) < j-k},
\]

\[
a_k = \inf_{A \in A_k} \sup_{z \in A} J(z), \quad b_k = \inf_{B \in B_k} \sup_{z \in B} J(z)
\]

\[
a_k(n) = \inf_{A \in A_k} \sup_{z \in A} J_n(z), \quad b_k(n) = \inf_{B \in B_k} \sup_{z \in B} J_n(z).
\]

Since $id \in \Gamma_j \cap A_j$ and $0 \in A \cap B$ $\forall A \in A_j, B \in B_k$, $-\infty < a_k, b_k, a_k(n), b_k(n) < +\infty$.
§4. Properties of sequences of minimax values.

We have

**Lemma 4.1.**

1°. \( \{a_k\} \) and \( \{b_k\} \) are increasing sequences.

2°. \( \{a_k(n)\} \) and \( \{b_k(n)\} \) are increasing sequences for fixed \( n \in \mathbb{N} \).

3°. \( a_k \leq b_k, a_k(n) \leq b_k(n) \quad \forall \ n, k \in \mathbb{N} \).

4°. \( a_k \leq a_{k+1} \leq a_k(n), \ b_k \leq b_{k+1} \leq b_k(n) \quad \forall \ n, k \in \mathbb{N} \).

**Proof.** 1° and 2° are due to the fact that \( A_{k+1} \subseteq A_k \) and \( B_{k+1} \subseteq B_k \).

For any \( B = \overline{h(D_{j+1}(X) \setminus Y)} \subseteq B_k \), let \( A = \overline{h(D_j(X) \setminus Y)} \), then \( A \subseteq B \) and \( A \in A_k \), so 3° holds. 4° follows from Corollary 2.17.

**Lemma 4.2.** For any fixed \( k \in \mathbb{N}, k > N+1 \).

1°. \( \lim_{n \to \infty} a_k(n) = a_k \).

2°. \( \lim_{n \to \infty} b_k(n) = b_k \).

**Proof.** We only prove 1°. 2° can be done similarly.

Given any \( \varepsilon > 0 \), by the definition of \( a_k \), there is \( A_0 \in A_k \) such that \( \sup_{z \in A_0} J(z) < a_k + \frac{\varepsilon}{3} \). So

\[
(4.3) \quad a_k(n) < \sup_{z \in A_0} J(z) < a_k + \frac{\varepsilon}{2} + \sup_{z \in A_0} D_n(z) \quad \forall \ n \in \mathbb{N}
\]

where \( D_n(z) = J_n(z) - J(z) \). Since \( A_0 \) is bounded in \( E \), by Lemma 2.3,

\[
\sup_{z \in A_0} D_n(z) < +\infty. \quad \text{Thus for every} \quad n \in \mathbb{N}, \quad \text{there exists} \quad z_n \in A_0 \quad \text{such that}
\]

\[
(4.4) \quad \sup_{z \in A_0} D_n(z) < D_n(z_n) + \frac{\varepsilon}{3}.
\]

Since \( E \) is compactly imbedded into \( L^1(S^1, \mathbb{R}^{2N}) \), \( \{z_n\} \) possesses a
subsequence \( \{z_{n_j}\} \) which converges to some \( z_0 \) in \( L^1(S^1, \mathbb{R}^{2N}) \). From real analysis (cf. [14]) \( \{z_{n_j}\} \) has a subsequence which converges to \( z_0 \) almost everywhere. We still denote it by \( \{z_{n_j}\} \).

Let \( Q = \{ t \in [0, 2\pi] \mid |z_0(t)| < \infty \} \cap \{ t \in [0, 2\pi] \mid z_{n_j}(t) + z_0(t) \text{ as } j \rightarrow \infty \} \), then \( Q \) has Lebesgue measure \( 2\pi \). For any \( t \in Q \), there is \( N_1(t) > 0 \) such that \( |z_{n_j}(t)| < |z_0(t)| + 1, \forall j > N_1(t) \). Choose \( N_2(t) > N_1(t) \) such that \( K_{n_j} > |z_0(t)| + 1, \forall j > N_2(t) \) where \( \{K_n\} \) is defined in Proposition 2.5. Then \( H_{n_j}(z_{n_j}(t)) = H(z_{n_j}(t)), \forall j > N_2(t) \). This shows that \( H(z_{n_j}) - H_{n_j}(z_{n_j}) = 0 \) almost everywhere as \( j \rightarrow \infty \). Thus \( H(z_{n_j}) - H_{n_j}(z_{n_j}) = 0 \) in measure as \( j \rightarrow \infty \).

Since \( \{z_{n_j}\} \) are bounded in \( E \), by (H3) and Lemma 2.3, \( \|H(z_{n_j}) - H_{n_j}(z_{n_j})\| \mid j \in \mathbb{N} \) are bounded in \( L^2(S^1, \mathbb{R}^{2N}) \). So by a theorem of De La Vallée-Poussin (Theorem VI.3.7 [14]) with \( \theta(u) = u \), \( \{H(z_{n_j}) - H_{n_j}(z_{n_j})\} \mid j \in \mathbb{N} \) have equi-absolute continuous integrals.

Now we can apply D. Vitali's theorem (Theorem VI.3.2 [14]) and get a constant \( N_3 > 0 \) such that

\[
\int_0^{2\pi} |H(z_{n_j}) - H_{n_j}(z_{n_j})| \, dt < \frac{2\varepsilon}{9}, \forall j > N_3.
\]

Combining with (4.4) and Lemma 2.14 we get

\[
0 < D_{n_j}(z_{n_j}) < \frac{\varepsilon}{3}, \forall j > N_3.
\]

Let \( N_0 = n_{N_3} \), then combining with (4.3) yields \( a_k(N_0) < a_k + \varepsilon \). By Lemma 4.1 we get

\[
a_k < a_k(n) < a_k(N_0) < a_k + \varepsilon, \forall n > N_0.
\]

This completes the proof.
§5. An upper estimate for the growth rate of \( \{a_k\} \)

By (2.27) we get

\[ \frac{1}{q_1} J(T_\theta z) < J(z) + \beta_9 \log (|J(z)| + 1) + 1 \quad \forall z \in E \quad \text{and} \quad \theta \in [0, 2\pi] \]

So there is \( M_1 > 0 \) depending only on \( \beta_9 \) and \( q_1 \) such that

\[ \frac{1}{q_1} J(z) + \beta_9 \log (|J(z)| + 1) + 1 < 0 \quad \text{if} \quad J(z) < -M_1 \]

We shall prove the following claim in §6: \( a_k \to \infty \) as \( k \to \infty \). So there is a \( k_0 \in \mathbb{N}, k_0 > N+1 \) such that

\[ a_k > M_1 + 1, \quad \forall k > k_0 \]

**Proposition 5.4.** Assume that there is \( k_1 > k_0 \) such that \( b_k = a_k \), \( \forall k > k_1 \). Then there is \( M = M(k_1) > 0 \) such that

\[ a_k < M \log k \frac{1}{q_1} \quad \forall k > k_1 \]

**Proof.** Assuming the following inequality for a moment

\[ \inf \sup( \max_{A \in \mathcal{A}_{k+1}} \sup_{\theta \in \mathcal{B}_{k}} J(T_\theta z)) < \inf \sup( \max_{A \in \mathcal{A}_{k+1}} \sup_{\theta \in \mathcal{B}_{k}} J(T_\theta z)) \]

For \( k > k_1 \) we get

\[ a_{k+1} < \inf \sup( \max_{B \in \mathcal{B}_{k}} J(T_\theta z)) \]

For any \( \varepsilon > 0 \), by the definition of \( b_k \), there is a \( B \in \mathcal{B}_{k} \) such that

\[ \sup_{\theta \in \mathcal{B}_{k}} J(z) < b_k + \varepsilon = a_k + \varepsilon \]

For this \( B \), using (5.1), (5.2), (5.3) we get \( \forall \theta \in [0, 2\pi] \) that

\[ \frac{1}{q_1} J(T_\theta z) < a_k + \varepsilon + \beta_9 \log (a_k + \varepsilon + 1) + 1 \]

Therefore there is \( M_2 > 0 \) depending only on \( \beta_9 \) and \( q_1 \) such that

\[ a_{k+1} < a_k + \varepsilon + M_2 \log (a_k + \varepsilon) + 1 \]

Letting \( \varepsilon \to 0 \) yields

\[ a_{k+1} < a_k + M_2 \log a_k + 1 \]

Write \( \delta_k = a_k (k \log^p k)^{-1} \), \( p = 1/q_1 \). We need to prove \( \{\delta_k\} \) is bounded. Now
(5.7) becomes

\[ \delta_{k+1}(k+1) \log^P(k+1) \leq \delta_k \log^P k + M_2(\log \delta_k + \log(k \log^P k)^P + 1). \]

If \( \delta_{k+1} > \delta_k \) we get

\[
\delta_k \log^{-P} \delta_k < M_2 \frac{(1 + \log(k \log^P k)^P)}{k^P(k+1) - \log^P(k+1) + \log^P(k+1)} + \log^P(k+1)
\]

If \( \delta_k > e \), then

\[
\delta_k \log^{-P} \delta_k < M_2 \frac{(1 + \log k + p \log \log k)^P + 1}{\log^P(k+1)}
\]

Thus there is \( M_3 > 0 \) depending only on \( P \) and \( M_2 \) such that \( \delta_k < M_3 \).

Then from (5.8) it is easy to see that there is a constant \( M_4 > 0 \) depending only on \( M_3 \) and \( p \) such that \( \delta_{k+1} < M_4 \).

