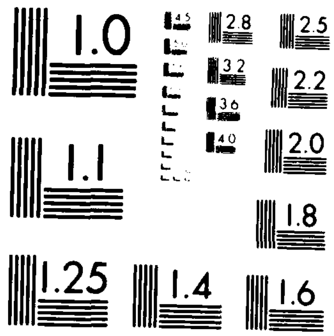
AD-A187 010 DETERMINISTIC EQUIVALENT FOR A CONTINUOUS LINEAR-CONVEX STOCHASTIC CONTRO..(U) FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS S SETHI ET AL. 01 SEP 87 1/1

UNCLASSIFIED AFOSR-TR-87-1400 AFOSR-87-0270 F/G 12/3 NL



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS [illegible] A

Deterministic Equivalent for a Continuous Linear-convex Stochastic Control Problem

by

S. Sethi*
Faculty of Management
University of Toronto
Toronto, Ontario
CANADA

and

M. I. Taksar**
Department of Statistics
Florida State University
Tallahassee, FL 32306

September, 1987

* This research is supported by NSERC.
** This research is supported by NSF Grant DMS 86-01510, and AFOSR Grant 87-0278.

87

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR TR 87-1480
6a. NAME OF PERFORMING ORGANIZATION Florida State University 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION AFOSR/NM
6c. ADDRESS (City, State, and ZIP Code) Department of Statistics Tallahassee, FL 32306-3033	7b. ADDRESS (City, State, and ZIP Code) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-87-0278
8c. ADDRESS (City, State, and ZIP Code) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO
61102F	2304		

11. TITLE (Include Security Classification)

Deterministic Equivalent for a Continuous Linear-Convex Stochastic Control Problem

12. PERSONAL AUTHOR(S)
S. Sethi and M. I. Taksar

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Technical	FROM ____ TO ____	9-1-87	17

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

We consider a finite horizon control model with additive input. There are two convex functions which discribe the running and the terminal costs within the system. The cost of input is proportional to input and can take both positive and negative values. It is shown that there exists a deterministic control problem whose optimal cost is the same as the one in the stochastic control problem. The optimal policy in the stochastic problem consists of keeping the process as close to the optimal deterministic trajectory as possible.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION
22a. NAME OF RESPONSIBLE INDIVIDUAL Maj. James M. Crowley	22b. TELEPHONE (Include Area Code) / 22c. OFFICE SYMBOL NM

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

Deterministic Equivalent for a Continuous Linear-convex Stochastic Control Problem

by

S. Sethi*

University of Toronto

and

M. I. Taksar**

Florida State University

ABSTRACT

We consider a finite horizon control model with additive input. There are two convex functions which describe the running and the terminal costs within the system. The cost of input is proportional to input and can take both positive and negative values. It is shown that there exists a deterministic control problem whose optimal cost is the same as the one in the stochastic control problem. The optimal policy in the stochastic problem consists of keeping the process as close to the optimal deterministic trajectory as possible.

* This research is supported by NSERC.

** This research is supported by NSF Grant DMS 86-01510, and AFOSR Grant 87-0278.

1. Introduction and Statement of Problem

We consider a stochastic linear system with additive "noise" and additive input which is under our control. The controlled process is described by a stochastic differential equation

$$\begin{aligned} dx(t) &= \alpha x(t)dt + \sigma dw(t) + d\nu(t), \\ x(0) &= x. \end{aligned} \tag{1.1}$$

Here $x(t) \in \mathbf{R}^1$ represents the coordinate of the system, $\sigma > 0$ and α are constants, $w(t)$ is a standard Wiener process on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $\nu(t)$ is $\mathcal{F}_t$-adapted process of bounded variation.

The running cost is described by a function $g(x,t)$ and the terminal cost by the function $G(x)$. A constant $c > 0$ represents a unit cost of input. The objective is to find

$$\min E\left\{\int_0^T g(x(t),t)dt + c\nu(T) + G(x(T))\right\} \tag{1.2}$$

where minimum is taken over all $\mathcal{F}_t$-adapted processes ν of finite expected variation.

