Decomposing Properties into Safety and Liveness using Predicate Logic

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Decomposing Properties into Safety and Liveness
using Predicate Logic*

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ABSTRACT

A new proof is given that every property can be expressed as a conjunction of safety and liveness properties. The proof is in terms of first-order predicate logic.

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1. Introduction

Two classes of properties are of particular interest when considering programs: safety properties and liveness properties. Informally, a safety property stipulates that "bad things" do not happen during execution of a program and a liveness property stipulates that "good things" do happen (eventually) [2]. Distinguishing between safety and liveness properties is useful because knowing whether a property is safety or liveness helps when deciding how to prove that the property holds for a program.

In [1], formal definitions of safety and liveness are given and it is proved that every property can be expressed as the conjunction of a safety property and a liveness property. The formal definitions of safety and liveness are given in terms of first-order predicate logic, but the proof that every property can be decomposed into safety and liveness is not—it uses topology. The purpose of this paper is to give a proof of this theorem using only first-order predicate logic.

2. Specifying Properties

A program state is a mapping from variables to values. An execution of a concurrent program can be viewed as an infinite sequence of program states
\[ \sigma = s_0 \ s_1 \ldots \]
which we call a history. In a history, \( s_0 \) is an initial state of the program and each subsequent state results from executing a single atomic action in the preceding state. (For a terminating execution, an infinite sequence is obtained by repeating the final state.) A property is a set of such sequences.

One way to specify a property is by using first-order predicate logic. For a state \( s \), define \( s.v \) to be the value of variable \( v \) in that state. A formula of first-order predicate logic where \( s \) is the only free variable defines a set of states. For example,
\[ (\forall i: 1 \leq i < N: s.a[i] \leq s.a[i+1]) \]
specifies the set of states in which the elements of array \( a[1:N] \) are sorted. Usually "\( s.\)" is implicit and therefore left out of such a formula, resulting in the more familiar use of first-order predicate logic as an assertion language.

A set of sequences of states—a property—can also be defined using first-order predicate logic. To facilitate such specifications, for any sequence \( \sigma = s_0 \ s_1 \ldots \) define for \( 0 \leq i \):
\[
\begin{align*}
\sigma[i] & = s_i, \\
\sigma[i:] & = s_0 \ s_1 \ldots s_{i-1}. \text{ The empty sequence if } i=0. \\
|\sigma| & = \text{ the length of } \sigma \text{ (}\omega \text{ if } \sigma \text{ is infinite).}
\end{align*}
\]
A formula of first-order predicate logic in which \( \sigma \) is the only free variable defines the set of sequences that satisfy the formula and therefore specifies a property. For example,
\[ (\forall i: 0 \leq i: \sigma[i] \ v = 0) \]
specifies the property in which the value of \( v \) remains 0 throughout execution.
We write $\alpha = P$ if $\alpha \in S^\omega$ is in the property specified by $P$. Thus,
\begin{align*}
\alpha = P &= P^\alpha, \\
\alpha \neq P &= \neg P^\alpha.
\end{align*}

3. Safety and Liveness

According to [1], a property $P$ is a safety property provided
\begin{equation}
\text{Safety: } (\forall \alpha: \sigma \in S^\omega: \sigma \# P \Rightarrow (\exists i: 0 \leq i: (\forall \beta: \beta \in S^\omega: \sigma[\ldots i \beta \# P])),
\end{equation}
where $S$ is the set of program states, $S^*$ the set of finite sequences of states, $S^\omega$ the set of infinite sequences of states, and juxtaposition is used to denote catenation of sequences. A property $P$ is a liveness property provided
\begin{equation}
\text{Liveness: } (\forall \alpha: \alpha \in S^*: (\exists \beta: \beta \in S^\omega: \alpha \beta \# P)).
\end{equation}

Given a property $P$, we are interested in defining properties Safe $(P)$ and Live $(P)$ such that
\begin{itemize}
    \item Safe $(P)$ is a safety property,
    \item Live $(P)$ is a liveness property, and
    \item $P = \text{Safe } (P) \land \text{Live } (P)$.
\end{itemize}

Observe that if
\begin{align*}
\text{Safe } (P) &= P \lor M_P \\
\text{Live } (P) &= P \lor \neg M_P
\end{align*}
then
\begin{align*}
\text{Safe } (P) \land \text{Live } (P) &= (P \lor M_P) \land (P \lor \neg M_P) \\
&= (P \land P) \lor (P \land \neg M_P) \lor (M_P \land P) \lor (M_P \land \neg M_P) \lor \neg M_P \\
&= P
\end{align*}
Hence, we have only to look for an $M_P$ that makes $P \lor M_P$ (i.e. Safe $(P)$) a safety property and $P \lor \neg M_P$ (i.e. Live $(P)$) a liveness property.