Therefore \( \delta_{k+1} < \max(\delta_k, M_4) \), \( \forall k > k_1 \). So \( \delta_k < \max(\delta_{k_1}, M_4) \), \( \forall k > k_1 \).

Let \( M = \max(\delta_{k_1}, M_4) \). This yields (5.5). Therefore we reduce to the proof of (5.6), i.e.

Lemma 5.9. If \( L \) is a continuous \( S^1 \)-invariant functional on \( E \), then

\[ \inf_{A \in \mathcal{A}_{k+1}} \sup_{z \in A} L(z) < \inf_{A \in \mathcal{A}_k} \sup_{z \in A} L(z) \quad \forall k > N+1. \]

Proof. Given any \( B \in \mathcal{B}_k \), by the definition, there are \( j > k \), \( h_1 \in A_j \), \( Y \in X \) with \( \gamma(Y) < j-k \) such that \( B = h_1(D_{j+1}(X) \setminus Y) \). Let

\[ U_j(x) = \{x' \in D_{j+1}(X) : x = x' + \rho_{j+1} \theta_{j+1}, x' \in V_j(x), \rho_{j+1} > 0 \} \]

and \( I \times E \subset R_{j+1} \).

By the definition of \( h_1 \), for any \( x \in U_j(X) \), \( P^{-1} h_1(x) = a_1(x) P^{-1} \text{id}(x) + \beta_1(x) \), where \( a_1 \) and \( \beta_1 \) are given by 3° of Definition 3.4. We define

\[
\alpha(x) = a_1(x), \quad \beta(x) = \beta_1(x) \quad \forall x \in U_j(x)
\]

\[
\alpha(T_\theta x) = a_1(x), \quad \beta(T_\theta x) = T_\theta \beta(x) \quad \forall x \in U_j(x) \text{ and } \theta \in [0, 2\pi).
\]
Note that \( D_{j+1}(X) = \bigcup_{\theta \in [0, 2\pi]} T_{\theta}U_{j}(X) \) and for any given \( y \in D_{j+1}(X) \), \( \hat{T}_{\theta}x = y \) possesses a unique solution \((x, \theta)\) in \( U_{j}(X) \times [0, 2\pi] \). So \( \alpha \) and \( \beta \) are well defined, \( \alpha \in C(D_{j+1}(X), [1, \alpha_1]) \) is \( S^1 \)-invariant, \( \beta \in C(D_{j+1}(X), E) \) is compact and \( S^1 \)-equivariant.

Define

\[
\begin{align*}
    h^{-}(x) &= \alpha(x)P_{\id(x)} + \beta(x) \quad \forall x \in D_{j+1}(X) \\
    h^{+}(x) &= P^{+}h_{1}(x), \quad h^{0}(x) = P^{0}h_{1}(x) \quad \forall x \in U_{j}(X) \\
    h^{+}(\hat{T}_{\theta}x) &= \hat{T}\hat{T}_{\theta}h^{+}(x), \quad h^{0}(\hat{T}_{\theta}x) = h^{0}(x) \quad \forall x \in U_{j}(X) \quad \text{and} \quad \theta \in [0, 2\pi]
\end{align*}
\]

and

\[
h(x) = h^{+}(x) + h^{-}(x) + h^{0}(x) \quad \forall x \in D_{j+1}(X).
\]

Then \( h \in \Gamma_{j+1}, h = h_{1} \) on \( U_{j}(X) \) and \( h(\hat{T}_{\theta}x) = \hat{T}_{\theta}h(x) \quad \forall x \in U_{j}(X) \quad \text{and} \quad \theta \in [0, 2\pi] \). Let \( A = \overline{h(D_{j+1}(X) \setminus Y)} \) then \( A \subset A_{k+1} \) and we have that

\[
\sup_{z \in \overline{A}} L(z) = \sup_{z \in \overline{h_{1}(U_{j}(X) \setminus Y)}} \left( \max_{\theta \in [0, 2\pi]} L(\hat{T}_{\theta}x) \right) < \sup_{z \in B} L(z).
\]

This completes the proof of Lemma 5.9 and then Proposition 5.4.

\[\square\]

Remark. Proposition 5.4 is a variant of Lemma 1.64 [18] in \( S^1 \)-setting.

Lemma 5.9 is new. The space \( X \) is introduced to get the unique expression \( y = \hat{T}_{\theta}x \) for given \( y \in D_{j+1}(X) \) in terms of \((x, \theta)\) in \( U_{j}(X) \times [0, 2\pi] \), which is crucial in the proof of Lemma 5.9.

\[\text{§6. A lower estimate for the growth rate of } \{a_{k}\}\]

In this section, we shall prove the following estimate on \( \{a_{k}\} \).

Proposition 6.1. There are constants \( \lambda > 0, k_{0} > N+1 \) such that
We shall carry out the proof in several steps.

Step 1. We consider a Hamiltonian system

\[ \dot{z} = JVF(|z|) \equiv \frac{F'(|z|)}{z} J z \]

and its corresponding Lagrangian functional

\[ \phi_F(z) = \frac{1}{2} A(z) - \int_0^{2\pi} F(|z|) dt \quad \forall z \in C^1(S^1, \mathbb{R}^{2N}) . \]

By direct computation we have

**Lemma 6.4.** If the function \( F \) satisfies

(F1) \( F \in C^1((0, +\infty), \mathbb{R}) \).

(F2) Let \( g_F(t) = \frac{F'(t)}{t} \), then \( g_F(0) = 0 \), \( \lim_{t \to +\infty} g_F(t) = +\infty \), and \( g_F(t) \) is strictly increasing.

(F3) Let \( h_F(t) = F'(t) t - 2F(t) \), then \( h_F(g_F^{-1}(1)) > 0 \) and \( h_F(t) \) is strictly increasing for \( t > g_F^{-1}(1) \).

Then

1°. The solutions of (6.3) are all in \( E^+ \) and of the following form

\[ z_k(t) = \begin{pmatrix} \cos(kt) & \sin(kt) \\ \sin(kt) & \cos(kt) \end{pmatrix} v_k \]

for any \( v_k \in \mathbb{R}^{2N} \) with \( |v_k| = \gamma_k(F) \), \( k \in \mathbb{N} \) and \( v_0 = 0 \) where \( \gamma_k(F) = g_F^{-1}(k) \) satisfies \( \gamma_k(F) \to +\infty \) as \( k \to +\infty \) and \( 0 = \gamma_0(F) < \gamma_k(F) < \gamma_{k+1}(F) \) for \( k \in \mathbb{N} \), \( I \) is the identity matrix on \( \mathbb{R}^N \).

2°. Let \( d_0(F) = 0 \), \( d_k(F) = \phi_F(z_k) \) \( \forall k \in \mathbb{N} \), then \( d_k(F) = \pi h_F(\gamma_k) > 0 \), is strictly increasing in \( k \).

Step 2. We define a function \( G : [0, +\infty) \to \mathbb{R} \) by

\[ G(t) = \alpha \sum_{k=n}^{\infty} \frac{t^k}{k!} q = \alpha t^q q \sum_{k=0}^{n} \frac{t^k}{k!} \]

where \( \alpha = 2a_2, \tau = \tau_2, q = q_2, a_2, \tau_2, q_2 \) are given by (H3), and \( n \) is the smallest positive integer such that \( n > \left\lfloor \frac{5}{q} \right\rfloor + 1 \) and \( \tau q_2 \left( \frac{q_2}{4} \right)^{2/q} \sum_{k=n}^{\infty} \frac{1}{k!} \left( \frac{t}{q} \right)^k < 1 \).
Then it is easy to see $G$ satisfies (F1) and (F2). Since

\[(6.5) \quad h_G(t) = G'(t)t - 2G(t) = aq \sum_{n=1}^{n+1} t^{n+1}q(kn+1)q + a(tqt^q - 2) \sum_{k=n+1}^{\infty} \frac{1}{k!} t^k q^k \]

when $t > \left(\frac{2}{\tau q}\right)^{1/q}$, $h_G(t) > 0$ and is strictly increasing.

Write $\gamma_k = \gamma_k(G)$ for $k \in \mathbb{N}$, we claim that $\gamma_1 > (\frac{2}{\tau q})^{1/q}$. For otherwise we have the following contradiction,

\[ 1 = q_G(\gamma_1) = aq \sum_{k=n+1}^{\infty} \frac{k}{(k-1)!} \gamma_k k^{-2} < aq(\frac{\tau q}{4})^{2/q} \sum_{k=n}^{\infty} \frac{1}{k!} \left(\frac{4}{q}\right)^k < 1. \]

Therefore $G$ satisfies (F3), and we also have that

\[(6.6) \quad G'(t) - 4G(t) = aq \sum_{n=1}^{n+1} t^{n+1}q(n+1)q + a(tqt^q - 4) \sum_{k=n+1}^{\infty} \frac{1}{k!} t^k q^k > 0 \]

and is strictly increasing for $t > (\frac{4}{\tau q})^{1/q}$. It is also easy to see that there is $t_1 > 0$ such that

\[(6.7) \quad 0 < G(t) < t^4 \quad \forall t \in [0, t_1]. \]

We consider the Hamiltonian system

\[(6.8) \quad \dot{z} = J\nabla G(|z|) \equiv \frac{G(|z|)}{|z|} Jz \]

and write $\phi = \phi_G$, $d_k = d_k(G) \quad \forall k \in \{0\} \cup \mathbb{N}$. Besides properties described in Lemma 6.4 we also have