Parallel to the above stochastic problem, we consider a deterministic control problem

$$\begin{aligned} dy(t) &= \alpha y(t)dt + dU(t) \\ y(0) &= x. \end{aligned} \tag{1.3}$$

with an objective to find

$$\min_U \left(\int_0^T g(y(t),t)dt + cU(T) + G(y(T)) \right). \tag{1.4}$$

It will be shown that there exists an optimal path $y^*(\cdot)$ such that, whatever is the initial state, the optimal policy consists of following this path exerting minimal control necessary for that.

In stochastic problem the optimal policy looks similar to the deterministic one. It is necessary to follow $y^*(\cdot)$ as close as possible. The optimal policy, however, in this case does not exist, because the control which forces a Brownian motion into a deterministic path is of unbounded variation.

We will also consider deterministic and stochastic problems with bounded control rates. In these problems U is subject to

$$U(t) = \int_0^t u(s)ds \quad \text{with} \quad |u(s)| \leq M. \tag{1.5}$$

It will be shown that when $M \to \infty$ the optimal cost in these problems converges to the optimal cost of the original problem. The optimal control is bang-bang that is u is equal to either $+M$ or $-M$.

It is interesting to contrast our results with the discrete-time analog of this problem treated in Bes and Sethi [1987]. While in both cases, it is possible to obtain equivalent deterministic problems there are certain important differences between them. In the descrete-time case, the optimal feedback control can be explicitly constructed from the optimal control of the equivalent deterministic problem and the optimal state trajectory arising from the feedback control is not deterministic in general. In the continuous-time case, on the other hand, the optimal state trajectory is deterministic in general and there exists no optimal policy yielding that trajectory.

The paper is structured as follows. In Section 2 we study the deterministic problems and find the equation for the optimal path $y^*(\cdot)$. We show that the optimal cost of the bounded control rate problem converges to the optimal cost (1.4). In Section 3 we prove that the optimal cost (1.2) is equal to that of (1.4) and we construct an ε-optimal policy by keeping the controlled process within a narrow strip around $y^*(\cdot)$ and reflecting it at the boundaries of the strip.

2. Deterministic model.

We start with a controlled process governed by the following equation

$$y(t) = x + \int_0^t \alpha y(s)ds + U(t), \quad 0 \le t \le T \tag{2.1}$$

Here α is a constant and $U(t), t \le T$ is a right continuous process of bounded variation. We denote the set of all such processes by $\mathcal{A}$.

Let $G(x)$ be a nonnegative continuously differentiable strictly convex function such that

$$G'_x(x) \to \infty \text{ as } |x| \to \infty. \tag{2.2}$$

Let $g(x,t)$ be a twice continuously differentiable function of two arguments such that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \le \frac{\partial^2 g(x,t)}{\partial x^2}, \tag{2.3}$$

$$\left| \frac{\partial^2 g(x,t)}{\partial x \partial t} \right| \le c_2 \tag{2.4}$$

With each $U \epsilon \mathcal{A}$ we associate a cost funtional

$$J_x(U) = \int_0^T g(y(t),t)dt + G(y(T)) + cU(T) \tag{2.5}$$

The objective is to find

$$v(x) = \min_{U \epsilon \mathcal{A}} J_x(U) \tag{2.6}$$

and U^* such that

$$v(x) = J_x(U^*) \tag{2.7}$$

Let $y(t)$ be any trajectory given by (2.1). Consider it as a continuous contour Y in a two dimensional plane $\mathbf{R}^2 = (y, t)$ (If $y(s) \neq y(s-)$ then we connect the points $(y(s-), s)$ and $(y(s), s)$ with a segment). Then, using (2.1) for representing $U(t)$,

$$J_x(U) = \int_Y (g(y,t) - c\alpha y)dt + c\int_Y dy + cx + G(y(T)). \tag{2.8}$$

Let U_1 and U_2 be two control functional which yield trajectories y_1 and y_2 such that $y_1(T) = y_2(T)$. Assume for a moment that $y_1(t) \geq y_2(t)$ for all $t \leq T$ and let S be a closed region formed by the contours Y_1 and Y_2. Then, by virtue of (2.8)

$$J_x(U_1) - J_x(U_2) = \oint_{\partial S} (g(y,t) - c\alpha y)dt + \oint_S c dy = \int\int_S \frac{\partial g(y,t)}{\partial x} - c\alpha dt dy. \tag{2.9}$$

(The last equality in (2.9) is due to Green's formula. Note that $\oint$ stands for the integral taken in the countercloclwise direction). Formula (2.9) suggests the equation for the optimal trajectory.

