

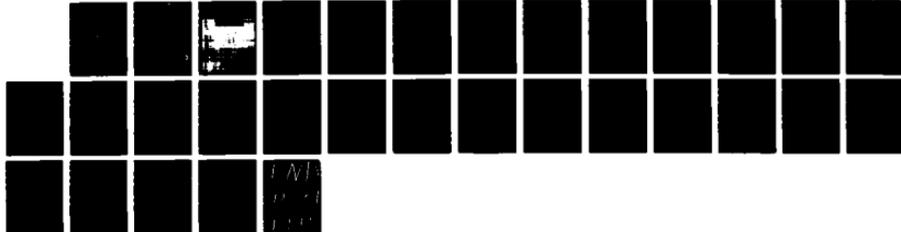
NO A105 004

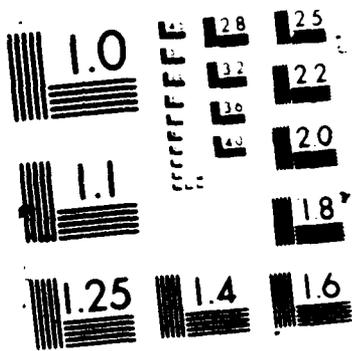
SHADOW SYSTEMS AND ATTRACTORS IN REACTION-DIFFUSION  
EQUATIONS. (U) BROWN UNIV PROVIDENCE RI LEFSCHETZ  
CENTER FOR DYNAMICAL SYSTEMS. J K HALE ET AL. APR 87  
LCDS/CCS-87-28 AFOSR-TR-87-1526 F/G 12/1

1/1

UNCLASSIFIED

NL







SECURITY

IDENTIFICATION PAGE

Form Approved OMB No. 0704-0188

1a RE <b>AD-A185 804</b>		1b RESTRICTIVE MARKINGS	
2a SECURITY CLASSIFICATION		3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.	
2b DECLASSIFICATION/DOWNGRADING SCHEDULE		5 MONITORING ORGANIZATION REPORT NUMBER(S) <b>AFOSR-TR-87-1526</b>	
4 PERFORMING ORGANIZATION REPORT NUMBER(S)		7a NAME OF MONITORING ORGANIZATION	
6a NAME OF PERFORMING ORGANIZATION	6b OFFICE SYMBOL (if applicable)	7b ADDRESS (City, State, and ZIP Code)	
6c ADDRESS (City, State, and ZIP Code)		9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER <b>AFOSR-84-0376</b>	
8a NAME OF FUNDING/SPONSORING ORGANIZATION	8b OFFICE SYMBOL (if applicable) <b>nm</b>	10 SOURCE OF FUNDING NUMBERS	
8c ADDRESS (City, State, and ZIP Code)		PROGRAM ELEMENT NO.	PROJECT NO.
11 TITLE (Include Security Classification)		TASK NO.	WORK UNIT ACCESSION NO.
12 PERSONAL AUTHOR(S)			
13a TYPE OF REPORT	13b TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
16 SUPPLEMENTARY NOTATION			
17 COSATI CODES		18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB-GROUP	
19 ABSTRACT (Continue on reverse if necessary and identify by block number)			
20 DISTRIBUTION/AVAILABILITY OF ABSTRACT <input type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS		21 ABSTRACT SECURITY CLASSIFICATION	
22a NAME OF RESPONSIBLE INDIVIDUAL		22b TELEPHONE (Include Area Code)	22c OFFICE SYMBOL

**DTIC SELECTED**  
**NOV 06 1987**  
**S E D**

AFOSR-TR. 87-1526

SHADOW SYSTEMS AND ATTRACTORS  
IN REACTION-DIFFUSION EQUATIONS

by

Jack K. Hale and Kunimochi Sakamoto

April 1987

LCDS/CCS #87-28



Lefschetz Center for Dynamical Systems  
and  
Center for Control Sciences

Division of Applied Mathematics

Brown University Providence RI 02912

87 14 418

SHADOW SYSTEMS AND ATTRACTORS  
IN REACTION-DIFFUSION EQUATIONS

by

Jack K. Hale and Kunimochi Sakamoto

April 1987

LCDS/CCS #87-28

Accession For	
DTIC GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	



**SHADOW SYSTEMS AND ATTRACTORS  
IN REACTION-DIFFUSION EQUATIONS**

by

**Jack K. Hale**

and

**K. Sakamoto**

Lefschetz Center for Dynamical Systems  
Division of Applied Mathematics  
Brown University  
Providence, RI 02912

April 1987

This research was supported in part by the Air Force Office of Scientific Research under contract #AFOSR-84-0376, the National Science Foundation under contract #DMS-8507056, and the Army Research Office under contract #DAAL03-86-K-0074.

### Abstract

For a pair of reaction diffusion equations with one diffusion coefficient very large, there is associated a reaction diffusion equation coupled with an ordinary differential equation (the shadow system) with nonlocal effects which has the property that it contains all of the essential dynamics of the original equations.

*For a pair of reaction diffusion equations with one diffusion coefficient very large, there is associated a reaction diffusion equation coupled with an ordinary differential equation (the shadow system) with nonlocal effects which has the property that it contains all of the essential dynamics of the original equations.*

## 1. Introduction

Many models of chemical, biological and ecological problems involve systems of reaction-diffusions in a bounded domain  $\Omega$  with Neumann boundary conditions. Of major concern is an understanding of the mechanism for the creation of stable patterns; that is, stable solutions which are spatially dependent (see, for example, Turing [1952], Nicolis and Prigogine [1973]). For the understanding of how stable patterns occur, it is obviously of interest to characterize those situations for which stable patterns do not exist and, even more particularly, those systems for which the flow is essentially determined by an ordinary differential equation. This situation was studied rather extensively by Conway, Hoff and Smoller [1978] for the situation where the system of partial differential equations had an invariant region and by Hale [1986] for the general situation. Due to the generality of the methods in the latter work, the theory applies equally as well to functional differential equations or delay equations with diffusion. The basic result is that no patterns exist if the diffusion coefficients are sufficiently large.