**Lemma 6.9.** There are $\lambda_1 > 0$, $k_1 \in \mathbb{N}$ such that

\[(6.10) \quad d_k > \lambda_1 k (\log k)^{2/q} \quad \forall k > k_1. \]

**Proof.** From $q_G(\gamma_k) = k$ we have

\[ \gamma_k q^{-2} \gamma_k q = aq \sum_{j=1}^{n} \frac{1}{(j-1)!} \gamma_k j^{-2} = k \]

so

\[ \tau q \gamma_k > \log k + (2-q)\log \gamma_k - \log(a\tau q). \]

Thus there is $k_2 \in \mathbb{N}$ such that

\[ \gamma_k > \left(\frac{1}{2t} \log k\right)^{1/q} \quad \forall k > k_2. \]
Since \( G(t) = \frac{G'(t)}{t} - \tau qt \frac{(n+1)q-2}{n} \frac{1}{\tau q} t^{2-q} \), we get

\[
d_k = \pi(G'(\gamma_k)\gamma_k - 2G(\gamma_k)) = \pi(k\gamma_k^2 - \frac{2k}{\tau q} \gamma_k^{2-q}) = \pi k\gamma_k^2(1 - \frac{2}{\tau q} \gamma_k^{-q}).
\]

So there is \( k_1 > k_2 \) such that for any \( k > k_1 \)

\[
d_k > \frac{\pi}{2} k\gamma_k^2 > \lambda_1 k (\log k)^{2/q}
\]

where \( \lambda_1 = \frac{\pi}{2} (\frac{1}{2^2})^{2/q} \).

For \( k \in \mathbb{N}, k > N+1 \) we define

\[ c_k = \inf \sup \phi(z). \]

Since \( \phi \in C(E, \mathbb{R}), (6.6) \) and \( 0 \in \cap \Lambda_k \), we get \(-\infty < c_k < \infty\). From \( \Lambda_{k+1} \subset \Lambda_k \)

we get

\[ (6.11) \]

\[ c_k < c_{k+1} \quad \forall \ k > N+1. \]

By (H3) and the definition of \( G \), there is a constant \( \zeta_1 > 0 \) such that

\[ H(z) + \frac{1}{2} \ |z|^2 < G(|z|) + \zeta_1 \quad \forall \ z \in \mathbb{R}^{2N}. \]

For \( z \in E \), by Hölder's inequality we get for \( \zeta_2 = \zeta_1 + \frac{1}{2} \|f\|^2_{L^2} \).

\[ J(z) > \frac{1}{2} A(z) - \int_0^{2\pi} (H(z) + \frac{1}{2} |z|^2) dt - \frac{1}{2} \|f\|^2_{L^2} > \phi(z) - \zeta_2. \]

So we get

\[ (6.12) \]

\[ a_k > c_k - \zeta_2 \quad \forall \ k > N+1. \]

**Step 3.** Let \( m_0 = [\gamma_1] + 1 \) and define for \( m > m_0 \)

\[
G_m(t) = \begin{cases} 
G(t) & \text{if } 0 < t < m \\
\frac{G'(m)}{m^4} t^4 + G(m) - \frac{m}{4} G'(m) & \text{if } m < t 
\end{cases}
\]

Then \( G_m \) satisfies (F1) and (F2). For \( n < t \)

\[ (6.13) \]

\[ G_m'(t)t - 2G_m(t) = \frac{G'(m)}{m^3} t^4 + \frac{1}{2} (G'(m)m - 4G(m)). \]

By (6.6) and the properties of \( G \), \( G_m \) satisfies (F3). So Lemma 6.4 holds for \( G_m \). Since
\[ G_m(t) - 4G_m(t) = \frac{G'(m)}{m^3} t^4 + (G'(m)m - 4G(m)), \forall t > m, \]

We get that
\[ 0 < G_m(t) < G_{m+1}(t) < G(t), \forall t > 0 \]
and
\[ 0 < 4G_m(t) < G_m(t)t, \forall t > r_1 \equiv \left(\frac{4}{r_0}\right)^{1/q}. \]

From the definition of \( G_m \) and (6.7), there is a constant \( \sigma_m > 0 \) depending on \( m \) such that
\[ (6.14) \quad G_m(t) < \sigma_m t^4, \forall t > 0. \]

We consider
\[ (6.15) \quad z = J \mathcal{V} G_m(|z|) \equiv \frac{G'(|z|)}{|z|} Jz \]
and write \( \phi_m = \phi_{G_m}, d_k(m) = d_k(G_m) \). Then \( \phi_m \in C^1(E,R) \) and satisfies (P.S) condition. Define
\[ c_k(m) = \inf_{A \in \mathcal{A}_m} \sup_{z \in A} \phi_m(z), \forall k > N+1, m > m_0. \]

Then we have \(-\infty < c_k(m) < +\infty\). To get more accurate estimates on \( c_k(m) \), we need

**Lemma 6.16.** Let \( N+1 < j < m, 0 < \rho < \rho_m, h \in \Gamma_m \) and
\[ Q \equiv \{ x \in D_m(X) | h(x) \in \partial B_{\rho} (E) \cap V_{j-1}(E) \}. \]

Then \( Q \) is compact and \( \gamma(Q) > m-j+1 \).

**Proof.** This Lemma is a variant of Proposition 1.19 [19]. Note that firstly \( \text{id}(x^\perp_1) = P^- \text{id}(x^\perp_1) \) by 5° of Lemma 3.3, \( \{P^- \text{id}(x^\perp_1)\} \) being convergent implies that \( \{x^\perp_1\} \) has a convergent subsequence. This yields that \( Q \) is compact. Secondly, by the definition of \( h \in \Gamma_m \), if we let \( E_k = E^+_{N+1,k} \oplus E^-_{N+1,k} \oplus E^0 \),
\[ X_k = X^+_{N+1,k} \oplus X^-_{N+1,k} \oplus X^0 \]
and \( P_k : E \oplus E_k \) be the orthogonal projector, then \( P_k h(x) = z \) for \( z \sim x \in X^0 \cap D_m(X) \) and \( P_k h(\partial B_{\rho_m}(X) \cap V_{m}(X) \cap X_k) = \).
$P_{\text{kid}}(\partial B_m(X) \cap V_m(X) \cap X_k) = \partial B_m(E) \cap V_m(E) \cap E_k$. This allows us to apply Borsuk-Ulam theorem [8]. Therefore we can go through the proof of Proposition 1.19 [19]. We omit the details here.

**Corollary 6.17.** Let $N+1 < j < m$, $0 < \rho < R_m$, $h \in \Gamma_m$. For any $Y \subset X$ with $y(Y) < m-j$

$$h(D_m(X) \setminus Y) \cap \partial B_\rho(E) \cap V_{j-1}^\perp(E) \neq \emptyset$$

**Lemma 6.18.** $c_k(m) > 0 \quad \forall k > N+1, m > m_0$.

**Proof.** Fix $m > m_0$, $k > N+1$, by Corollary 6.17 for any $A \in A_k$ and $0 < \rho < R_k$ there is a $z \in A \cap \partial B_\rho \cap E^t$. Let $C$ denote the embedding constant from $E$ into $L^4$, then by (6.14)

$$\phi_m(z) > \frac{1}{2} \lambda(z) - \sigma_m \int_0^{2\pi} |z|^4 \, dt > \frac{1}{2} \sigma^2 - \sigma_m C \rho^4 = \frac{1}{2} \rho^2 (1 - \sigma_m C \rho^2) .$$

Choose $\rho_m = \min\{1, (2\sigma_m C)^{-1/2}\}$, we get $c_k(m) > \frac{1}{2} \rho_m^2 > 0$.

**Lemma 6.19.** For any $k > N+1$, $m > m_0$

1°. $c_k(m)$ is a critical value of $\phi_m$.

2°. Any critical point of $\phi_m$ corresponding to $c_k(m)$ lies in $E \setminus E_0$.

3°. If $c_{k+1}(m) = \cdots = c_{k+j}(m) \equiv c$ and $K \equiv (\phi_m)^{-1}(0) \cap \phi_m^{-1}(c)$, then $\gamma(K) > j$.

**Proof.** 1° and 3° follow from the standard argument, we refer to [19]. 2° follows from 1°, Lemma 6.4 and Lemma 6.18. We omit details here.

**Lemma 6.20.** For $k \in \mathbb{N}$, $k > N+1$, $m > m_0$, $c_k < c_k(m+1) < c_k(m)$ and

$$\lim_{m \to +\infty} c_k(m) = c_k$$
Proof. The Lemma follows from the proofs of Lemmas 4.1 and 4.2.


Fix \( k > N+1 \), for any \( m > m_0 \), by 1\(^o\) of Lemma 6.19 and Lemma 6.18, \( c_k(m) = d_j(m) \) for some \( j > 0 \). So

\[
c_k(m) = \phi_m(z_j) = \pi(G'_m(y_j)y_j - 2G_m(y_j)) > \min\{\pi \frac{G'(m)}{m^3} y_j, \pi aq \frac{t^{n+1}}{n!} y_j^{(n+1)q}\},
\]

here we used (6.5) and (6.13). Since by definition of \( G \), \( \frac{G'(t)}{t^3} \) is strictly increasing for \( t > 0 \), by Lemma 6.20 we get

\[
c_k(m_0) > c_k(m) > \min\{\pi \frac{G'(m_0)}{m_0^3} y_j, \pi aq \frac{t^{n+1}}{n!} y_j^{(n+1)q}\}.
\]

So there exists \( M_1 > 0 \) independent of \( m \) such that if \( z \) is a critical point of \( \phi_m \) corresponding to \( c_k(m) \) with \( m > m_0 \), then \( |z| < M_1 \). Thus \( C_m(|z|) = G(|z|) \) for \( m > m_1(k) \equiv \max\{m_0, [M_1] + 1\} \), and then there exists \( j(m) \in \mathbb{N} \) depending on \( m \) such that

\[
(6.21) \quad c_k(m) = d_j(m) \quad \forall \ m > m_1(k).
\]

By Lemma 6.20, \( 0 < d_k < d_{k+1} \), and (6.10), we get \( c_k = d_j \) for some \( j \in \mathbb{N} \).

