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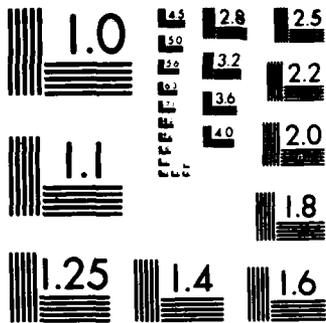
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THE CONSTRUCTION OF IMPLICIT AND EXPLICIT SOLITARY WAVE SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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THE CONSTRUCTION OF IMPLICIT AND EXPLICIT SOLITARY WAVE SOLUTIONS  
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ABSTRACT

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→ By means of easy examples, such as the Korteweg-de Vries, the Harry Dym, the sine-Gordon equations, and the Hirota coupled system, it is shown how nonlinear partial differential equations can be exactly solved by a direct algebraic method.



The physical concept, on which the method relies, is one of generation and mixing of the real exponential solutions of the underlying linear equations.

This approach leads in a straightforward way to single solitary waves of pulse, kink and cusp shape.

The extension of the method towards the construction of multi-soliton solutions and the connections with other direct methods are outlined.

AMS (MOS) Subject Classification: 35G20

Key Words: soliton theory, solitary waves, coupled KdV, evolution equations, direct methods, Harry Dym, sine-Gordon

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THE CONSTRUCTION OF IMPLICIT AND EXPLICIT SOLITARY WAVE SOLUTIONS  
OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Willy Hereman \*

1. INTRODUCTION

In 1978, a new direct method for generating exact solutions of nonlinear partial differential equations (PDEs) of evolution and wave type was independently established by Rosales<sup>(1)</sup> and Korpel<sup>(2)</sup>.

The first investigator focused on the mathematical applicability of a standard perturbation scheme to construct single and multi solitary wave solutions of many famous evolution and wave equations. Korpel looked at solitary wave formation from an engineering point of view. Applying physical concepts borrowed from e.g. nonlinear optics, he came to the conclusion that a pulse shaped solitary wave (e.g.  $\text{sech}^2$ -type) can be decomposed into an obviously convergent infinite series with real exponential terms. These exponentials are nothing else than the subsequent harmonics of the real exponential solution, characteristic of the linear dispersive medium.

Recently, Hereman et al<sup>(3,4)</sup> aimed at unifying the mathematical rigorous, but less transparent perturbation method and the heuristic physics/engineering approach toward soliton construction. The attentive reader will recognize the existing isomorphisms between Rosales' iterative scheme and our recursive system, between the solution techniques and the summation procedures applied to the resulting infinite series expansions.

In this paper we illustrate by means of rather easy examples how the physical interpretation deepens the understanding of solitary wave formation. Lack of space does not permit us to elaborate on the complementary mathematical details, which can be learned from earlier work<sup>(1,4,5)</sup> on the subject.

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Advocating a pedagogical approach, we first construct the well-known hyperbolic secant squared solution of the Korteweg-de Vries equation (KdV), working our way up to the derivation of the multi-soliton solution of the sine-Gordon equation (SG) as a finale. In between, we solve the Harry Dym (HD), the coupled Korteweg-de Vries (cKdV) and the sine-Gordon equation, for which the construction of single solitary waves is of intermediate level of difficulty.

Wherever appropriate we indicate connections with various other direct methods (Trace method<sup>(6)</sup>, Hirota's bilinear formalism<sup>(7)</sup> and its clones<sup>(8,9)</sup>, Fredholm determinant method<sup>(10,11)</sup>, direct linearization<sup>(12)</sup>, direct integration<sup>(13)</sup>, etc.). In this manner we outline the general framework in which this contribution fits.

## 2. THE KORTEWEG-DE VRIES EQUATION

The celebrated KdV<sup>(14-16)</sup>,

$$u_t + \alpha u u_x + u_{3x} = 0, \quad \alpha \in \mathbb{R}, \quad (1)$$

wherein subscripts indicate partial derivatives (e.g.  $u_{3x} = \partial^3 u / \partial x^3$ ), describes shallow water waves, ion-acoustic waves in plasmas, the dynamics of a nonlinear lattice, etc.

