STRONG LAW FOR MIXING SEQUENCE

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ABSTRACT

In this note we present some theorems on the strong law for the mixing sequence which is not necessarily stationary, and the mixing coefficient involving only a pair of variables in the sequence.

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Key words and phrases: mixing coefficient, stationary sequence, strong law of large numbers.

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1. INTRODUCTION

In this article we present some results concerning the strong law of a mixing sequence \( \{X_n, n \geq 1\} \). We do not assume that \( \{X_n\} \) is stationary, and we use mixing coefficients involving only a pair of variables \( X, Y \) (in that order): The Rosenblatt mixing coefficient

\[
\alpha(X,Y) = \sup \{ |P(X \in A, Y \in B) - P(X \in A)P(Y \in B)| : A \in B', B \in B' \}
\]

and the Ibragimov mixing coefficient

\[
\beta(X,Y) = \sup \{ |P(Y \in B|X \in A) - P(Y \in B)| : A \in B', B \in B', P(X \in A) > 0 \}
\]

where \( B' \) is the \( \sigma \)-field of all Borel sets in \( \mathbb{R}^\prime \).

THEOREM 1. Suppose that \( \{X_n, n \geq 1\} \) is a sequence of random variables, and for some \( p > 1 \) the following conditions are satisfied:

1. \( \sup_{n} E|X_n|^p < \infty. \) \hspace{1cm} (1)

2. There exists \( \varepsilon > 0 \) such that as \( |i-j| \to \infty, \)

\[
\alpha(X_i,X_j) \leq \rho(|i-j|) = \begin{cases} 0(|i-j|^{-p/(2p-2) \varepsilon}), & 1 < p < 2, \\ 0(|i-j|^{-2/p \varepsilon}), & p \geq 2. \end{cases} \hspace{1cm} (2)
\]

Then

\[
\lim_{n \to \infty} (S_n - ES_n)/n = 0, \text{ a.s.} \hspace{1cm} (3)
\]

Here and in the sequel \( S_n = \sum_{i=1}^{n} X_i \).

THEOREM 2. Suppose that \( \{X_n, n \geq 1\} \) is a sequence of random variables, and one of the following conditions are satisfied:

(1) \( \sum_{n=1}^{\infty} \text{var}(X_n)/n^2 < \infty, \sup_{n} E|X_n| < \infty, \) and

\[
\beta(X_i,X_j) \leq \nu(|i-j|), \hspace{1cm} \sum_{n=0}^{\infty} \frac{1}{n^{1/2}} < \infty; \hspace{1cm} (4)
\]
(II) \( \sup_n \text{var}(X_n) < \infty \) and there exists \( \varepsilon > 0 \) such that

\[
\sum_{i=1}^{n} \mu_{1/2}(i) = O(n/(\log n)^{1+\varepsilon});
\]  

(III) (4) holds, \( X_1, X_2, \ldots \) are identically distributed and \( \mathbb{E}|X_1| < \infty \) (the existence of variance is not assumed). Then (3) is true.

Remarks:

1. Part (I) of Theorem 2 can be compared with a result of Blum et al [1], who assumes that \( \{X_n\} \) is a *-mixing sequence instead of (4). Note that this assumption does not follow from (4). We can easily construct a pairwise independent sequence which is not *-mixing.

2. Parts (I) and (II) of Theorem 2 can also be compared with some results (see Theorem 3.7.2 and Theorem 3.7.4 of Stout [5]) derived from Serfling [4]. The conditions of these results involve correlation coefficients between two variables in the sequence.