Once the results are known to be valid for large diffusion coefficients, the next step is to try to understand the occurrence of qualitative changes in the flow through bifurcations as the diffusion coefficients become smaller. If all diffusion coefficients are allowed to be completely arbitrary, many complications occur and it is therefore natural as a first step to allow some diffusion coefficients to become small while others remain very large. For a system of two equations with special types of nonlinearities that occur in biological and ecological models, such a theoretical investigation has been made for the bifurcation and stability of equilibrium solutions by Nishiura [1981] and

Nishiura and Fujii [1985]. With the aid of numerical methods, the bifurcation of equilibria for the same system for arbitrary diffusions has been discussed by Fujii, Mimura and Nishiura [1982].

In the work of Nishiura [1982] and Nishiura and Fujii [1985], if  $d_1, d_2$  are the diffusion coefficients, an important role was played by a limiting system called a "shadow" system which is obtained by letting  $d_2 \rightarrow \infty$  and consists of a single reaction-diffusion equation with diffusion coefficient  $d_1$  coupled with an ordinary differential equation. It was shown that the existence and stability properties of equilibrium for the shadow system were reflected in the original pair of reaction-diffusion equations for the diffusion coefficient  $d_2$  large. It is the purpose of this paper to carry this idea further by showing that the existence of a compact attractor for the shadow system implies the existence of a compact attractor for the original system if the diffusion coefficient  $d_2$  is large.

Let us now be more precise in the statement of the results. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ , with  $\partial\Omega$  smooth and consider the system of reaction-diffusion equations

$$(1.1) \quad \begin{aligned} \partial u / \partial t &= D_1 \Delta u + f(u, v) \\ \partial v / \partial t &= D_2 \Delta v + g(u, v) \quad \text{in } \Omega \end{aligned}$$

subject to Neumann boundary conditions

$$(1.2) \quad \partial u / \partial n = 0, \quad \partial v / \partial n = 0 \quad \text{on } \partial\Omega$$

where  $u \in \mathbb{R}^m$ ,  $v \in \mathbb{R}^n$  are vectors,  $D_1 = \text{diag}(d_{11}, \dots, d_{1m})$ ,  $D_2 = \text{diag}(d_{21}, \dots, d_{2n})$  where each  $d_{jk} > 0$ ,  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,

$g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are  $C^2$ -functions.

Let  $X = L^2(\Omega, \mathbb{R}^m)$ ,  $Y = L^2(\Omega, \mathbb{R}^n)$ . Using the operator  $A = -\Delta$ , one can define the usual fractional power spaces  $X^\alpha$ ,  $Y^\alpha$ . For convenience in notation, let us assume that  $N \leq 3$  and choose  $\frac{1}{2} < \alpha < 1$ . The latter choice of  $\alpha$  is made to ensure that  $X^\alpha \subset W^{1,2}(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$ ,  $Y^\alpha \subset W^{1,2}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ . For  $N = 1$ , we can take  $\alpha = \frac{1}{2}$ . One can then show (see, for example, Henry [1980]) that, for any  $(u_0, v_0) \in X^\alpha \times Y^\alpha$ ,  $\alpha > \frac{1}{2}$ , there is a unique solution  $(u(t, \cdot, u_0, v_0), v(t, \cdot, u_0, v_0)) \in X^\alpha \times Y^\alpha$  of (1.1), (1.2) through  $(u_0, v_0)$  at  $t = 0$  which is continuous in  $(t, u_0, v_0)$ .

A set  $A \subset X^\alpha \times Y^\alpha$  is invariant under (1.1), (1.2) if  $(u(t, \cdot, A), v(t, \cdot, A)) = A$  for  $t \geq 0$ . The set  $A$  is a compact attractor if it is compact, invariant and there is a neighborhood  $U$  of  $A$  such that the  $\omega$ -limit set of  $u$  is  $A$ . The  $\omega$ -limit set  $\omega(U)$  of  $U$  is defined as

$$\omega(U) = \bigcap_{T \geq 0} \text{cl} \bigcup_{t \geq T} (u(t, \cdot, U), v(t, \cdot, U)).$$

By the shadow system of (1.1), (1.2), we mean the system

$$(1.3) \quad \begin{aligned} \partial u / \partial t &= D_1 \Delta u + f(u, \xi) \\ d\xi / dt &= |\Omega|^{-1} \int_{\Omega} g(u(\cdot, x), \xi) dx \quad \text{in } \Omega \end{aligned}$$

with the boundary condition

$$(1.4) \quad \partial u / \partial n = 0 \quad \text{on } \partial \Omega.$$

The concept of a compact attractor in  $X^\alpha \times \mathbb{R}^n$  for (1.3), (1.4) is defined in a manner analogous to the one for (1.1), (1.2).

If  $d_1^0 > 0$  is a given positive constant, we use the notation  $D_1 \geq d_1^0 I_n$ . In general, we let  $N(\delta, A)$  be the  $\delta$ -neighborhood of a set  $A$  of a Banach space.

We will impose the following hypothesis:

(H) Suppose there is a compact set  $K \subset X^\alpha \times R^n$  and a constant  $\delta_0 > 0$  such that (1.3), (1.4) has a compact attractor  $A_{D_1} \subset K$ ,  $\omega(N(\delta_0, A_{D_1})) = A_{D_1}$  for every  $D_1 \geq d_1^0 I_m$ .

**Theorem 1.** *If (H) is satisfied, then there is a  $d_2^0$  such that, if  $D_2 \geq d_2^0 I_n$ , then there is a compact attractor  $A_{D_1, D_2} \subset X^\alpha \times Y^\alpha$  for (1.1), (1.2) and, for  $\epsilon > 0$ , there is an  $n > 0$  such that  $A_{D_1, D_2} \subset N(\epsilon, A_{D_1} \times \{0\})$  if  $D_2 \geq n I_n$ , where  $\{0\}$  in  $A_{D_1} \times \{0\}$  means  $\{0\} \subset Y_\alpha^\perp$  with  $Y^\alpha = R^n \times Y_\alpha^\perp$*

The last statement of the theorem asserts that the attractor  $A_{D_1, D_2}$  cannot be much larger than  $A_{D_1} \times \{0\}$  if  $D_2$  is large. On the other hand, it could be smaller if we make no further hypotheses about the flow on  $A_{D_1}$  of (1.3), (1.4).

In Section 2, we prove Theorem 1. The proof follows some of the ideas in Hale [1986] except that a new argument must replace the use of Lyapunov functions to obtain *a priori* bounds of the solutions of the full system. These functions seem to be of little use due to the fact that one cannot obtain information about a perturbed partial differential equation using the derivative of the Lyapunov function.

