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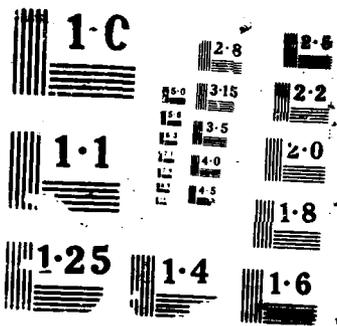
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ROYAL AIRCRAFT ESTABLISHMENT

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Technical Report 87038

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**BOUNDARY ELEMENTS COMBINED WITH  
SINGULAR FIELDS FOR  
THREE-DIMENSIONAL CRACKED SOLIDS**

by

D. P. Rooke  
D. J. Cartwright\*  
M. H. Aliabadi\*

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17. Abstract The boundary element method is used to obtain weight functions for three-dimensional solids containing cracks. A singular displacement field analogous to the two-dimensional Bueckner field is derived for a planar rectilinear crack subjected to equal and opposite localized forces on the crack surface. This field is prescribed around a small cylindrical cavity along the crack front and the boundary element method is used to obtain weight functions for the stress intensity factor. The advantages of the method are demonstrated for an edge cracked rectangular bar. (Keywords: Great Britain; fracture mechanics; stress analysis).					

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SUMMARY

The boundary element method is used to obtain weight functions for three-dimensional solids containing cracks. A singular displacement field analogous to the two-dimensional Bueckner field is derived for a planar rectilinear crack subjected to equal and opposite localized forces on the crack surface. This field is prescribed around a small cylindrical cavity along the crack front and the boundary element method is used to obtain weight functions for the stress intensity factor. The advantages of the method are demonstrated for an edge cracked rectangular bar.

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1 INTRODUCTION

The application of the principles of fracture mechanics to the safe and economic design of engineering structures, which may develop cracks in service, requires that the stress intensity factors of the cracks be known. In any analysis a complicated structure must usually be modelled as a three-dimensional body in which several boundaries interact with the crack tip. The stress intensity factor will now vary with position on the crack front and cannot be accurately represented by a two-dimensional model. It is therefore necessary to develop efficient, accurate and economic methods for determining stress intensity factors in three-dimensional cracked bodies.

Techniques based on weight functions are efficient since once known they can be used to find the stress intensity factor for any arbitrary loading from a simple summation or integration. Weight functions for three-dimensional symmetrical crack problems have been determined<sup>1</sup> using the finite element method (FEM). In this work it was observed that for three-dimensional problems the boundary element method (BEM) may lead to faster solutions because of the reduction in the number of elements that would be possible. The BEM has been used<sup>2</sup> to obtain stress intensity factors for a semi-elliptical crack subjected to a polynomial distribution of tractions over the crack surfaces.

Recently it has been shown<sup>3</sup> that an application of Betti's theorem allows the stress intensity factor, for arbitrary applied stresses, to be derived efficiently and economically from certain 'weight functions'. These weight functions are derived from the boundary displacements of the body, resulting from point forces applied near the tip of the crack; the crack-tip region is modelled by a Bueckner singular field<sup>4</sup>. This approach was developed for two-dimensional bodies containing cracks where it was shown that a boundary method has considerable advantages over domain methods<sup>5</sup>; these advantages will be greater in modelling three-dimensional bodies. The use of a Bueckner singular field in the determination of weight functions reduces the number of boundary integrations needed in the BEM calculations.

In the present application to three-dimensional cracked bodies a singular displacement field, analogous to the two-dimensional Bueckner field, is derived from the analysis<sup>6</sup> for a planar, rectilinear crack subjected to equal and opposite localised forces on the crack surface. The singular displacement field is then prescribed over a small cylindrical cavity along the crack front and the BEM is used to obtain weight functions. The advantages of the method are demonstrated for an edge-cracked rectangular bar by calculating the stress intensity factors

are calculated, from the weight functions, at points on the crack front for uniform normal tractions on the ends of the bar. Even for relatively coarse meshes the results are in good agreement with those determined from other methods.

## 2 FORMULATION OF METHOD

For an elastic body Betti's reciprocal theorem can be written as

$$\int_s \underline{u}^{\dagger} \underline{t}^* ds = \int_s \underline{t}^{\dagger} \underline{u}^* ds, \quad (1)$$

where  $\underline{t}^{\dagger}$ ,  $\underline{u}^{\dagger}$  are the self-equilibrated tractions and displacement field, respectively, corresponding to one state (state 1) of loading and  $\underline{t}^*$ ,  $\underline{u}^*$  are those corresponding to another independent state (state 2) of loading. The choice of  $(\underline{t}^{\dagger}, \underline{u}^{\dagger})$  and  $(\underline{t}^*, \underline{u}^*)$  are arbitrary except that they must separately satisfy the equations of equilibrium, the compatibility conditions within the body and the boundary conditions on the body.