Therefore \( \{c_k\} \) is a subset of \( \{d_k\} \).

We claim that \( c_{k+N} > c_k \ \forall \ k > N+1 \). If not, by (6.11) we get \( c = c_k = \cdots = c_{k+N} \). By (6.21) there exists \( m > m_0 \) depending on \( k \) such that \( c = c_k(m) = \cdots = c_{k+N}(m) \). Let \( K = (\phi_m)^{-1}(0) \cap \phi_m^{-1}(c) \). By 3\(^o\) of Lemma 6.19, \( y(K) > N+1 \). But 1\(^o\) of Lemma 6.4 and 4\(^o\) of Lemma 3.2 shows that \( y(K) = N \). This contradiction proves the claim.

Assume \( c_{N+1} = d_{\ell} \) for some \( \ell \in \mathbb{N} \), then by the above discussion and (6.10), for \( k > \max(k_1, 6N) \).
\[ c_k > \frac{c}{N+1} \left( \frac{k-N-2}{N} \right)^N - \frac{d}{N+1} \left( \frac{k-N-2}{N} \right) > \lambda_1 \left( \lambda + \left( \frac{k-N-2}{N} \right) \right) \log \frac{2}{q} (\lambda + \left( \frac{k-N-2}{N} \right) ) \]
\[ > \lambda_1 \left( \lambda + \frac{k}{N} - 3 \right) \log \frac{2}{q} (\lambda + \frac{k}{N} - 3) > \lambda_1 \frac{k}{2N} \log \frac{2}{q} (\frac{k}{2N}) \]
\[ > \lambda k (\log k)^{2/q} \]
for some \( \lambda > 0 \). Combining with (6.12) we get (6.2).

The proof of Proposition 6.1 is complete.

§7. The existence of critical values of \( J_n \)

Fix \( n, k \in \mathbb{N}, k > N+1 \), we have

**Proposition 7.1.** Suppose \( b_k(n) > a_k(n) > \beta_8 \). Let \( \delta_k(n) \in (0, b_k(n) - a_k(n)) \) and

\[ B_k(n, \delta_k(n)) = \{ h(D_j(X) \setminus Y) | J_n(h(x)) < a_k(n) + \delta_k(n) \text{ for } x \in D_j(X) \setminus Y \} . \]

Let

\[ b_k(n, \delta_k(n)) = \inf_{B \in B_k(n, \delta_k(n))} \sup_{z \in B} J_n(z) . \]

Then \( b_k(n, \delta_k(n)) \) is a critical value of \( J_n \).

**Remark.** \( b_k(n, \delta_k(n)) > b_k(n) \). By Lemma 3.6, \( B_k(n, \delta_k(n)) \neq \emptyset \), and \( b_k(n, \delta_k(n)) < +\infty \).

For the proof of Proposition 7.1, we need the following "Deformation Theorem", which was proved in [19].

**Lemma 7.2.** Let \( J_n \) be as above, then if \( b > \beta_8, \varepsilon > 0 \) and \( b \) is not a critical value of \( J_n \), there exist \( \varepsilon \in (0, \varepsilon) \) and \( n \in C([0,1] \times E, E) \) such that
1. \( n(t,z) = z \) if \( z \notin J^{-1}_n(b_\varepsilon,b_+\varepsilon) \)

2. \( n(0,z) = z \quad \forall z \in E \)

3. \( n \), \( (J_n)^{-1} \subset (J_n)^{-1} \), where \( (J_n)^a = \{ z \in E \mid J_n(z) < a \} \).

4. \( \beta_1(z) = a_1(z)z^- + \beta_1(z) \quad \forall z \in E \), where \( a_1 \in C(E,[1,2]) \), \( \beta_1 \in C(E,E^-) \) and \( \beta \) is compact.

5. \( n(t,*) \) is a bounded map from \( E \) to \( E \) for \( t \in [0,1] \).

**Proof of Proposition 7.1.** Let \( \varepsilon = \frac{1}{2} \left( b_k(n) - a_k(n) \right) > 0 \). If \( b_k(n,\delta_k(n)) \) is not a critical value of \( J_n' \), then there exist \( \varepsilon \) and \( n \) as in Lemma 7.2. Choose \( B \in B_k(n,\delta_k(n)) \) such that

\[
\sup_{z \in B} J(z) < b_k(n,\delta_k(n)) + \varepsilon \quad .
\]

Then there exist \( j > k \), \( h_0 \in A_j \), \( Y \in X \) with \( \gamma(Y) < j-k \) such that \( B = h_0(D_{j+1}(X) \setminus Y) \). Define

\[
\begin{align*}
h(x) &= n(1,h_0(x)) \quad \forall x \in D_{j+1}(X) \setminus Y \equiv \Omega_1, \\
h(x) &= h_0(x) \quad \forall x \in B_{j+1}(X) \cap V_j(X) \cap Y \equiv \Omega_2, \\
h(x) &= \text{id}(x) \quad \forall x \in 3B_{j+1}(X) \cap V_{j+1}(X) \equiv \Omega_3.
\end{align*}
\]

Denote \( \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \). For \( x \in D_j(X) \setminus Y \), \( J_n(h_0(x)) < a_k(n) + \delta_k(n) < b_k(n) - 2\varepsilon < b_k(n,\delta_k(n)) - \varepsilon \).

Thus

\[
(7.3) \quad h(x) \equiv n(1,h_0(x)) = h_0(x) \quad \forall x \in D_j(X) \setminus Y .
\]

For \( x \in \Omega_3 \cup (B_{j+1}(X) \setminus V_j(X)) \), \( J_n(h_0(x)) < 0 < b_k(n,\delta_k(n)) - \varepsilon \), thus \( n(1,h_0(x)) = h_0(x) = \text{id}(x) \). So \( h \in C(\Omega,E) \).

For \( x \in \Omega_4 \equiv (D_{j+1}(X) \cap V_j(X)) \cup \Omega_3 \), \( P^n(h(x)) = P^n h_0(x) = a_0(x)P^n \text{id}(x) + \beta_0(x) \) where \( a_0, \beta_0 \) are defined for \( h_0 \) in 3° of Definition 3.5. For \( x \in \Omega \setminus \Omega_4 \),

-30-
\[ P^{-h}(x) = a_1(h_0(x))a_0(x)P^{-id}(x) + a_1(h_0(x))\beta_0(x) + \beta_1(h_0(x)) \]

Define

\[
a(x) = \begin{cases} 
a_0(x) & \text{if } x \in Q_4, \\
a_1(h_0(x))a_0(x) & \text{if } x \in Q \setminus Q_4.
\end{cases}
\]

\[
\beta(x) = \begin{cases} 
\beta_0(x) & \text{if } x \in Q_4, \\
a_1(h_0(x))\beta_0(x) + \beta_1(h_0(x)) & \text{if } x \in Q \setminus Q_4.
\end{cases}
\]

Then \( a \in C(Q, [1, e^{2\alpha_0}]), \beta \in C(Q, E^-) \) is compact and

\[ P^{-h}(x) = a(x)P^{-id}(x) + \beta(x) \quad \forall \ x \in Q. \]

Let \( W = (D_{j+1}(X) \cap Y) \setminus V_j(X) \), then \( \partial W \subset Q \) where "\( \partial \)" is taken within \( V_j+1(X) \). Since \( a, \beta, P^+h, \) and \( P^0h \) are continuously defined on \( \partial W \), we may use the Dugundji extension theorem ([7]) to extend them to \( W \), then define \( P^{-h}(x) = a(x)P^{-id}(x) + \beta(x), \) and \( h(x) = P^+h(x) + P^{-h}(x) + P^0h(x). \)

We have \( h \in A_j \). Thus \( D = h(D_{j+1}(X) \setminus Y) \in S_k. \) By (7.3)

\[ J_n(h(x)) = J_n(h_0(x)) < a_k(n) + \delta_k(n) \quad \forall \ x \in D_j(X) \setminus Y. \]

Thus \( D \in S_k(n, \delta_k(n)). \) Now 3\(^\circ\) of Lemma 7.2 yields

\[ \sup_{z \in D} J_n(z) < b_k(n, \delta_k(n)) - \epsilon. \]

This contradicts to the definition of \( b_k(n, \delta_k(n)) \). Therefore the proof is complete.
§8. The proofs of the main theorems

We prove Theorem 1.2 by contradiction. Assume that the functional $I$ is bounded from above by $M_1 > 0$ on $S$, the solution set of (1.1).

Since $q_2 < 2q_1$, Propositions 5.4 and 6.1 show that there exists $k \in N$ such that

$$b_k > a_k > \max \{b_g, M_1\}.$$ 

Let $\varepsilon = \frac{1}{5} (b_k - a_k)$. By Lemmas 4.1 and 4.2, there exists $n_1 > 0$ such that

$$b_k(n) - a_k(n) > 4\varepsilon \quad \text{and} \quad a_k(n) < a_k + \varepsilon \quad \forall \ n > n_1.$$ 

Let $\delta_k(n_1) = \varepsilon$, and $\delta_k(n) = 2\varepsilon$ for $n > n_1$. Then by Proposition 7.1, $b_k(n, \delta_k(n))$ is a critical value of $J_n$ for $n > n_1$.