3. Part (III) of Theorem 2 extends Theorem 1 of Etemadi [2]. The assumption that \( \{X_i\} \) is identically distributed can be somewhat relaxed, for example, it can be replaced by the condition that there exists a random variable \( Y \) such that \( \mathbb{P}(|X_n| \geq x) < \mathbb{P}(|Y| \geq x) \) for all \( n \geq 1 \) and \( x > 0 \). We also mention a related result of Blum et al [1] Theorem 1. They assume that \( \{X_n\} \) is identically distributed, the distribution of \( X_1 \) has a moment generating function in the neighborhood of zero and that \( \{X_n\} \) is *-mixing. Under these more stronger conditions they prove that \( \mathbb{P}(|S_n - \mathbb{E}X_n|/n \geq \varepsilon) \) tends to zero exponentially.
2. PROOF OF THE THEOREMS

In deducing our results we shall borrow a trick from Etemadi [2].

The following well-known facts concerning \( a(X,Y) \) and \( \beta(X,Y) \) will be used:

\[
|\text{cov}(X,Y)| \leq 10(\alpha(X,Y))^{5/(2+\delta)}(E|X|^{2+\delta}E|Y|^{2+\delta})^{1/(2+\delta)}, \quad \delta > 0 \quad (6)
\]

\[
|\text{cov}(X,Y)| \leq 2(\beta(X,Y)\text{var}(X)\text{var}(Y))^{1/2}. \quad (7)
\]

For a proof, see Ibragimov and Linnik [3]. Also it is trivially true that

\[
\alpha\left(X|\mathcal{C}(X), Y|\mathcal{D}(Y)\right) \leq \alpha(X,Y), \quad \beta\left(X|\mathcal{C}(X), Y|\mathcal{D}(Y)\right) \leq \beta(X,Y) \quad (8)
\]

\[
\alpha(X-a, Y-b) = \alpha(X,Y), \quad \beta(X-a, Y-b) = \beta(X,Y), \quad (9)
\]

where \( \mathcal{C} \) and \( \mathcal{D} \) are Borel sets in \( \mathbb{R}' \) and \( a, b \) are constants.

Proof of Theorem 1. In view of (9), by defining \( X^+_n = X_n I(X_n > 0) \), \( X^-_n = -X_n I(X_n < 0) \), \( n \geq 1 \), we can assume without loss of generality that \( X_n > 0 \), \( n \geq 1 \). Define

\[
Y_n = (X_n - EX_n)I(|X_n - EX_n| < n^{1/p+\epsilon_1}), \quad n \geq 1,
\]

\[
S^*_n = \sum_{i=1}^n (Y_i - EY_i),
\]

where \( \epsilon_1 > 0 \) is a constant to be chosen later.

From condition (1) we have \( \sum_{n=1}^\infty P(X_n - EX_n \neq Y_n) < \infty \) and \( \lim_{n \to \infty} EY_n = 0 \).

Therefore, (3) is equivalent to

\[
\lim_{n \to \infty} S^*_n/n = 0, \quad \text{a.s.} \quad (11)
\]

Now fix \( \alpha > 1 \) and let \( k_n = [\alpha^n] \). For positive integer \( m \) sufficiently large, there exists \( n \) such that \( k_n < m < k_{n+1} \), and \( n \to \infty \) as \( m \to \infty \). From (1) we have
\[
\sup_n E|Y_n| \equiv C < \infty. \quad (12)
\]

Here and in the sequel C is an unimportant constant which is allowed to change. Since \( Y_n \geq 0 \), it follows that

\[
S_m^* - S_{k_n}^* \geq -(m - k_n)C, \quad \text{when } S_m^* < S_{k_n}^*,
\]

\[
S_m^* - S_{k_n}^* \leq S_{k_{n+1}}^* - S_{k_n}^* + (k_{n+1} - m)C, \quad \text{when } S_m^* \geq S_{k_n}^*.
\]

Hence

\[
|S_m^*/m - S_{k_n}^*/k_n| \leq \left| \frac{k_{n+1}}{k_n} S_{k_{n+1}}^*/k_{n+1} - \frac{k_n}{k_{n+1}} S_{k_n}^*/k_n \right| + \frac{k_{n+1} - k_n}{k_n} C. \quad (13)
\]

From (13) it follows that if we have shown that

\[
\lim_{n \to \infty} S_{k_n}^*/k_n = 0, \quad \text{a.s.} \quad (14)
\]

Then we would have

\[
\lim \sup_{m \to \infty} |S_m^*/m| \leq (\alpha - 1)C, \quad \text{a.s.}
\]

For any \( \alpha > 1 \), hence (11).