It is assumed that state 1 corresponds to the desired solution of a crack in the body whose surface is subjected to a set of self-equilibrated tractions  $\underline{t}^{\dagger}$ . The leading terms in the general expansion of displacements near the crack tip can be obtained from Sih and Liebowitz<sup>7</sup> in the form

$$u_1^{\ell} = \frac{(1 + \nu)}{E} \left( \frac{2r}{\pi} \right)^{\frac{1}{2}} \left\{ K_{II} \cos\left(\frac{\theta}{2}\right) \left[ (1 - 2\nu) + \sin^2\left(\frac{\theta}{2}\right) \right] + K_{III} \sin\left(\frac{\theta}{2}\right) \left[ 2(1 - \nu) + \cos^2\left(\frac{\theta}{2}\right) \right] \right\},$$

$$u_2^{\ell} = \frac{(1 + \nu)}{E} \left( \frac{2r}{\pi} \right)^{\frac{1}{2}} \left\{ K_{II} \sin\left(\frac{\theta}{2}\right) \left[ 2(1 - \nu) - \cos^2\left(\frac{\theta}{2}\right) \right] - K_{III} \cos\left(\frac{\theta}{2}\right) \left[ (1 - 2\nu) - \sin^2\left(\frac{\theta}{2}\right) \right] \right\},$$

$$u_3^{\ell} = 2 \frac{(1 + \nu)}{E} \left( \frac{2r}{\pi} \right)^{\frac{1}{2}} K_{III} \sin\left(\frac{\theta}{2}\right), \quad \dots\dots (2)$$

where  $u_j^{\ell}$  ( $j = 1, 2, 3$ ) are the respective local displacements in the local  $x_j^{\ell}$  coordinate directions parallel to the normal, the binormal and the tangent to the crack. That is the displacement vector  $\underline{u}$  for state 1 can be written

$$\underline{u}^{\dagger} = u_1^{\ell} \hat{x}_1^{\ell} + u_2^{\ell} \hat{x}_2^{\ell} + u_3^{\ell} \hat{x}_3^{\ell}, \quad (3)$$

where  $\hat{x}_j^\ell$  are the local unit vectors in the  $x_j^\ell$  directions ( $j = 1, 2, 3$ ). The coordinates  $(r, \theta)$  are local cylindrical polar coordinates in the  $x_1^\ell, x_2^\ell$  plane, centred on the crack front at  $x_3^\ell = 0$ . The stress intensity factor  $K_N$  ( $N = I, II, III$ ) corresponds to the opening, sliding and tearing modes respectively;  $E$  and  $\nu$  are the elastic modulus and Poisson's ratio respectively. Since the only loading to be considered in state 2 is on the crack surface, it is convenient to simplify equation (2) to obtain the displacement components on the crack surfaces  $\theta = \pm\pi$  at a distance  $x_1^\ell$  behind the crack front. These are given from equation (2) by

$$u_2^\ell = \frac{2(1 - \nu^2)}{E} \left[ \frac{2x_1^\ell}{\pi} \right]^{\frac{1}{2}} K_I, \quad (4)$$

$$u_1^\ell = \frac{2(1 - \nu^2)}{E} \left[ \frac{2x_1^\ell}{\pi} \right]^{\frac{1}{2}} K_{II}, \quad (5)$$

$$u_3^\ell = \frac{2(1 + \nu)}{E} \left[ \frac{2x_1^\ell}{\pi} \right]^{\frac{1}{2}} K_{III}. \quad (6)$$

The state 2 is taken to be that corresponding to a solution of a body containing a crack which is subjected to pairs of concentrated symmetrical forces located on the crack surfaces near the crack tip as shown in Fig 1. The tractions  $\underline{t}^*$  are represented in terms of point forces in each of the three orthogonal directions and are given by the vector  $\underline{P} = P_1 \hat{x}_1^\ell + P_2 \hat{x}_2^\ell + P_3 \hat{x}_3^\ell$  where  $P_j$  ( $j = 1, 2, 3$ ) are the magnitudes of the point force in the  $x_j$  directions.

For point forces located at  $x_1^\ell = -c$ ,  $x_2^\ell = x_3^\ell = 0$  the traction vector  $\underline{t}^*$  can be written as

$$\underline{t}^* = \delta(x_1^\ell + c) \underline{P}, \quad (7)$$

where  $\delta(x_1^\ell + c)$  represents the Dirac delta function. Substituting equations (3) and (7) in equation (1) gives

$$P_1 K_{II} + P_2 K_I + \frac{P_3 K_{III}}{(1 - \nu)} = \frac{E\sqrt{\pi}}{2\sqrt{2}(1 - \nu^2)\sqrt{c}} \int_S \underline{t}^* \cdot \underline{u}^* ds. \quad (8)$$