If $h(D_{j+1}(X) \setminus Y) \in B_k(n_1, 2\varepsilon)$ then for any $x \in D_j(X) \setminus Y$

$$J_n(h(x)) < J_{n_1}(h(x)) < a_k(n_1) + \varepsilon < a_k + 2\varepsilon.$$ 

So $B_k(n_1, \varepsilon) \subset B_k(n_2, \varepsilon)$ for $n > n_1$. Therefore for $n > n_1$,

$$b_k(n, 2\varepsilon) < \inf_{B \in B_k(n_1, \varepsilon)} \sup_{z \in B} \left( \frac{1}{2} A(z) - \int_0^{2\pi} P_0(z) dt + \int_0^{2\pi} |f(z)| dt \right) \equiv b \leq 2.$$ 

where $P_0(z) = a_0 |z|^{\alpha} - \beta_0$ and we used (2.6).

Let $z_n$ be a critical point of $J_n$ corresponding to $b_k(n, 2\varepsilon)$ for $n > n_1$. Using (2.6), $f \in W^{1,2}(S^1, R^n)$ and the proof of Lemma 5.3 [2], we get

$$(8.1) \quad \|z_n\|_L^2 < \frac{M_2}{n} \quad \forall \ n > n_1$$

where the constant $M_2 > 0$ depending on $b$, but independent of $n$. Now we choose $n_2 > n_1$ such that $K_{n_2} > M_2$, where $\{K_n\}$ is defined in Proposition 2.5. From (8.1) and Lemma 2.19 we get that $H_{n_2}'(z_{n_2}) = H'(z_{n_2})$ on $[0, 2\pi]$ and $z_{n_2}$ is a solution of (1.1) i.e. $z_{n_2} \in S$. But

$$I(z_{n_2}) = I_{n_2}(z_{n_2}) = J_{n_2}(z_{n_2}) = b_k(n_2, 2\varepsilon) > b_k(n_2) > a_k(n_2) > a_k > M_1.$$
This contradicts to the definition of \( M_1 \), and completes the proof of Theorem 1.2.

The proof of Theorem 1.3 is similar. Instead of (5.5) and (6.2) we shall have \( a_k < \alpha k^{(p+1)/p_1}, \forall k > k_1 \) and \( a_k > \alpha k^{(p+1)/(p-1)} \), by (H4) they yield \( b_k > a_k \) for infinitely many \( k \). The proof is rather simpler than that of Theorem 1.2. For example, the corresponding

\[
\phi(z) = \frac{1}{2} A(z) - (a_2 + \frac{1}{2}) \int_0^{2\pi} |z|^{p_2 + 1} dt
\]

is \( C^2 \) and satisfies (P.S.) condition. So the lower estimate for \( a_k \) is quite straightforward. For the details we refer to [13].

In [15], Pisani and Tucci gave a result for (1.1),

**Theorem 8.2 (Theorem 1.1 [15])**. Let \( H \) satisfy (H1) and the following conditions

\[(H5)\]
\[
\lim_{|z| \to \infty} \frac{H'(z) \cdot z}{|z|^2} = +\infty .
\]

\[(H6)\] There are constants \( p > 1, a_1, \beta_1 > 0 \) such that

\[
\frac{1}{2} H'(z) \cdot z - H(z) \geq a_1 |z|^{p+1} - \beta_1 \quad \forall z \in \mathbb{R}^N .
\]

\[(H7)\] There are constants \( q \in [p, p+1) \) and \( a_2, \beta_2 > 0 \) such that

\[
|H'(z)| \leq a_2 |z|^q + \beta_2 \quad \forall z \in \mathbb{R}^N .
\]

Then the conclusion of Theorem 1.3 holds for given \( T > 0 \) and \( T \)-periodic function \( f \in L^2_{loc}(\mathbb{R}, \mathbb{R}^N) \).

Unfortunately, because of a difficulty caused by the \( S^1 \)-action on \( W^{1,2}(S^1, \mathbb{R}^N) \), their proof (Lemma 1.14 [15]) is not complete. Using the minimax idea introduced in §3, this difficulty can be overcome. The condition (H7) allows us to carry out the proof without doing any truncation on \( H \), so the function \( f \) can be allowed only in \( L^2 \) and the proof becomes rather...
simpler. We omit the details here.

We also refer readers to a related density result proved earlier.

Theorem 8.3 (Theorem 1.5 [11]). Let $H$ satisfy (H1) and (H5), then for any $T > 0$, there exists a dense set $D$ in the space of $T$-periodic functions in $L^2([0,T], \mathbb{R}^{2N})$ such that for every $f \in D$, (1.1) is solvable.

This theorem poses a natural question whether the condition (H3) or (H4) is necessary in Theorems 1.2 or 1.3.

§9. Results for general forced systems

In this section we consider the general Hamiltonian system

$$(9.1) \quad \dot{z} = J H_z^*(t,z).$$

Firstly we consider (9.1) with bounded perturbations. That is

Theorem 9.2. Let $\hat{H}$ satisfy the following conditions,

(G1) $\hat{H} \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ and $\hat{H}(t,z)$ is $T$-periodic in $t$.

(G2) There exists $H : \mathbb{R}^{2N} \to \mathbb{R}$ satisfying (H1), (H2) and constants $0 < q < 2$, $\alpha, \tau > 0$, $\beta > 0$ such that

1°. $H(z) < \alpha e^t \|z\|^q + \beta \forall z \in \mathbb{R}^{2N}$.

2°. $|\hat{H}(t,z) - H(z)| < \alpha \forall (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}$.

3°. $|\hat{H}_z(t,z) - H_z(z)| < \alpha (|z|^{p-1} + 1) \forall (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}$, where $1 < p < \mu$ and $\mu > 2$ is defined in (H2).

then the system (9.1) possesses infinitely many distinct $T$-periodic solutions.

Remark. Theorem 9.2 weakened conditions of Bahri and Berestycki's corresponding result, Theorem 10.1 [2], who required $\hat{H}$ satisfying the following conditions.
1°. \( H \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R}) \) and is \( T \)-periodic in \( t \).

2°. There exists \( H \in C^2(\mathbb{R}^{2N}, \mathbb{R}) \) satisfying (H2) and constants \( q > 1 \), \( a > 0 \) such that

\[
H(z) < a(|z|^{q+1} + 1) \quad \forall z \in \mathbb{R}^{2N}
\]

and

\[
C^{-1}(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R}) < H - H
\]

In order to prove Theorem 9.2, we let \( G(t,z) = \hat{H}(t,z) - H(z) \), and consider functionals

\[
J(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H(z) dt - \psi(z) \int_0^{2\pi} G(t,z) dt
\]

and

\[
J_n(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H_n(z) dt - \psi(z) \int_0^{2\pi} G(t,z) dt
\]

where \( H_n, \psi_n \) are defined in §2, and we can go through the proofs in §§2 - 7 with the following estimates for \( \{a_k\} \) from above.

**Lemma 9.3.** If \( b_k = a_k \quad \forall k > k_1 \), then there is \( M = M(k_1) > 0 \) such that

\[
(9.4) \quad a_k < Mk \quad \forall k > k_1
\]

**Proof.** Using 2° of (G2), instead of (2.27), we get that \( |J(z) - J(T_\theta z)| < 4\pi a \quad \forall z \in E \). So as in the proof of Proposition 5.4, we get that

\[
a_{k+1} = \inf \sup( \max J(T_\theta z)) < \inf \sup( \max J(T_\theta z)) \quad \text{(by Lemma 5.9)}
\]

\[
= \inf \sup \ J(z) + 4\pi a = b_k + 4\pi a = a_k + 4\pi a \quad \forall k > k_1
\]

Let \( \delta_k = \frac{a_k}{k} \). If \( \delta_{k+1} > \delta_k \), then from the above inequality

\[
(k+1)\delta_{k+1} < k\delta_k + 4\pi a < k\delta_{k+1} + 4\pi a
\]

so

\[\text{-35-}\]
\[ \delta_{k+1} < 4\pi a. \]

This shows that
\[ \delta_{k+1} < \max\{\delta_k, 4\pi a\}, \forall k > k_1. \]

Thus
\[ \delta_k < \max\{\delta_{k_1}, 4\pi a\}, \forall k > k_1. \]

Let \( M = \max\{\frac{a_{k_1}}{k_1}, 4\pi a\} \), we get (9.4), and completes the proof of Lemma 9.3.

Now the arguments in §8 yield Theorem 9.2.

Secondly, it is not difficult to get direct generalizations of Theorems 1.2 and 1.3 for (9.1).

**Theorem 9.5.** Let \( \hat{H} \) satisfy conditions (G1) and (G3) There exists \( H : \mathbb{R}^{2N} \to \mathbb{R} \) satisfying (H1), (H2) and either

1°. \( H \) satisfying (H3) and there are \( a > 0 \) and \( 1 < p < \min\{\frac{2q_1}{q_2}, \nu\} \) such that
\[
|\hat{h}(t,z) - H(z)| < a(|z|^{p+1}), \quad |\hat{h}_z(t,z) - H_z(z)| < a(|z|^{p-1} + 1), \quad \forall (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}.
\]
or

2°. \( H \) satisfying (H4) and there are \( a > 0 \) and \( 1 < q < \frac{2(p+1)}{p_2+1} \) such that
\[
|\hat{h}_z(t,z) - H_z(z)| < a(|z|^{q+1} + 1), \quad \forall (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}.
\]

Then the system (9.1) possesses infinitely many distinct T-periodic solutions.