In Section 3, we discuss the difficulties involved in obtaining  $A_{D_1, D_2}$  as a graph over  $(u, z)$ -space. Some restricted conditions for the existence of a graph also are given.

Under the assumption that the shadow system has a compact attractor with certain properties, Theorem 1 asserts the existence of a family of compact attractor  $A_{D_1, D_2}$  for the full system which are upper semicontinuous at  $d_2 = \infty$ . It is natural to ask the opposite questions: suppose that the full system has a compact attractor for each  $D_2 \geq d_2^0 I_n$ . Does the shadow system have a compact attractor  $A_\infty$  and is the set  $(A_{D_1, D_2} | D_2 \geq d_2^0 I_n, A_\infty)$  lower semicontinuous at  $d_2 = \infty$ ? Section 4 is devoted to a discussion of conditions which ensure that this is the situation.

In Section 5, we discuss the relationship between the shadow system and PDE's with nonlocal spatial effects and hereditary dependence.

Theorem 1 can be considered as a first attempt to understand the behavior of the solutions of systems of reaction-diffusion equations. Further information could be obtained in the following way. In Theorem 1, there is a restriction that  $D_1 \geq d_1^0 I_m$ . In any particular problem, one could first try to analyze the shadow system for all  $D_1 \geq 0$  and obtain a clear picture of the dynamics of the attractor. Taking the limit as  $D_1 \rightarrow 0$  gives some information about the types of singular solutions that can occur at  $D_1 = 0$ . Ideally, one would then hope to obtain an attractor  $A_{D_1, D_2}$  for  $D_2 \geq d_2^0 I_n$  and all  $D_1 > 0$ . This will involve a very difficult analysis of the existence and stability of large amplitude singular solutions near  $D_1 = 0$ . These solutions will play an important role in understanding the global flow for the original equations in the region in  $(D_1, D_2)$  --space where  $D_2$  is not very large. Such a program has been partially carried out analytically and numerically for a system of two equations modeling problems in ecology by several Japanese mathematicians (see

the survey article of Nishiura, Fujii and Hosono [1986]). They have analyzed in detail the bifurcation curves in  $(D_1, D_2)$ -space for the one and two point equilibrium solutions and have discussed the stability of these solutions. These results explain well the formation and coexistence of stable patterns. The manner in which the equilibria are dynamically connected has not been discussed and is certainly an important factor.

Various generalizations of Theorem 1 are possible. For example, different boundary conditions may be allowed. Consider the equation (1.1) with the boundary conditions

$$(1.5) \quad \begin{aligned} D_1 \partial u / \partial n + E_1(x) u &= 0 \\ D_2 \partial v / \partial n + E_2(x) v &= 0 \quad \text{in } \partial \Omega \end{aligned}$$

where  $E_1, E_2$  are diagonal matrices and let

$$(1.6) \quad \xi_2 = - |\Omega|^{-1} \int_{\partial \Omega} E_2(x) dx .$$

The appropriate shadow system for the system (1.1), (1.5) with  $D_2$  large is the system

$$(1.7) \quad \begin{aligned} \partial u / \partial t &= D_1 \Delta u + f(u, z) \quad \text{in } \Omega \\ dz / dt &= \xi_2 z + |\Omega|^{-1} \int_{\Omega} g(u, z) dx \end{aligned}$$

with the boundary conditions

$$(1.8) \quad D_1 \partial u / \partial n + E_1(x) u = 0 \quad \text{in } \partial \Omega .$$

Following essentially the same proof as below for Theorem 1 and the estimates on  $e^{D_2 \Delta t}$  from Hale and Rocha [1987a,b], one obtains  $(X_{E_1}^\alpha, Y_{E_2}^\alpha$

and  $\mathcal{M} = \mathcal{M}(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ .

**Theorem 2.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega$ . Let  $D_1, D_2$  be symmetric positive definite matrices. Let  $f, g$  be continuous functions. Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$  be continuous functions. Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$  be continuous functions.

It also will be clear from the proof of Theorem 1 that, as in Hale [1986], that the above results hold equally well for systems of functional differential equations with diffusion.

$$(1.9) \quad \begin{aligned} \partial u / \partial t &= D_1 \Delta u + f(u_t, v_t) \\ \partial v / \partial t &= D_2 \Delta v + g(u_t, v_t) \quad \text{in } \Omega \end{aligned}$$

with the boundary conditions (1.2) or (1.5) where the notation  $u_t, v_t$  designate  $u_t(\theta, x) = u(t+\theta, x)$ ,  $v_t(\theta, x) = v(t+\theta, x)$ ,  $\theta \in [-r, 0]$ . The functions  $u_t, v_t$  are supposed to belong to the spaces  $C([-r, 0], X^\alpha)$ ,  $C([-r, 0], Y^\alpha)$ . The shadow system is

$$(1.10) \quad \begin{aligned} \partial u / \partial t &= D_1 \Delta u + f(u_t, z_t) \quad \text{in } \Omega \\ dz / dt &= \xi_2 z(t) + |\Omega|^{-1} \int_{\Omega} g(u_t, z_t) dx \end{aligned}$$

i.e., a functional partial differential equation coupled with a functional ordinary differential equation. We do not state the precise result on the existence of an attractor for (1.9) which would be analogous to Theorem 2.

## 2. PROOF OF THEOREM 1

Let  $Z \subset Y^\alpha$  be the linear space of the constant functions,  $Y^\alpha = Z \oplus Y_\alpha^\perp$ ,  $v = z + w$  where  $z \in Z$ ,  $w \in Y_\alpha^\perp$ ,

$$(2.1) \quad z = |\Omega|^{-1} \int_{\Omega} v(x) dx, \quad \int_{\Omega} w(x) dx = 0.$$

We can identify  $Z$  with  $\mathbb{R}^n$  and therefore will consider  $z$  as an element of  $Z$  as well as a vector in  $\mathbb{R}^n$ .