It turns out that using
\begin{equation}
M_P: (\forall i: 0 \leq i: (\exists \beta: \beta \in S^\omega: \sigma[\ldots i \beta \# P]))
\end{equation}
suffices. First, we show formally that Safe $(P)$ satisfies definition (3.1) of safety. The proof that follows is a sequence of first-order predicate logic formulas with explanations interspersed (and delimited by « and ») of how each formula is derived from its predecessor.

Choose any $\sigma \in S^\omega$:
\begin{align*}
\sigma \# \text{Safe } (P)
\end{align*}
«by definition of Safe (P)»
\[ \sigma#(P) = (\forall i: 0 \leq i: (\exists \beta \in S^\omega: \sigma[i]\beta \implies P)) \]
«by definition of #»
\[ = - (P \vee (\forall i: 0 \leq i: (\exists \beta \in S^\omega: \sigma[i]\beta \implies P))) \]
«by substitution»
\[ = - (P \vee (\forall i: 0 \leq i: (\exists \beta \in S^\omega: \sigma[i]\beta \implies P))) \]
«by De Morgan's Laws»
\[ = - P \land (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P)) \]
«A \land B \Rightarrow B »
\[ = (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P)) \]
«because \((\forall x :: A) = (\forall x :: A \land (\forall y :: A_y))\)»
\[ = (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (\forall \gamma \in S^\omega: \sigma[i]\gamma \implies P))) \]
«because true \land P = P and \((\sigma[i]\beta)(i) = \sigma[i] \)»
\[ = (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (i=i) \land (\forall \gamma \in S^\omega: (\sigma[i]\beta)(i) \gamma \implies P))) \]
«by substitution»
\[ = (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (k=i) \land (\forall \gamma \in S^\omega: (\sigma[i]\beta)(k) \gamma \implies P))) \]
«by \exists -Generalization»
\[ \Rightarrow (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (\exists k: k=i: (\forall \gamma \in S^\omega: (\sigma[i]\beta)(k) \gamma \implies P))) \]
«by De Morgan's Law»
\[ = (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (k=i) \land (\forall \gamma \in S^\omega: (\sigma[i]\beta)(k) \gamma \implies P))) \]
«by definition of Safe (P)»
\[ = (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (\exists k: 0 \leq k: (\forall \gamma \in S^\omega: (\sigma[i]\beta)(k) \gamma \implies P))) \]
«by Range Widening»
\[ \Rightarrow (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (\forall k: 0 \leq k: (\forall \gamma \in S^\omega: (\sigma[i]\beta)(k) \gamma \implies P))) \]
«by De Morgan's Law»
\[ = (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (\forall k: 0 \leq k: (\exists \gamma \in S^\omega: (\sigma[i]\beta)(k) \gamma \implies P))) \]
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«by definition of Safe (P)»
\[ = (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (\exists k: 0 \leq k: (\exists \gamma \in S^\omega: (\sigma[k] \gamma \implies P))) \]
«by Range Widening»
\[ \Rightarrow (\exists i: 0 \leq i: (\forall \beta \in S^\omega: \sigma[i]\beta \implies P \land (\forall k: 0 \leq k: (\forall \gamma \in S^\omega: (\sigma[k] \gamma \implies P))) \]
«by De Morgan's Law»
\[ = (\forall \alpha : \alpha \in S^*: true) \]
«since true = A \lor \neg A »
\[ = (\forall \alpha : \alpha \in S^*: (\exists \beta \in S^\omega: \alpha \beta = P) \lor (\exists \beta \in S^\omega: \alpha \beta = P)) \]
«renaming bound variable \beta to \gamma»
\[ = (\forall \alpha : \alpha \in S^*: (\exists \beta \in S^\omega: \alpha \beta = P) \lor (\exists \gamma \in S^\omega: \alpha \gamma = P)) \]
«since \beta is not free in \((\exists \gamma \in S^\omega: \alpha \gamma = P)\)»
\[ = (\forall \alpha : \alpha \in S^*: (\exists \beta \in S^\omega: \alpha \beta = P \lor (\exists \gamma \in S^\omega: \alpha \gamma = P)) \]
«by De Morgan's Law»
\[ = (\forall \alpha : \alpha \in S^*: (\exists \beta \in S^\omega: \alpha \beta = P \lor (\forall \gamma \in S^\omega: \alpha \gamma = P)) \]
It is not surprising that Safe (P) is a safety property. If \(\sigma#Safe (P)\) then, by definition, \(\sigma#M_P\). However, this means there exists an \(i\) such that
\[ (\forall \beta : \beta \in S^\omega: \sigma[i]\beta \implies P) \]
We could consider prefix \(\sigma[i]\) to be a "bad thing". Thus, \(\sigma\) violates a safety property whenever \(\sigma#Safe (P)\).