Denote $\hat{y}(s)$ to be as a function for which

$$\frac{\partial g}{\partial x}(\hat{y}(s), s) = c\alpha. \tag{2.10}$$

By virtue of (2.3) formula (2.10) uniquely determines $\hat{y}(s)$ for each s. In view of (2.4)

$$\left|\frac{d\hat{y}(s)}{ds}\right| = \left|\frac{\partial^2 g(y(s), s)}{\partial x \partial t} \Big/ \frac{\partial^2 g(\hat{y}(s), s)}{\partial x^2}\right| \leq c_2/c_1. \tag{2.11}$$

In the remainder of this section we will prove that $\hat{y}(s)$ determined by (2.10) represents the optimal trajectory.

(2.12) Theorem. Let $\hat{y}(0)$ be determined by (2.10) and let a be the (unique) solution of

$$G'(a) = c. \tag{2.13}$$

Then the optimal control U^* is given by the formula

$$U^*(t) = \hat{y}(t) - x - \int_0^t \alpha \hat{y}(s) ds + 1_{t=T}(a - \hat{y}(t)). \tag{2.14}$$

The optimal trajectory y^* is then

$$y^*(t) = \begin{cases} \hat{y}(t), & \text{if } t < T, \\ a, & \text{if } t = T. \end{cases} \tag{2.15}$$

Proof. First notice that the strict convexity of G and (2.2) implies existence and uniqueness of the solution of (2.13). Also, a simple calculation shows that (2.15) follows from (2.14).

Note that the policy U^* moves the controlled process instantaneously from x to $\hat{y}(0)$, then follows the trajectory $\hat{y}(\cdot)$, and at the moment T moves the process instantaneously to point a.

Consider the contour Y^* associated with the trajectory y^* (to be specific we assume $x \le \hat{y}(0)$ and $a \le \hat{y}(T)$)

$$\begin{aligned} Y^* = \{(y,t) : y = \hat{y}(t), 0 \le t \le T\} \cup \{(y,t) : t = 0, x \le y \le \hat{y}(0)\} \\ \cup \{(y,t); t = T, a \le y \le \hat{y}(T)\} \end{aligned}$$

The contour Y^* consists of the graph of the function $\hat{y}$ and two segments one connecting the initial point x and $\hat{y}(0)$, the second connecting a and $\hat{y}(T)$.

Let U be any other control and y be the corresponding trajectory. Suppose $y(T) \neq a$. Consider

$$U_1(t) = U(t) + 1_{t=T}(a - y(T))$$

Then

$$J_x(U) - J_x(U_1) = G(y(T)) - G(a) - c(y(T) - a) \tag{2.16}$$

In view of (2.13) and stric convexity of G, the right hand side of (2.16) is strictly positive. Therefore we may consider only those controls U and the corresponding trajectories y for which

$$y(T) = a. \tag{2.17}$$

Let Y be the contour associated with y. This contour consists of the graph of the function $y(\cdot)$ and the vertical segments counnecting the discontinuities of this graph (including the segment connection x with $y(0)$). Using (2.8), we can write

$$\begin{aligned} J_x(U) - J_x(U^*) &= \int_Y [(g(y,t) - c\alpha y)dt + cdy] \\ &\quad - \int_{Y^*} [(g(y,t) - c\alpha y)dt + cdy] \\ &= \int_Y (g(y,t) - c\alpha y)dt - \int_{Y^*} (g(y,t) - c\alpha y)dt \end{aligned} \tag{2.18}$$

The last equality in (2.18) is due to the fact that $\int_Y cdy = \int_{Y^*} cdy = c(a - x)$.

Assume that there exist $k \geq 1$ and $0 = t_0 < t_2 < \ldots < t_k = T$ such that $y(t_i -) \leq y^*(t_i) \leq y(t_i)$ and $y^*(s) - y(s)$ does not change sign on $(t_{i-1}, t_i), i = 1, 2, \ldots, k$. The latter means that contours Y and Y^* have intersection at the points $(y^*(t_i), t_i)$ and on any interval (t_{i-1}, t_i) the graph of the function $y(\cdot)$ does not intersect the graph of the function $y^*(\cdot)$ so it is located above (or below) the graph of $y^*(\cdot)$. (The case in which k is infinite is considered similarly.)