We omit the details here.
Appendix. Monotone truncations of $H$ in $C^1(R^{2N}, R)$

In this appendix, we give a proof of Proposition 2.5.

Recall that $\sigma \in (0, 1)$, $\mu_0 > 2$, and $r_0 > 1$ (§2). Choose $\lambda \in (\sigma, 1)$ such that $u(\lambda - \sigma) < 1$. Define $K_1 = K_0 + 2$, $\tau_0 = 1$. For $n \in \mathbb{N}$, define inductively that

$$\tau_n = \max \{ \tau_{n-1} + 2, \frac{1}{K_n^0} \},$$

$$K_n = \max \{ K_n + 2, \frac{r_n}{1 - \lambda} \}.$$

Here $a_0 = \min |z| = r_0$ for $z \in \mathbb{R}$. For $K \in \mathbb{R}$, take $\chi(\cdot, K) \in C^\infty(\mathbb{R})$ such that $\chi(s, K) = 1$ for $s < K$, $\chi(s, K) = 0$ for $s > K + 1$ and $\chi'(s, K) < 0$ for $s \in (K, K+1)$. Then for $n \in \mathbb{N}$ set

$$M_n(z) = \chi(|z|, K_n)H(z) + (1 - \chi(|z|, K_n))\tau_n |z|^{\mu_\lambda} \quad \forall z \in \mathbb{R}^{2N}.$$ 

This kind of truncation functions was used by Rabinowitz in [17]. Since the $M_n$'s do not satisfy 4° of Proposition 2.5, we need to modify them. Direct computations (cf. [17]) show that

**Lemma A.3.** For $n \in \mathbb{N}$, $M_n \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ has the following properties,

$$M_n(z) = H(z) \quad \text{if} \quad |z| < K_n,$$

$$M_n(z) = \tau_n |z|^{\mu_\lambda}, \quad M_n'(z) = \tau_n \mu_\lambda |z|^{\mu_\lambda} \quad \text{if} \quad |z| > K_n + 1,$$

$$0 < \mu_\lambda M_n(z) < M_n'(z)z \quad \text{for} \quad |z| > r_0.$$ 

Integrating (A.6) we get that

$$a_0 |z|^{\mu_\lambda} < M_n(z) \quad \text{for} \quad |z| > r_0.$$

**Lemma A.8.** For $\rho > K_n + 1$ we have that

$$\max_{z \in S^{2N-1}} M_n'(\rho z)z \lesssim \min_{z \in S^{2N-1}} H'(\rho z)z$$

and

-37-
(A.10) \[
\max_{z \in S^{2N-1}} M'(pz) \cdot z < \min_{z \in S^{2N-1}} M'_{n+1}(pz) \cdot z.
\]

Proof. For any \( \zeta, \; z \in S^{2N-1} \), by (H2) and (2.4)
\[
H'(\rho \zeta) \cdot \zeta > \frac{\mu}{\rho} H(\rho \zeta) > \mu \alpha_0 \rho^{u-1}
\]
\[
\geq \mu \lambda \rho^{u-1} \sigma_0 \rho^{1-\lambda}
\]
\[
\geq \mu \lambda \rho^{u+1} \tau
\]
(by (A.2))
\[
= M'_n(pz) \cdot z. \quad \text{(by (A.5))}
\]
This proves (A.9).

For any \( \zeta, \; z \in S^{2N-1} \),
\[
M'_{n+1}(\rho \zeta) \cdot \zeta = \chi(\rho, K_{n+1}) H'(\rho \zeta) \cdot \zeta + (1 - \chi(\rho, K_{n+1})) \tau_{n+1} \mu \lambda \rho \mu \lambda^{-1} \]
\[
\geq \min\{H'(\rho \zeta) \cdot \zeta, \tau_{n+1} \mu \lambda \rho \mu \lambda^{-1}\}
\]
(by (A.1) and the definition of \( \chi \))
\[
M'_n(pz) \cdot z \quad \text{(by (A.9) and (A.1))}.
\]
This proves (A.10).

We now introduce the spherical coordinates \((r, \theta)\) on \(\mathbb{R}^{2N}\). For \( z = (z_1, \ldots, z_{2N}) \in \mathbb{R}^{2N} \), write \( z = rz(\theta), \; r = |z|, \; z(\theta) = \frac{z}{|z|}, \; \theta = (\theta_1, \ldots, \theta_{2N-1}) \),
\[
\begin{align*}
z_1 &= r \cos \theta_1 \\
z_2 &= r \sin \theta_1 \cos \theta_2 \\
&\vdots \\
z_{2N-1} &= r \sin \theta_1 \cdots \sin \theta_{2N-2} \cos \theta_{2N-1} \\
z_{2N} &= r \sin \theta_1 \cdots \sin \theta_{2N-2} \sin \theta_{2N-1}
\end{align*}
\]
(A.11)
where \( r > 0, \; \theta_1 \in [0, \pi], \; \theta_i \in \mathbb{R} \) for \( i = 2, \ldots, 2N-1 \). We also write \( d\theta = \)
\[ d\theta_1 \cdots d\theta_{2N-1}, \text{ and } d\tilde{\theta}_i = d\theta_1 \cdots d\theta_{i-1} d\theta_{i+1} \cdots d\theta_{2N-1} \text{ for } i = 1, \ldots, 2N-1. \]

Let \( \Omega = [0, \pi] \times \mathbb{R}^{2N-2} \). For \( \theta \in \Omega, \rho > 0 \) define

\[ U(\theta, \rho) = \left[ [\theta_1 - \pi \sqrt{\rho}, \theta_1 + \pi \sqrt{\rho}] \cap [0, \pi] \right] \times \prod_{i=2}^{2N-1} [\theta_i - \sqrt{\rho}, \theta_i + \sqrt{\rho}] . \]

Since \( H \) and \( M_n \) are uniformly continuous on \( \{K_n < |z| < K_{n+1}\} \), there is a constant \( \delta_n \in (0, 1) \) such that

\[ |H(z) - H(\tilde{z})| + |M_n(z) - M_n(\tilde{z})| < (\lambda - \sigma) d_n \]

for any \( z, \tilde{z} \in \{K_n < |z| < K_{n+1}\} \) and \( |z - \tilde{z}| < \delta_n \). There is a constant \( \epsilon_n \in [0, 1] \) such that

\[ |rz(\theta) - rz(\xi)| < \epsilon_n \]

for any \( r \in [K_n, K_{n+1}], \theta \in \Omega \) and \( \xi \in U(\theta, \epsilon_n) \). Define

\[ \nu_n(t) = \min\{t, \sqrt{\epsilon_n}\} \text{ for } t > 0 . \]

For \( n \in \mathbb{N}; \ i = 2, \ldots, 2N-1; \ k = 1, 2; \ \theta \in \Omega; \ \rho > K_n \) define

\[ \omega_n,1(\theta_1, \rho) = [\theta_1 - \frac{1}{2} \theta_1 v_n(\rho - K_n), \theta_1 + \frac{1}{2} (\pi - \theta_1) v_n(\rho - K_n)] , \]

\[ \omega_n,i(\theta_i, \rho) = [\theta_i - \frac{1}{2} \theta_i v_n(\rho - K_n), \theta_i + \frac{1}{2} \theta_i v_n(\rho - K_n)] , \]

\[ \omega_n,1^{(k)}(\theta_1, \rho) = [\theta_1 - \frac{k}{2} \pi v_n(\rho - K_n), \theta_1 + \frac{k}{2} \pi v_n(\rho - K_n)] \cap [0, \pi] , \]

\[ \omega_n,i^{(k)}(\theta_i, \rho) = [\theta_i - \frac{k}{2} \pi v_n(\rho - K_n), \theta_i + \frac{k}{2} \pi v_n(\rho - K_n)] , \]

and for \( j = 1, \ldots, 2N-1, \) define

\[ \Omega_n(\theta, \rho) = \prod_{j=1}^{2N-1} \omega_n,1^{(j)}(\theta_j, \rho) , \]

\[ \Omega_n,j(\theta, \rho) = \prod_{1 \leq k \leq 2N-1, k \neq j} \omega_n,k^{(j)}(\theta_k, \rho) , \]

\[ \Omega_n^{(k)}(\theta, \rho) = \prod_{j=1}^{2N-1} \omega_n,j^{(k)}(\theta_j, \rho) . \]
Then direct computations show that

\[ \Omega_{n, j}^{(k)}(\theta, \rho) = \prod_{1 \leq i \leq 2N-1, \ i \neq j} \omega_{n, i}(\theta_i, \rho) . \]

To simplify the notations we write \( V_n(\rho) = |\Omega_n(\theta, \rho)| . \) These sets satisfy:

**Lemma A.16.** For \( n \in \mathbb{N}, \ \theta \in \Omega, \ \rho > K_n, \)

1°. \( \beta \in \Omega_n(\theta, \rho) \) implies \( \theta \in \Omega_n^{(1)}(\beta, \rho). \)

2°. \( \beta \in \Omega_n(\theta, \rho) \) and \( \gamma \in \Omega_n^{(1)}(\beta, \theta) \) imply \( \gamma \in \Omega_n^{(2)}(\theta, \rho). \)

**Proof.** 1°. If \( \beta_1 \in \omega_{n, 1}(\theta_1, \rho), \) we have

\[ \theta_1 - \frac{1}{2} \nu_n(\rho - K_n) < \beta_1 < \theta_1 + \frac{1}{2} (\pi - \theta_1) \nu_n(\rho - K_n) , \]

then

\[ \beta_1 - \frac{1}{2} \nu_n(\rho - K_n) < \theta_1(1 - \frac{1}{2} \nu_n(\rho - K_n)) < \theta_1 \]

\[ < \theta_1 + \frac{1}{2} (\pi - \theta_1) \nu_n(\rho - K_n) < \beta_1 + \frac{\pi}{2} \nu_n(\rho - K_n) . \]

Since \( \theta_1 \in [0, \pi], \) \( \theta_1 \in \omega_{n, 1}(\beta_1, \rho). \) Similarly \( \beta_i \in \omega_{n, i}(\theta_i, \rho) \) implies \( \theta_i \in \omega_n^{(i)}(\beta_i, \rho) \) for \( i = 2, \ldots, 2N-1. \) Therefore 1° holds.