Using (1) as a paradigm, we sketch our direct algebraic method in ten steps, confining ourselves to a single solitary wave.

- (1) Searching for a stationary solution, we introduce a moving frame of reference by assuming  $u(x,t) = \phi(\xi)$ , where  $\xi = x - vt$ . The constant  $v$  refers to the anticipated travelling wave velocity. Doing so, PDE (1) can be replaced by the ODE

$$-v\phi_{\xi} + \alpha\phi\phi_{\xi} + \phi_{3\xi} = 0 . \quad (2)$$

(ii) For mathematical convenience, we next integrate (2), yielding

$$-v\phi + (\alpha/2)\phi^2 + \phi_{2\xi} + cC = 0 , \quad (3)$$

where  $cC$  is some integration constant.

(iii) We will allow travelling wave solutions to have a constant term  $c$ . Upon substitution of  $\phi = c + \hat{\phi}$  into (3), we get

$$(\alpha c - v)\hat{\phi} + (\alpha/2)\hat{\phi}^2 + \hat{\phi}_{2\xi} + c(-v + (\alpha c/2) + C) = 0 . \quad (4)$$

(iv) Adhering to the intuitively simple model<sup>(2)</sup> of solitary wave formation, we solve the linearized version of (4), (i.e. ignore the second term) for real exponential solutions in the form  $\hat{\phi}(\xi) = \exp(k(v)\xi)$ . Obviously the dispersion law,  $k^2(v) = v - \alpha c - 2C - v$ , follows provided  $c = (2/\alpha)(v - C)$ . Observe that both decaying and rising exponentials are possible. For the remainder we will work with the decaying solution, denoted by

$$g(\xi) = \exp(-k\xi), \quad k = \sqrt{2C - v} . \quad (5)$$

(v) Appropriate scaling by  $\hat{\phi} = (2/\alpha)(2C - v)\tilde{\phi} = (2k^2/\alpha)\tilde{\phi}$ , simplifies (4) into

$$-\bar{\phi} + \bar{\phi}^2 + (1/k^2)\bar{\phi}_{2\xi} = 0 . \quad (6)$$

(vi) According to Korpel's model<sup>(2)</sup>, the quadratic nonlinearity in (6) will square the linear solution  $g$ . Subsequently,  $g$  and  $g^2$  now being present, in the next step the nonlinearity generates  $g^3$  and  $g^4$ . Proceeding in this way, any integer power of  $g$  will be created. In the optics terminology,  $a_n g^n = a_n \exp(nkx - n\omega t)$ , with wave number  $k$  and angular frequency  $\omega = kv$ , represents the  $n^{\text{th}}$  harmonic wave with amplitude  $a_n$ . For  $n = 1$ , we refer to  $a_1 g$  as the fundamental (wave). This heuristic principle gave impetus to the search for an exact solution (to the nonlinear equation) in the form

$$\bar{\phi} = \sum_{n=1}^{\infty} a_n g^n(\xi) . \quad (7)$$

(vii) To determine the coefficients  $a_n$ , we substitute (7) into (6) and apply Cauchy's rule<sup>(3,4)</sup> to recollect equal powers in  $g$ . This results in an infinite hierarchy of equations:

$$(n^2 - 1)a_n + \sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} = 0, \quad n \geq 2, \quad (8)$$

starting with an arbitrary (positive) constant  $a_1$ . To the physicist, the recursion relation (8) for the amplitudes merely describes the mechanism of energy transfer into successive modes.

(viii) Solving the system (8) recursively, we recognize that

$$a_n = 6n(-1)^{n+1}a^n, \quad a = a_1/6 > 0, \quad n \in \mathbb{N}. \quad (9)$$

In an earlier paper<sup>(4)</sup> we discussed various techniques to solve systems similar to (8), how to recognize the general solution and how to verify it.