Let the set of integers I_1 (the set I_2) be the set of all i for which $y(s) \geq y^*(s)(y(s) \leq y^*(s))$ for $s \in (t_{i-1}, t_i)$. Let ∂S_i be a closed loop formed by $Y^* \cap (\mathbf{R} \times [t_{i-1}, t_i])$ and the part of $Y \cap (\mathbf{R} \times [t_{i-1}, t_i])$ which lies above (below) of $Y^* \cap (\mathbf{R} \times [t_{i-1}, t_i])$ if $i \in I_1$ (if $i \in I_2$). Note that if $y(t_{i-1}-) = y(t_{i-1})$ and $y(t_i-) = y(t_i)$ then $\partial S_i = (Y \cup Y^*) \cap (\mathbf{R} \times [t_{i-1}, t_i])$. Let S_i be the set enclosed by ∂S_i.

Using (2.18), we can write

$$\begin{aligned} J_x(U) - J_x(U^*) = &- \sum_{i \in I_1} \oint_{\partial S_i} (g(y,t) - c\alpha y) dt \\ &+ \sum_{i \in I_2} \oint_{\partial S_i} (g(y,t) - c\alpha y) dt \end{aligned} \tag{2.19}$$

Using Green's formula we transform (2.19) into

$$\sum_{i \in I_1} \int \int_{S_i} \frac{\partial g(y,t)}{\partial x} - c\alpha dy dt - \sum_{i \in I_2} \int \int_{S_i} \frac{\partial g(y,t)}{\partial x} - c\alpha dy dt. \tag{2.20}$$

In view of (2.3) $\frac{\partial g}{\partial x}$ is an increasing function of x, hence $\frac{\partial g(y,t)}{\partial x} \geq c\alpha$ for all $y > \hat{y}(t)$. Since $y \geq \hat{y}(t)$ for every $(y,t) \in S_i$ such that $i \in I_1$ and $0 < t < T$, we get nonnegativity of every integrand in the first sum in (2.20). Likewise every integrand in the second sum in (2.20) is nonpositive. The later implies

$$J_x(U) - J_x(U^*) \geq 0,$$

which proves the theorem.

(2.21) Corollary. The optimal cost $v(x)$ is given by the formula

$$v(x) = c(a - x) + G(a) + \int_0^T g(\hat{y}(t), t) dt - \int_0^T c\alpha \hat{y}(t) dt$$

Let $\mathcal{A}_M$ be the set of all $U \in \mathcal{A}$ subject to (1.5). Denote

$$v_M(x) = \sup_{U \epsilon \mathcal{A}_M} J_x(U)$$

It is obvious that $v_M(x)$ is an increasing function of M and $v_M(x) \leq v(x)$.

Let

$$\tau_1^M = \begin{cases} \min\{t : x + Mt = \hat{y}(t)\}, & \text{if } x \leq \hat{y}(0), \\ \min\{t : x - Mt = \hat{y}(t)\}, & \text{if } x > \hat{y}(0), \end{cases}$$

$$\tau_2^M = \begin{cases} \max\{t : a + (T-t)M = \hat{y}(t)\}, & \text{if } a \leq \hat{y}(T), \\ \max\{t : a - (T-t)M = \hat{y}(t)\}, & \text{if } a > \hat{y}(T). \end{cases}$$

Let N be such that for each $M > N$

$$\tau_1^M < \tau_2^M.$$

Let $N_1 = \alpha \max(|\hat{y}(t)|, 0 \leq t \leq T) + c_1/c_2$, where c_1, c_2 are given by (2.3), (2.4). For any $M \geq N_1 \vee N$ put

$$u_M(t) = \begin{cases} M \ \text{sign} \ (\hat{y}(0) - x), & \text{if } t \leq \tau_1^M, \\ \frac{d\hat{y}(t)}{dt} - \alpha\hat{y}(t), & \text{if } \tau_1^M < t < \tau_2^M, \\ M \ \text{sign} \ (a - \hat{y}(T), & \text{if } \tau_2^M \leq t \leq T. \end{cases}$$