2°. If \( \beta \in \Omega_n(\theta, \rho), \gamma \in \Omega_n^{(1)}(\beta, \rho) \) then

\[ \theta_1 - \frac{1}{2} \nu_n(\rho - K_n) < \beta_1 < \theta_1 + \frac{1}{2} (\pi - \theta_1) \nu_n(\rho - K_n) \]

and
\[
\theta_1 - \pi n (\rho - K_n) < \gamma_1 < \beta_1 + \pi n (\rho - K_n) \quad .
\]

So
\[
\theta_1 - \pi n (\rho - K_n) < \theta_1 - \frac{1}{2} (\pi + \theta_1) n (\rho - K_n) < \beta_1 - \pi n (\rho - K_n) < \gamma_1 ,
\]

and
\[
\gamma_1 < \beta_1 + \pi n (\rho - K_n) < \theta_1 + \pi n (\rho - K_n) - \frac{1}{2} \theta_1 n (\rho - K_n) < \theta_1 + \pi n (\rho - K_n) .
\]

Since \( \gamma_1 \in [0, \pi] \), we get that \( \gamma_1 \in \omega_n(1)(\theta_1, \rho) \). Similarly \( \gamma_i \in \omega_n(1)(\theta_i, \rho) \) for \( i = 2, \ldots , 2N-1 \). Thus 2° holds and the proof is complete.

\[
\square
\]

For \( n \in \mathbb{N}, \rho > K_n, z \in S^{2N-1} \) define
\[
F_n(J, z) = \min \{ M_n(J z) \cdot z, H_n(J z) \cdot z \} .
\]

Note that \( F_n \) is continuous in its arguments. Define for \( z = rz(\theta) \in \mathbb{R}^{2N} \),
\[
r = |z|, \quad z(\theta) = \frac{z}{|z|},
\]

\[
G_n(z) = \int_{|z|}^{\infty} \frac{1}{V_n(\rho, z(\theta))} \min_{\gamma \in \Omega_n(1)(\beta, \rho)} F_n(\rho, z(\gamma)) d\beta d\rho
\]

and
\[
H_n(z) = \begin{cases} H(\rho), & \text{if } |z| < K_n, \\
G_n(z) + H(K_n z(\theta)), & \text{if } K_n < |z| .
\end{cases}
\]

Note that when \( r = |z| > K_{n+1} \), by (A.5) and (A.9),
\[
G_n(z) = \int_{K_n}^{K_{n+1}} \frac{1}{V_n(\rho, \Omega_n(\theta, \rho))} \min_{\gamma \in \Omega_n(1)(\beta, \rho)} F_n(\rho, z(\gamma)) d\beta d\rho
\]

(A.17)
\[
+ \tau_n (\rho^{\mu \lambda} - K_{n+1}^{\mu \lambda} ) .
\]

Lemma A.18. For \( n \in \mathbb{N}, \hat{H} \in C^1(\mathbb{R}^{2N}, \mathbb{R}) \).
Proof. Since $H, H_n \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ and in the formula of $G_n$ all the variables $r, \theta$ only appear linearly in the integration limits, $H_n$ is $C^1$-continuous on $\{ |z| < K_n \}$ and $\{ |z| > K_n \}$. We only need to verify the $C^1$-continuity of $H_n$ at every $\zeta \in \mathbb{R}^{2N}$ with $|\zeta| = K_n, \overline{\zeta} = \frac{\zeta}{|\zeta|}$.

For $z = rz(\theta)$ with $r = |z| > K_n$, $\overline{z}(\theta) = \frac{z}{|z|}$,

$$\frac{\partial H_n(z)}{\partial r} = \frac{\partial G_n(z)}{\partial r} = \frac{1}{V_n(r)} \int_{\Omega_n(\theta,r)} \min_{\gamma \in \Omega_n^{(1)}(\beta,r)} F_n(r, z(\gamma)) d\gamma,$$

(A.19)

$$= \min_{\gamma \in \Omega_n^{(1)}(\xi,r)} F_n(r, z(\gamma)),$$

for some $\xi \in \Omega_n(\theta,r)$ by the mean value theorem of integration. Thus

$$\lim_{z \to \zeta} \frac{\partial H_n(z)}{\partial r} = \lim_{z \to \zeta} \min_{|z| > |\zeta| = K_n} F_n(r, z(\gamma)) = F_n(K_n, \overline{z}(\zeta)) = H'(K_n, \overline{\zeta}) = \frac{\partial H(z)}{\partial r}.$$

From the definition of $G_n$ we get that

$$\frac{\partial G_n(z)}{\partial \theta_1} = \int_{K_n} \frac{1}{V_n(r)} \int_{\bar{\Omega}_n(\theta_1, \theta, \rho)} (1 - 1/2 \nu_n(\rho - K_n)) \left[ \min_{\gamma \in \Delta_1} F_n(r, z(\gamma)) \right] d\gamma d\rho,$$

(A.20)

$$- \min_{\gamma \in \Delta_2} F_n(r, z(\gamma))$$

where $\Delta_1 = \omega_n^{(1)}(\theta_1 + 1/2 (\pi - \theta_1) \nu_n(\rho - K_n), \rho) \times \Omega_n^{(1)}(\beta, \rho)$, $\Delta_2 = \omega_n^{(1)}(\theta_1 - 1/2 \nu_n(\rho - K_n), \rho) \times \Omega_n^{(1)}(\beta, \rho)$. So for $K_n < r < K_n + \varepsilon_n$, we get that

$$\left| \frac{\partial G_n(z)}{\partial \theta_1} \right| < \frac{2}{\pi} \frac{1}{2 M_n} \int_{K_n} \frac{1}{\sqrt{\rho - K_n}} d\rho \quad \text{(by (A.15))}$$

$$= \frac{\varepsilon}{\pi} \frac{1}{M_n \sqrt{r - K_n}} \quad \text{as } r + K_n.$$
where \( \bar{M_n} = \max(\theta, \rho) \in \Omega \times [K_n, K_{n+1}] \). Thus

\[
\lim_{z \to \xi} \frac{\partial \hat{H}(z)}{\partial \theta_i} = \lim_{z \to \xi} \frac{\partial \hat{G}(z)}{\partial \theta_i} + \frac{\partial \hat{H}(\bar{z}(\theta))}{\partial \theta_i} = \frac{\partial \hat{H}(\xi)}{\partial \theta_i}.
\]

Similarly we have that

\[
\lim_{z \to \xi} \frac{\partial \hat{H}(z)}{\partial \theta_i} = \frac{\partial \hat{H}(\xi)}{\partial \theta_i} \quad \text{for } i = 2, \ldots, 2N-1.
\]

This completes the proof.

\[\square\]

**Lemma A.21.** For \( n \in \mathbb{N} \) and \( z \in \mathbb{R}^{2N} \)

(A.22) \[ \hat{H}_n(z) < \hat{H}_{n+1}(z) < H(z) \]

**Proof.** \( 1^\circ \). We prove that \( \hat{H}_n(z) < H(z) \).

If \( |z| < K_n \), this is true since \( \hat{H}_n(z) = H(z) \).

If \( K_n \leq |z| \), write \( z = rz(\theta) \), then

\[
\hat{H}_n(z) \leq \int_{K_n}^{\infty} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta, \rho)} \min_{\gamma \in \Omega_n^{(1)}(\beta, \rho)} H'(\rho z(\gamma)) z(\gamma) d\beta d\rho \]

\[
+ H(K_n z(\theta)).
\]

By \( 1^\circ \) of Lemma A.16, \( \beta \in \Omega_n(\theta, \rho) \) implies that \( \theta \in \Omega_n^{(1)}(\beta, \rho) \), so

\[
\hat{H}_n(z) \leq \int_{K_n}^{\infty} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta, \rho)} H'(\rho z(\theta)) z(\theta) d\beta d\rho \]

\[
+ H(K_n z(\theta))
\]

\[
= \int_{K_n}^{\infty} H'(\rho z(\theta)) z(\theta) d\rho + H(K_n z(\theta))
\]

\[
= H(z).
\]

\( 2^\circ \). We prove that \( \hat{H}_n(z) < \hat{H}_{n+1}(z) \).
If $|z| < K_{n+1}$, this is a consequence of $1^\circ$ since $\hat{H}_{n+1}(z) = H(z)$.

