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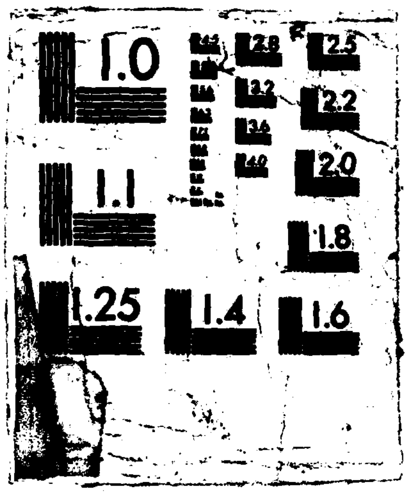
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Boundary value problems are studied for ordinary and functional differential equations. Emphasis is made on equations with piecewise constant delay important in control theory. Existence-uniqueness theorems are proved, stability behaviour investigated, unusually interesting oscillatory and periodic properties discovered.

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BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL AND FUNCTIONAL DIFFERENTIAL EQUATIONS

FINAL REPORT

JOSEPH WIENER

AUGUST 31, 1987

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A. STATEMENT OF THE PROBLEM STUDIED

Boundary-value problems for differential and functional differential equations have been investigated. In case of ordinary differential equations, the major concern was proving existence of solutions to nonlinear third-order boundary-value problems. In the area of functional differential equations, the main thrust was in the direction of differential equations with piecewise constant argument (EPCA.) The theory of functional differential equations with continous argument is well developed and has numerous applications in natural and engineering sciences. It was the main purpose of the project to extend this theory to some classes of differential equations with discontinuous argument deviations. The investigation of EPCA has been originated earlier in [1-4]. These equations represent a hybrid of continuous and discrete dynamical systems and combine properties of both differential and difference equations. Hence their importance in control theory and in certain biomedical problems [5]. All EPCA are closely related to impulse and loaded equations of control theory and, especially, to difference equations of a discrete argument. Within intervals of certain length EPCA have the structure of continuous dynamical systems. Continuity of a solution at a point joining any two consecutive intervals implies recursion relations for the values of the solution at such points. Therefore, EPCA are intrinsically closer to difference rather than to differential equations. A typical EPCA is of the form

$$x'(t) = f(t, x(t), x(h(t))),$$

where the argument $h(t)$ has intervals of constancy. In [2] , equations with $h(t) = [t]$, $h(t) = [t-n]$, $h(t) = t-n[t]$ have been considered, where $[t]$ designates the greatest-integer function and n is integer. In [1] , EPCA of advanced type have been studied, and equations with both delays and advances of the argument have been tackled in [6].

The aim of the project was to establish existence-uniqueness theorems for broad classes of EPCA, obtain significant results on stability and asymptotic behavior of solutions, conduct a thorough analysis of oscillatory and periodic properties of solutions to these equations and systems of equations.

B. SUMMARY OF THE MOST IMPORTANT RESULTS

1. The studies conducted according to the project considerably clarified the place of equations with piecewise constant argument in the general theory of functional differential equations. This research shows that all types of EPCA (retarded, neutral, advanced) share similar characteristics. First of all, it is natural to pose initial-value and boundary-value problems for such equations not on an interval but at a number of individual points. Secondly, in ordinary differential equations with a continuous vector field the solution exists to the right and left of the initial argument value. For retarded functional differential equations this is not necessarily the case. Moreover, it appears that advanced equations, in general, lose their margin of smoothness, and the method of successive integration shows that after several steps to the right from the initial interval the solution may even not exist. However, two-sided solutions do exist for all types of EPCA. Finally, of particular importance was the study of oscillatory aud periodic solutions to EPCA, since it enabled one to explore interesting properties which are caused by the deviating argument and which do not appear in the corresponding ordinary differential equations.