By virtue of (2.11)

$$U_M(s) = \int_0^s u(s)ds \ \epsilon \ \mathcal{A}_M$$

It is easy to see that $\tau_1^M \to 0$ and $\tau_2^M \to T$ as $M \to \infty$. Hence

$$
\begin{aligned}
J_x(U_M) =& c(a-x) + G(a) + \int_{\tau_1^M}^{\tau_2^M} g(\hat{y}(t),t)dt \\
&+ \int_0^{\tau_1} g(x \pm Mt, t)dt + \int_{\tau_2}^{T} g(a \pm M(T-t), t)dt \\
&- \int_{\tau_1}^{\tau_2} c\alpha\hat{y}(t)dt \to v(x).
\end{aligned}
$$

The latter shows

$$v^M(x) \to v(x) \text{ as } M \to \infty.$$

3. Stochastic case.

Let V stand for the set of all $\mathcal{F}_t$-adapted processes ν with

$$E\{|\nu|(T)\} < \infty \tag{3.1}$$

where $|\nu|$ stand for the variation of the process $\nu(\cdot)$. For each $\nu \in V$ we define the process $x(\cdot)$ satisfying the following equation

$$x(t) = x + \int_0^t \alpha x(s)ds + \sigma w(t) + \nu(t), \tag{3.2}$$

where $\sigma > 0$, α is the same as in section 2 and $w(t)$ is a standard Wiener process adapted to $\mathcal{F}_t$. With each $\nu \in V$ is associated the following cost

$$J_x(\nu) = E\Big\{\int_0^T g(x(t),t)dt + G(x(T)) + c\nu(T)\Big\}. \tag{3.3}$$

Similarly, we define

$$F(x) = \inf_{\nu \in V} J_x(\nu). \tag{3.4}$$

Let V_M stand for all $\nu \in V$ such that

$$\nu(t) = \int_0^t \eta(s)ds, \quad |\eta(s)| \le M \ \text{ for all } \ 0 \le s \le T, \tag{3.5}$$

and

$$F_M(x) = \inf_{\nu \epsilon V_M} J_x(\nu). \tag{3.6}$$

(3.7) <u>Theorem.</u> For every x

$$F(x) \ge v(x) \tag{3.8}$$

$$F_M(x) \ge v_M(x) \tag{3.9}$$

<u>Proof</u>. For $\nu \epsilon V$ put

$$U_\nu(t) = E\{\nu(t)\} \tag{3.10}$$

By virtue of (3.1) the right hand side of (3.10) is finite. Also if $0 = t_0 < t_1 < \ldots < t_k = T$, then

$$\sum_{i=1}^{k} |U_\nu(t_i) - U_\nu(t_{i-1})| = \sum_{i=1}^{k} |E\{\nu(t_i) - \nu(t_{i-1})\}|$$

$$\le \sum_{i=1}^{k} E\{|\nu(t_i) - \nu(t_{i-1})|\} \le |E\Big\{\sum_{i=1}^{k} |\nu|(t_i) - |\nu|(t_{i-1})\Big\} = E\{|\nu|(T)\}.$$

This shows that $|U_\nu|(T)$ is finite. Let $x(t)$ be given by (3.2). Let $y_\nu(t)$ satisfies (2.7) with $U = U_\nu$. It is obvious that $y_\nu(t) = E\{x(t)\}$. By Jensen's inequality $E\{g(x(t),t)\} \ge g(y_\nu(t),t)$ and $E\{G(x(T)\} \ge G(y_\nu(T))$, therefore

$$\begin{aligned} J_x(\nu) &= \int_0^T E\{g(x(t),t)\}dt + E\{G(x(T)\} + E\{\sigma w(T)\} \\ &+ cE\{\nu(T)\} \\ &\ge \int_0^T g(y_\nu(t),t)dt + G(y_\nu(T)) + cU_\nu(T) \\ &= J_x(U_\nu) \end{aligned} \tag{3.11}$$

Inequality (3.11) implies (3.8). The proof of (3.9) is similar.