Thus the stress intensity factor  $K_N$  for any mode ( $N = I, II, III$ ) can be determined independently by making  $P_j$  corresponding to that mode non-zero, and all other  $P_j$  zero. The displacements  $\underline{u}^*$  will be those, on the surface  $S$ , corresponding to  $P_j$  acting on the crack near the tip. Following Bueckner's suggestion<sup>4</sup> for two-dimensional fields, the components of  $\underline{u}^*$  are called weight functions, since the integral in equation (8) represents a weighted average of the tractions over the surface  $S$ . Because  $\underline{u}^*$  depends only on the displacements on the surface of the cracked body, it follows that a boundary method of analysis, such as the BEM, should be a more efficient procedure than a domain method. The boundary integral equation to be solved is

$$c_{ij}(\underline{x}')u_j(\underline{x}') + \int_S T_{ij}(\underline{x}, \underline{x}')u_j(\underline{x})ds(\underline{x}) = \int_S U_{ij}(\underline{x}, \underline{x}')t_j(\underline{x})ds(\underline{x}) \quad , \quad (9)$$

where  $u_j, t_j$  are general displacements and tractions ( $i, j = 1, 2, 3$ ) respectively on the boundary  $S$ ,  $U_{ij}(\underline{x}, \underline{x}')$  and  $T_{ij}(\underline{x}, \underline{x}')$  are the Kelvin's fundamental solution in an infinite elastic (uncracked) body and  $c_{ij}$  is a known constant.

In order to determine  $\underline{u}^*$  it is necessary to solve the boundary integral equation (9) for the cracked body subjected to state 2 loading; for which equation (9) becomes

$$c_{ij}(\underline{x}')u_j^*(\underline{x}') + \int_S T_{ij}(\underline{x}, \underline{x}')u_j^*(\underline{x})ds(\underline{x}) = \int_{S_0} U_{ij}(\underline{x}, \underline{x}')t_j^*(\underline{x})ds(\underline{x}) \quad , \quad (10)$$

where the surface integral on the right-hand side of equation (10) is over a small near-tip region  $S_0$  only, since  $t_j^*$  are zero everywhere except on  $S_0$ . Thus the computing time to evaluate the right-hand side integral is less since the number of elements on  $S_0$  is much less than the total number of elements on  $S$ .

### 3 STRAIGHT FRONTED CRACK: MODE I

Equations (8) and (9) are now applied to the determination of weight functions and opening mode stress intensity factors for a straight-fronted planar crack. For this case  $P_1 = P_3 = 0$  in equation (8). The traction  $t_j^*$  now becomes the applied force  $P_2$ . In order to avoid numerical difficulties

due to the resulting singular field, this force is modelled by taking  $S_0$  to be a cylindrical cavity of radius  $r_0$  centred on the crack front with a displacement field  $v_j$  prescribed over its surface.

This avoids numerical difficulties due to the singular field resulting from  $P_2$  and, if  $r_0$  is small enough, adequately represents the presence of the force  $P_2$  near the crack tip. The displacement field is taken to be that for a semi-infinite planar crack subjected to forces perpendicular to the crack surface and has been derived from the general solutions given by Kassir and Sih<sup>6</sup>. The displacement field on a cylindrical surface  $r = r_0$  near the crack line may be written, using the equation (3) in the Appendix, with  $u_0 = R_I / (\sqrt{2\mu} r_0^{3/2})$ ,  $\rho_0 = \rho / r_0$ ,  $z_0 = x_3 / r_0$  and  $R_I = \lim_{c \rightarrow 0} P_2 \sqrt{c} / \pi^2$ , as

$$\left. \begin{aligned} v_1^I &= -u_0 \left\{ (1 - 2\nu) r_0^{3/2} A - \frac{\sin \theta \sin(\frac{\theta}{2})}{\sqrt{2} \rho_0^3} \left( \rho_0 + \frac{4 \cos \theta}{\rho_0} \right) \right\} , \\ v_2^I &= -u_0 \left\{ \frac{-2\sqrt{2}(1 - \nu) \sin(\frac{\theta}{2})}{\rho_0^2} + \frac{\sin^2 \theta}{2\sqrt{2} \rho_0^3 \sin(\frac{\theta}{2})} \left[ \rho_0 - \frac{4}{\rho_0} (1 - \cos \theta) \right] \right\} , \\ v_3^I &= -u_0 \left\{ (1 - 2\nu) r_0^{3/2} B - \frac{2\sqrt{2} z_0 \sin \theta \sin(\frac{\theta}{2})}{\rho_0^4} \right\} . \end{aligned} \right\} \quad (11)$$

Equation (10) is solved for  $u_j^*$  on  $S$  and  $t_j^*$  on  $S_0$  for a body subjected to the displacement conditions on the cylindrical cavity given by equation (11). The opening mode stress intensity factor is given from equation (8) by

$$K_I = \frac{1}{4(1 - \nu)(\pi r_0)^{3/2}} \int_S \underline{t}^* \cdot \underline{U}^I ds , \quad (12)$$

where  $\underline{U}^I = u^* / u_0$  is the normalised weight function for the body and  $\underline{t}^*$  are the applied tractions for which  $K_I$  is to be determined.