2. Although some oscillatory properties of EPCA were mentioned in [1,2,3] , the first consistent attempt in this direction was made in [4]. Recently, new results on oscillatory and periodic solutions of retarded EPCA have been discovered in [7] , and of advanced EPCA - in [8] . Currently, there are few results on

the oscillation of linear systems with delay. The first study of oscillatory properties of linear systems with piecewise constant argument [t] has been initiated in [9]. Paper [4] is concerned with the oscillatory properties of the equations

$$x'(t) + a(t)x(t) + p(t)x([t]) = 0$$

and

$$x'(t) + a(t)x(t) + q(t)x([t+1]) = 0,$$

where a(t), p(t), and q(t) are continuous on $[0,\infty)$, and [t] is the greatest-integer function. Sufficient conditions are given under which these equations have oscillatory solutions. The conditions are the "best possible" in the sense that when a,p, and q are constants the conditions reduce to

$$p > a/(e^a - 1) \text{ and } q < -ae^a/(e^a - 1),$$

which are necessary and sufficient conditions. Sufficient conditions under which linear differential-difference equations with several argument deviations have oscillatory solutions only have been established in [4] , too. These theorems generalize the corresponding results obtained in [10-13] . Finally, oscillatory properties of solutions to linear equations with linear transformations of the argument have been also discussed in [4] . Such equations have been studied earlier in a number of other works, including [14-19].

3. One of the significant achievements of the project was the discovery and deep investigation of a new type of differential equations with piecewise constant argument-equations of alternately retarded and advanced type. A comprehensive study of the equation

$$x'(t) = ax(t) + bx([t + ½])$$

has been given in [20] . This equation is of considerable interest since the argument deviation

$$T(t) = t - [t + \tfrac{1}{2}]$$

changes the sign in each interval $(n-\frac{1}{2}, n+\frac{1}{2})$, with integer n. Indeed, $T(t) < 0$ for $n - \frac{1}{2} < t < n$ and $T(t) > 0$ for $n < t < n + \frac{1}{2}$, which means that the above equation is alternately of advanced and retarded type. The function T(t) is periodic of period 1, and $T(t) = t$ for $t \in [-\frac{1}{2}, \frac{1}{2})$. We see that the given equation is of advanced type on each interval $[n - \frac{1}{2}, n)$ and is of retarded type on each interval $(n, n + \frac{1}{2})$. This complicates the asymptotic behavior of the solutions, generates two essentially different conditions for oscillations in each interval $(n-\frac{1}{2}, n+\frac{1}{2})$, and leads to interesting properties of periodic solutions. Some other equations of alternately retarded and advanced type have been tackled in [21]. The conclusions of [20] and [21] have been extended to broader classes of EPCA in [22]. Exploration of such extremely interesting equations of alternately retarded and advanced argument should be continued.

4. New interesting results on the existence of periodic solutions to linear delay differential equations with piecewise constant deviating argument have been obtained in [7]. The type of equation studied there is

$$x'(t) + a(t)x(t) + b(t)\,x([t-1]) = 0,$$

where a (t) and b(t) are continuous functions on $[0, \infty)$. A sufficient condition is given under which the equation has oscillatory solutions, and this condition is the "best possible" in the sense that when a and b are constants the condition reduces to

$$b > ae^{-a}/4(e^{a} - 1)$$

which is a necessary and sufficient condition. In case of constant coefficients conditions are found under which oscillatory solutions are periodic. Let $b > 0$, then every oscillatory solution of the above equation is periodic of period k if and only if

$$b = ae^{a}/(e^{a}-1) \text{ and } a = -\ln\left(2\cos\frac{2\pi m}{k}\right),$$

where m and k are relatively prime and $m = 1,2,\ldots, [(k-1)/4]$. Let λ_1, λ_2 be the roots of the equation.

$$\lambda^2 - e^{-a}\lambda + \frac{b}{a}(1 - e^{-a}) = 0$$

and let $x(0) = c_0$, $x(-1) = c_{-1}$. If $b<0$ and $c_0 = \lambda_2 c_{-1}$, then every oscillatory solution is periodic of period 2 if and only if

$$b = -a(e^{a} + 1)/(e^{a} - 1).$$

The most important conclusion in [7] is the following: if $b>0$, then for given c_o and c_{-1} the set of all equations of the above type having periodic solutions is countable. These results were obtained with the implicit assumption $a \neq 0$. If $a = 0$, then

$$b = \lim_{a \to 0} ae^{a}/(e^{a} - 1) = 1.$$

In this case, the equation

$$x(t) + x(t - 1) = 0$$

has periodic solutions of period 6.