Let $\hat{y}(t)$ be the function defined by (2.10) and let a be given by (2.13). Without loss of generality we can assume $x \leq \hat{y}(0)$ and $a \leq y(T)$. Fix $\varepsilon > 0$. Let $y_1(t)$ and $y_2(t)$ be three times continuously differentiable functions such that

$$\hat{y}(t) - \varepsilon \leq y_1(t) \leq \hat{y}(t) \leq y_2(t) \leq \hat{y}(t) + \varepsilon, \text{ if } \varepsilon \leq t \leq T - \varepsilon, \tag{3.12}$$

$$y_1(0) = a - \varepsilon, \quad y_2(0) = a + \varepsilon, \tag{3.13}$$

$$y_{1,2}(t) = y_{1,2}(0) + t(y_{1,2}(\varepsilon) - y_{1,2}(0)) \quad \text{if }, \ 0 \leq t \leq \varepsilon, \tag{3.14}$$

$$y_1(T) = a - \varepsilon, \quad y_2(T) = a + \varepsilon, \tag{3.15}$$

$$y_{1,2}(t) = y_{1,2}(T - \varepsilon) + (T - t)(y_{1.2}(T) - y_{1,2}(T - \varepsilon)), \text{ if } T - \varepsilon \leq t \leq T. \tag{3.16}$$

The graphs of $y_1(t)$ and $y_2(t)$ form a "tube" of the width not exceeding 2ε. This tube encloses the initial point x, the endpoint a, and on the interval $(\varepsilon, T - \varepsilon)$ it contains the graph of $\hat{y}(t)$. Construction of such functions $y_1(\cdot)$ and $y_2(\cdot)$ is rather elementary and we omit it.

Let $k_\varepsilon(t) \epsilon V$ be a functional such that

$$x_\varepsilon(t) = x + \int_0^t \alpha x_\varepsilon(s) ds + \sigma w(t) + k_\varepsilon(t), 0 \leq t \leq T, \tag{3.17}$$

$$y_1(t) \leq x_\varepsilon(t) \leq y_2(t) \text{ for all } 0 \leq t \leq T, \tag{3.18}$$

$$k_\varepsilon(t) = \int_0^t 1_{y_1(s)}(x_\varepsilon(s)) d|k_\varepsilon|(s) - \int_0^t 1_{y_2(s)}(x_\varepsilon(s)) d|k_\varepsilon|(s) \tag{3.19}$$

The functional $k_\varepsilon(\cdot)$ is the so called solution of the Skorokhod problem for the Brownian motion with drift αx and diffusion σ. Its effect results in reflection of the Brownian motion from the time dependent boundaries $y_1(\cdot)$ and $y_2(\cdot)$. The existence of such a functional follows easily from Lions and Sznitman [1984].

(3.20) <u>Theorem.</u> As $\varepsilon \to 0$

$$J_x(k_\varepsilon) \to v(x). \tag{3.21}$$

Proof. Let $D = \max(|a|, |x|, \sup\{|\hat{y}(s)|, 0 \le s \le T\}) + 1$ and let

$$N = \max_{\substack{|y| \le D \\ 0 \le t \le T}} |g(y,t)|,$$

$$\delta = \max_{\substack{|x_1|,|x_2| \le D, |x_1 - x_2| \le \varepsilon \\ 0 \le t \le T}} |g(x_1,t) - g(x_2,t)|,$$

$$\delta_1 = \max_{|y-a| \le \varepsilon} |G(y) - G(a)|.$$

Then

$$J_x(k_\varepsilon) = E\Big\{\int_0^T g(x_\varepsilon(s),s)ds\Big\} + E\{G(x_\varepsilon(T))\} + cE\{k_\varepsilon(T)\} = I_1 + I_2 + I_3.$$

Consider

$$\begin{aligned} |I_1 - \int_0^T g(\hat{y}(s),s)ds| &\le |E\int_0^\varepsilon g(x_\varepsilon(s),s)ds| \\ &+ \Big|\int_0^\varepsilon g(\hat{y}(s),s)ds| + |E\int_{T-\varepsilon}^T g(x_\varepsilon(s),s)ds| + \int_{T-\varepsilon}^T g(\hat{y}(s),s)ds| \\ &+ E\{\int_\varepsilon^{T-\varepsilon} |g(x_\varepsilon(s),s)ds - g(\hat{y}(s),s)|ds\} \end{aligned} \tag{3.22}$$

In view of (3.12)-(3.16) and (3.18), $|x_\varepsilon(s)| \le D$ if $\varepsilon < 1$. Therefore, each of the four terms in the right hand side of (3.22) does not exceed $N\varepsilon$. Applying (3.18) to the integrand in the last term of (3.22), we see that it does not exceed δ. Therefore, (3.22) does not exceed $4N\varepsilon + T\delta$. Since $\delta \to 0$ as $\varepsilon \to 0$, the right hand side of (3.22) converges to 0.