#### 4 SOLUTION STABILITY

It is necessary to choose an optimum value for the radius  $r_0$  which must be sufficiently small to represent adequately the concentrated forces near the crack front but not so small that the cavity cannot be modelled without using an excessive number of boundary elements on its surface. The stability of the solution, as  $r_0$  varies, is studied using an edge cracked bar of rectangular cross-section, thickness  $t$ , width  $b$  and total length  $2h$ . The bar contains an edge crack of uniform depth  $a$  across the thickness  $t$ .

The element distribution on the surface of the bar is shown in Fig 2, for  $a = 0.5b$ ,  $t = 1.5b$ ,  $h = 1.5b$ . There are a total of 68 elements, 46 on the rectangular surfaces and 22 on the cylindrical cavity. The elements on the cavity were clustered in the vicinity of the origin of the singular field in order to model accurately the rapid variation of the field in that region. Because of the symmetry of the bar about the crack-line, only half of the configuration needs to be modelled.

Weight functions were determined from the solution of equation (10). The stress intensity factor at the mid-point on the crack front was calculated from equation (12) for a uniform stress applied over the ends of the bar as shown in Fig 3. The variation of the normalised stress intensity factor  $K_I / \sigma \sqrt{\pi a}$  with  $r_0/t$  for  $0.003 < r_0/t < 0.01$  is shown in Fig 4.

Results for various values of  $t/a$  and  $h/a$  are compared with the known plane strain value<sup>8</sup> which they should approach for large  $t/a$ . It can be seen that the present results differ from the plane strain value by less than 2% for  $r_0/t < 0.006$ , and for any given configuration the maximum variation is 12% over the whole range of  $r_0/t$ . For subsequent calculations a value of  $r_0/t = 0.004$  was chosen.

#### 5 VARIATION OF THE STRESS INTENSITY FACTOR ALONG THE CRACK FRONT

In order to illustrate the versatility of the method developed in this paper, it is applied to the determination of stress intensity factors at different positions along the crack front for the edge cracked bar shown in Fig 3, with the same dimensions as given in the previous section. This problem has also been solved by Raju and Newman<sup>9</sup> using the FEM and by Mendelson and Alam<sup>10</sup> using the method of lines (MOL). The boundary element equation (10) is solved for the weight functions with the coordinate origin of the singular field at  $x_3/t = 0.0, 0.15, 0.3, 0.4$  and  $0.45$ . These weight functions are used in equation (12) with  $\underline{t}^{\#} = \sigma \hat{x}_2$  where  $\sigma$  is the uniform tensile stress acting

on the ends of the bar. The resulting normalised stress intensity factor  $K_I / \sigma \sqrt{\pi a}$  is shown in Fig 5 as a function of position along the crack front. These results are compared with those determined by Raju and Newman<sup>9</sup>: other results<sup>10</sup> show similar trends. Also shown in Fig 5 is the plane strain solution for the case  $t = \infty$ . This is close to the three-dimensional solution near the centre of the body but differs significantly in the region  $x_3/t > 0.4$ , where the traction-free boundary conditions on the faces of the bar affect the stress intensity factor.

The boundary element results were determined using 68 surface elements compared with 432 solid elements in the finite element analysis<sup>9</sup>. Thus the BEM coupled with a singular field, as developed herein, represents a substantial saving in data preparation and computer storage. Furthermore since the method produces weight functions, the stress intensity factors along the crack front can be determined for any symmetrical loading on the body without re-solving the boundary element equations.

## 6 CONCLUSIONS

A three-dimensional singular field has been derived and used to model point forces acting at a crack tip in a boundary element analysis. The tip is replaced by a cylindrical boundary over which the field acts; an optimum value is suggested for the radius of the cylinder.

This field combined with the boundary element method can be used to determine weight functions on all surfaces of a body simultaneously, and hence the stress intensity factors for arbitrary loadings can be calculated.

This combination is more efficient than the standard boundary element procedure because it requires less boundary integrations: it also requires much less data preparation and storage than domain methods. Even with relatively coarse meshes, accurate values of the stress intensity factor were obtained for a cracked bar.

