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A Simultaneous M-ary Channel Hypothesis Test with Least-Mean-Square Signal Amplitude Estimation

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This paper presents a new strategy for detecting and estimating multiple linearly independent signals immersed in additive random noise. This new approach is a Maximum Log-likelihood Ratio Test (MLRT) and determines the presence of one or more signals through a single filter operating on a received input data vector. In addition, it performs a signal-amplitude estimation when a positive response occurs. This test is a Uniformly Most Powerful Invariant Test and reduces the computationally intensive work to only those situations where important data exists. The estimation technique determines the signal amplitude of all possible input signals in a least-mean-square sense with a variance inversely proportional to the inherent signal-to-noise ratio of each linearly independent library waveform.

Computer simulations verify the theory and demonstrate near-optimum performance of the MLRT for small signal libraries. The MLRT is close to optimum for larger signal sets, and is clearly optimum for complex multiwaveform signals.

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INTRODUCTION

Most problems in surveillance and communications involve the specific determination of whether or not one of a library of complex signals is sent within a fixed observation interval T. In radar, this library is composed of time-displaced versions of a certain known waveform, while in communications the library is either a set of linearly independent waveforms or unique linear combinations of those waveforms (references 1, 2).

Figure 1 illustrates the determination of signal existence which is known as **M-ary Signal Hypothesis Testing** and its most common detection strategy. A time-varying voltage v(t) from a receiving device such as an antenna or hydrophone is input to a set of signal correlators, whose outputs are sent to an "optimum" processor designed to establish signal presence(s). The hypotheses being evaluated are the null hypothesis H_o where the voltage v(t) is just random background and/or electronic noise. The signal presence hypothesis H_1 where the input voltage is random noise plus one or more of M possible waveforms $\{\phi_j(t);\ j = 1, M\}$.

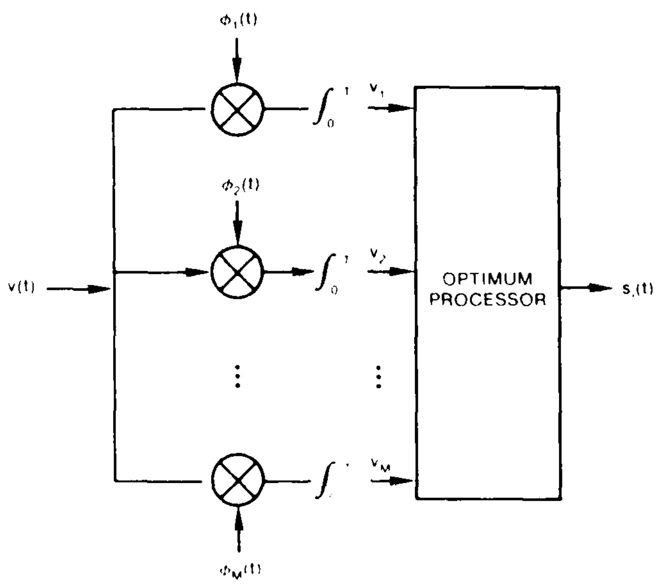

Figure 1 Receiver configuration for a M ary signal filter bank

The specific form of the signal-presence processor depends on the application, the type of random noise involved,and the particular performance criteria adopted by the system designer (references 1, 2). However, the correlator bank approach is always computationally intensive for large signal libraries, no matter what the application or noise process(es) involved.

The intent of this paper is to describe a **Maximum Likelihood Ratio Test (MLRT)** which determines signal presence(s) in a near-optimal fashion and with minimal computational load. Specifically, a log-likelihood ratio test is presented which preprocesses input data with a single filter developed from a M-ary signature library. This test indicates whether one or more of the M possible signal waveforms is present in the input.

The above type of test is analogous to the Uniformly Most Powerful Invariant Test developed by Scharf for unknown amplitude M-ary signal detection in noise (reference 3). Here, this test is developed independently from both a signal detection and estimation point-of-view. When signal presence is indicated, the input voltage train is processed by a signal vector estimation filter, whose output is a Least-Mean-Square (LMS) or Maximum Likelihood (ML) estimate of the amplitudes of the linearly independent signals detected.

In this paper the above detection strategy is tested against computer-generated noise-only, signal-plus-noise data sequences, and the results validate the theory. In addition, comparisons are made between MLRT and correlator bank receivers for single signal presence detection in Additive White Gaussian Noise (AWGN). The results indicate that there is only a small decibel advantage for the latter when M is large. However, this advantage disappears for multiple signal detection scenarios, independent of the size of M.

MAXIMUM LIKELIHOOD RATIO TEST OF LINEARLY SUPERIMPOSED SIGNALS

Consider an L-element sample set of received radar, communication, or lexicographically mapped image data having the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_L \end{bmatrix} = [x_1, x_2, \ldots, x_L]^T . \tag{1}$$

Without loss of generality, the analysis to come assumes only real data sets are involved. One can extend the development to complex data fields by simply substituting Hermitian conjugate or adjoint operations for matrix/vector transpose operations where they occur in the subsequent text (reference 4).

Let $\mathbf{S}$ denote the signature library matrix, composed of the M possible signal vectors which might be sent through the channel. Specifically, one writes

$$\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_M] \tag{2}$$

where

$$\mathbf{s}_j = [s_{1j}, s_{2j}, \ldots, s_{Lj}]^T \tag{3}$$

is the j^{th} signal vector. In this form, $\mathbf{S}$ is an LxM matrix. For this development the M signals are assumed to be a linearly independent set (reference 4).

For the L-sample observation period, consider the possible inputs to the receiver to be either noise only or a noise plus weighted sum of the signal vectors. In particular, let the signal portion of the input to the receiver be written as the column vector

$$\mathbf{s}_I = \mathbf{Sa} = \left[\sum_{i=1}^{M} S_{ji} a_i \right] \tag{4}$$

with

$$\mathbf{a} = [a_1, a_2, \ldots, a_M]^T$$

denoting a M-element column vector of weight coefficients. If the noise incident to the receiver is a zero-mean Gaussian process **n** with autocovariance $\mathbf{K} = E(\mathbf{n}\mathbf{n}^T)$, the input vector **x** has the following form for hypotheses H_o and H_1, respectively

$$H_0: \quad \mathbf{x} = \mathbf{n} \tag{5}$$

under the noise-only hypothesis and

$$H_1: \quad \mathbf{x} = \mathbf{n} + \mathbf{S}\mathbf{a} \tag{6}$$

under the signal-plus-noise hypothesis. In the latter situation, the exact composition of the weight vector **a** is assumed to be known. The accepted way to process this data is through a log-likelihood ratio test. In the following subsections this approach is reviewed, and with it a maximum likelihood ratio test is developed for determining the presence or absence of a library of possible signals and an estimate $\hat{\mathbf{a}}$ of the weight vector **a**. In addition, some important properties and aspects of this test are derived. The first subsection describes classical Neyman-Pearson hypothesis testing (reference 1).

