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ITERATED LOGARITHM FOR A CLASS OF STOCHASTIC
PROCESSES RELATED TO SYMMETRIC STATISTICS

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FREIDLIN-WENTZELL TYPE ESTIMATES AND THE LAW OF THE ITERATED LOGARITHM FOR A CLASS OF STOCHASTIC PROCESSES RELATED TO SYMMETRIC STATISTICS

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Abstract

Analogues of Freidlin and Wentzell's estimates for diffusion processes and the functional law of the iterated logarithm are obtained for a class of stochastic processes represented by multiple Wiener integrals with respect to two parameter Wiener processes, which arise as the limit processes of sequences of normalized symmetric statistics.

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1. Introduction and results. Let \( h = h(u_1, \ldots, u_m) \) be a square integrable symmetric function on \([0,1]^m\) and assume that \( h \) is canonical, i.e., it satisfies the condition
\[
\int_0^1 h(u_1, u_2, \ldots, u_m) \, du_1 = 0 \quad \text{for all } u_2, \ldots, u_m \in [0,1].
\]
Let \( \{X_j\} \) be a sequence of independent identically distributed random variables uniformly distributed over \([0,1]\). Consider the following random sequence of normalized symmetric statistics
\[
Y_n(t) = \frac{1}{n} \sum_{1 \leq i_1 < \cdots < i_m \leq \lfloor nt \rfloor} h(X_{i_1}, \ldots, X_{i_m}), \quad 0 \leq t \leq 1,
\]
in \( D[0,1] \), the space of right continuous functions on \([0,1]\) having left limits with Skorohod's \( J_1 \) topology. A. Mandelbaum and M.S. Taqqu \([3]\) showed that the random sequence \( \{Y_n(t)\} \) converges weakly in \( D[0,1] \) to the following process \( X \):
\[
X(t) = \int \ldots \int h(u_1, \ldots, u_m) 1_{[0,t]}(u_1) \ldots 1_{[0,t]}(u_m) \, W(du_1, \ldots, du_m), \quad 0 \leq t \leq 1.
\]
where the right hand side is an \( m \)-ple Wiener integral with respect to a two parameter Wiener process \( \{W(u,v), 0 \leq u, v \leq 1\} \) and \( 1_{[0,t]}(\cdot) \) is the indicator function of \([0,t]\). \( X \) has continuous paths a.s. and note also that it can be written as
\[
X(t) = \int \ldots \int h(u_1, \ldots, u_m) W_t(du_1) \ldots W_t(du_m), \quad 0 \leq t \leq 1,
\]
with \( W_t(u) = W(u,t) \).

The purpose of this note is firstly to prove certain large deviations results, i.e., asymptotic estimates of Freidlin-Wentzell type, for the above process \( X \), and secondly to remark that the functional law of the iterated logarithm for \( X \) can be derived by the same arguments.

Let \( C_N = C([0,1]; \mathbb{R}^N) \) be the space of \( \mathbb{R}^N \)-valued continuous functions \( x \) on \([0,1]\) vanishing at the origin, with the norm \( ||x||_C = \sup_{0 \leq t \leq 1} |x(t)| \) and the metric
\[
d(\cdot, \cdot), \quad \text{where } |\cdot| \text{ stands for the Euclidean norm in } \mathbb{R}^N.
\]
$0 \leq t \leq 1$, be an $N$-dimensional standard Brownian motion with $B(0) = 0$, and let $H_N$ denote the reproducing kernel Hilbert space (RKHS) associated with $B$, i.e., the Hilbert space consisting of absolutely continuous functions $\varphi$ on $[0,1]$ such that $\varphi(0) = 0$ and its derivative $\dot{\varphi}$ is square integrable. Its norm $||\varphi||_{H}$ is given by $||\varphi||_{H} = ||\dot{\varphi}|| = (\int_{0}^{1} \dot{\varphi}^2(t) dt)^{1/2}$. $H_N$ is a subspace of $C_N$ and the sets $K_r = \{\varphi \in H_N : ||\varphi||_{H} \leq r\}$, $r > 0$, are compact in $C_N$.