Appendix  
DERIVATION OF SINGULAR FIELD

From Ref 6 the displacement field for a semi-infinite planar crack subjected to opening forces  $\pm P_2$  perpendicular to the crack are given (see Fig 1 for  $P_j = 0, j = 1,3$  and  $x_j^k = x_j, j = 1,2,3$ ) by

$$v_1^I = (1 - 2\nu) \int_{\infty}^{x_2} \frac{\partial f^I}{\partial x_1} dx_2 + x_2 \frac{\partial f^I}{\partial x_1},$$

$$v_2^I = -2(1 - \nu) f^I + x_2 \frac{\partial f^I}{\partial x_2}, \quad (A-1)$$

$$v_3^I = (1 - 2\nu) \int_{\infty}^{x_2} \frac{\partial f^I}{\partial x_3} dx_2 + x_2 \frac{\partial f^I}{\partial x_3},$$

where

$$f^I = -\frac{P_2}{2\mu\pi^2\rho} \tan^{-1} \left[ \frac{2\sqrt{cr}}{\rho} \sin\left(\frac{\theta}{2}\right) \right].$$

The constants  $\mu$  and  $\nu$  are the shear modulus and Poisson's ratio respectively,  $(r, \theta)$  are the polar coordinates in the  $x_1, x_2$  plane,  $\rho$  is defined by

$$\rho^2 = (x_1 + c)^2 + x_2^2 + x_3^2,$$

and the coordinate system  $x_j, j = 1,2,3$  has an origin on the crack line with the  $x_j$  directions parallel to the normal, the binormal and the tangent to the crack front.

In the limit as  $c \rightarrow 0$ ,  $f^I = -\frac{R_I}{\sqrt{2}\mu\rho^2} \sqrt{r} \sin\left(\frac{\theta}{2}\right),$

where

$$R_I = \lim_{c \rightarrow 0} \left( \frac{P_2 \sqrt{c}}{\pi^2} \right)$$

and now

$$\rho^2 = x_1^2 + x_2^2 + x_3^2 .$$

Writing

$$f^I = - \frac{R_I}{\sqrt{2\mu}} g^I ,$$

where

$$g^I = \frac{\sqrt{r - x_1}}{\rho^2} ,$$

equations (A-1) become

$$\left. \begin{aligned} v_1^I &= - \frac{R_I}{\sqrt{2\mu}} \left\{ (1 - 2\nu)A + x_2 \frac{\partial g^I}{\partial x_1} \right\} , \\ v_2^I &= \frac{R_I}{\sqrt{2\mu}} \left\{ 2(1 - \nu)g^I + x_2 \frac{\partial g^I}{\partial x_3} \right\} , \\ v_3^I &= - \frac{R_I}{\sqrt{2\mu}} \left\{ (1 - 2\nu)B + x_2 \frac{\partial g^I}{\partial x_3} \right\} , \end{aligned} \right\} \quad (A-2)$$

where the derivatives are given by

$$\frac{\partial g^I}{\partial x_1} = - \frac{\sqrt{r - x_1}}{2\rho^3} \left( \frac{\rho}{r} + \frac{4x_1}{\rho} \right) ,$$

$$\frac{\partial g^I}{\partial x_2} = \frac{x_2}{2\rho^3 \sqrt{r - x_1}} \left( \frac{\rho}{r} - \frac{4(r - x_1)}{\rho} \right) ,$$

$$\frac{\partial g^I}{\partial x_3} = - \frac{2x_3 \sqrt{r - x_1}}{\rho^4} .$$

The functions A and B are then given by

$$A = -\frac{1}{2} \int_{-\infty}^{x_2} \frac{\left[ (x_1^2 + x_2^2)^{\frac{1}{2}} - x_1 \right]^{\frac{1}{2}}}{(x_1^2 + x_2^2)^{\frac{1}{2}} (x_1^2 + x_2^2 + x_3^2)} dx_2 - 2x_1 \int_{-\infty}^{x_2} \frac{\left[ (x_1^2 + x_2^2)^{\frac{1}{2}} - x_1 \right]^{\frac{1}{2}}}{(x_1^2 + x_2^2 + x_3^2)^2} dx_2$$

and

$$B = -2x_3 \int_{-\infty}^{x_2} \frac{\left[ (x_1^2 + x_2^2)^{\frac{1}{2}} - x_1 \right]^{\frac{1}{2}}}{(x_1^2 + x_2^2 + x_3^2)^2} dx_2 .$$

Substitution of  $x_1^2 + x_2^2 = t^2$  enables the integrals to be evaluated and the final expression for the singular field becomes

$$\left. \begin{aligned} v_1^I &= -\frac{R_I}{\mu\sqrt{2}} \left\{ (1 - 2\nu)A - \frac{r^{3/2} \sin \theta \sin\left(\frac{\theta}{2}\right)}{\sqrt{2} \rho^3} \left[ \frac{\rho}{r} + \frac{4r \cos \theta}{\rho} \right] \right\} , \\ v_2^I &= -\frac{R_I}{\mu\sqrt{2}} \left\{ \frac{-2\sqrt{2}(1 - \nu)r^{\frac{1}{2}} \sin\left(\frac{\theta}{2}\right)}{\rho^2} + \frac{r^{3/2} \sin^2 \theta}{2\sqrt{2} \rho^2 \sin\left(\frac{\theta}{2}\right)} \left[ \frac{\rho}{r} - \frac{4r(1 - \cos \theta)}{\rho} \right] \right\} , \\ v_3^I &= -\frac{R_I}{\mu\sqrt{2}} \left\{ (1 - 2\nu)B - \frac{2\sqrt{2} x_3 r^{3/2} \sin \theta \sin\left(\frac{\theta}{2}\right)}{\rho^4} \right\} , \end{aligned} \right\} \quad (A-3)$$