If  $u(t, \cdot)$ ,  $v(t, \cdot)$  are solutions of (1.1), (1.2) and  $v(t, \cdot) = z(t) + w(t, \cdot)$ ,  $z(t) \in Z$ ,  $w(t, \cdot) \in Y_\alpha^\perp$ , then

$$\partial u / \partial t = D_1 \Delta u + f(u, z + w)$$

$$(2.2) \quad dz/dt = |\Omega|^{-1} \int_{\Omega} g(u, z + w) dx$$

$$\partial w / \partial t = D_2 \Delta w + g(u, z + w) - |\Omega|^{-1} \int_{\Omega} g(u, z + w) dx \quad \text{in } \Omega$$

with

$$(2.3) \quad \partial u / \partial n = 0, \quad \partial w / \partial n = 0 \quad \text{in } \partial \Omega.$$

We are going to consider this equation as a perturbation of the system

$$\partial u / \partial t = D_1 \Delta u + f(u, z)$$

$$(2.4) \quad dz/dt = |\Omega|^{-1} \int_{\Omega} g(u, z) dx$$

$$\partial w / \partial t = D_2 \Delta w \quad \text{in } \Omega$$

with the boundary conditions (2.3). By hypothesis (H), (2.3), (2.4) has a compact attractor  $A_{D_1} \times \{0\}$ .

Let us rewrite (2.2) as

$$\begin{aligned}
 \partial u / \partial t &= D_1 \Delta u + f(u, z) + F(u, z, w) \\
 (2.5) \quad dz / dt &= |\Omega|^{-1} \int_{\Omega} g(u, z) dx + Z(u, z, w) \\
 \partial w / \partial t &= D_2 \Delta w + g(u, z) - |\Omega|^{-1} \int_{\Omega} g(u, z) dx + G(u, z, w) .
 \end{aligned}$$

Then using the fact that  $\alpha > \frac{1}{2}$  and standard types of estimates (see, for example, Hale [1986] or Hale and Rocha [1986a]), there is a constant  $k_6$  such that

$$\begin{aligned}
 (2.6) \quad & \|F(u, z, w)\|_{L^2} , \quad |Z(u, z, w)| , \quad |G(u, z, w)|_{L^2} \leq k_6 \|w\|_{Y_{\alpha}^1} \\
 & \|g(u, z) - |\Omega|^{-1} \int_{\Omega} g(u, z) dx\|_{L^2} \leq k_6
 \end{aligned}$$

for  $(u, z, w) \in N(\delta, K \times \{0\})$  .

There are  $\mu > 0$  ,  $k_1 > 0$  , such that, for  $d_{2j} \geq d > 0$  , we have

$$\begin{aligned}
 (2.7) \quad & \|e^{D_2 \Delta t} w\|_{Y_{\alpha}^1} \leq k_1 e^{-d\mu t} |w|_{Y_{\alpha}^1} , \quad t \geq 0 \\
 & \|e^{D_2 \Delta t} w\|_{Y_{\alpha}^1} \leq k_1 e^{-d\mu t} t^{-\alpha} |w|_{L^2} , \quad t \geq 0 .
 \end{aligned}$$

(see, for example, Hale [1986]).

If we now use the variation of constants formula on the equation for  $w$  in (2.5), then, as long as  $(u(t), z(t), w(t)) \in N(\delta, A_{D_1} \times \{0\})$  , we can use the relations (2.6), (2.7) to obtain

$$\begin{aligned} \|w(t)\|_{Y_{\alpha}^{\perp}} &\leq k_1 e^{-\mu t} \|w(0)\|_{Y_{\alpha}^{\perp}} + k_1 k_{\delta} \int_0^t (t-s)^{-\alpha} e^{-\mu d(t-s)} ds \\ &\quad + k_1 k_{\delta} \int_0^t (t-s)^{-\alpha} e^{-\mu d(t-s)} \|w(s)\|_{Y_{\alpha}^{\perp}} ds \end{aligned}$$

Choose  $0 < \sigma < \mu$  and let  $\eta(t) = e^{\sigma t} \|w(t)\|_{Y_{\alpha}^{\perp}}$ ,  $y(t) = \sup\{\eta(s), 0 \leq s \leq t\}$  and

$$L = \int_0^{\infty} s^{-\alpha} e^{-(1-\sigma/\mu)s} ds .$$

Then

$$\eta(t) \leq k_1 e^{-(\mu-\sigma)t} \eta(0) + k_1 k_{\delta} e^{\sigma t} L(\mu d)^{\alpha-1} + k_1 k_{\delta} L(\mu d)^{\alpha-1} y(t)$$

Now choose  $d$  so that

$$(2.8) \quad \theta = 1 - k_1 k_{\delta} L(\mu d)^{\alpha-1} > 0 .$$

Then

$$y(t) \leq \theta^{-1} k_1 e^{-(\mu-\sigma)t} \eta(0) + \theta^{-1} k_1 k_{\delta} e^{\sigma t} L(\mu d)^{\alpha-1} .$$

Since  $\eta(t) = e^{\sigma t} \|w(t)\|_{Y_{\alpha}^{\perp}}$ , we have

$$(2.9) \quad \|w(t)\|_{Y_{\alpha}^{\perp}} \leq \theta^{-1} k_1 e^{-\mu t} \|w(0)\|_{Y_{\alpha}^{\perp}} + \theta^{-1} k_1 k_{\delta} L(\mu d)^{\alpha-1} .$$

Remember that (2.9) is valid for all  $t \in [0, \tau]$  if  $(u(t), z(t), w(t)) \in N(\delta, A_{D_1} \times \{0\})$  for  $t \in [0, \tau]$ .

If  $(u(t), z(t), w(t)) \in N(\delta, A_{D_1} \times \{0\})$  for all  $t \geq 0$ , then inequality (2.9) implies that

$$\|w(t)\|_{Y_\alpha^\perp} \leq \theta^{-1} k_1 \|w(0)\|_{Y_\alpha^\perp} + \theta^{-1} k_1 k_5 L(\mu d)^{\alpha-1}, \quad t \geq 0$$

(2.10)

$$\limsup_{t \rightarrow \infty} \|w(t)\|_{Y_\alpha^\perp} \leq \theta^{-1} k_1 k_5 L(\mu d)^{\alpha-1}$$

Therefore,  $w(t)$  can be made as small as desired by taking  $w(0)$  small and  $d$  large and the  $\limsup$  of  $\|w(t)\|_{Y_\alpha^\perp}$  can be made small by making  $d$  large.