MAXIMUM LOG-LIKELIHOOD RATIO TEST

The log-likelihood ratio test is based on the logarithm of the ratio of the probability density functions associated with the two possible simple hypotheses, H_o, the noise-only hypothesis and H_1, the signal-plus-noise hypothesis, assuming the weight vector **a** is known. When **a** is known, the optimum test is

$$L(\mathbf{x}) = f(\mathbf{x},\mathbf{a}) = \log \left[\frac{p_1(\mathbf{x},\mathbf{a})}{p_0(\mathbf{x})}\right] \begin{array}{c} H_1 \\ \geq \\ < \\ H_0 \end{array} \begin{array}{c} L_0 \\ L_0 \end{array} \tag{7}$$

where

$$p_1(\mathbf{x},\mathbf{a}) = \frac{1}{(2\pi)^{M/2}} \,||\mathbf{K}||^{-1/2} \exp\{-1/2(\mathbf{x} - \mathbf{S}\mathbf{a})^T \mathbf{K}^{-1}(\mathbf{x} - \mathbf{S}\mathbf{a})\} \tag{8}$$

and

$$p_o(x) = \frac{1}{(2\pi)^{M/2}} \,||\mathbf{K}||^{-1/2} \exp\{-1/2\mathbf{x}^T \mathbf{K}^{-1}\mathbf{x}\} \tag{9}$$

$$= p_1(\mathbf{x},\ 0)\ . \tag{10}$$

When the weight vector **a** is unknown, an estimate of **a** is substituted in equation 8 and used in the log-likelihood test given in equation 7. The simplest estimate is the **Maximum Likelihood Estimate** (MLE) of **a** under hypothesis H_1. Using this estimate, the log-likelihood ratio test becomes

$$L(\mathbf{x}) = \max_{\mathbf{a}} f(\mathbf{x},\mathbf{a}) = f(\mathbf{x},\hat{\mathbf{a}}) \underset{H_0}{\overset{H_1}{\substack{\geq L_0 \\ < L_0}}} \tag{11}$$

where

$$f(\mathbf{x},\mathbf{a}) = -1/2(\mathbf{x} - \mathbf{Sa})^T K^{-1}(\mathbf{x} - \mathbf{Sa}) + 1/2\mathbf{x}^T K^{-1}\mathbf{x}$$

$$= (\mathbf{Sa})^T K^{-1}\mathbf{x} - 1/2(\mathbf{Sa})^T K^{-1}(\mathbf{Sa}) \tag{12}$$

since

$$(K^{-1})^T = [E\{nn^T\}^{-1}]^T = K^{-1} . \tag{13}$$

Equation 11 is known as the **Maximum Log-Likelihood Ratio Test**.

The autocovariance K is a positive-definite symmetric matrix. Hence, it has a square-root factorization of the form

$$K = K^{1/2}K^{1/2} \tag{14}$$

where

$$K^{-1/2}KK^{-1/2} = I_L \tag{15}$$

with I_L denoting the LxL identity matrix. The matrix $K^{1/2}$ is used to "whiten" the input data before processing. Mathematically, we write

$$y = K^{-1/2}\mathbf{x} \tag{16}$$

$$n_K = K^{-1/2}n \tag{17}$$

$$S_K = K^{-1/2}S \tag{18}$$

as the "prewhitened" input vector, noise vector, and signal library. Since this transformation is linear, the initial multivariate gaussian nature of the data is retained and the log-likelihood test given in equation 11 becomes

$$f(\mathbf{x},\mathbf{a}) = (S\mathbf{a})^T K^{-1}\mathbf{x} - 1/2(S\mathbf{a})^T K^{-1}(S\mathbf{a})$$

$$= \mathbf{a}^T S_K^T y - 1/2\mathbf{a}^T(S_K^T S_K)\mathbf{a} \underset{H_0}{\overset{H_1}{\gtrless}} L_0 \quad . \qquad (19)$$

By completing the square in equation (19), one obtains

$$f(\mathbf{x},\mathbf{a}) = \mathbf{a}^T S_K^T y - 1/2\left([(S_K^T S_K)^{1/2}]^T \mathbf{a}\right)^T (S_K^T S_K)^{1/2}\mathbf{a}$$

$$= -1/2||(S_K^T S_K)^{1/2}\mathbf{a} - (S_K^T S_K)^{-1/2} S_K y|| + 1/2y^T S_K (S_K^T S_K)^{-1} S_K^T y \qquad (20)$$

$$\leq 1/2y^T S_K (S_K^T S_K)^{-1} S_K^T y \qquad (21)$$

with equality if and only if

$$\mathbf{a} = (S_K^T S_K)^{-1} S_K^T y = \hat{a}(\mathbf{x}) \ . \qquad (22)$$

Equation (22) is the least-squares estimate of the weight vector **a** one obtains from the general theory of regression (reference 5). The existence of the inverse matrix $(S_K^T S_K)^{-1}$ in that expression is guaranteed by the assumption of the independence of the M signal vectors. It is clear that equality holds in equation 21 if and only if the estimate $\hat{a}(\mathbf{x})$ in equation 22 is substituted into $f(\mathbf{x},\mathbf{a})$. This implies that

$$f(\mathbf{x},\hat{a},(\mathbf{x})) \underset{H_o}{\overset{H_1}{\gtrless}} L_o$$

is the required Maximum Log-Likelihood Ratio Test. Hereafter in this paper, it is referred to as the **Maximum Likelihood Ratio Test (MLRT)**.

As noted earlier, the above test is a Uniformly Most Powerful Invariant Test and, as a consequence, is a very reasonable way of detecting the presence of a linear combination of M signals (reference 3). In addition, it accomplishes the signal-presence/signal-absent test using a single filter operation before threshold comparison. In the next subsection, the rank of this processing filter is obtained.

RANK OF THE FILTER MATRIX

In the last subsection the maximum log-likelihood ratio test for determining whether one or more signals are present in a particular data sequence is shown to be

$$L(\mathbf{x}) = 1/2\mathbf{y}^T S_K (S_K^T S_K)^{-1} S_K^T \mathbf{y}$$

$$= 1/2\mathbf{x}^T K^{-1} S(S^T K^{-1} S) S^T K^{-1} \mathbf{x} \quad \overset{H_1}{\underset{H_0}{\gtrless}} L_0 \tag{23}$$

where

$$\hat{\mathbf{a}}(\mathbf{x}) = (S^T K^{-1} S)^{-1} S^T K^{-1} \mathbf{x} \ . \tag{24}$$

As noted above, the estimate $\hat{\mathbf{a}}(\mathbf{x})$ can be used after detection for the discrimination of a signal with one linear combination of the s_j's from the set of signals with other linear combinations of the linearly independent waveforms.