Define a mapping $A$ from $L^2[0,1] \otimes H_1$ (the tensor product of $L^2[0,1]$ and RKHS $H_1$) to $C_1$ by

$$ Af(t) = \int \ldots \int h(u_1, \ldots, u_m)f(u_1, t) \ldots f(u_m, t) du_1 \ldots du_m, \quad 0 \leq t \leq 1 $$

for $f \in L^2[0,1] \otimes H_1$, and let $G$ denote the class of functions

$$ G = \{g = Af, f \in L^2[0,1] \otimes H_1\}. $$

Let $||| \cdot |||$ be the norm of $L^2[0,1] \otimes H_1$, which is given, e.g., by $|||f|||^2 = \int_{0}^{1} \int_{0}^{1} (\frac{\partial}{\partial t} f(u, t))^2 du dt$. Define

$$ D(g) = \inf \{|||f||| : g = Af, f \in L^2[0,1] \otimes H_1\} \quad \text{for } g \in G, $$

and

$$ G_r = \{g = Af ; |||f||| \leq r\}, \quad r > 0. $$

The main result of this note is the following

**Theorem 1.** (i) For any $g \in G$ and for any $\delta, \delta' > 0$, there is a number $a_1 = a_1(\delta, \delta')$, $D(g)$ such that

$$ P(||X/\alpha - g||_C < \delta) \geq \exp\left[-(\frac{\alpha^2}{m}/2)(D^2(g) + \delta')\right] $$

for all $\alpha \geq a_1$, and

(ii) for any $\delta, \delta', r > 0$, there is a number $a_2 = a_2(\delta, \delta', r)$ such that

$$ P(d(X/\alpha, G_r) > \delta) \leq \exp\left[-(\frac{\alpha^2}{m}/2)(r^2 - \delta')\right] $$

for all $\alpha \geq a_2$.

In the proof of Theorem 1 we need the following Freidlin-Wentzell type
estimates for N-dimensional Brownian motion $B$.

**Theorem A** (1) For any $\varphi \in H_N$ and for any $\delta, \delta' > 0$, there is a number $a_1 = a_1(\delta, \delta', ||\varphi||_H)$ such that

$$P(||B/\alpha - \varphi||_C < \delta) \geq \exp[-(a_1^2/2)(||\varphi||_H^2 + \delta')]$$

for all $\alpha \geq a_1$ and

(ii) for any $\delta, \delta', r > 0$, there is a number $a_2 = a_2(\delta, \delta', r)$ such that

$$P(d(B/\alpha, K_r) > \delta) \leq \exp[-(a_2^2/2)(r^2 - \delta')]$$

for all $\alpha \geq a_2$.

Theorem A is a special case of a general result on Gaussian processes, and we shall use the following easy consequences of Theorem A (see [2]).

Let $F$ be a continuous mapping from $C_N$ to $C_1$, homogeneous with degree $p > 0$, i.e., satisfying the condition $F(c\cdot) = c^p F(\cdot)$ for any $c > 0$.

**Theorem B** (1) For any $\varphi \in H_N$ and for any $\delta, \delta' > 0$,

$$P(||F(B)/\alpha - F(\varphi)||_C < \delta) \geq \exp[-(a^2/p)(||\varphi||_H^2 + \delta')]$$

for all sufficiently large $\alpha$, and

(ii) for any $\delta, \delta', r > 0$,

$$P(d(F(B)/\alpha, F(K_r)) > \delta) \leq \exp[-(a^2/p)(r^2 - \delta')]$$

for all sufficiently large $\alpha$.

**Theorem C**

$$\lim_{\alpha \to \infty} (1/\alpha^{2/p}) \log P(||F(B)||_C > \alpha) = -b^2/2.$$  

where

$$b^2 = \inf\{||\varphi||_H^2: ||F(\varphi)||_C > 1\}$$

$$= \sup\{r^2: \sup(||F(\varphi)||_C: \varphi \in K_r) < 1\}.$$
The arguments used in the proof of Theorem 1, combined with Strassen's law of the iterated logarithm for Brownian motion, yield also the following functional iterated logarithm law for the process X.

Theorem 2. Define the random sequence \( \{Z_n\} \) in \( C_{1} \) by
\[
Z_n(t) = X(nt)/(2n \log(2n)^{m/2}, \ 0 \leq t \leq 1, \ n \geq 3,
\]
where \( \log_{2} = \log \log \). Then, with probability 1 \( \{Z_n\} \) is relatively compact and the set of its limit points coincides with \( C_{1} \).