5. Stability problems for differential equations with certain piecewise continuous delays have been explored in [23]. Equations considered are of the form

$$x'(t) = ax(t) + \sum_{i=0}^{n} a_i x([t - ir]_r),$$

where a and a_i are real constants, and $r > 0$. The symbol $[t]_r$ denotes an extension of the usual greatest-integer function and is defined by

$$[t]_r = n, \text{ for } nr \leq t < (n + 1)r, \ n = 0, 1, 2, \ldots .$$

That is, $[t]_r$ is a step function whose value increases by one when t is an integral multiple of r. For the case r = 1, this equation was studied in [2], and it was shown that the zero solution is asymptotically stable if and only if all roots of a certain associated characteristic polynomial have moduli less than one. Here an extra parameter r is introduced, and attention is directed to the way in which stability depends on r, as well as on the coefficients a, a_i. General sufficient conditions for the stability of the above differential equation and of the equation

$$x'(t) = ax(t) = \sum_{i=0}^{n} a_i x([t - idr]_{dr}),$$

for all d, $0 < d < \infty$, are obtained. The "first-order" equation with N=0 is examined and the stability region in the (a, a_0) parameter space is precisely described. This is compared with the stability region for the first-order differential-difference equation with constant lag r. The "second-order" equation with N=1 is also investigated and a set of (a, a_0, a_1) found for which there is asymptotic stability for every positive r. A general theorem on the stability of equations with piecewise constant delays is presented which is analogous to theorems in [24, 25, 26] relating to stability of linear differential-difference equations.

Let

$$b_i = (e^{ra} - 1)a^{-1}a_i \quad (i = 1,\ldots,N),$$
$$b_o = e^{ra} + (e^{ra} - 1)a^{-1}a_o ,$$
$$f(\lambda, r, A) = \lambda^{n+1} - b_o\lambda^n - b_1\lambda^{n-1} - \ldots - b_n$$

The symbol $f(\lambda, r, A)$ is used to indicate that f is a polynomial in λ

whose coefficients depend on r and on the set of numbers A= $\{a_o, a_1, \ldots, a_n\}$. It then follows that a necessary and sufficient condition for asymptotic stability of the zero solution $x(t) = 0$ is that all roots of $f(\lambda,r.A) = 0$ satisfy $|\lambda| < 1$. Let $r > 0$ and assume

$$(H_1) \qquad a + \sum_{i=o}^{n} a_i < 0.$$

Then there exists a maximal interval $(0,d_o)$, with $0 < d_o \leq \infty$, such that all roots of $f(\lambda,dr,A)$ lie in $|\lambda| < 1$ for $d \in (0,d_o)$, and therefore the zero solution $x(t) = 0$ is asymptotically stable. Assume that (H_1) holds and also (H_2) $f(\lambda,dr,A) \neq 0$, for $|\lambda| = 1, 0 < d < \infty$. Then the given differential equation is asymptotically stable for every positive d. Assume that (H_1) holds and also

$$(H_3) \qquad f(\lambda,dr,A) \neq 0, \text{ for } |\lambda| = 1,\ 0 \leq d < 1$$

Then the differential equation is asymptotically stable for $0 \leq d < 1$. Precise results were obtained in [23] for the equation

$$x'(t) = ax(t) + a_o x([t]_r),\ a \neq 0$$

which can be transformed into

$$y(s) = ray(s) + ra_o y([s]),$$

and then

$$f(\lambda,r,A) = \lambda - b_o.$$

The necessary and sufficient condition of asymptotic stability of the given differential equation is $|b_o| < 1$, which can be written as

$$-2a/(e^{ra} - 1) < a + a_o < 0.$$

The function

$$F(a) = -a - \frac{2a}{e^{ra} - 1}$$

is readily seen to be increasing for $a < 0$, and decreasing for $a > 0$, with $F(0) = -2/r$. It is asymptotic to $-a$ for $a \to +\infty$, and to $+a$ as $a \to -\infty$, and lies below those lines. Therefore, the stability region has the appearance indicated in Fig. 1. We note that if a parameter is varied in such a way that the line $a + a_0 = 0$ is crossed, the root $\lambda = b_0$ crosses + 1, and nonoscillatory instability arises. If the lower boundary in Fig. 1 is crossed, the root $\lambda = b_0$ crosses -1, and oscillatory instability arises. Observe that as $r \to 0^+$, the stability region expands to cover the region $a + a_0 < 0$, whereas as $r \to +\infty$, the stability region reduces to a quarter plane. In fact, we have the following result. The given differential equation is asymptotically stabe if and only if $-2a/(e^{ra} - 1) < a + a_0 < 0$. The region of stability in the (a, a_0) -plane decreases as r increases. The region of stability for all delays r, $0 < r < \infty$, is the set