Now let us obtain *a priori* bounds on  $u(t)$ ,  $z(t)$ ; namely, the solutions of

$$\partial u / \partial t = D_1 \Delta u + f(u, z) + F(u, z, w(t))$$

(2.11)

$$dz/dt = |\Omega|^{-1} \int_{\Omega} g(u, z) dx + Z(u, z, w(t)).$$

Choose constants  $\epsilon, \eta$ ,  $0 < 2\eta < \epsilon < \delta$ , and let  $t_0(\eta)$  be the constant such that the solution  $(u(t), z(t))$  of (1.3) (1.4) with  $(u(0), z(0)) \in N(\delta, A_{D_1})$  satisfies  $(u(t), z(t)) \in N(\eta, A_{D_1})$  for  $t \geq t_0(\eta)$ , which is ensured to exist by (H). There is a constant  $\bar{\epsilon}$  such that, if  $\|F(u, z, w)\|_{L^2}$ ,  $|Z(u, z, w)| < \bar{\epsilon}$ , then any solution  $(u(t), z(t))$  of (2.11) with  $(u(0), z(0)) \in N(\epsilon, A_{D_1})$  must stay in  $N(\delta, A_1)$  for  $0 \leq t \leq t_0(\eta)$  and satisfy  $(u(t_0(\eta)), z(t_0(\eta))) \in N(2\eta, A_{D_1})$ . Therefore, the same will be true for  $t \in [t_0(\eta), 2t_0(\eta)]$ , etc., and the solution will remain in  $N(\delta, A_{D_1})$  for  $t \geq 0$ . To obtain this estimate on  $F$ ,  $Z$ , choose  $\|w(0)\|_{Y_\alpha^\perp}$  so small and  $d_2^0$  so large that the right-hand side of the first inequality in (2.10) is less than  $\bar{\epsilon}/k_5$ .

Therefore, we have shown that there is a neighborhood  $U$  of  $A_{D_1} \times \{0\}$  such that the solution  $(u(t), z(t), w(t))$  of (2.2), (2.3) with initial data in  $U$

and  $d_2^0$  sufficiently large stays in  $N(\delta, A_1 \times \{0\})$  for  $t \geq 0$ . Thus,  $\omega(U)$  is compact (see Henry [1980], Th. 3.3.6) and  $\omega(U) = A_{D_1, D_2}$  is a compact attractor for all  $D_1 \geq d_1^0 I_m$  and  $D_2 \geq d_2^0 I_n$ .

From the second inequality in (2.10), we can use an argument similar to the one above to show that, for any sequence  $\eta_j \rightarrow 0$  as  $j \rightarrow \infty$ , there is a sequence  $d_{2j}^0 \rightarrow \infty$  as  $j \rightarrow \infty$  such that

$$A_{D_1 D_2} = \omega(U) \in N(\eta_j, A_{D_1} \times \{0\}) .$$

This implies the last assertion in Theorem 1 and the proof is complete.

### 3. THE ATTRACTOR AS A GRAPH

In this section, we discuss the possibility of the attractor  $A_{D_1, D_2}$  being a graph over  $(u, z)$ -space. It is tempting to try to obtain such a graph by applying the method of center manifolds to the system (2.2), (2.3) attempting to obtain an invariant manifold which contains  $A_{D_1, D_2}$  and is defined by  $w = h(u, z)$  for some function  $h$ . After appropriately re-defining  $f$ ,  $g$  outside a neighborhood of the attractor so that they are bounded functions, such an integral manifold would be required to satisfy the equations

$$\partial u / \partial t = D_1 \Delta u + f(u, z + h(u, \xi))$$

$$dz / dt = |\Omega|^{-1} \int_{\Omega} g(u, z + h(u, \xi)) dx$$

$$u(0) = u_0, \quad z(0) = z_0$$

$$h(u_0, z_0) = (Th)(u_0, z_0) \stackrel{\text{def}}{=} \int_{-\infty}^0 e^{-D_2 \Delta s} \left[ g(u(s), z(s) + h(u(s), z(s))) - |\Omega|^{-1} \int_{\Omega} g(u(s, x), z(s), h(u(s, x), z(s))) dx \right] ds .$$

To define this integral operator  $Th$  requires that  $u(s)$  must be defined on  $(-\infty, 0]$ . One does not expect such solutions to exist for all values of  $u_0$  because of the smoothing properties of the solutions of  $\partial u / \partial t = D_1 \Delta u$ .

These remarks seem to indicate that the standard method of center manifolds will not apply to this problem.

Another approach that could make use of center manifold techniques is to assume that  $A_{D_1}$  for the extended  $f$ ,  $g$  lies in a finite dimensional invariant subspace which is normally hyperbolic for the shadow system. This

can be accomplished, for example, if the spectrum of the Laplacian has sufficiently large gaps. The proof can be supplied as in Mallet-Paret and Sell [1986] for inertial manifolds or one can use the integral equation approach for center manifolds in a form similar to that mentioned above. This implies, for example, that the attractor will be a graph if the equation for  $u$  acts in only one space variable.

#### 4. LOWER SEMICONTINUITY OF THE ATTRACTORS

In this section, we assume the existence of compact attractors  $A_{D_1, D_2}$  for (1.1), (1.2) satisfying certain properties and conclude that the shadow system (1.3), (1.4) has a compact attractor.

**Lemma 1.** For  $\epsilon > 0$ ,  $\tau > 0$ , there exists  $d_2 = d_2(\epsilon, \tau)$  such that: if  $(u(t), v(t))$  and  $(w(t), \xi(t))$  are solutions of (1.1), (1.2) and (1.3), (1.4) respectively with  $u(0) = w(0)$ ,  $\bar{v}(0) = \xi(0)$  then, for  $D_2 \geq d_2(\epsilon, \tau)I_n$ , the following are valid:

$$\sup_{t \in [0, \tau]} |u(t) - w(t)|_{X^\alpha} < \epsilon, \quad \sup_{t \in [0, \tau]} |\bar{v}(t) - \xi(t)| < \epsilon$$

$$\sup_{t \in [0, \tau]} |v(t) - \bar{v}(t)|_{Y_\perp^\alpha} < \epsilon$$

where  $\bar{v}(t) = |\Omega|^{-1} \int_{\Omega} v(t) dx$ .