Remark. The above theorem is an improvement of a recent result of H. Dehling [1], in the sense that a moment condition on \( h \) is weakened. However, it should be noted that Dehling [1] proves more generally the functional law of the iterated logarithm for \( \{Y_n(t)\} \). The methods of proofs are different.

2. Proof of Theorem 1

Let \( \{e_i, \ i \geq 0\} \) be a complete orthonormal sequence (CONS) in \( L^2[0,1] \) with \( e_0 = 1 \). Then \( \{e_{i_1}(u_1) \ldots e_{i_m}(u_m), \ i_1, \ldots, i_m \geq 0\} \) is a CONS in \( L^2([0,1]^m) \) and \( h \in L^2([0,1]^m) \), symmetric and canonical, can be expanded as
\[
h(u_1, \ldots, u_m) = \sum_{1 \leq i_1, \ldots, i_m \leq m} c_{i_1, \ldots, i_m} e_{i_1}(u_1) \ldots e_{i_m}(u_m), \ c_{i_1, \ldots, i_m} \in \mathbb{R}.
\]

For \( N \geq 1 \) let
\[
h_N(u_1, \ldots, u_m) = \sum_{1 \leq i_1, \ldots, i_m \leq N} c_{i_1, \ldots, i_m} e_{i_1}(u_1) \ldots e_{i_m}(u_m),
\]
and define the process \( X_N = \{X_N(t), \ 0 \leq t \leq 1\} \) by
\[
X_N(t) = \int \ldots \int h_N(u_1, \ldots, u_m) \ W_t(du_1) \ldots W_t(du_m).
\]

[0,1]^m

Then we have
Lemma 1

\[ P(\|X-X_N\|_C > z) \leq C \exp(-M_N z^{2/m}) \]

for all sufficiently large \( z > 0 \), where \( C \) is a finite constant and the positive constant \( M_N \) can be made arbitrarily large by taking \( N \) large.

Proof. The lemma follows from a result of Plikusas [5] on multiple Wiener integrals and Lemmas 6.2 and 6.3 of [4]. Indeed it is enough to note that the exponential bound obtained by Plikusas holds also for multiple Wiener integrals with respect to two parameter Wiener processes and that, putting \( Z_N = X - X_N \), we have

\[ E|Z_N(t+s) - Z_N(t)|^2 \leq C_N s \quad \text{for } 0 \leq t < t + s \leq 1, \]

with a finite constant \( C_N \) and

\[ ||E|Z_N(t)|^2||_C = m! \ ||h-h_N||_2^2, \]

where \( || \cdot ||_2 \) is the norm of \( L^2([0,1]^m) \). Now

\[ X_N(t) = \sum_{1 \leq i, \ldots, i_m \leq N} c_{i_1 \ldots i_m} \int_{[0,1]^m} e_{i_1}(u_1) \ldots e_{i_m}(u_m) W_t(du_1) \ldots W_t(du_m) \]

and, since \( \{e_i\} \) is orthonormal, each term of the right hand side can be written as a product of Hermite polynomials, i.e.,

\[ \int_{[0,1]^m} e_{i_1}(u_1) \ldots e_{i_m}(u_m) W_t(du_1) \ldots W_t(du_m) \]

\[ = H_{p_1}^{e_{q_1}(\cdot)}(u) W_t(du) \ldots H_{p_r}^{e_{q_r}(\cdot)}(u) W_t(du), \]

if there are \( p_1 e_{q_1}(\cdot), \ldots, p_r e_{q_r}(\cdot) \) among \( e_{i_1}(\cdot), \ldots, e_{i_m}(\cdot) \) with \( p_1 + \ldots + p_r = m, 0 < p_1, \ldots, p_r \), where \( H_p(\cdot) \) is the \( p \)-th Hermite polynomial with leading coefficient 1. Note also that

\[ B = (\int_0^1 e_1(u) W_t(du), \ldots, \int_0^1 e_N(u) W_t(du)) \]

is an \( N \)-dimensional Brownian motion.