By virtue of (3.15) and (3.18), $|x_\varepsilon(T) - a| < \varepsilon$. Thus,

$$|I_2 - G(a)| \le \delta_1 \to 0 \text{ as } \varepsilon \to \infty. \tag{3.23}$$

Formula (3.17) shows

$$E\{k_\varepsilon(T)\} = E\{x_\varepsilon(T)\} - x - E\Big\{\int_0^T \alpha x_\varepsilon(s)ds\Big\}. \tag{3.24}$$

Formula (3.15) and (3.18) show that the first term in the right hand side of (3.24) converges to a. Likewise, using (3.12)-(3.16) and (3.18), one can show that the last term in the right hand side of (3.24) converges to $\int_0^T \alpha\hat{y}(s)ds$. Therefore (3.24) converges to $U^*(T)$. This fact along with (3.23) and the convergence of (3.22) to zero proves (3.21).

(3.25) Corollary. F(x) = v(x).

The proof follows from Theorem (3.7) and Theorem (3.20).

Let $y_1(t)$ and $y_2(t)$ satisfy (3.12)-(3.16). Consider the process $x_{\varepsilon,M}(s)$ defined by the following stochastic differential equation

$$\begin{aligned} dx_{\varepsilon,M}(t) =& \alpha x_{\varepsilon,M}(t)dt + M1_{x_{\varepsilon,M}(t)<y_1(t)}dt \\ & -M1_{x_{\varepsilon,M}(t)>y_2(t)}dt + \sigma dw(t), \\ x_{\varepsilon,M}(0) =& x. \end{aligned}$$

Let

$$\eta_{\varepsilon,M}(s) = \begin{cases} M, & \text{if } x_{\varepsilon,M}(s) < y_1(s), \\ -M, & \text{if } x_{\varepsilon,M}(s) > y_2(s). \end{cases}$$

and $\nu_{\varepsilon,M}(t) = \int_0^t \eta_{\varepsilon,M}(s)ds$. It is obvious that $\nu_{\varepsilon,M} \in V_M$ and $x_{\varepsilon,M}$ is the solution of (3.2) with $\nu = \nu_{\varepsilon,M}$. Simple calculations show that

$$J_x(\nu_{\varepsilon,M}) \to J_x(k_\varepsilon) \text{ as } M \to \infty.$$

This implies

$$F_M(x) \to v(x) \text{ as } M \to \infty$$

(3.26) Remark. Although we have identified trajectory $y^*(\cdot)$ which is optimal for both deterministic and stochastic cases, there is no optimal policy in the latter case. Any functional which keeps Brownian motion "stuck" to a deterministic trajectory has a.s. infinite variation on any finite interval.

REFERENCES

Bertsekas, D. P. (1986). *Dynamic programming and stochastic control.* Academic Press, New York.

Bes, C. and Sethi, S. (1987). Solution of a Stochastic linear-convex control problems using deterministic equivalents. To appear in *JOTA.*

Fleming, W. M. and Rishel, R. W. (1975). *Deterministic and stochstic optimal control.* Springer, New York.

Harrison, J. M. and Taksar, M. I. (1983). Instantaneous control of Brownian motion. *Math of Oper. Res..* **8**, 439-453.

Lions, P. L. and Sznitman, A. S. (1984). Stochastic differential equation with reflection boundary conditions. *Comm. Pure and Appl. Math.* **37**, 511-537.

Mendaldi, J. L. and Taksar, M. I. (1987). Singular control of multidimensional Brownian motion. To appear in *Proceedings of the Xth IFAC Congress, Munich, Germany 1987.*

Sethi, S, and Thompson, G. L. (1981). *Optimal control theory: Application to management science.* Martinus Nijhoff Publishing Co.. Boston.

Taksar, M.I. (1987). Singular control in a multidimensional space with control costs proportional to displacement, *Proceedings of the International Conference on Optimization, Singapore, April 1987*, 314-323.

END
FILMED
FEB. 1988
DTIC