(ix) The last but one step aims at the summation of the resulting infinite series

$$\tilde{\phi} = 6 \sum_{n=1}^{\infty} (-1)^{n+1} n (ag)^n, \quad (10)$$

which converges only for sufficiently large  $\xi$ , into its closed form

$$\tilde{\phi} = 6ag/(1+ag)^2, \quad (11)$$

which is valid everywhere.

(x) Finally, writing (11) in the original variables  $x$  and  $t$ , gives

$$u(x,t) = (2/\alpha)(v-C) + (3/\alpha)(2C-v) \operatorname{sech}^2\left\{\frac{1}{2}\sqrt{2C-v}(x-vt) + \delta\right\}. \quad (12)$$

The single solitary wave solution (12) depends on three arbitrary, unrelated constants  $v, C$  and  $\delta = \frac{1}{2} \ln(1/a)$ . Observe that for the particular choices  $C - v > 0$  and  $C - v/4 < 0$ , we obtain the familiar pulse-type solution,

$$u(x,t) = (3v/\alpha)\operatorname{sech}^2\left\{\frac{1}{2}\sqrt{v}(x-vt) + \delta\right\} , \quad (13)$$

and well-type solution

$$u(x,t) = (3v/2\alpha)\operatorname{tanh}^2\left\{\frac{1}{2}\sqrt{-v/2}(x-vt) + \delta\right\} . \quad (14)$$

Of course, finding the solitary wave form (12) is neither original nor a great achievement; it has been obtained by half a dozen different methods<sup>(14-17)</sup>, but this method clearly reveals the physical mechanism behind solitary wave formation.

### 3. THE HARRY DYM EQUATION

The prototype of equation for so-called cusp (or spiky) solitary wave solutions is the Harry Dym equation<sup>(18-21)</sup>,

$$u_t + (1-u)^3 u_{3x} = 0 , \quad (15)$$

or any equivalent form<sup>(19)</sup>, which occurs in connection with the classical string problem.

The presence of a cubic nonlinearity in the coefficient of the dispersive term  $u_{3x}$  drastically changes the nature of the most elementary particular solution<sup>(20,21)</sup>, i.e.

$$u(x,t) = \text{sech}^2\left\{\frac{1}{2}\sqrt{v}[x-vt+\delta(x,t)]\right\} , \quad (16)$$

where the time and space dependent phase  $\delta$  is governed by the transcendental equation

$$\delta(x,t) = c + (2/\sqrt{v})\tanh\left\{\frac{1}{2}\sqrt{v}[x-vt+\delta(x,t)]\right\} , \quad (17)$$

where  $c$  is an arbitrary constant. This implicit solution was obtained through the inverse scattering technique (IST).

Straightforward application of our method to (15), which accidentally has the same linearized version as the KdV (1), does not lead to any solution. Obviously, we must relax our approach by introducing the new variable.

$$f(x,t) = k[x-vt+\delta(x,t)] , \quad (18)$$

and then search for solutions  $u(x,t) = F(f)$ , with  $k\delta(x,t) = G(f)$ , where both  $F$  and  $G$  remain to be determined.

Through the operator relations

$$\frac{\partial}{\partial t} = \frac{-kv}{1-G_f} \frac{d}{df} , \quad \frac{\partial}{\partial x} = \frac{k}{1-G_f} \frac{d}{df} \quad (19)$$

(15) is transformed into

$$\begin{aligned} -kvF_f + k^3(1-F)^3(1-G_f)^{-4} \{ (F_f G_{3f} + 3F_{2f} G_{2f})(1-G_f) \\ + F_{3f}(1-G_f)^2 + 3F_f G_{2f}^2 \} = 0 . \end{aligned} \quad (20)$$

Clearly,  $G_f - F$  is the right choice to simplify (20) to

$$(-v/k^2)(1-F)F_f + [4F_f F_{2f}(1-F) + F_{3f}(1-F)^2 + 3F_f^3] - 0. \quad (21)$$