**Proof.** By  $g_\perp(\phi(\cdot), \psi(\cdot))$ ,  $\phi \in X^\alpha$ ,  $\psi \in Y^\alpha$ , we denote the projection of the function  $g(\phi(\cdot), \psi(\cdot))$  onto  $Y_\perp^\alpha$  along  $Z$ . Let us first give an estimate on  $v_\perp(t) = v(t) - \bar{v}(t)$ , which satisfied the following

$$v_\perp(t) = \int_0^t e^{D_2 \Delta(t-s)} g_\perp(u(s), \bar{v}(s) + v_\perp(s)) ds$$

By using the estimates in (2.7), one obtains

$$(4.1) \quad \begin{aligned} \|v_{\perp}(t)\|_{Y^{\alpha}} &\leq k_1 \int_0^t (t-s)^{-\alpha} e^{-\mu dz(t-s)} \|g_{\perp}(u(s), v(s))\|_{Y^0} ds \\ &\leq k_1 M \Gamma(1-\alpha) / (\mu dz)^{1-\alpha} \quad \text{for } t \in [0, \tau] \end{aligned}$$

where  $M = \sup\{\|g_{\perp}(u(s), v(s))\|_{Y^0} \mid 0 \leq s \leq \tau\}$ .

Now rewrite the equations for  $(u, \nabla)$  as follows:

$$\begin{cases} \partial u / \partial t = D_1 \Delta u + f(u, \nabla) + R_1(v_{\perp}) \\ d \nabla / dt = |\Omega|^{-1} \int_{\Omega} g(u(y, t), \nabla(t)) dy + R_2(v_{\perp}) \end{cases}$$

where  $R_1(v_{\perp}) = f(u, \nabla + v_{\perp}) - f(u, \nabla)$

and

$$R_2(v_{\perp}) = |\Omega|^{-1} \int_{\Omega} [g(u(y, t), \nabla(t) + v_{\perp}(y, t)) - g(u(y, t), \nabla(t))] dy .$$

By using the estimate in (4.1), one can find a constant  $K > 0$  such that

$$\sup_{t \in [0, \tau]} \|R_i(v_{\perp})\|_{Y^0} \leq K d_2^{\alpha-1}, \quad i = 1, 2 .$$

These estimates prove the existence of  $d_2 = d_2(\epsilon, \tau)$  in the lemma.

Let us denote by  $T_{D_1, D_2}(t)$  the semiflow on  $X^{\alpha} \times Y^{\alpha}$  generated by (1.1), (1.2) and make the following hypotheses:

(H-1)  $T_{D_1, D_2}(t)$  has a compact attractor  $A_{D_1, D_2} \neq \emptyset$  in  $X^{\alpha} \times Y^{\alpha}$ .

(H-2) For any bounded set  $B \subset X^{\alpha} \times Y^{\alpha}$  and any  $\epsilon > 0$ , there is a  $t_0 = t_0(\epsilon, B)$  such that:

$$T_{D_1, D_2}(t)B \subset N(\epsilon, A_{D_1, D_2}), \quad \forall t \geq t_0$$

(H-3)  $\bigcup_{D_2 \geq d_2^0} A_{D_1, D_2}$  is compact for some  $d_2^0 > 0$ .

**Theorem 3.** Under the hypotheses (H-1)-(H-3), the shadow system (1.3), (1.4) has a compact attractor  $A_{D_1, \infty}$  and, moreover,

$$A_{D_1, \infty} = \bigcap_{\delta \geq d_2^0} \bigcup_{D_2 \geq \delta I_n} C_l \cup A_{D_1, D_2} .$$

**Proof.**  $\bigcap_{\delta \geq d_2^0} \bigcup_{D_2 \geq \delta I_n} C_l \cup A_{D_1, D_2}$  is non-empty and compact because of (H-1) and (H-3).

Let us denote by  $T_{D_1, \infty}(t)$  the semiflow on  $X^\alpha \times Z$  generated by the shadow system (1.3), (1.4).

Firstly, we show that  $T_{D_1, \infty}(t)$  has a compact attractor  $A_{D_1, \infty}$  and that  $A_{D_1, \infty} \subset \bigcap_{\delta \geq d_2^0} \bigcup_{D_2 \geq \delta I_n} C_l \cup A_{D_1, D_2}$  holds. For any bounded set  $B \subset X^\alpha \times Z$ , the set  $\{T_{D_1, D_2}(t)B \mid t \geq 0\}$  is bounded by virtue of (H-1). We therefore repeat the arguments in the proof of Lemma 1 uniformly in the initial data  $(\phi, \psi) \in B$ , namely, for any  $\epsilon > 0$  and  $\tau > 0$ , there exists a constant  $\delta_0 = \delta_0(\epsilon, \tau, B)$  such that

$$\text{dist}_{X^\alpha \times Y^\alpha}(T_{D_1, \infty}(t)B, T_{D_1, D_2}(t)B) < \epsilon$$

for  $t \in [0, \tau]$ ,  $D_2 \geq \delta_0(\tau, \epsilon, B)I_n$ . By the hypothesis (H-2), there is a  $t_0(\epsilon, B)$  such that  $T_{D_1, D_2}(t)B \subset N[\epsilon, A_{D_1, D_2}]$  for  $t \geq t_0(\epsilon, B)$ ,  $D_2 \geq d_2^0 I_n$ . For any increasing sequence  $\{\tau_j\}_{j=1}^\infty$ ,  $\tau_j \uparrow \infty$ ,  $j \uparrow \infty$ , let  $j_0 = j_0(\epsilon)$  be the least index  $j$  for which  $\tau_j > t_0(\epsilon, B)$  and

$$N\left[\epsilon, \bigcap_{\delta \geq d_2^0} \bigcup_{D_2 \geq \delta I_n} C_l \cup A_{D_1, D_2}\right] \supset C_l \cup \left\{A_{D_1, D_2} \mid D_2 \geq \delta_0(\epsilon, \tau_j, B)I_n\right\}$$

hold.