$$\{ (a, a_0) : a < 0, a \leq a_0 < |a| \}$$

The region in Fig. 1 may be compared with the stability region for the differential-difference equation

$$x'(t) = ax(t) + a_0 x(t - r).$$

This region is defined by $g(a) < a_0 < -a$, $a < 1/r$, where g is a certain function for which $g(0) = \pi/2r$, $g(1/r) = -1/r$, and $g(a)$ is asymptotic to the line $a_0 = a$ as $a \to -\infty$. This region is similar to that in Fig. 1, but for fixed r does not extend to the right of $a = 1/r$.

6. In [27] the study of boundary-value problems for differential equations with reflection of the argument was initiated for the firt time. This is one of the important contributions of the project, and the work on such problems was continued in [28, 29, 30]. Equations with reflection of the argu-

ment represent a particular case of differential equations with involutions which were discovered in [31-34]. Important in their own right, they have applications in the investigation of stability of differential-difference equations [35-39]. Initial-value problems for equations with involutions have been considered in numerous papers. A survey of results in this direction was given in [40]. However, research on boundary-value problems for such equations remains developed insufficiently. Paper [27] is concerned with existence and uniqueness of solutions of

$$y'(x) = f(x, y(x), y(-x))$$

with the boundary conditions

$$y(-a) = y_0, \quad y(a) = y_1$$

or

$$y'(-a) - hy(-a) = 0, \quad y'(a) + ky(a) = 0.$$

The method used is the Schauder fixed point theorem in the case of general f, and in the case when f is linear the equation with reflection is changed to a higher-order ordinary differential equation. Currently, we are exploring third-order boundary-value problems for differential equations with reflection of the argument.

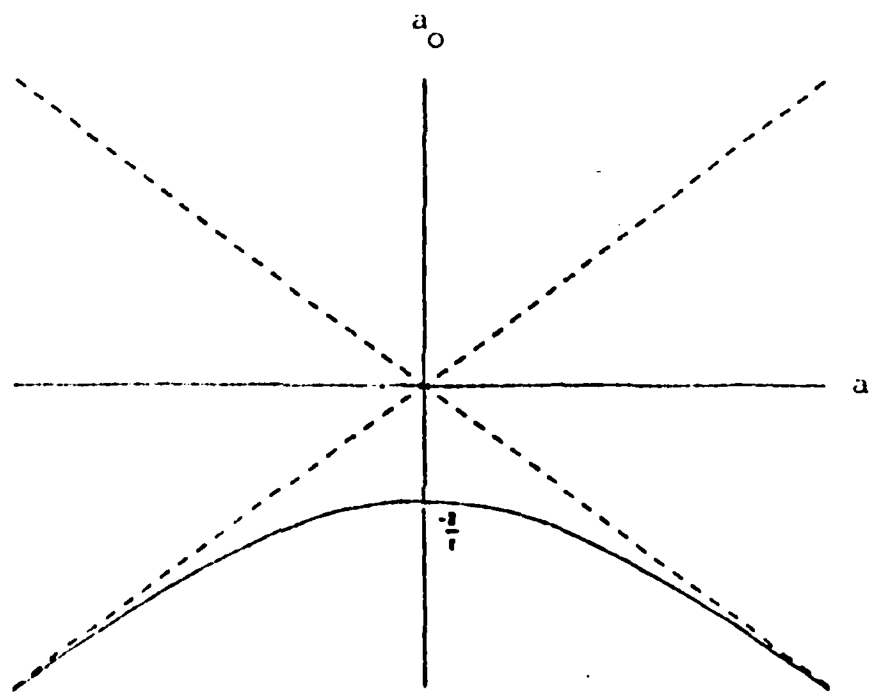

Fig. 1. The stability region lies below the line $a_0 + a = 0$, and above the curve $a_0 + a = -2a/(e^{ra} - 1)$

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