After division throughout by  $(1-F)^4$ , followed by a first integration (introducing constant  $c_1$ ), a subsequent multiplication by  $F_f$  and another integration (constant  $c_2$ ), we arrive at

$$F_f^2 = (1-F)[(v/k^2 - c_2) + (c_2 - c_1)F + c_1 F^2]. \quad (22)$$

Seeking for a solitary wave, we add the boundary conditions  $F, F_f, F_{2f} \rightarrow 0$  as  $|f| \rightarrow \infty$ . Therefore, set  $c_1 = c_2 = v/k^2$ , and readily integrate (22), yielding

$$F(f) = \operatorname{sech}^2\{\sqrt{v}/2k(f+c_3)\}, \quad (23)$$

where  $c_3$  is the final integration constant. With the definition (18) of  $f$ , we thus obtain solution (16). Regarding the choice  $G_f - F$ ,  $G = k\delta$ ; after integration of (23) we retrieve (17), wherein  $c = (c_3 + c_4)/k$  and  $c_3$  is absorbed in  $\delta$ . Other relevant solutions of (22), as listed e.g. by Drazin<sup>(22)</sup> are presently under investigation<sup>(19)</sup>.

Although we decided in favor of direct integration of (21), application of our direct method would effortlessly have led to (16)-(17). Hence, in conclusion, a slight generalization of our method broadens the class of retrievable solutions to implicit ones.

#### 4. THE COUPLED KORTEWEG-DE VRIES EQUATIONS

To extract further information about the applicability of our technique let us investigate how it would fare on a famous coupled system<sup>(23-29)</sup>.

$$u_t - \alpha(6uu_x + u_{3x}) - 2\beta ww_x = 0, \quad (24)$$

$$w_t + 3uw_x + w_{3x} = 0, \quad \alpha, \beta \in \mathbb{R}. \quad (25)$$

These equations describe the interaction of two long waves with different dispersion laws. The coupled system is often referred<sup>(24,25)</sup> to as the coupled KdV equations, as for  $w = 0$  it reduces to the KdV in  $u$ . Sometimes (24)-(25) is quoted<sup>(26-28)</sup> as the Hirota-Satsuma system after the two investigators that first solved it using a quite ingenious bilinear formalism<sup>(7,24)</sup>.

In a forthcoming paper<sup>(29)</sup> we will prove that if  $u$  is of travelling wave form, say  $u(x,t) = \phi(\xi)$ , with  $\xi = x - vt$ , then  $w$  exhibits the same form, hence  $w(x,t) = \psi(\xi)$ . Thus we must solve

$$v\phi + 3\alpha\phi^2 + \alpha\phi_{2\xi} + \beta\psi^2 = 0, \quad (26)$$

$$-v\psi_{\xi} + 3\phi\psi_{\xi} + \psi_{3\xi} = 0, \quad (27)$$

ignoring integration constants here. Substitution of  $g(\xi) = \exp(-k\xi)$  into the linear parts of (26)-(27), leads to two dispersion laws,  $v = -\alpha k^2$  and  $v = k^2$ . According to our philosophy, the nonlinear

solutions  $\phi$  and  $\psi$  can only be built up from the same real exponential if  $\alpha = -1$ . A profound study<sup>(29)</sup> reveals that either one of the dispersion laws leads to the same exact solutions. Hence, let us proceed with  $k = \sqrt{v}$ ,  $v > 0$ .