where

$$A = \frac{1}{2x_3\sqrt{R}} \left[ \frac{L}{2} \sin \phi_0 + T \cos \phi_0 \right] - \frac{x_1}{x_3} B$$

and

$$B = \frac{\sqrt{2} x_3 r^{3/2} (1 - \cos \theta) \cos\left(\frac{\theta}{2}\right)}{R^2 \rho^2} - \frac{x_3}{4R^{5/2}} \left( \cos \phi_0 + \frac{x_1}{x_3} \sin \phi_0 \right) L + \frac{x_3}{2R^{5/2}} \left( \sin \phi_0 - \frac{x_1}{x_3} \cos \phi_0 \right) T ,$$

$$L = \ln \left[ \frac{r(1 + \cos \theta) + R - 2\sqrt{2Rr} \cos \phi_0 \cos\left(\frac{\theta}{2}\right)}{r(1 + \cos \theta) + R + 2\sqrt{2Rr} \cos \phi_0 \cos\left(\frac{\theta}{2}\right)} \right] ,$$

$$T = \tan^{-1} \left[ \frac{2\sqrt{2Rr} \sin \phi_0 \cos\left(\frac{\theta}{2}\right)}{r(1 + \cos \theta) - R} \right] ,$$

where  $\phi_0 = \frac{1}{2} \tan^{-1} \left( \frac{x_3}{x_1} \right) , \quad R = \sqrt{x_1^2 + x_3^2} ,$

$\theta = \tan^{-1} \left( \frac{x_2}{x_1} \right) , \quad \rho = \sqrt{x_1^2 + x_2^2 + x_3^2} .$