There are two things to note about the above expressions: 1. $L(\mathbf{x})$ is a quadratic form in the input vector $\mathbf{x}$ and 2. $\hat{\mathbf{a}}(x)$ is an unbiased estimate of the weight/amplitude vector $\mathbf{a}$. The processing filter which is applied to the whitened input data $\mathbf{y}$ is given by

$$H = S_K (S_K^T S_K)^{-1} S_K^T = S_K G_K^T \tag{25}$$

where

$$G_K^T = (S_K^T S_K)^{-1} S_K^T = [g_1, g_2, \cdots, g_L] \tag{26}$$

with

$$g_j = [g_{1j}, \cdots, g_{Mj}]^T.$$

Rewriting the filter H in more specific form, it is apparent that

$$H = [S_K g_1, S_K g_2, \cdots, S_K g_L] \tag{27}$$

implies that every column of this filter matrix is a linear combination of the M independent column vectors of S_K. Thus the rank of H must be less than or equal to M, i.e., rank $(H) \leq M$.

Consider the product

$$S_K G_K^T S_K = H S_K = S_K (S_K^T S_K)^{-1} S_K^T S_K = S_K \ .$$

This expression clearly shows that

$$HS_{K(j)} = S_{K(j)} \tag{28}$$

for all j = 1, 2, •••, M. Recall that $S_{K(j)}$ represents a whitened version of one of the M independent column vectors of **S**. Equation 28 has the form of an eigenvalue equation. Equation 23 demonstrates that the column vectors of $\mathbf{S}_K$ are distinct and independent eigenvectors of the matrix operator **H** with equal eigenvalues of magnitude one for all eigenvectors. This implies that the rank of **H** must be greater than or equal to M, i.e., rank (**H**) $\geq$ **M**. If one combines the two inequalities for the rank of **H**, the rank of **H** must equal M. Thus, the matrix operator H has M eigenvalues equal to one and (L-M) eigenvalues equal to zero. The impact of this is that even though each of the signals has L components, the driving factor of the MLRT in terms of degrees of freedom is the number M of signals in the library matrix. In the next section the probabilities of false alarm and detection are determined for the MLRT.

DETECTION AND FALSE ALARM PROBABILITIES FOR MLRT

From the above discussion, the maximum likelihood ratio test L(x) can be rewritten as

$$L(\mathbf{x}) = L(\mathbf{y}) = \mathbf{y}^T\mathbf{H}\mathbf{y} \underset{H_0}{\overset{H_1}{\gtrless}} L_0 \ . \tag{29}$$

Let **U** be an orthogonal unitary matrix which diagonalizes **H**. Specifically, **U** is defined to be

$$U = \left[\frac{s_{K1}}{|s_{K1}|} , \frac{s_{K2}}{|s_{K2}|} , \ldots , \frac{s_{KM}}{|s_{KM}|} , u_{M+1} , \ldots , u_L \right]^T \tag{30}$$

which gives

$$UHU^T = \begin{bmatrix} I_M & 0 \\ 0 & 0_{L-M} \end{bmatrix} = \Lambda_M \ . \tag{31}$$

(See appendix A for the proof.) By defining a new input vector by the relation

$$\mathbf{y} = \mathbf{U}\mathbf{z} \tag{32}$$

the decision strategy for simultaneous multisignal detection becomes

$$\mathbf{y}^T\mathbf{H}\mathbf{y} = \mathbf{z}^T\mathbf{U}\mathbf{H}\mathbf{U}^T\mathbf{z} = \mathbf{z}^T\Lambda_M\mathbf{z} = \sum_{j=1}^{M} z_j^2 \quad \begin{array}{c} H_1 \\ \geq 2L_0 \\ \\ < 2L_0 \\ H_1 \end{array} \quad . \tag{33}$$

Rewriting equation 32, one has

$$\mathbf{z} = \mathbf{U}^T\mathbf{y} \ .$$

The probability density function of this vector is similar to that of the original measurement vector $\mathbf{x}$, except this new distribution has a new mean and covariance values (reference 3). Specifically we have

$$p(\mathbf{z}) = \frac{1}{(2\pi)^{1/2}} \, ||K'||^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{z} - \mathbf{d})^T K'^{-1}(\mathbf{z} - \mathbf{d})\right\} \tag{34}$$

where

$$\mathbf{d} = E\{z\} \tag{35}$$

and

$$\mathbf{K}' = E\{(z - \mathbf{d})(\mathbf{z} - \mathbf{d})^T\} \ . \tag{36}$$

The average value of the vector $\mathbf{z}$, under hypothesis H_1 is equal to

$$\begin{aligned} \mathbf{d} &= E\{\mathbf{z}|H_1\} = E\{U^T n_K + U^T S_K \mathbf{a}\} \\ &= U^T S_K \mathbf{a} \ . \end{aligned} \tag{37}$$

Its covariance, under the same hypothesis, is given by

$$\begin{aligned} E\{\mathbf{z} - \mathbf{d})(\mathbf{z} - \mathbf{d})^T|H_1\} &= E\{U^T n_K n_K^T U\} \\ &= U^T E\{n_K n_K^T\} U \\ &= I_L \ . \end{aligned} \tag{38}$$

Equation 38 also holds for hypothesis H_0. However, the mean vector $\mathbf{d}$ for H_0 is the null vector.

Referring to equation 33, it is clear that the MRLT decision strategy depends on the norm of the vector **z**. Under hypothesis H_1, the probability density function for this situation is the noncentral chi-squared density

$$p_1(r) = d(r/d)^{M/2} e^{-(r^2+d^2)/2} I_{\frac{(M-2)}{2}}(rd) \tag{39}$$

where by equation 33,

$$r = \left(\sum_{j=1}^{M} z_j^2 \right)^{1/2} = [(\mathbf{x}^T K^{-1} S)(S^T K^{-1} S)^{-1}(S^T K^{-1} \mathbf{x})]^{1/2} \tag{40}$$

and

$$\begin{aligned} d^2 = |d|^2 &= \mathbf{a} S_K^T U U^T S_K \mathbf{a} \\ &= \mathbf{a}(S^T K^{-1} S)\mathbf{a} \end{aligned} \tag{41}$$

is the peak signal-to-noise ratio (reference 6).

The probability density function under hypothesis H_0 is central chi-squared ~~and~~ is derived as follows: The modified Bessel function of the first kind can be written as

$$I_\nu(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{\nu+2n}}{n!\Gamma(\nu + n + 1)} \; . \tag{42}$$

Equation 39 then becomes

$$\begin{aligned} p_1(r) &= d(r/d)^{M/2} e^{-(r^2+d^2)/2} I_{\frac{(M-2)}{2}}(rd) \\ &= (2)^{M/2} e^{-(r^2+d^2)/2} (r/2)^{M-1} \sum_{n=0}^{\infty} \frac{(1/2rd)^{2n}}{n!\Gamma(M/2 + n)} \; . \end{aligned} \tag{43}$$

This implies that

$$p_o(r) = \lim_{d\to 0} p_1(r) = 2^{M/2} (r/2)^{M-1} e^{-r^2/2} \frac{1}{\Gamma(M/2)} \tag{44}$$

is the probability density function for the null hypothesis. Equation 44 is a chi-squared density function with M/2 degrees of freedom.