Upon substitution of the scaled series representations,

$$\phi = \frac{v}{3} \sum_{n=1}^{\infty} a_n g^n(\xi), \quad \psi = \frac{v}{\sqrt{3|\beta|}} \sum_{n=1}^{\infty} b_n g^n(\xi), \quad (28)$$

into (26)-(27), we arrive at

$$(1+\alpha n^2)a_n + \sum_{\ell=1}^{n-1} (\alpha a_{\ell} a_{n-\ell} + e b_{\ell} b_{n-\ell}) = 0, \quad n \geq 2, \quad (29)$$

$$n(n^2-1)b_n + \sum_{\ell=1}^{n-1} \ell b_{\ell} a_{n-\ell} = 0, \quad n \geq 2, \quad (30)$$

together with  $(1+\alpha)a_1 = 0$ ,  $b_1$  arbitrary, and where  $e = +1$  ( $e = -1$ ) if  $\beta > 0$  ( $\beta < 0$ ).

Two interesting cases pop up:

(1) For  $\alpha \neq -1$ , so  $a_1 = 0$ , we straightforwardly obtain

$$a_{2n} = 24(-1)^{n+1} n A^n, \quad b_{2n} = 0, \quad n \in \mathbb{N} \setminus \{0\}, \quad (31)$$

$$a_{2n+1} = 0, \quad b_{2n+1} = (-1)^n b_1 A^n, \quad n \in \mathbb{N}, \quad (32)$$

requiring  $A = -eb_1^2/24(4\alpha+1) > 0$ , hence  $\beta$  and  $4\alpha + 1$  must have opposite signs. With (31)-(32) the solutions (28) then become

$$\phi = 8v \sum_{n=1}^{\infty} (-1)^{n+1} b_n (Ag^2)^n = \frac{8vAg^2}{(1+Ag^2)^2}, \quad (33)$$

$$\psi = \frac{v}{\sqrt{3|\beta|}} \sum_{n=0}^{\infty} (-1)^n b_n A^n g^{2n+1} = \frac{vb_1}{\sqrt{3|\beta|}} \frac{g}{(1+Ag^2)}, \quad (34)$$

Returning to the original variables, we obtain

$$u(x,t) = 2v \operatorname{sech}^2[\sqrt{v}(x-vt)+\delta], \quad (35)$$

$$w(x,t) = \pm v\sqrt{-2(4\alpha+1)/\beta} \operatorname{sech}[\sqrt{v}(x-vt)+\delta], \quad (36)$$

With  $\delta = \frac{1}{2} \ell n |24(4\alpha+1)/b_1^2|$ . Note that the cKdV indeed remains invariant for reversing the sign of  $w$ . For  $\alpha = -1/4$ , clearly  $b_n = 0, \forall n \in \mathbb{N}$ ; so  $w = 0$  and the cKdV reduces to the KdV (1) with  $\alpha = 6$ . Replacing  $t$  by  $t/4$ ,  $v$  by  $4v$ , from (13) we get (35). We should remark that (36) is a special case of a solution obtained by the dressing operator technique<sup>(27)</sup>. Furthermore, for  $\alpha = 1/2$  the cKdV are known to be completely integrable<sup>(25)</sup>, it has the Painlevé property<sup>(26)</sup> and a  $N$ -soliton solution<sup>(24)</sup>.

(11) For  $\alpha = -1$ , apparently  $a_1$  is arbitrary and so is  $b_1$ .

Recursively, one can calculate all  $a_n$  and  $b_n$  in the hope to obtain the general closed form, which we do not know yet.

However, for  $b_1^2 = a_1^2/2$  and  $\beta > 0$  (so  $e = 1$ ) we obtained  $a_n = 12n(-1)^{n+1}a_1^n$ ,  $b_n^2 = a_n^2/2$ , with  $a = a_1/12$ . Substitution into (28), leads to

$$u(x,t) = v \operatorname{sech}^2\left[\frac{1}{2}\sqrt{v}(x-vt)+\delta\right] \quad (37)$$

$$w(x,t) = (3/\sqrt{6|\beta|}) u(x,t) = (3v/\sqrt{6|\beta|}) \operatorname{sech}^2\left[\frac{1}{2}\sqrt{v}(x-vt)+\delta\right], \quad (38)$$

with  $\delta = \frac{1}{2} \ln(12/a_1)$ . Observe that for  $w = 3u\sqrt{6\beta}$  both equations (24)-(25) become identical to the KdV (1) with  $\alpha = 3$ , (13) then being the same as (37).