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Fig 1

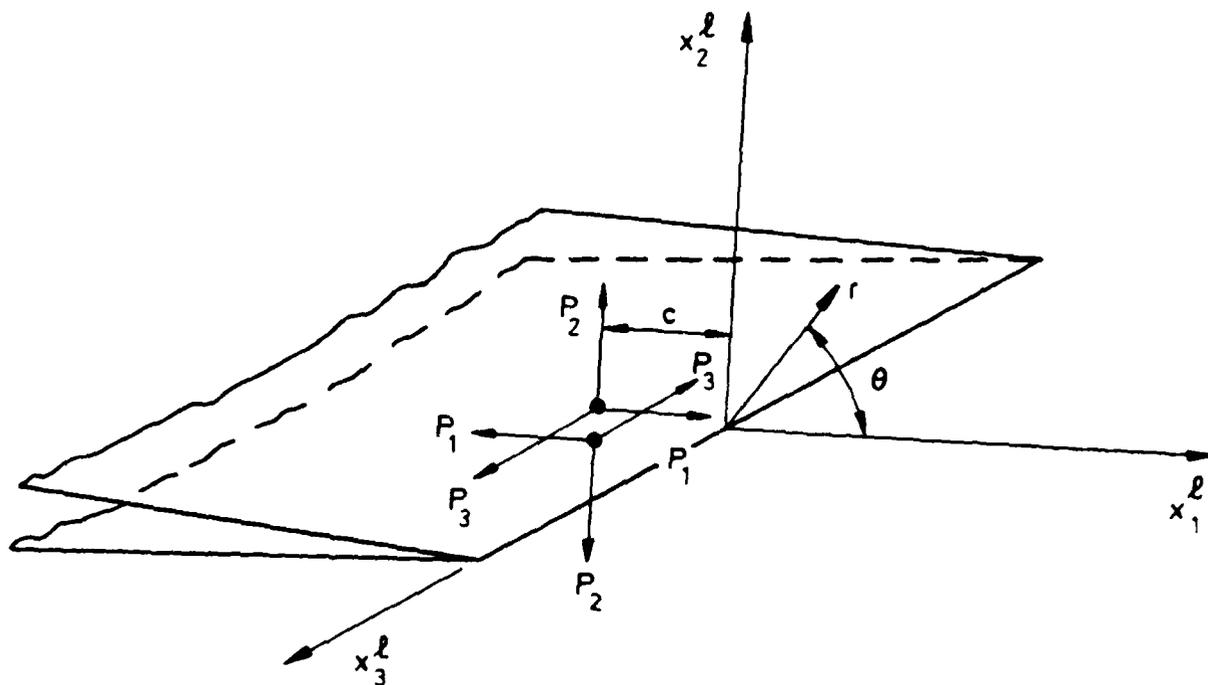


Fig 1 Concentrated forces near the tip of a planar crack

Fig 2

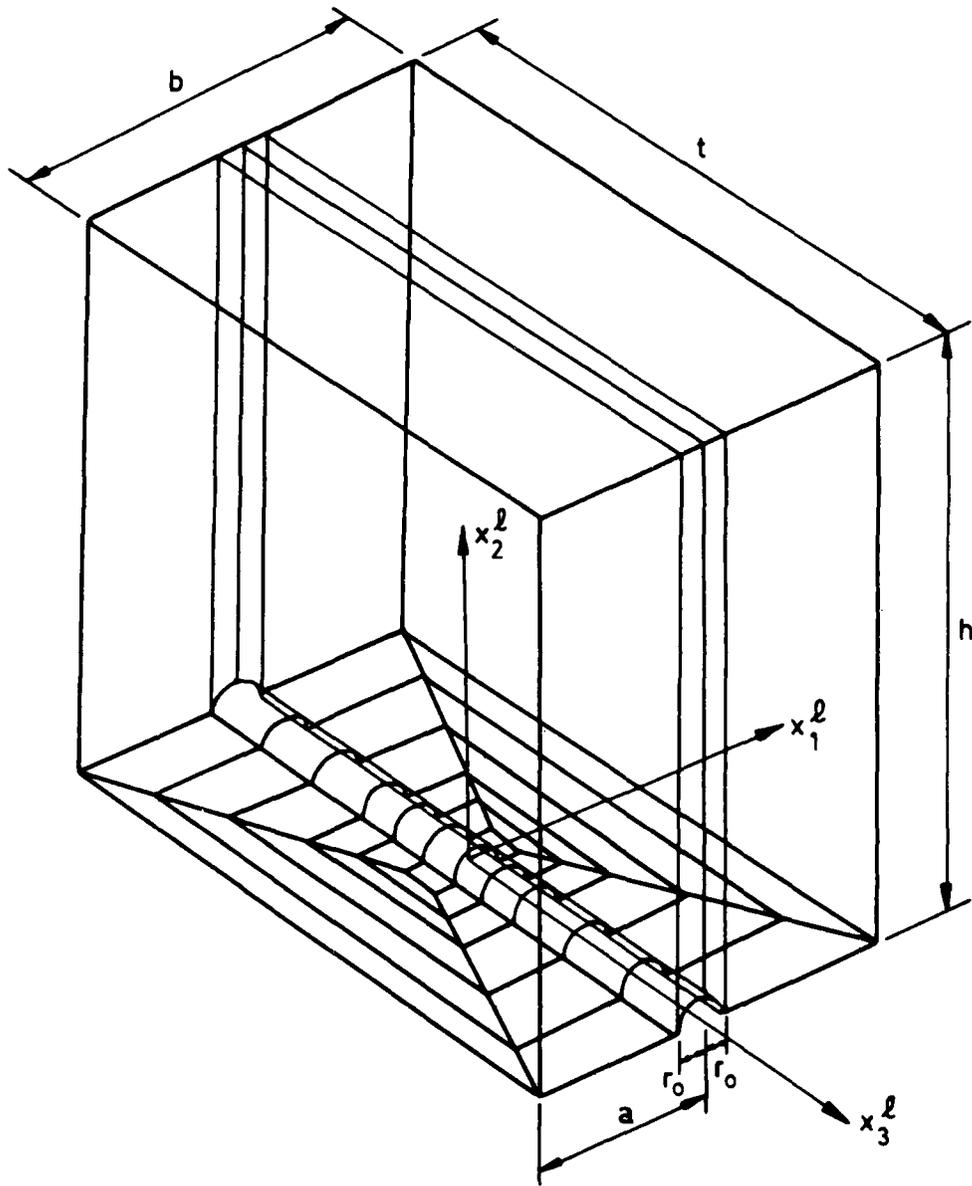


Fig 2 Representative boundary element distribution

Fig 3