Based or the above, one can write

$$p_d = d \int_{b_0}^{\infty} (r/d)^{M/2} e^{-(r^2+d^2)/2} I_{\frac{(M-2)}{2}}(rd) \quad dr \tag{45}$$

as the probability of detection for the maximum likelihood ratio test, and

$$p_{fa} = \int_{b_0}^{\infty} \frac{r^{M-1} e^{-r^2/2}}{2^{M/2-1}\Gamma(M/2)} dr \tag{46}$$

as its probability of a false alarm. In these two equations b_0 is the MLRT threshold level in normalized units. If one refers to the usual analysis of a multiple signal detection (e.g., reference 1), equations 45 and 46 indicate that MLRT has one half the number of degrees of freedom that would normally be expected for this type of detection strategy.

COMPUTER SIMULATIONS

With the above detection strategy defined, a validation of the maximum likelihood ratio test was performed. In addition, the quality of the weight vector estimation under positive processor response was concurrently assessed. Three computer simulations were performed which assumed the number of signals in the library to be of lengths 2, 4, and 6.

In the following subsection, the performance of the MLRT is summarized for the six-signal library simulation. For simplicity, the input noise is assumed to be Additive White Gaussian Noise (AWGN).

The simulations were performed on a AT-clone Personal Computer with an 80287 Co-processor. The computer programs themselves were written in Microsoft FORTRAN, Version 3.31, and required externally generated random numbers to create the simulated noise sequences.

The uniform random deviates are generated from the RAN1 and RAN3 subroutines developed by Knuth and others (reference 7). These routines create uniformly distributed numbers between 0.0 and 1.0 with negligible sequential correlations. These numbers are translated into normal deviates (zero mean and unity variance) using a technique suggested by Dillard (reference 8).

The linearly independent signals chosen for the simulations were binary (+1,-1) pulse sequences with various power-of-two frequencies. The computation of the theoretical false alarm and detection probabilities were performed using a recursive relation for the generalized Q-function described by Dillard (reference 9).

The six-signal library simulation was performed as follows. Six linearly independent signal vectors were generated. The selected signal cycles were 2, 4, 8, 16, 32, and 64 Hz. The length of each sample vector was 128. This is chosen to accommodate two cycles of the lowest frequency waveform. Under the AWGN assumption, the processing filter **H** described in equation 25 was created and stored, as well as the estimating filter

$$\hat{\mathbf{H}} = (\mathbf{S}^T\mathbf{K}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{K}^{-1} \quad . \tag{47}$$

Ten thousand null-hypothesis sequences and ten thousand signal-present data sequences were independently generated and applied to both filters. The filtered results were stored and later statistically analyzed.

Figure 2 shows the theoretical and simulation-derived false alarm probabilities for the six-signal library MLRT case. The error bars were computed using the analytical expression developed by Clopper and Pearson (reference 10). Figure 3 compares theoretical and computer-generated results for the probability of detection as a function of threshold for input signal-to-noise ratios equal to 128, 288, and 512. Both figures illustrate good agreement between simulation and theory. This good agreement is similarly found for the two- and four-signal library simulations not discussed.

The LMS estimate of the weight vector **a** gave good results in predicting the presence and amplitude of the appropriate signal when hypothesis H_1 is true. The covariance of the prediction can be shown (reference 5) to be given by

$$\sigma^2_{\hat{a}} = E(\mathbf{a}\mathbf{a}^T|H_1) - [E(\mathbf{a}|H_1)\ E(\mathbf{a}^T|H_1)] = (\mathbf{S}^T\mathbf{K}^{-1}\mathbf{S})^{-1} \quad . \tag{48}$$

All three simulations exhibited this variation in the statistics of the weight vector estimate. In the six-signal library case, the normalized standard deviation of the estimate was ~9% as expected. This expression shows that the accuracy of the estimation is inversely proportional to the inherent signal-to-noise ratio of each waveform in the signal library. This is not an

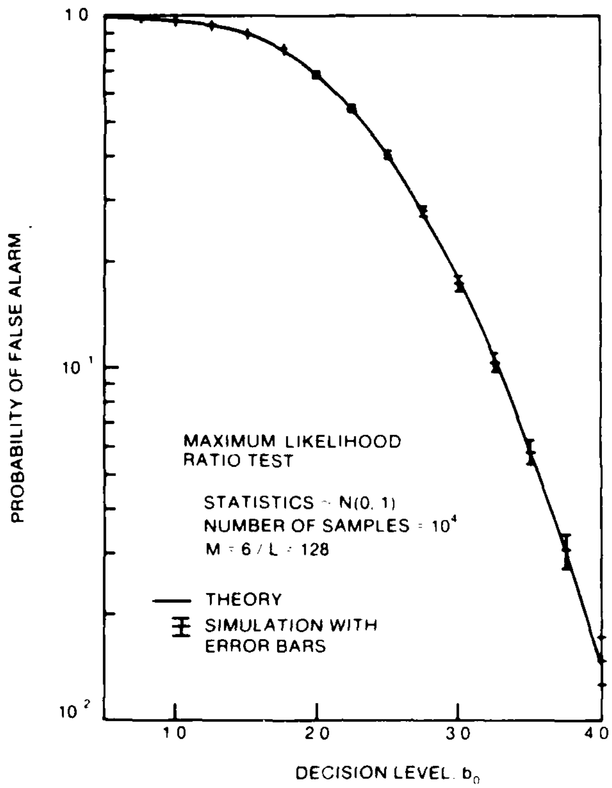

Figure 2. A comparison of the false alarm probability for a six-signal library MLRT with computer simulation-derived data

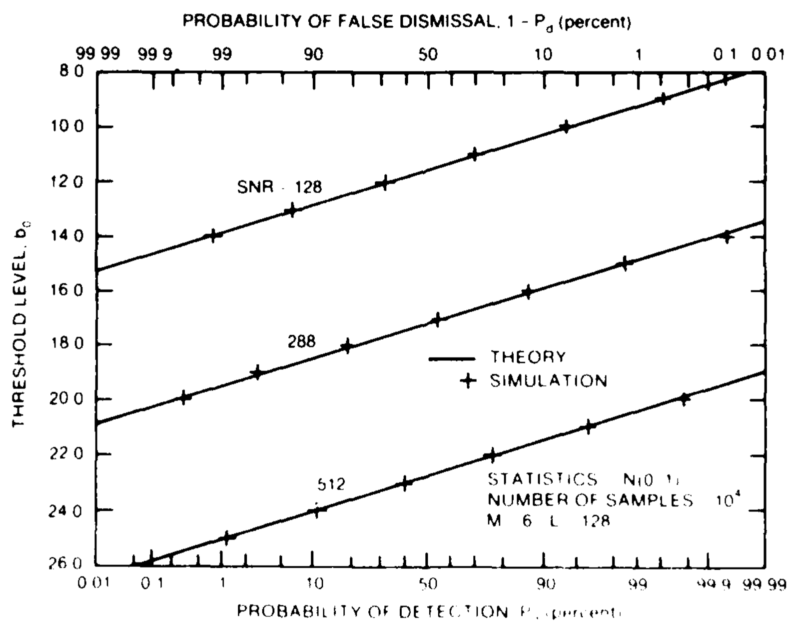

Figure 3. A comparison of the detection probability for a six signal library MLRT with computer simulation derived data at signal to noise ratios of 128, 288 and 512

unexpected result. Parameter estimation is always better with strong signals (reference 5).