### 5. THE SINH-GORDON EQUATION

The sG equation, in light cone coordinates,

$$u_{xt} = \sin u, \quad (39)$$

describes the propagation of crystal dislocation, superconductivity in a Josephson junction, ultrashort optical pulse propagation in a resonant medium, etc. (15,16,22). For the mathematician, (39) is long known in the differential geometry of surfaces of constant negative curvature (15,16).

At the cost of dealing with one equation with a transcendental (1) nonlinearity, we rather transform the sG into a nonlinear coupled system with strictly polynomial terms:

$$\phi_{xt} - \phi - \phi\psi = 0, \quad (40)$$

$$2\psi + \psi^2 + \phi_t^2 = 0, \quad (41)$$

where  $\phi = u_x$ ,  $\psi = (\cos u) - 1$ .

To construct a single solitary wave, we proceed as in section 4, focusing on steady solutions  $\phi(\xi) = \phi(x,t)$ ,  $\psi(\xi) = \psi(x,t)$  with  $\xi = x-vt$ . Expanding the scaled functions as

$$\phi = (1/\sqrt{-v}) \sum_{n=1}^{\infty} a_n g^n(\xi), \quad \psi = \sum_{n=1}^{\infty} b_n g^n(\xi), \quad (42)$$

with  $g(\xi) = \exp(-k\xi)$ , we obtain

$$(n^2-1)a_n - \sum_{\ell=1}^{n-1} a_\ell b_{n-\ell} = 0, \quad n \geq 2, \quad (43)$$

$$2b_n + \sum_{\ell=1}^{n-1} (b_\ell b_{n-\ell} + \ell(n-\ell)a_\ell a_{n-\ell}) = 0, \quad n \geq 2, \quad (44)$$

where we used the dispersion law  $k^2 = -1/v$ ,  $v > 0$ , to simplify. Iterative calculation suggests

$$a_{2n} = 0, \quad b_{2n} = 8(-1)^n n a^{2n}, \quad n \in \mathbb{N} \setminus \{0\}, \quad (45)$$

$$a_{2n+1} = 4(-1)^n a^{2n+1}, \quad b_{2n+1} = 0, \quad n \in \mathbb{N}, \quad (46)$$

where  $a = a_1/4 > 0$ . Upon substitution into (42), we find

$$\phi = (4/\sqrt{-v}) \sum_{n=0}^{\infty} (-1)^n (ag)^{2n+1} = \frac{4\sqrt{-v} ag}{1+(ag)^2}, \quad (47)$$

$$\psi = -8 \sum_{n=1}^{\infty} (-1)^{n+1} n (ag)^{2n} = \frac{-8(ag)^2}{[1+(ag)^2]^2}, \quad (48)$$

In the variables  $x$  and  $t$ , we thus get

$$(\cos u(x,t)) - 1 = 1 - 2 \operatorname{sech}^2[(1/\sqrt{-v})(x-vt)+\delta], \quad (49)$$

$$\begin{aligned} u(x,t) &= \pm \frac{2}{\sqrt{-v}} \int \operatorname{sech}[(1/\sqrt{-v})(x-vt)+\delta] dx \\ &= \pm 4 \arctan\{\exp[(1/\sqrt{-v})(x-vt)+\delta]\}, \end{aligned} \quad (50)$$

with  $\delta = \ln(4/a_1)$ . This is the well-known kink-type solution of the sine-Gordon equation<sup>(11,15,16)</sup>.

## 6. N-SOLITON SOLUTIONS

The most effective techniques to construct  $N$ -soliton solutions are inverse scattering<sup>(15-18,22)</sup> and Hirota's method<sup>(7,15,16)</sup>, the latter being closely linked<sup>(1,26,30)</sup> to all other (often iterative)

procedures listed in the introduction. Neither of these methods is very transparent in explaining why N-solitons are the way they are. In the spirit of an earlier paper<sup>(4)</sup>, where we showed how the N-soliton solution to the KdV could be built up from N real exponentials, we construct the N-soliton solution of the sG (39), using (40) and (41). Emphasis is again on physical relevance, more than on mathematical rigor.