For this choice of  $j_0(\epsilon)$ , we have:

$$(1) \text{ dist}(T_{D_1, D_2}(\tau_{j_0(\epsilon)})B, T_{D_1, \infty}(\tau_{j_0(\epsilon)})B) < \epsilon \quad \text{for } D_2 \geq \delta_0(\epsilon, \tau_{j_0(\epsilon)}, B)I_n.$$

$$(2) T_{D_1, D_2}(\tau_{j_0(\epsilon)})B \subset N(\epsilon, A_{D_1, D_2}), \quad \forall D_2 \geq d_2^0 I_n.$$

$$(3) N(\epsilon, \bigcap_{\delta \geq d_2^0} Cl \cup A_{D_1, D_2}) \supset Cl \cup \{A_{D_1, D_2} | D_2 \geq \delta_0(\epsilon, \tau_{j_0(\epsilon)}, B)I_n\}.$$

From these three properties, it follows that

$$T_{D_1, \infty}(\tau_{j_0(\epsilon)})B \subset N(3\epsilon, \bigcap_{\delta \geq d_2^0} Cl \cup A_{D_1, D_2}).$$

Let  $j_k = j_0(1/k)$ ,  $k \geq 1$ . Hence, the limit

$$\lim_{k \rightarrow \infty} T_{D_1, \infty}(\tau_{j_k})B \subset \bigcap_{\delta \geq d_2^0} Cl \cup A_{D_1, D_2} \quad \text{exists.}$$

Since the sequence  $\{\tau_j\}_{j=1}^{\infty}$ ,  $\tau_j \uparrow \infty$ ,  $j \uparrow \infty$  is arbitrary, we have thus established

$$(4.2) \quad \omega_{T_{D_1, \infty}}(B) = \bigcap_{t \geq 0} \bigcup_{s \geq t} T_{D_1, \infty}(s)B \subset \bigcap_{\delta \geq d_2^0} Cl \cup A_{D_1, D_2}$$

This, in particular, implies that  $T_{D_1, \infty}(t)$  is bounded dissipative and  $(T_{D_1, \infty}(t)B | t \geq 0)$  is bounded for  $B \subset X^\alpha \times Z$  bounded. One can also show that  $T_{D_1, \infty}(t)$  is compact for  $t > 0$  (see Hale [1985] for detail). These three properties are sufficient to guarantee that  $T_{D_1, \infty}(t)$  has a compact attractor  $A_{D_1, \infty}$  (see Hale [1985]). From (4.2), it follows that

$$A_{D_1, \infty} \subset \bigcap_{\delta \geq d_2^0} Cl \cup A_{D_1, D_2}.$$

In order to prove  $A_{D_1, \infty} \supset \bigcap_{\delta \geq d_2^0} C I \cup_{D_2 \geq \delta I_n} A_{D_1, D_2}$ , it suffices to note that Lemma 1 actually shows that the semiflow  $T_{D_1, D_2}(t)$  is continuous in  $D_2$  at infinity. This fact together with the existence of  $A_{D_1, \infty}$  and (H-3) is sufficient to ensure that  $A_{D_1, D_2}$  is upper-semicontinuous in  $D_2$  at infinity, namely,

$$A_{D_1, \infty} \supset \bigcap_{\delta \geq d_2^0} C I \cup_{D_2 \geq \delta I_n} A_{D_1, D_2} .$$

This completes the proof of Theorem 3.

## 5. SHADOW SYSTEMS AND FUNCTIONAL DEPENDENCE

For a system of two reaction-diffusion equations with one diffusion coefficient large relative to the other, we have seen that the flow on the attractor can be reduced to the discussion of the shadow system consisting of a PDE coupled with an ODE with nonlocal terms. In some situations, the flow of the shadow system can be reduced to the discussion of a scalar PDE with nonlocal effects in the spatial variables. In this section, we give some illustrations of this fact.

Suppose the shadow system is given by

$$(5.1) \quad \begin{aligned} u_t &= d\Delta u + f(u, \xi) \\ \xi_t &= |\Omega|^{-1} \int_{\Omega} g(u(\cdot, x), \xi) dx \quad \text{in } \Omega. \end{aligned}$$

with the boundary condition

$$(5.2) \quad \partial u / \partial n = 0 \quad \text{in } \partial \Omega.$$

Suppose that (5.1), (5.2) has a compact attractor  $A_d$  and that

$$(5.3) \quad g(u, v) = -\lambda[v - h(u)]$$

where  $\lambda > 0$  is a positive constant and  $h(u)$  is a  $C^2$ -function.

If  $(u(t), \xi(t)) \in A_d$  for  $t \in \mathbb{R}$ , then, in particular,  $\xi(t)$  is a solution of the equation

$$\xi_t = -\lambda \xi(t) + \lambda |\Omega|^{-1} \int_{\Omega} h(u(t, x)) dx$$

which is bounded on  $\mathbb{R}$ . Therefore,  $\xi(t)$  is uniquely defined by the formula

$$(5.4) \quad \xi(t) = \lambda \int_{-\infty}^t e^{-\lambda(t-s)} H[u](s) ds$$

where

$$(5.5) \quad H[u](t) = |\Omega|^{-1} \int_{\Omega} h(u(t,x)) dx .$$

This implies that  $u(t)$  is a solution of the equation

$$(5.6) \quad u_t = d\Delta u + f \left[ u(t, \cdot), \lambda \int_{-\infty}^0 e^{\lambda s} H[u](t+s) ds \right] \quad \text{in } \Omega$$

with the boundary condition (5.2). This is a retarded PDE with nonlocal effects in the spatial terms.

In summary, if  $g$  satisfied (5.3), then the flow on the attractor for the shadow system (5.1), (5.2) is equivalent to discussing the flow on the attractor for (5.6), (5.2) making use of (5.4). Of course, we must make certain that (5.6), (5.2) defines a semigroup in some appropriate space.