By Borel-Cantelli lemma, in order to prove (14), we have only to show that

\[
\sum_{n=1}^{\infty} \text{var}(S_{k_n}^*)/k_n^2 < \infty. \quad (15)
\]

By (6), (8) and (9), we have for any \( \delta > 0 \):

\[
\text{Var}(S_{k_n}^*) = \sum_{k,j=1}^{k_n} \text{cov}(Y_i, Y_j) \leq C \sum_{i,j=1}^{k_n} \left( \alpha(X_i, X_j) \right)^{\delta/(2+\delta)} (E|Y_i|^{2+\delta} E|Y_j|^{2+\delta})^{1/(2+\delta)}. \quad (16)
\]
From (1) it follows that

\[ E|Y_i|^{2+\delta} \leq Cn^{(2+\delta-p)(1/p+\varepsilon_1)} \quad n = 1, 2, \ldots \quad (17) \]

First consider the case \( p > 2 \). From (2), (16) and (17) we obtain

\[
\begin{align*}
\text{var}(S_{k_n}^*) & \leq C \sum_{i,j=1}^{kn} (\alpha(X_i, X_j))^{\delta/(2+\delta)} (ij) \\
& \leq C \sum_{i,j=1}^{kn} (\alpha(X_i, X_j))^{\delta/(2+\delta)} 2^{(2+\delta-p)(1/p+\varepsilon_1)/(2+\delta)} \\
& \leq C \sum_{i,j=1}^{kn} i^{-(2/p+\varepsilon)\delta/(2+\delta)} k_n 2^{(2+\delta-p)(1/p+\varepsilon_1)/(2+\delta)} \\
& \leq C \sum_{i=1}^{kn} i^{-(2/p+\varepsilon)\delta/(2+\delta)} \sum_{i=1}^{kn} 2^{(2+\delta-p)(1/p+\varepsilon_1)/(2+\delta)}.
\end{align*}
\]

(18)

Noticing \( 2/p < 1 \), we can assume that \( 2/p + \varepsilon < 1 \). Hence from (18) we have

\[
\text{var}(S_{k_n}^*) \leq C k_n^{2-\eta}.
\]

(19)

This inequality holds for any \( \delta > 0 \). Now we choose \( \varepsilon_1 \in (0, \varepsilon/2) \), then

\[
\lim_{\delta \to \infty} \{ -(2/p+\varepsilon)\delta/(2+\delta) + 2^{(2+\delta-p)(1/p+\varepsilon_1)/(2+\delta)} \} = -\varepsilon + 2\varepsilon_1 = 0.
\]

Therefore, choosing \( \delta \) sufficiently large, from (19) we obtain

\[
\text{var}(S_{k_n}^*) \leq C k_n^{2-\eta}. \quad \text{Hence (15) is true in view of } \sum_{n=1}^{\infty} k_n^{-\eta} < \infty.
\]

Next assume that \( p = 2 \). Again, choose \( \varepsilon_1 \in (0, \varepsilon/2) \). Choose \( \delta > 0 \) sufficiently small, such that \( (1+\varepsilon)\delta/(2+\delta) < 1 \). We still have (19), with \( p = 2 \). Since

\[
-(1+\varepsilon)\delta/(2+\delta) + 2\delta(1/2+\varepsilon_1)/(2+\delta) = -(\varepsilon - 2\varepsilon_1)\delta/(2+\delta) < 0,
\]

(15) holds again.