A COMPARISON OF MLRT AND THRESHOLDED/SIGNAL-CORRELATION RECEIVERS

To demonstrate the power of the MLRT procedure, a comparison is made between this new detection strategy and a strategy based on the approach shown in figure 1. One of the most common techniques for M-ary hypothesis testing is to use a filter bank of M signal correlators with a threshold signal-presence detection criteria (reference 1). The outputs of the M correlators are compared to a predetermined value and those exceeding this value are said to indicate signal presence during that observation time. The null hypothesis occurs when none of the correlator outputs exceed the predetermined threshold. The particular threshold value selected is based on the desired probability of error.

In this section a comparison is made between a MLRT receiver and the above-cited filter bank receiver for the case of 6, 200, and 400 linearly independent signals at two fixed false alarm rates. These examples illustrate the relative performance of these two M-ary detection techniques, but more importantly they scope the kind of penalties encountered from the use of the less computationally intensive MLRT detection strategy. In other words, these comparisons demonstrate how close to optimum the MLRT really is.

Assume that a receiver filter bank has been created to determine which of M possible signals is sent. The M waveforms of interest are all of equal energy. For this comparison, AWGN with zero mean and unit variance again is assumed. In order to compare the detection performance of the filter bank receiver with a MLRT receiver, analytical expressions for the probabilities of false alarm and detection of the correlation receiver must be determined.

The probability of the correct "reception" of the null hypothesis is

$$p[\text{correct}|H_0] = p_r\{\text{all } \mathbf{s}_j^T\mathbf{K}^{-1}\mathbf{x} < b_0;\ j = 1, 2, 3, \cdots, M|H_0\} \tag{49}$$

$$= \prod_{j=1}^{M} p_r\{s_j^T K^{-1}\mathbf{x} < b_0\} \tag{50}$$

$$= \left(1 - \int_{b_0}^{\infty} p_n(r)\,dr\right)^M . \tag{51}$$

The integral

$$\int_{b_0}^{\infty} p_n(r)dr \tag{52}$$

represents the probability of choosing the signal presence hypothesis after thresholding when the null hypothesis is true. This is the false alarm probability or the Error of the First Kind. For AWGN, equation 52 is defined as

$$\mathrm{erfc}(b_0) = (2\pi)^{-1/2} \int_{b_0}^{\infty} \exp(-t^2/2)\ dt \ . \tag{53}$$

As before, b_0 is the receiver threshold setting.

The probability of false alarm is the difference between equation 51 and unity. Specifically, the false alarm probability for a M-ary filter bank with threshold detection is given by

$$p_{fa} = 1 - p[\mathrm{correct}|H_0] = 1 - (1 - \mathrm{erfc}(b_0))^M \ . \tag{54}$$

Rewriting equation 52, one has

$$\mathrm{erfc}(b_0) = 1 - (1 - p_{fa})^{1/M} \ . \tag{55}$$

This last expression can be used to determine the threshold setting for a fixed false alarm probability using an inverse error-function integral relation (reference 11).

Since the assumed waveforms have equal energy, their integrated power levels are also equal. The probability of detection is, therefore, the same for each of the M signal correlators and is given by

$$p_d \simeq 1 - (1 - \mathrm{erfc}(b_0 - \sqrt{SNR}))^{b_1} \tag{56}$$

with SNR representing the expected signal-to-noise ratio for each and b_1 being the number of waveforms per complex signal (reference 12). Equation 56 assumes that the probability of false alarm at every filter contributes nothing to the detection of any signal. For this analysis, the number of waveforms per signal is 1 and equation 56 reduces to

$$p_d \simeq \mathrm{erfc}(b_0 - d) \ . \tag{57}$$

Figure 4 compares the probability of detection for MLRT and filter bank receivers, given six linearly independent signals, as a function of signal-to-noise ratio for false alarm probabilities of 1E-04 and 1E-06. The MLRT results are computed using equations 45 and 46, and the filter bank numbers were computed using equations 55 and 57. It is apparent that the filter bank performs better than the MLRT for all signal-to-noise ratios shown. However, MLRT suffers only a 1 dB penalty over this same range and is less computationally intensive for situations when the null hypothesis is the most frequently occurring event. This supports the previous contention that MLRT is a near-optimum M-ary detection strategy for AWGN, or at least for small signal library scenarios. The signal-to-noise ratio penalty for large signal library detection is investigated next.

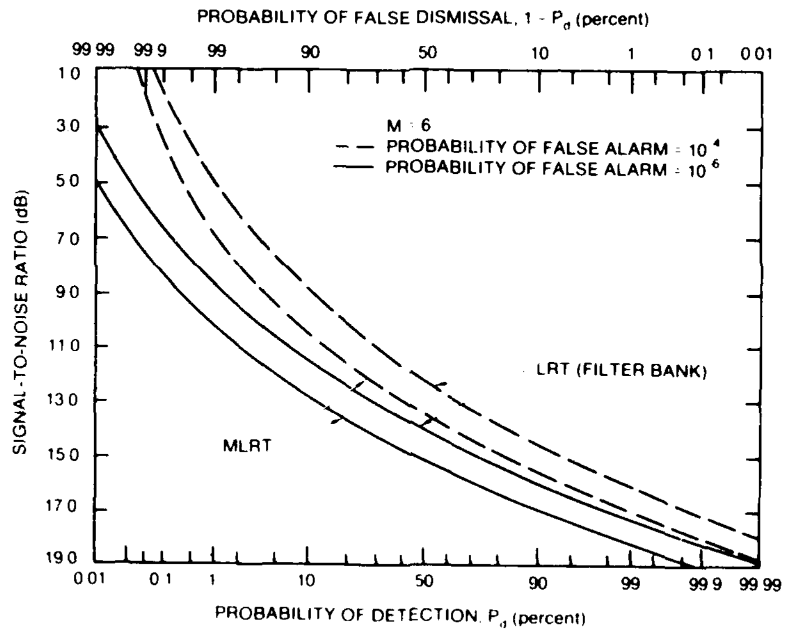

Figure 4 The probability of detection false dismissal as a function of signal to noise ratio for MLRT and filter bank receivers at false alarm probabilities of 1 E 04 and 1 E 06, and with M equal to 6