To obtain a solution of (5.6), one must specify a function  $\varphi: (-\infty, 0] \rightarrow X^\alpha$  and then attempt to use (5.6) to extend  $\varphi$  to a function  $u(t, \varphi)$  defined on  $(-\infty, \infty)$  with  $u(t, \varphi) = \varphi(t)$  for  $t \leq 0$ . There are several natural spaces for the  $\varphi$ 's; for example, for any  $0 < \gamma < \lambda$ , the space

$$C^\gamma = \{ \varphi \in C((-\infty, 0), X^\alpha); \varphi(\theta) e^{\gamma\theta} \rightarrow \text{limit as } \theta \rightarrow -\infty \}$$

with the sup norm  $\|\cdot\|_{C^\gamma}$ , or the space

$$L_{\gamma,p}((-\infty, 0), X^\alpha) \times X^\alpha$$

$$L_{\gamma,p}((-\infty, 0), X^\alpha) = \left\{ \varphi: (-\infty, 0) \rightarrow X^\alpha; \varphi \text{ measurable and} \right.$$

$$\left. \int_{-\infty}^0 e^{\gamma p \theta} |\varphi(\theta)|^p d\theta < \infty \right\}$$

with the norm

$$|(\varphi, u_0)|_{\gamma, p} = \left[ |u_0|^p + \int_{-\infty}^0 e^{\gamma p \theta} |\varphi(\theta)|^p d\theta \right]^{1/p} .$$

With either of these spaces, one can follow the usual procedures to obtain the local existence and uniqueness of solutions of the initial value problem for (5.6), (5.2).

Let us restrict our discussion to  $C^\gamma$ . If  $A_d \subset X^\alpha \times R$  is a compact attractor for (5.1), (5.2), then (5.6), (5.2) has a compact attractor  $\bar{A}_d \subset C^\gamma$  and  $(u(t), \xi(t)) \in A_d$  if and only if  $\xi(t)$  is given by (5.4) and  $\tau(t)u \in \bar{A}_d$  where  $\tau(t)u(\theta) = u(t+\theta)$ ,  $-\infty < \theta \leq 0$ .

It is clear that properties of the solutions of equations of the form (5.6) need to be investigated in more detail. In this paper, we are content with a few remarks.

If we let  $\Omega = (0, \pi)$  and let  $u_0$  be the equilibrium point of (5.6), then the eigenvalues of the linear variational equation about  $u_0$  are

$$\lambda_n = \alpha - n^2 d, \quad n \geq 1$$

(5.7)

$$\lambda_0 = \alpha + \beta(1 + \lambda_0)^{-1}$$

with corresponding eigenfunctions  $e^{\lambda_n \theta} \cos nx$ ,  $\theta \in (-\infty, 0]$ ,  $x \in [0, \pi]$  where  $\alpha = f_u(u_0, \xi_0)$ ,  $\beta = f_v(u_0, \xi_0)$ ,  $\xi_0 = h(u_0)$ . Now suppose that

$$(5.8) \quad 0 < \alpha < 1, \quad \beta < 0, \quad \alpha + \beta < 0 .$$

These inequalities are the usual ones corresponding to the Turing conditions for the destabilization of equilibria in the original pair of reaction diffusion equations (see, for example, Nishiura [1982]).

If (5.7) is satisfied, then  $\operatorname{Re} \lambda_0 < 0$ . If we consider  $d$  as a parameter, then each  $\lambda_n < 0$  if  $n$  is large. If we decrease  $d$ , then there is a bifurcation at  $d = \alpha$  and another equilibrium will arise which is spatially nonhomogeneous. This seems to be typical of the Turing mechanism. The nonlocal spatial effects make the eigenfunction corresponding to the dominant eigenvalue spatially dependent.

Another interesting equation is obtained by taking a limiting situation in (5.4). If  $\lambda \rightarrow \infty$ , then the equation (5.6) should have the dynamics given by the simpler equation

$$(5.9) \quad u_t = d\Delta u + f \left[ u, |\Omega|^{-1} \int_{\Omega} h(u(t,x)) dx \right]$$

with the boundary condition (5.2). Equations of this type have been encountered by Chafee [1981], Levin and Segal [1982], [1985], where they also observed that stable patterns could be generated by nonlocal spatial effects.

REFERENCES

- Chafee, N. [1981], The electric ballast resistor: homogeneous and nonhomogeneous equilibria, 97-127 in *Nonlinear Differential Equations* (Eds. P. de Mottoni and L. Salvadori) Academic Press.
- Conway, E., Hoff, D. and J. Smoller [1978], Large time behavior of solutions of systems of reaction diffusion equations. *SIAM J. Appl. Math.* 35, 1-16.
- Fujii, H., Mimura, M. and Y. Nishiura [1982], A picture of the global bifurcation diagram in ecological interacting and diffusing systems. *Physica* 5D, 1-42.
- Hale, J. K. [1986], Large diffusivity and asymptotic behavior in parabolic systems. *J. Math. Anal. Appl.* 118, 455-466.
- Hale, J. K. and C. Rocha [1987a], Varying boundary conditions with large diffusivity. *J. Math. pures et appl.*, 66.
- Hale, J. K. and C. Rocha [1987b], Interaction of diffusion and boundary conditions. *Nonlinear Anal. TMA.* To appear.
- Hale, J. K. [1985]. Asymptotic behavior and dynamics in infinite dimensions, in *Nonlinear Differential Equations* (Eds. Hale, Martinez-Amores) Research Note in Mathematics 132, Pitman.
- Henry, D. [1981], Geometric theory of semilinear parabolic equations. *Lect. Notes Math.*, Vol. 840, Springer-Verlag.
- Levin, S. A. and L. A. Segal [1982], Models of the influence of predation on aspect diversity in prey populations. *J. Math. Biol.* 14, 253-285.
- Levin, S. A. and L. A. Segal [1985], Pattern generation in space and aspect. *SIAM Review* 27, 45-67.
- Mallet-Paret, J. and G. Sell [1986], Inertial manifolds for a parabolic equation. Preprint.
- Nicolis, G. and I. Prigogine [1973], *Self-Organization in Non-Equilibrium Systems*, J. Wiley and Sons.
- Nishiura, Y. [1982], Global structure of bifurcating solutions for reaction diffusion systems. *SIAM J. Math. Anal.* 13, 555-593.
- Nishiura, Y. and H. Fujii [1985], Stability of singularly perturbed solutions to systems of reaction diffusion equations. Preprint. KSU/ICS 85-05. Inst. Comp. Sci., Kyoto Sangyo Univ.

Nishiura, Y., Fujii, H. and Y. Hosono [1986], On the structure of multiple existence of stable stationary solutions in systems of reaction diffusion equations--a survey, *in* Patterns and Waves (Eds. Nishida, Mimura and Fujii) North-Holland, 157-220.

Turing, A. M. [1952], The chemical basis of morphogenesis, Phil. Trans. Roy. Soc. B 237, 37-72.

END

12-87

DTIC