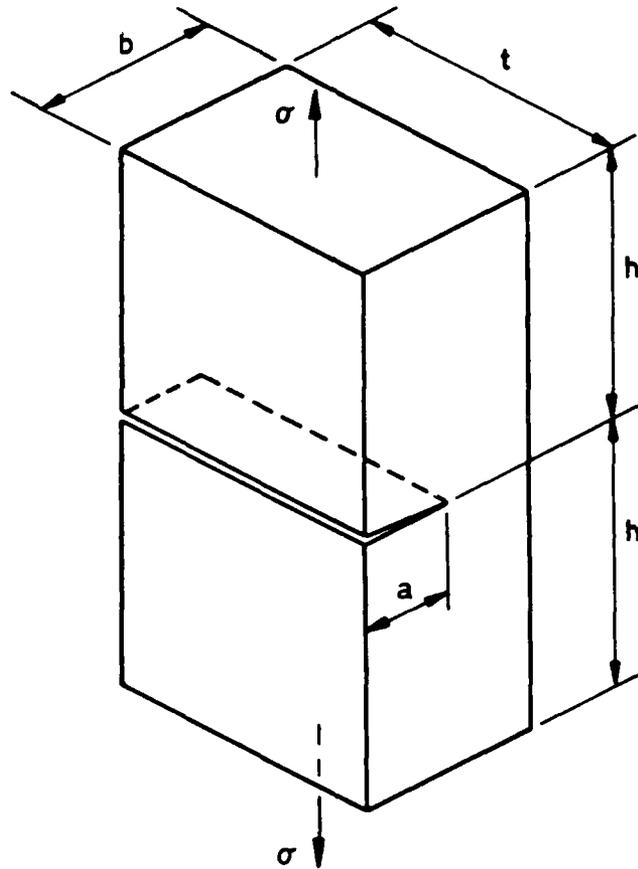


Fig 3 Edge cracked rectangular bar subjected to uniaxial uniform tensile stress

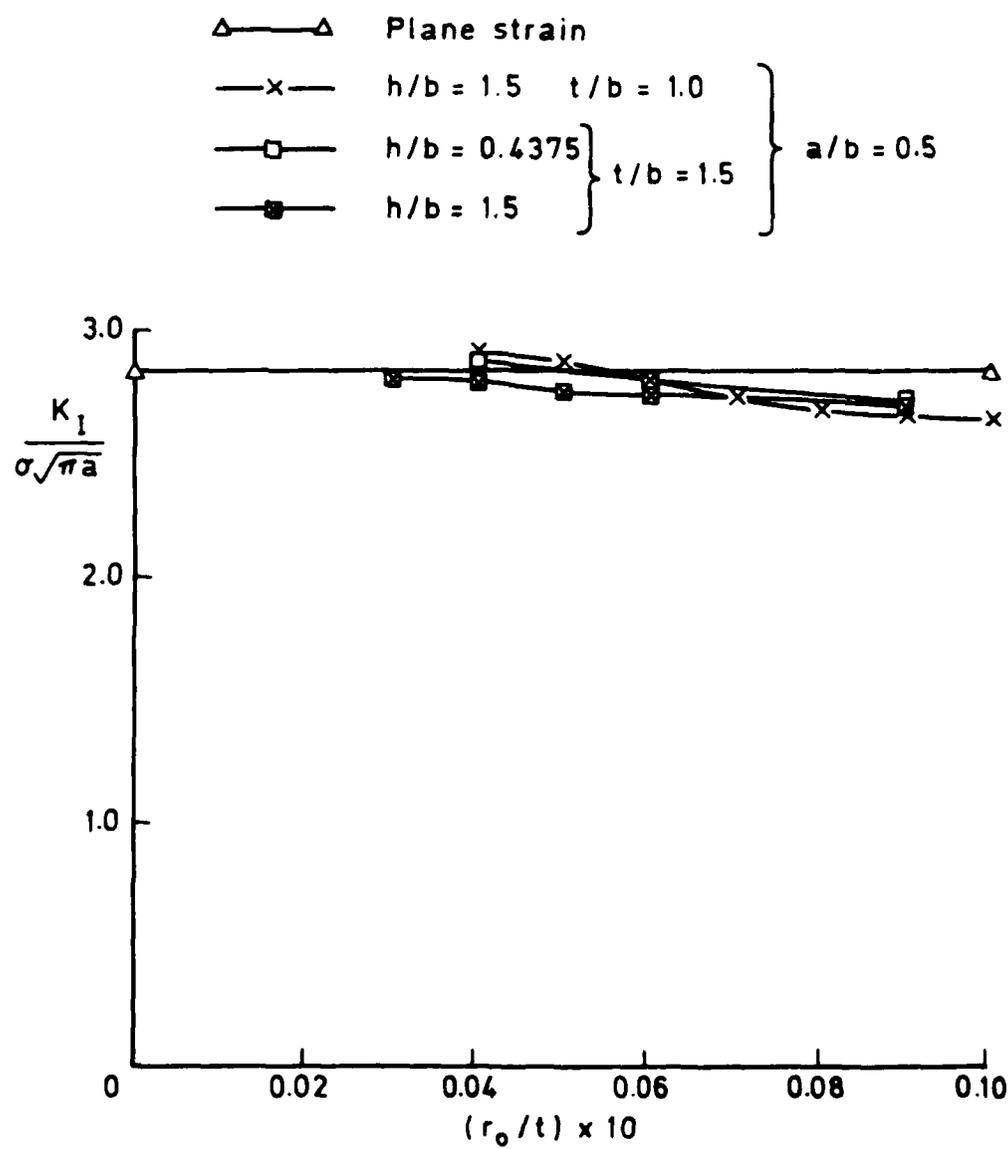


Fig 4 Effect of radius  $r_0$  on the stress intensity factor at mid-thickness



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