For large numbers of signals, the probabilities of false alarm and detection for the MLRT reduce to

$$P_{fa} \approx \mathrm{erfc}([b_0^2 - M]/\sqrt{2M}) \tag{58}$$

and

$$P_d \approx \mathrm{erfc}\,(b_0^2 - M - d^2)/\sqrt{2}(M + 2d^2) \tag{59}$$

respectively, for values of M greater than 200 (see appendix B). The false alarm and detection probabilities for the filter bank/threshold receiver are

$$p_{fa} \approx M \text{ erfc } (b_0) \qquad (60)$$

and

$$p_d \approx \text{erfc } (b_0 - d) \ . \qquad (61)$$

Equation 60 assumes that $\text{erfc}(b_0) << 1$. Figure 5 compares the probabilities of detection for the MLRT and the filter bank/threshold receivers for M = 200 and false alarm probabilities of 1E-04 and 1E-06. As expected, the filter bank/threshold receiver performs better than the MLRT receiver. In fact, it is around 4.5 to 5 dB better for reasonable detection probabilities. This is really not too bad considering the reduced computational load requirements one has with the latter. Figure 6 depicts the same comparison as figure 5, except for M = 400. This figure shows the MLRT receiver has a 1 to 1.5 dB increase in SNR penalty for the factor of two increase in the number of signals. Again, this is not a significant penalty for the reduced data processing load. However, the choice depends on the application.

When the signals of interest are linear combinations of linearly independent waveforms, the MLRT detection strategy is the definitely preferred approach. This is because the MLRT filter causes the linear combination of waveforms to incoherently add their respective energies before thresholding. As an example, figure 7 shows the probability of detection for false alarm probabilities of 1E-04 and 1E-06 where each complex signal is assumed to be composed of two linear independent waveforms. It is clear in this figure that MLRT performs better over most of the signal-to-noise ratio range shown at both false alarm probabilities. MLRT has the effect of linearly increasing the total signal-to-noise ratio within the test by the total number of signals in the linear combination. In this case, the linear increase in signal-to-noise ratio is a factor of 2 or 3 dB. On the other hand, the correlator bank still looks for transmitted signals on an individual linearly independent waveform detection basis and is not able to take advantage of the total energy of the complex signal. The probability of detection for the filter bank is

$$P_d \approx 1 - (1 - \text{erfc}(b_0 - d))^2 \qquad (62)$$

with d again being the signal-to-noise ratio of the individual waveforms.

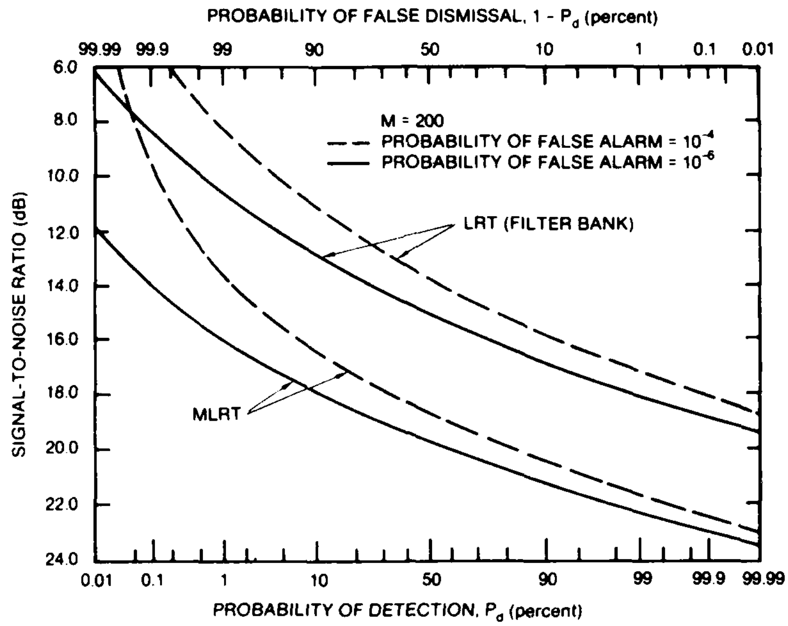

Figure 5. The probability of detection/false dismissal as a function of signal-to-noise ratio for MLRT and filter bank receivers at false alarm probabilities of 1.E-04 and 1.E-06, and with M equal to 200.

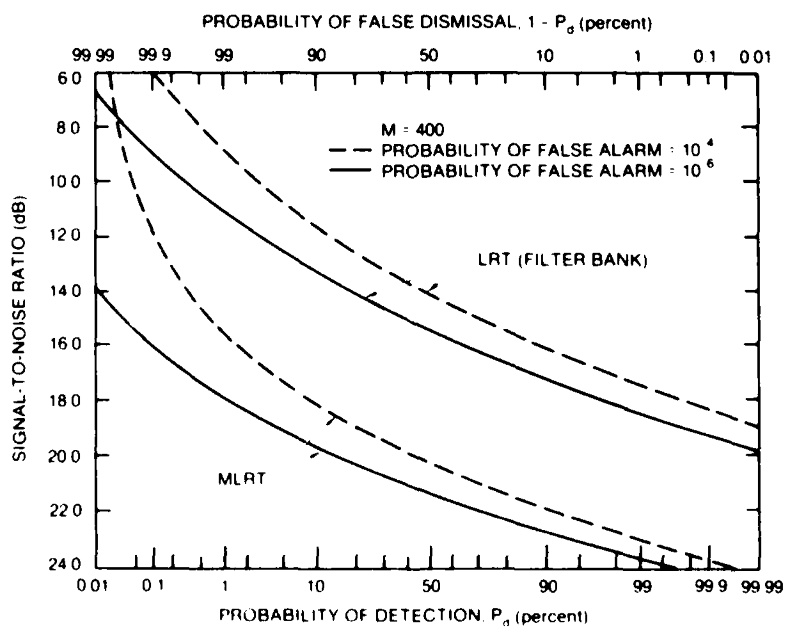

Figure 6 The probability of detection false dismissal as a function of signal to noise ratio for MLRT and filter bank receivers at false alarm probabilities of 1 E 04 and 1 E 06, and with M equal to 400

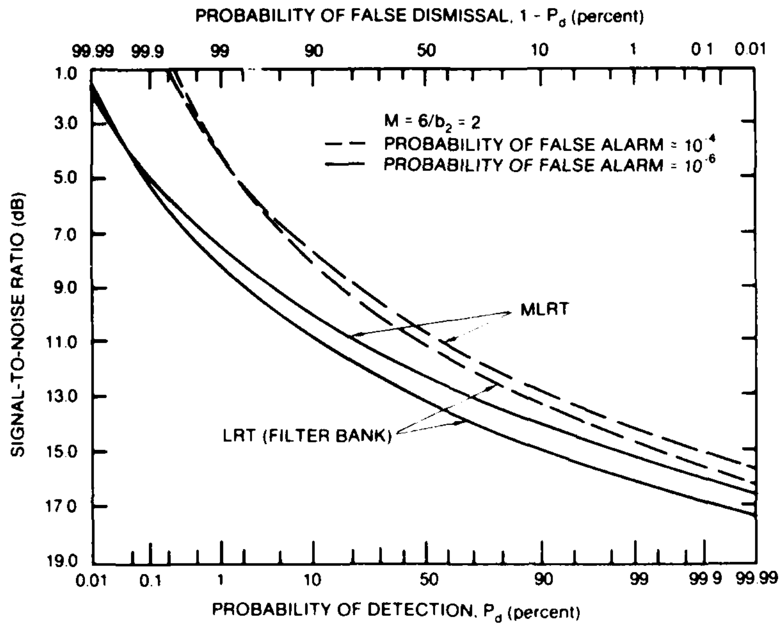

Figure 7 The probability of detection/false dismissal as a function of signal-to-noise ratio for two-signal-combination MLRT and filter bank receivers at false alarm probabilities of 1.E-04 and 1.E-06, and with M equal to 6.