Finally, consider the case \( 1 < p < 2 \). In this case we have, instead of (18),
Write $\delta_0 = 2(p/(2p-2) - 1 + \epsilon)^{-1}$. Since $1 < p < 2$, we have $\delta_0 > 0$. Choose $\epsilon_1 > 0$ sufficiently small, such that

$$0 < \delta < \delta_0 \Rightarrow 2(2+\delta-p)(1/p+\epsilon_1)/(2+\delta) \leq 1 - n$$

where $n > 0$ does not depend on $\delta$, as long as $0 < \delta < \delta_0$. Because $(p/(2p-2)+\epsilon)\delta/(2+\delta) < 1$ for $0 < \delta < \delta_0$ and $(p/(2p-2)+\epsilon)\delta_0/(2+\delta_0) = 1$, one can find $\delta \in (0, \delta_0)$, such that

$$1 - n/2 < (p/(2p-2)+\epsilon)\delta/(2+\delta) < 1.$$

For this $\delta$ we have, by (20),

$$\text{var}(S^*_{kn}) \leq Ck^{-n/2} + 1 + (1-n) + 1 \leq Ck^{-n/2}.$$

So we obtain (15) again. Theorem 1 is proved.

**Proof of Theorem 2.** Part (I): Again we can assume $X_n \geq 0$. Write $Y_n = X_n - EX_n$ and $S^*_n = \sum_{i=1}^{n} Y_i$. From $\sup E|X_n| < \infty$ we have $\sup E|Y_n| < \infty$. Using the same argument employed in proving Theorem 1, we reduce the proof of (11) to that of (15). From (4), (7) and (9),

$$\sum_{n=1}^{\infty} \text{var}(S^*_{kn})/k_n^2 = \sum_{n=1}^{\infty} k_n^{2-1} \sum_{i,j=1}^{k_n} \text{cov}(Y_i, Y_j)$$

$$\leq C \sum_{n=1}^{\infty} k_n^{2-1} \sum_{i,j=1}^{k_n} (\mu|i-j|) \text{var}(X_i) \text{var}(X_j))^{1/2}$$

$$\leq C \sum_{n=1}^{\infty} k_n^{2-1} \mu^{1/2} (\sum_{i=0}^{k_n} \text{var}(X_i))$$

$$\leq C \sum_{n=1}^{\infty} k_n^{2-1} \sum_{i=1}^{k_n} \text{var}(X_i)$$

$$\leq C \sum_{n=1}^{\infty} \text{var}(X_n)/n^2 < \infty.$$

(21)

(22)
Part (II) is proved in much the same way as Part (I), only that we replace \( C_k n \) for \( \sum_{i=1}^{k} \text{var}(X_i) \) and \( C_k n / (\log n)^{1+\epsilon} \) for \( \sum_{i=1}^{k} 1/2(i) \) in (21) to obtain (22). Part (III) is proved by truncating \( X_n \) at \( n \) and combining the reasoning above and that of Etemadi [2].

3. AN EXAMPLE

Consider the autoregression model
\[
X_n = a_1 X_{n-1} + \ldots + a_m X_{n-m} + e_n, \quad n = 0, \pm 1, \pm 2, \ldots.
\] (23)

We want to show that under certain conditions it is true that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0, \quad \text{a.s.}
\] (24)
for any solution of (23). Suppose that the following conditions are satisfied:

1. \( \{e_n, n = 0, \pm 1, \ldots\} \) is a sequence of independent real random variables, and \( E e_n = 0, \quad n = 0, \pm 1, \ldots \)

\[
\sup_{-\infty < n < \infty} E|e_n|^p = C < \infty \quad \text{for some } p > 1.
\] (25)

where, as before, \( C \) is an unimportant constant which is allowed to change.

2. \( e_n \) has a density \( f_n \) satisfying the Lipschitz condition over \( \mathbb{R}^t \):
\[
|f_n(x) - f_n(y)| \leq C|x - y|, \quad n = 0, \pm 1, \pm 2, \ldots
\] (26)

where \( C \) does not depend on \( n \).