Motivated by the fact<sup>(4)</sup> that multi-soliton solutions evolve from the mixing of real exponential solutions of the underlying linear equation, we substitute

$$\phi^{(1)} = \sum_{i=1}^N c_i g_i(x,t) = \sum_{i=1}^N c_i a_i \exp(k_i x - \omega_i t) , \quad (51)$$

into the linear part of (40), implying  $\omega_i = -1/k_i$  ( $i = 1, 2, \dots, N$ ). The constants  $c_i(k_i)$  will be fixed later. Observe that the starting term in the expansion of  $\psi$ , say  $\psi^{(2)}$ , must be of the form

$$\psi^{(2)} = \sum_{i=1}^N \sum_{j=1}^N d_{ij} g_i g_j , \quad (52)$$

so that  $-2\psi^{(2)}$  balances the term

$$\phi_t^{(1)2} = \sum_{i=1}^N \sum_{j=1}^N c_i c_j \omega_i \omega_j g_i g_j \quad (53)$$

in (41). Thus, with the dispersion law  $\omega_1 = -1/k_1$ , we find

$$d_{1j} = -\frac{1}{2} c_1 c_j \omega_1 \omega_j = -\frac{1}{2} \frac{c_1 c_j}{k_1 k_j} = \frac{1}{2} c_1 c_j \frac{\omega_1 + \omega_j}{k_1 + k_j}. \quad (54)$$

Note that any term in the expansion of  $\phi$  ( $\psi$ , respectively) will only have an odd (even, respectively) number of  $g$ 's, which is in agreement with (47) and (48).

The analogue of the first term in (43), i.e.  $(n^2-1)a_n$ , will result from the action of the linear operator

$$L = \frac{\partial^2}{\partial x \partial t} - 1. \quad (55)$$

on the  $(2n+1)$ th term in the expansion of  $\phi$ , namely

$$\phi^{(2n+1)} = \underbrace{\sum_{i=1}^N \sum_{j=1}^N \cdots \sum_{s=1}^N}_{2n+1 \text{ summations}} c_{1j\dots s} g_1 g_j \cdots g_s, \quad n \in \mathbb{N} \setminus \{0\}. \quad (56)$$

The analogue to the second term in (43) will, in its most symmetric form, look like

$$\begin{aligned} \frac{1}{2} \sum_{\ell=0}^{n-1} \phi^{(2\ell+1)} \underbrace{(k_1, k_j, \dots, k_0)}_{2\ell+1 \text{ arguments}} \psi^{(2n-2\ell)} \underbrace{(k_p, k_q, \dots, k_s)}_{2(n-\ell) \text{ arguments}} \\ + \psi^{(2n-2\ell)} \underbrace{(k_1, k_j, \dots, k_\ell)}_{2(n-\ell) \text{ arguments}} \phi^{(2\ell+1)} \underbrace{(k_m, k_n, \dots, k_s)}_{2\ell+1 \text{ arguments}}. \end{aligned} \quad (57)$$

The analogue to (44) reads

$$\begin{aligned}
 \psi^{(2n)} &= \underbrace{\sum_{i=1}^N \sum_{j=1}^N \cdots \sum_{r=1}^N}_{2n \text{ summations}} d_{ij\dots r} g_1 g_j \cdots g_r \\
 &= -\frac{1}{2} \sum_{\ell=1}^{n-1} \underbrace{\psi^{(2\ell)}(k_1, k_j, \dots, k_n)}_{2\ell \text{ arguments}} \underbrace{\psi^{(2n-2\ell)}(k_0, k_p, \dots, k_r)}_{2(n-\ell) \text{ arguments}} \\
 &+ \sum_{\ell=0}^{n-1} \underbrace{\phi_t^{(2\ell+1)}(k_1, k_j, \dots, k_0)}_{2\ell+1 \text{ arguments}} \underbrace{\phi_t^{(2n-2\ell-1)}(k_p, k_q, \dots, k_s)}_{2(n-\ell)-1 \text{ arguments}}, \quad n \geq 1, \quad (58)
 \end{aligned}$$

allows to subsequently determine the coefficients  $d_{ij\dots r}$ .