We now show formally that Live (P) satisfies definition (3.2) of liveness.
\[ (\forall \alpha : \alpha \in S^*: true) \]
«since true = A \lor \neg A »
\[ = (\forall \alpha : \alpha \in S^*: (\exists \beta \in S^\omega: \alpha \beta = P) \lor (\exists \gamma \in S^\omega: \alpha \gamma = P)) \]
«renaming bound variable \beta to \gamma»
\[ = (\forall \alpha : \alpha \in S^*: (\exists \beta \in S^\omega: \alpha \beta = P) \lor (\exists \gamma \in S^\omega: \alpha \gamma = P)) \]
«since \beta is not free in \((\exists \gamma \in S^\omega: \alpha \gamma = P)\)»
\[ = (\forall \alpha : \alpha \in S^*: (\exists \beta \in S^\omega: \alpha \beta = P \lor (\exists \gamma \in S^\omega: \alpha \gamma = P)) \]
We now show formally that Live (P) satisfies definition (3.2) of liveness.
\[
(\forall \alpha \in S^*: \exists \beta \in S^\omega: \alpha \beta \models P \lor (\forall \alpha \models P))
\]

«since true \land A = A»

\[
(\forall \alpha \in S^*: \exists \beta \in S^\omega: \alpha \beta \models P \land (\forall \gamma \in S^\omega: \alpha \gamma \models P))\]

«by substitution, since (\alpha \beta)[i = \alpha] = \alpha»

\[
(\forall \alpha \in S^*: \exists \beta \in S^\omega: \alpha \beta \models P \lor ((i = \alpha) \land (\forall \gamma \in S^\omega: (\alpha \beta)[i \gamma] P))\]

<<by 3-Generalization»

\[
(\forall \alpha \in S^*: \exists \beta \in S^\omega: \alpha \beta \models P \lor (\exists i = \alpha: (\forall \gamma \in S^\omega: (\alpha \beta)[i] P))\]

«by Range Widening»

\[
(\forall \alpha \in S^*: \exists \beta \in S^\omega: \alpha \beta \models P \lor (\exists i = \alpha: (\forall \gamma \in S^\omega: (\alpha \beta)[i] P))\]

<<by De Morgan's Law»

\[
(\forall \alpha \in S^*: \exists \beta \in S^\omega: \alpha \beta \models P \lor (\forall i = \alpha: (\exists \gamma \in S^\omega: (\alpha \beta)[i] P))\]

«by definition of \alpha \beta = A»

\[
(\forall \alpha \in S^*: \exists \beta \in S^\omega: \alpha \beta \models P \lor (\forall i = \alpha: (\exists \gamma \in S^\omega: (\alpha \beta)[i] P))\]

<<by definition of Live (P)>>

\[
(\forall \alpha \in S^*: \exists \beta \in S^\omega: \alpha \beta \models P \lor (\forall i = \alpha: (\exists \gamma \in S^\omega: (\alpha \beta)[i] P))\]

«by Liveness definition (3.2)»

\[
Live (P) \text{ is liveness.}
\]

An informal justification that Live (P) is liveness is the following. If \(\sigma \models \neg \text{Live (P)}\) then, by definition, \(\sigma \models \text{M} \rho\). From, \(\sigma \models \text{M} \rho\), we conclude that it always remains possible for some "good thing" (i.e. \(\beta \in \text{M} \rho\)) to happen. This is the defining characteristic of liveness, so \(\sigma\) violates a liveness property whenever \(\sigma \models \text{Live (P)}\).

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References


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