3. \( a_1, a_2, \ldots, a_m \) are real constants, and the equation \( 1 - a_1 z - \ldots - a_m z^m = 0 \) has all its root outside the unit circle.

Under the condition 1 and 3, the general real solution of (23) has
the form

$$X_n = \sum_{t=0}^{\infty} b_t e_{n-t} + \sum_{j=0}^{J} \sum_{n^j}^{m^j-1} n^j_{s} (\epsilon_j \cos \omega_j + \eta_j \sin \omega_j) \equiv X_n^*$$

(27)

where \( b_0 = 1, b_2, b_3, \ldots \) are real constants such that

$$|b_t| \leq C H^t, \quad t = 0, 1, 2, \ldots \quad \text{for some } H \in (0, 1).$$

(28)

\( \rho_j \) and \( \omega_j, j = 1, \ldots, J, \) are real constants, \( 0 < \rho_j < 1, j = 1, \ldots, J, \)

\( m_1 + \ldots + m_j = m, \) and \( \epsilon_j, \eta_j, \xi_j, \kappa_j = 1, \ldots, m_j, j = 1, \ldots, J, \) are arbitrary random variables. From (25), (27) and (28) it follows that

$$E X_n = 0, \quad n = 0, 1, 2, \ldots, \quad \sup_{-\infty < n < \infty} E|X_n|^P = C < \infty.$$ 

(29)

Let \( n, N \) be positive integers, \( n < N. \) Define

$$Y_{nN} = \sum_{t=0}^{N-n-1} b_t e_{N-t}, \quad Z_{nN} = \sum_{t=N-n}^{\infty} b_t e_{N-t}.$$ 

Since \( b_0 = 1, \) from (26) it follows that the density \( g_{nN} \) of \( Y_{nN} \) obeys Lipschitz's condition with the same constant \( C \) as in (26). Also

$$\sup(E|Y_{nN}|^P: 1 \leq n < N < \infty) = C < \infty.$$ 

(30)

Now let \( q_1 \) be a positive constant, \( q_2 = 2q_1. \) Define the event

$$D_{nN} = \{|Z_{nN}| \geq (N-n)^{-q_2}\}.$$ 

(31)

(25) entails \( \sup_{-\infty < n < \infty} E|e_n| = C < \infty. \) Hence

$$P(D_{nN}) \leq C(N-n)^{q_2} \sum_{t=N-n}^{\infty} H^t \leq C(N-n)^{q_2} H^{N-n}.$$ 

(32)

Let \( G \) be a Borel set in \( \mathbb{R}', h \) be a constant. \( G - h \) is defined as the
set \{ g - h : g \in H \}. Write $$G = G \cap \{ u : |u| \leq (N-n)^{q_1} \}, \ G^* = G \setminus G. \ \text{If} \ |h| < 1, \ \text{we have}
$$

$$|P(Y_{nN} \in G) - P(Y_{nN} \in G - h)| \leq |P(Y_{nN} \in G) - P(Y_{nN} \in G - h)| + P(Y_{nN} \in G^*) + P(Y_{nN} \in G^* - h)$$

$$\leq \int_{G} |g_n(u) - g_n(u - h)|\, du + P(|Y_{nN}| > (N-n)^{q_1}) + P(|Y_{nN}| > (N-n)^{q_1} - 1)$$

$$\leq C(N-n)^{q_1} h + C(N-n)^{-q_1} + C[(N-n)^{q_1} - 1]^{-1}$$

$$\leq C(N-n)^{q_1} h + C(N-n)^{-q_1}. \quad (33)$$

Now let A and B be two Borel sets in \( R' \). We proceed to estimate

$$|P(\tilde{X}_n \in A, \tilde{X}_n \in B) - P(\tilde{X}_n \in A)P(\tilde{X}_n \in B)|. \ \text{From} \ (32), \ (33) \ \text{and the independence of} \ e_1, e_2, ..., \ \text{we have}$$