If $K_{n+1} < |z|$, write $z = rz(\theta)$, then by the definition of $\hat{H}_n$,

$$\hat{H}_n(z) = \int_{K_{n+1}}^z \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma \in \Omega_n(\beta,\rho)} F_n(\rho, z(\gamma)) d\beta dp$$

$$+ \hat{H}_n(K_{n+1}z(\theta))$$

$$< \int_{K_{n+1}}^z \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \max_{\gamma \in \Omega_n(\beta,\rho)} M_n(\rho z(\gamma)) \cdot z(\gamma) d\beta dp$$

$$+ \hat{H}_n(K_{n+1}z(\theta))$$

$$= \int_{K_{n+1}}^z \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \max_{\gamma \in \Omega_n(\beta,\rho)} M_n(\rho z(\gamma)) \cdot z(\gamma) d\beta dp$$

$$+ \hat{H}_n(K_{n+1}z(\theta))$$

here we used (A.9), (A.10), and $1^\circ$ of this lemma. Thus

$$\hat{H}_n(z) < \hat{H}_{n+1}(z) \text{ if } K_{n+1} < |z|$$

and this completes the proof.

\[\square\]

**Lemma A.23.** For every $n \in \mathbb{N}$

(A.24) \[0 < \mu_0 \hat{H}_n(z) < \hat{H}_n(z) \cdot z, \quad \forall |z| > r_0.\]

**Proof.** Write $z = rz(\theta)$, $r = |z|$, $z(\theta) = \frac{z}{|z|}$.

If $r_0 < |z| < K_n$, (A.24) holds by (H2) and $\hat{H}_n(z) = H(z)$.

If $K_{n+1} < |z|$, then by the definition of $\hat{H}_n$ and (A.17) $\hat{H}_n(z) \cdot z = r_n \mu z \lambda^\mu \lambda$. So
\[ H'_n(z) \cdot z = \mu \hat{H}_n(z) - \mu \int_{K_n} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta, \rho)} \min_{\gamma \in \Omega_n(1)(\beta, \rho)} F_n(\rho, \gamma) d\beta d\rho \\
- \mu \sigma H(\kappa z(\theta)) + \tau_n \mu \lambda r^\mu \]
\[ > \mu \hat{H}_n(z) - \mu \sigma \int_{K_n} M'(\rho)(z(\theta)) \cdot z(\theta) d\rho \\
- \mu \sigma H(\kappa z(\theta)) + \tau_n \mu \lambda r^\mu \]

(by 1° of Lemma A.16)
\[ = \mu \hat{H}_n(z) + \tau_n \mu \lambda r^\mu - \mu \sigma r^\mu + \mu \sigma M_n(\kappa z(\theta)) \\
- \mu \sigma H(\kappa z(\theta)) \]
\[ > \mu \hat{H}_n(z) , \]
since \( \lambda > \sigma \) and \( M_n(\kappa z(\theta)) = H(\kappa z(\theta)) \).

If \( K_n < |z| < K_{n+1} \), by 2° of Lemma A.16,
\[ H'_n(z) \cdot z = \frac{r}{V_n(\rho)} \int_{\Omega_n(\theta, \rho)} \min_{\gamma \in \Omega_n(1)(\beta, \rho)} F_n(\rho, \zeta(\gamma)) d\beta \\
> \min_{\gamma \in \Omega_n(2)(\theta, \rho)} r F_n(\rho, \zeta(\gamma)) \\
= r F_n(\rho, \zeta(\xi)) , \]
for some \( \xi \in \Omega_n^{(2)}(\theta, \rho) \), by the compactness of \( \Omega_n^{(2)}(\theta, \rho) \). So we get that by 1° of Lemma A.16
\[ H'_n(z) \cdot z > \mu \hat{H}_n(z) + r F_n(\rho, \zeta(\xi)) - \mu \sigma \int_{K_n} F_n(\rho, \zeta(\xi)) d\rho \\
(\text{A.25}) \\
- \mu \sigma H(\kappa z(\theta)) . \]

We consider two cases:

\textbf{Case 1.} \( F_n(\rho, \zeta(\xi)) = M'_n(\rho z(\xi)) \cdot z(\xi) \).
Then from (A.25) and 1° of Lemma A.16

\[ \hat{H}_n'(z) \cdot z > \mu_0 H_n'(z) + M_n'(rz(\xi)) \cdot rz(\xi) \]

\[ - \mu \sigma \int_{K_n} M_n'(pz(\theta)) \cdot z(\theta) d\theta - \mu_0 H_n(K_n z(\theta)) \]

\[ > \mu_0 H_n'(z) + \mu_0 M_n(rz(\xi)) - \mu_0 M_n(rz(\theta)) \]

(by (A.4) and (A.6))

\[ > \mu_0 H_n'(z) + \mu(\lambda - \sigma)M_n(rz(\xi)) - \mu_0 M_n(rz(\theta)) - M_n(rz(\theta)) \]

(by (A.7))

\[ > \mu_0 H_n'(z) \]

In the last inequality, we used (A.12), (A.13), (A.15), and that \( \xi \in \Omega_n(2)(\theta, \tau) \).

**Case 2.** \( F_n(r, rz(\xi)) = H_n'(rz(\xi)) \cdot rz(\xi) \).

Then from (A.25) and 1° of Lemma A.16,

\[ \hat{H}_n'(z) \cdot z > \mu_0 H_n'(z) + H_n'(rz(\xi)) \cdot rz(\xi) - \mu_0 \int_{K_n} H_n'(pz(\theta)) \cdot z(\theta) d\theta \]

\[ - \mu_0 K_n z(\theta) \]

\[ > \mu_0 H_n'(z) + \mu H(rz(\xi)) - \mu_0 H(rz(\theta)) \]

(by (H2))

\[ > \mu_0 H_n'(z) + \mu(1 - \sigma)\alpha_n K_n^\mu - \mu_0 H(rz(\xi)) - H(rz(\theta)) \]

(by (2.4))

\[ > \mu_0 H_n'(z) , \]

here we used (A.12), (A.13), (A.15), and that \( \xi \in \Omega_n(2)(\theta, \tau) \).

Thus \( \hat{H}_n'(z) \cdot z > \mu_0 H_n'(z) \) if \( K_n < |z| < K_{n+1} \).

Finally from (A.6) and (H2), \( \hat{H}_n'(z) > 0 \) for \( |z| > r_0 \), and this completes the proof of (A.24).

\[ \square \]

To get 6° of Proposition 2.5, we modify \( \hat{H}_n's \) again.

For \( n \in \mathbb{N}, z = rz(\theta), \) from (A.17) if \( r > K_{n+1} \), we get that
\[^{\hat{}}\text{H}_n(z) = G_n(K_{n+1}z) + \tau_n(z^{\mu\lambda} - x_{n+1}^{\mu\lambda}) + H(K_nz)\]

Set
\[C_n = \max_{z \in S} \left| G_n(K_{n+1}z) + H(K_nz) - \tau_n x_{n+1}^{\mu\lambda} \right| + 1.\]

Then we have that
\[^{\hat{}}\text{H}_n(z) < \tau_n|z|^{\mu\lambda} + C_n\]  
(A.26)
\[
< (\tau_n + 1)|z|^{\mu\lambda} + |z|^{\mu\lambda} + C_n + 1 \quad \text{(by (A.1))}
\]
and
\[^{\hat{}}\text{H}_{n+1}(z) > \tau_{n+1}|z|^{\mu\lambda} - C_{n+1}\]  
(A.27)
\[
> (\tau_n + 1)|z|^{\mu\lambda} + |z|^{\mu\lambda} + C_{n+1} \quad \text{(by (A.1))}
\]
\[
> (\tau_n + 1)|z|^{\mu\lambda} + |z|^{\mu\lambda} + C_{n+1} \quad \text{(by (A.1))}
\]

Let \(\hat{K}_n = \max\{K_{n+2}, C_n, C_{n+1}\}\) and define
\[^{\hat{}}\text{H}_n(z) = \chi(|z|, \hat{K}_n)H_n(z) + (1 - \chi(|z|, \hat{K}_n))(\tau_n + 1)|z|^{\mu\lambda} \quad \forall z \in \mathbb{R}^{2N}.
\]

Then we have that \(^{\hat{}}\text{H}_n \in C^1(\mathbb{R}^{2N}, \mathbb{R})\) possesses the following properties,
\[^{\hat{}}\text{H}_n(z) = \hat{H}_n(z) \quad \text{for} \quad |z| < \hat{K}_n,
\]
(A.28)
\[^{\hat{}}\text{H}_n(z) = (\tau_n + 1)|z|^{\mu\lambda} \quad \text{for} \quad |z| > \hat{K}_n + 1,
\]
(A.29)
\[0 < \mu \hat{H}_n(z) < \hat{H}_n(z) \quad \forall z \in \mathbb{R}^{2N};
\]
(A.30)
\[0 < \mu \hat{H}_n(z) < \hat{H}_n(z) \quad \forall z \in \mathbb{R}^{2N}.
\]

From (A.26) - (A.29) and the definition of \(^{\hat{}}\text{H}_n\) we also have that
\[^{\hat{}}\text{H}_n(z) < \hat{H}_n(z) < \hat{H}_{n+1}(z) \quad \forall z \in \mathbb{R}^{2N}.
\]

Now we can give the

Proof of Proposition 2.5. \(1^\circ - 3^\circ\) are true from the definitions of \(K_n, \hat{K}_n, H_n\) and Lemma A.18. Lemma A.21 and (A.31) yield \(4^\circ\). (A.30) gives \(5^\circ\). \(6^\circ\) is a consequence of (A.29) by letting \(\lambda_0 = \frac{\mu\lambda}{\mu\lambda + 1}\).

The proof of Proposition 2.5 is complete.
REFERENCES