SUMMARY

This paper has presented a new strategy for detecting and estimating multiple linearly independent signals immersed in additive random noise. This technique is most useful for communication and surveillance applications. This new approach determines the presence of one or more signals through a single filter operating on the received input vector, and performs a signal-amplitude estimation when a positive response o^curs. It is a Uniformly Most Powerful Invariant Test and reduces the compu' .tionally intensive work to only those situations where important data may exis.. The estimation technique determines the signal amplitude of all the possible input signals in an LMS sense with variance inversely proportional to the inherent signal-to-noise ratio of each linearly independent waveform.

Computer simulations verify the theory and demonstrate near-optimum performance of MLRT for small signal libraries. The MRLT is close to optimum for larger signal sets and is clearly optimum for complex multiwaveform signals.

APPENDIX A: THE UNITARY TRANSFORMATION MATRIX U WHICH DIAGONALIZES H

It is known that any arbitrary $L \times N$ matrix F of rank M can be decomposed into the sum of a weighted set of unit rank $L \times N$ matrices by a singular value decomposition (SVD) (references 13, 14). Application of this concept to the MLRT processing filter H will be used to determine the unitary transformation matrix U.

According to the SVD matrix decomposition, there exists an $L \times L$ unitary matrix U for which

$$UHU^T = \Lambda^{1/2} \tag{A-1}$$

where

$$U = [u_1, u_2, u_3, \ \ldots, \ u_L]^T \tag{A-2}$$

$$H = S_K(S_K^T S_K)^{-1} S_K^T \tag{A-3}$$

$$\Lambda^{1/2} = \begin{array}{c} \left[\begin{array}{ccc:c} \lambda^{1/2}(1) & & & \\ & \ddots & & 0 \\ & & \lambda^{1/2}(M) & \\ \hdashline & 0 & & 0 \end{array}\right] \\ \begin{array}{cc} \underbrace{\hphantom{\lambda^{1/2}(1)\qquad\lambda^{1/2}(M)}}_{M} & \underbrace{\hphantom{000}}_{L-M} \end{array} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} M \\ \left.\vphantom{a}\right\} L - M \\ \vphantom{\underbrace{a}_{M}} \end{array} \tag{A-4}$$

The matrix $\Lambda^{1/2}$ is called the singular value of the matrix H.

Since U is an unitary matrix, $U^TU = UU^T = I_L$. Consequently,

$$H = U^T \Lambda^{1/2} U \ . \tag{A-5}$$

The rows of the unitary matrix U are composed of the eigenvectors of the product matrix HH^T. The defining relation is

$$
UHH^TU^T = \begin{array}{c}
\left[\begin{array}{ccc|c}
\lambda(1) & & & \\
 & \ddots & & 0 \\
 & & \lambda(M) & \\
\hline
 & & 0 & 0
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} M \\
\left.\vphantom{a}\right\} L - M
\end{array} \\
\underbrace{\hphantom{\lambda(1)\ \ddots\ \lambda(M)}}_{M}\ \underbrace{\hphantom{0}}_{L-M}
\end{array}
\tag{A-6}
$$

where $\lambda(j)$ are the nonzero eigenvalues of the matrix HH^T (reference 13). Since H is a symmetric matrix,

$$
\begin{aligned}
HH^T &= S_K(S_K^TS_K)^{-1}S_K^TS_K(S_K^TS_K)^{-1}S_K^T \\
&= S_K(S_K^TS_K)^{-1}S_K^T \\
&= H \ .
\end{aligned}
\tag{A-7}
$$

Substituting equation A-7 into equation A-6, we obtain

$$
UHU^T = \begin{array}{c}
\left[\begin{array}{ccc|c}
\lambda(1) & & & \\
 & \ddots & & 0 \\
 & & \lambda(M) & \\
\hline
 & & 0 & 0
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} M \\
\left.\vphantom{a}\right\} L - M
\end{array} \\
\underbrace{\hphantom{\lambda(1)\ \ddots\ \lambda(M)}}_{M}\ \underbrace{\hphantom{0}}_{L-M}
\end{array}
\tag{A-8}
$$

The MLRT processing filter H has been shown in the main text to have M unity eigenvalues with associated eigenvectors equal to the columns of S_K. This implies that equation A-2 becomes

$$
\begin{aligned}
U &= [u_1, u_2, \ldots, u_L]^T \\
&= \left[\frac{s_{K1}}{|s_{K1}|}, \frac{s_{K2}}{|s_{K2}|}, \ldots, \frac{s_{KM}}{|s_{KM}|}, u_{M+1}, \ldots, u_L\right]^T
\end{aligned}
\tag{A-9}
$$

taking into account the unitary nature of U. The vectors $(u_{M+1}, \ldots, u_L)$ remain arbitrary, subject only to the unitary requirement of U (reference 15). Substituting the eigenvalues of H into equation A-8 yields

$$UHU^T = \begin{array}{c} \left[\begin{array}{cccc|c} 1 & & & & \\ & \cdot & & & \\ & & \cdot & & 0 \\ & & & \cdot & \\ & & & 1 & \\ \hline & & & & \\ & & 0 & & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{array}}\right\} M \\ \left.\vphantom{\begin{array}{c}0\\ 0\end{array}}\right\} L - M \end{array} \\ \underbrace{\hphantom{1\ \cdot\ \cdot\ \cdot\ 1}}_{M} \ \underbrace{\hphantom{0\ 0}}_{L - M} \end{array} \qquad \text{(A-10)}$$

which completes the derivation of U.

APPENDIX B: APPROXIMATE EXPRESSIONS FOR THE PROBABILITIES OF DETECTION AND FALSE ALARM

When the number of waveforms M in the signal library of MRLT is large, the probability density function (pdf) of the statistic $r^2 = \sum_{j=1}^{M} z_j^2$ is nearly Gaussian by virtue of the Central Limit Theorem. Following Helstrom (reference 1), an asymptotic expression for the pdf $p_1(r^2)$ is derived in this appendix which exhibits this limiting Gaussian character and is used to establish approximate expressions for the probabilities of detection and false alarm associated with this new log-likelihood ratio test for large M.