$$|P(\tilde{X}_n \in B|e_n, e_{n-1}, ..., e_{n-2}) - P(Y_{nN} \in B)| = |P(Y_{nN} \in B - Z_{nN}|Z_{nN}) - P(Y_{nN} \in B)|$$

$$\leq C(N-n)^{-q_2} + C(N-n)^{-q_1}$$

$$\leq C(N-n)^{-q_1}, \quad (34)$$

when \( D_{nN} \) does not occur. But

$$|P(\tilde{X}_n \in B) - P(Y_{nN} \in B)| = |P(Y_{nN} \in B - Z_{nN}) - P(Y_{nN} \in B)|$$

$$= |P(D^C_{nN})P(Y_{nN} \in B - Z_{nN}) + P(D_{nN})P(Y_{nN} \in B - Z_{nN}|D_{nN}) - P(Y_{nN} \in B)|$$

$$\leq P(D_{nN}) + |P(Y_{nN} \in B - Z_{nN}|D_{nN}^C) - P(Y_{nN} \in B)| + P(D_{nN})$$

$$\leq 2P(D_{nN}) + C(N-n)^{-q_1} \leq C(N-n)^{q_2} h^{N-n} + C(N-n)^{-q_1}$$

$$\leq C(N-n)^{-q_1}. \quad (35)$$
From (34) and (35) we get
\[ |P(\tilde{X}_N \in B | e_n^e_{n-1}, \ldots) - P(\tilde{X}_N \in B) | \leq C(N-n)^{-q_1} \]
when \(D_{nn}\) does not occur. If \(P(\tilde{X}_n \in B) \geq C(N-n)^{-q_1}\), then from (33) and (35) we obtain
\[ P(\tilde{X}_n \in A, \tilde{X}_N \in B) \geq [P(\tilde{X}_n \in B) - C(N-n)^{-q_1}][P(\tilde{X}_n \in A) - C(N-n)^2 H N-n]. \tag{36} \]
Also
\[ P(\tilde{X}_n \in A, \tilde{X}_N \in B) \leq [P(\tilde{X}_n \in B) + C(N-n)^{-q_1}][P(\tilde{X}_n \in A) + C(N-n)^2 H N-n]. \tag{37} \]
From (36) and (37) we have
\[ |P(\tilde{X}_n \in A, \tilde{X}_N \in B) - P(\tilde{X}_n \in A)P(\tilde{X}_N \in B) | \leq C(N-n)^{-q_1} + C(N-n)^2 H N-n + C(N-n)^{q_1} H N-n \leq C(N-n)^{-q_1}, \tag{38} \]
where \(C\) does not depend on \(A, B\). (38) is proved when \(P(\tilde{X}_n \in B) \geq C(N-n)^{-q_1}\). If \(P(\tilde{X}_n \in B) < C(N-n)^{-q_1}\), (38) is trivially true. Therefore we get
\[ \alpha(\tilde{X}_n, \tilde{X}_N) \leq C(N-n)^{-q_1}. \tag{39} \]
Now choose \(q_1 = p/(2p-2) + 2\). From (39) we see that the condition (2) is satisfied. This, together with (29), gives, by Theorem 1,
\[ \lim_{n \to \infty} \sum_{i=1}^{n} \tilde{X}_i/n = 0, \ a.s. \tag{40} \]
From the expression of \(X_n^*\), it is readily seen that
\[ \lim_{n \to \infty} \sum_{i=1}^{n} X_i^*/n = 0, \ a.s. \tag{41} \]
From (27), (40) and (41), we obtain (24).
The conclusion (40) does not follow from the ergodic theorem of stationary process, since \( \{e_n\} \) is not assumed to be identically distributed, so \( \{X_n\} \) may not be a strictly stationary process.

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