To make this less obscure, let us give an example.  $\phi^{(1)}$  and  $\psi^{(2)}$  being computed, we equate

$$\begin{aligned}
 L\phi^{(3)} &= \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N [-(\omega_i + \omega_j + \omega_k)(k_i + k_j + k_k) - 1] c_{ijk} g_i g_j g_k \\
 &= \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N (k_i + k_j)(k_j + k_k)(k_k + k_i) (c_{ijk} / k_i k_j k_k) g_i g_j g_k \quad (59)
 \end{aligned}$$

to

$$\frac{1}{2}[\phi^{(1)}(k_1)\psi^{(2)}(k_j, k_k) + \psi^{(2)}(k_1, k_j)\phi^{(1)}(k_k)]$$

$$= -\frac{1}{4} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N (k_1+k_j)(k_j+k_k)(k_k+k_1)(c_1 c_j c_k / k_1 k_j k_k) g_1 g_j g_k. \quad (60)$$

therefore  $c_{ijk} = -1/[4(k_1+k_j)(k_j+k_k)]$  if we set  $c_1 = 1$ . After some lengthy calculations, partly carried out with MACSYMA (i.e. a large scale computer program that performs algebraic manipulations), we obtain

$$c_{1j\dots s} = (-1)^{n-1} 4^{-n} [(k_1+k_j)(k_j+k_k)\dots(k_r+k_s)]^{-1}, \quad n \in \mathbb{N},$$

$$d_{1j\dots r} = (-1)^{n-1} 2^{1-2n} (\omega_1+\omega_j+\dots+\omega_r) [(k_1+k_j)(k_j+k_k)\dots(k_q+k_r)]^{-1},$$

$$n \in \mathbb{N} \setminus \{0\}. \quad (61)$$

The final objective is then to write

$$\phi = \sum_{n=0}^{\infty} \phi^{(2n+1)} = \sum_{i=1}^N g_i + \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \left(-\frac{1}{4}\right) \frac{g_i g_j g_k}{(k_1+k_j)(k_j+k_k)} + \dots$$

$$+ \sum_{i=1}^N \sum_{j=1}^N \dots \sum_{s=1}^N (-1)^n \left(\frac{1}{4}\right)^n \frac{g_i g_j \dots g_s}{(k_1+k_j)(k_j+k_k)\dots(k_r+k_s)} + \dots, \quad (62)$$

and a similar expression for  $\psi$ , in their closed forms. Various authors<sup>(1,5,6,11,31)</sup> have shown that this can be done by introducing the  $N \times N$  matrices  $I$  (unity) and  $B$ , with elements

$$B_{1j} = \frac{1}{2} \frac{\sqrt{a_1 a_j}}{(k_1 + k_j)} \exp \frac{1}{2} [(k_1 + k_j)x - (\omega_1 + \omega_j)t]. \quad (63)$$

The N-soliton solution to the sine-Gordon equations (40)-(41) is then found to be

$$\phi(x,t) = 4[\text{Tr}(\arctan B)]_x, \quad (64)$$

$$\psi(x,t) = -2[\ln(\det(I+B^2))]_{xt}, \quad (65)$$

while, regarding  $\phi = u_x$ ,

$$u(x,t) = \pm 4 \text{Tr}(\arctan B) = \pm \frac{2}{i} \text{Tr} \left[ \ln \left( \frac{I+iB}{I-iB} \right) \right] \quad (66)$$

satisfies (39).

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