Recall that the logarithm for the characteristic function h(z) of a random variable x can be expressed in a power series of the form

$$\ln h(z) = iz\bar{x} + \sigma^2 (iz)^2/2 + \sum_{k=3}^{\infty} \eta_k (iz)^k/k! \tag{B-1}$$

where

$$\bar{x} = E\{x\} \ ,$$

$$\sigma^2 = E\{(x - \bar{x})^2\} \ ,$$

and η_k is the "k-th cumulant" of the random variable x. This implies that

$$h(z) = \exp\{iz\bar{x} + \sigma^2 (iz)^2/2!\} \cdot \left[1 + \sum_{k=3}^{\infty} C_k (iz)^h\right] \tag{B-2}$$

after collecting terms with like-powers of (iz). The coefficients C_k contained in equation B-2 have the form

$$C_3 = \eta_3/3!$$

$$C_4 = \eta_4/4!$$

$$C_5 = \eta_5/5!$$

$$C_6 = (\eta_6 + 10\eta_3^2)/6!$$

and so on. The pdf of x is the inverse Fourier transform of the characteristic function h(z) and is equal to the following:

$$p(x) = 2\pi\sigma^2)^{-1/2} \exp\{-(x - \bar{x})^2/2\sigma^2\} \left[1 + \sum_{k=3}^{\infty} (C_k/\sigma^k) h_k(u)\right] \tag{B-3}$$

$$= \sigma^{-1}\left[\phi^o(u) + \sum_{k=0}^{\infty} (-1)^k (C_k/\sigma^k)\phi^k(u)\right] \tag{B-4}$$

with

$$(-1)^k e^{-u^2/2} h_k(u) = \frac{d^k}{du^k}(e^{-u^2/2}) = (2\pi)^{1/2}\phi^k(u) \tag{B-5}$$

$$\phi^k(u) = (-1)^k \phi^k(-u) \quad . \tag{B-6}$$

The functions $h_k(u)$ in equation B-3 denote Hermite Polynomials which are defined by equation B-5.

Evaluating equation B-3 when

$$p_1(L) = 1/2(L/S)^{(N-1)/2} \exp\{-(L + S)/2\} \, I_{N-1} \, (\sqrt{SL}) \tag{B-7}$$

yields

$$p_1(L) = \sigma_L^{-1}[\phi^0(y) - \frac{2N + 3S}{6(N + S)^{3/2}} \phi^3(y)$$

$$+ \frac{N + 2S}{4(N + S)^2} \phi^4(y) + \frac{12(N + 3S) + (2N + 3S)^2}{72(N + S)^3} \phi^0(y)$$

$$+ \cdot \cdot \cdot] \tag{B-8}$$

$$y = \frac{L - 2N - S}{2(N + S)^{1/2}} \quad . \tag{B-9}$$

The probability of detection is given by the termwise integration of this density function. The probability of a false alarm is also obtainable from equation B-8 by setting S = 0 before the termwise integration.

When M is large, only the first two terms of equation B-8 need be retained. The probabilities of false alarm and detection in this case after termwise integration are

$$P_{fa} \approx \mathrm{erfc}(L_0') \tag{B-10}$$

and

$$P_d \approx \mathrm{erfc}(L_0'') \tag{B-11}$$

with

$$L_0' = N^{-1/2}(1/2\ L_0 - N) \tag{B-12}$$

and

$$L_0'' = (L_0' - 1/2\ SN^{-1/2})\left(\frac{N}{N + S}\right)^{1/2} \quad . \tag{B-13}$$

In this paper's nomenclature, equations B-12 and B-13 are

$$L_0' = \left(\frac{M}{2}\right)^{-1/2}(1/2\ b_0^2 - M/2) = (b_0^2 - M)/\sqrt{2M} \tag{B-14}$$

and

$$L_0'' = [(b_0^2 - M)/\sqrt{2M} - 1/2\ d^2(M/2)^{-1/2}]\left[\frac{\frac{M}{2}}{\frac{M}{2} + d^2}\right]^{1/2}$$

$$= (b_0^2 - M - d^2)\left(\frac{1}{\sqrt{2M}}\right)\left[\frac{\frac{M}{2}}{\frac{M}{2} + d^2}\right]^{1/2}$$

$$= (b_0^2 - M - d^2)/\sqrt{2(M + 2d^2)} \quad . \tag{B-15}$$

REFERENCES

1. C.W. Helstrom, 1968. *Statistical Theory of Signal Detection*, Second Edition, Pergamon Press, New York.
2. H. . Van Trees, 1968. *Detection, Estimation, and Modulation Theory*, Wiley and Sons, New York, Vol. I.
3. L. Scharf, *Statistical Signal Processing: Detection, Estimation, and Time Series*, Section 4.9; To be published.
4. J. Mathews and R.L. Walker, 1970. *Mathematical Methods of Physics*, Second Edition, W.A. Benjamin, New York, Chapter Six, pp. 139-166.
5. M.G. Kendall and A. Stuart, 1961. *The Advanced Theory of Statistics*, Hafner Publishing Company, New York, Volume II, Chapter 28, pp. 354-355.
6. K. . Miller, 1964. *Multidimensional Gaussian Distributions*, John Wiley and Sons, New York.
7. W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, 1986. *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, New York, Chapter 7, pp. 192-199.
8. G.M. Dillard, 1967. "Generating Random Numbers Having Probability Distributions Occurring in Signal Detection Problems," *IEEE Transactions on Information Theory*, IT-13, Number 4, pp. 616-617, October.
9. G.M.Dillard, 1973. "Recursive Computation of the Generalized Q Function," *IEEE Transactions on Aerospace and Electronic Systems*, Correspondence pp. 614-615, July.
10. C.J. Clopper and E.S. Pearson, 1934. "The use of confidence or fiducial limits illustrated in the case of the Binomial," *Biometrika: A Journal for the Statistical Study of Biological Problems*, Cambridge University Press, Cambridge, Volume XXVI, pp. 404-413.
11. M. Abramowitz and I.A. Segun, 1970. *Handbook of Mathematical Functions*, National Bureau of Standards.
12. R.A. Dillard, 1986. "Detection of Covert Signals," Naval Ocean Systems Center Technical Document, TD965, August, Section 3.4.4, pp. 3.4-8 to 3.4-12.
13. W.K. Pratt, 1978. *Digital Image Processing*, John Wiley and Sons, New York, Chapter 5, pp. 126-127.
14. H.C. Andrews and B.R. Hunt, 1977. *Digital Image Restoration*, Prentice Hall, Englewood Cliffs, New Jersey, Chapter 2, pp. 30-42.
15. E. Kreyszig, 1967. *Advanced Engineering Mathematics*, John Wiley and Sons, New York, Chapter 7, pp. 388-389.