Define a mapping \( T_N \) from \( C_N \) to \( C_1 \) by
Then
\[
X_N(t) - T_N(B)(t) = \sum_{1 \leq i_1, \ldots, i_m \leq N} c_{i_1} \cdots c_{i_m} \left( \int_0^1 e_{i_1}(u) \mathcal{W}_t(du) \cdots \int_0^1 e_{i_m}(u) \mathcal{W}_t(du) \right) \mathcal{W}_t(du) - \left( \int_0^1 e_{i_1}(u) \mathcal{W}_t(du) \right) \cdots \left( \int_0^1 e_{i_m}(u) \mathcal{W}_t(du) \right)
\]
i.e., $X_N - T_N(B)$ is a finite linear combination of polynomials of degree $\leq m - 2$.

Applying Theorem C to each term, we obtain

**Lemma 2.** For any $\delta > 0$ we have
\[
P((1/\alpha) ||X_N - T_N(B)||_C > \delta) \leq C' \cdot \exp[-C'' \alpha^2/(m-2)]
\]
for sufficiently large $\alpha$, where $C'$ and $C''$ are finite constants.

Since $T_N$ is clearly continuous and homogeneous with degree $m$, we get from Theorem B

**Lemma 3.** (i) For any $\phi \in H_N$ and for any $\delta, \delta' > 0$,
\[
P(||T_N(B)/\alpha - T_N(\phi)||_C < \delta) \geq \exp[-(\alpha^{2/(m-2)})(||\phi||^2_\mathcal{H} + \delta')]
\]
for sufficiently large $\alpha$, and

(ii) for any $\delta, \delta', r > 0$,
\[
P(d(T_N(B)/\alpha, T_N(K_r)) > \delta) \leq \exp[-(\alpha^{2/m}/2)(r^2 - \delta')]
\]
for sufficiently large $\alpha$.

Let $\{\psi_j, j \geq 0\}$ be a CONS in $H_1$. Then $\{e_{ij}\psi_j, i, j \geq 0\}$ is a CONS in
\[L^2[0,1] \otimes H_1\] and any $f \in L^2[0,1] \otimes H_1$ can be expanded as
\[
f(u, t) = \sum_{i, j = 0}^{\infty} c_{ij} e_{i}(u) \psi_j(t)
\]
with $\sum_{i, j = 0}^{\infty} |c_{ij}|^2 < \infty$. Put $\varphi_i(t) = \sum_{j = 0}^{\infty} c_{ij} \psi_j(t)$. Then $\varphi_i \in H_1$ for all $i \geq 0$.
\[ f(u, t) = \sum_{i=0}^{\infty} e_i(u)\varphi_i(t) \text{ and } \left\| f \right\|^2 = \left\| \sum_{i=0}^{\infty} e_i\right\|^2 = \sum_{i=0}^{\infty} \left\| e_i \right\|^2 \left\| \varphi_i \right\|^2 = \sum_{i=0}^{\infty} \left\| \varphi_i \right\|^2. \]

Now, let \( g = Af \in \mathcal{G} \) with \( f \in L^2_{[0,1]} \otimes H_1 \) and define \( g_N(t) = \int_0^1 f(u, t) du \). By Schwarz's inequality,

\[ |g(t) - g_N(t)| \leq \left\| h-h_N \right\|_2 \left( \int_0^1 f(u, t) du \right)^{m/2} \]

\[ = \left\| h-h_N \right\|_2 \left( \sum_{i=0}^{\infty} \left\| \varphi_i \right\| \right)^{m/2}. \]

and so

\[ \left\| g-g_N \right\| \leq \left\| h-h_N \right\|_2 \left( \sum_{i=0}^{\infty} \left\| \varphi_i \right\| \right)^{m/2} \]

\[ \leq \left\| h-h_N \right\|_2 \left( \sum_{i=0}^{\infty} \left\| \varphi_i \right\| \right)^{m/2} \]

\[ = \left\| h-h_N \right\|_2 \left\| f \right\|_m. \]

Note that

\[ g_N(t) = \sum_{1 \leq i_1, \ldots, i_m \leq N} c_{i_1} \ldots c_{i_m} \int_0^1 e_{i_1}(u_1) \ldots e_{i_m}(u_m) f(u_1, t) \ldots f(u_m, t) du_1 \ldots du_m \]

\[ = \sum_{1 \leq i_1, \ldots, i_m \leq N} c_{i_1} \ldots c_{i_m} \left( \int_0^1 e_{i_1}(u)f(u, t) du \right) \ldots \left( \int_0^1 e_{i_m}(u)f(u, t) du \right) \]

\[ = \sum_{1 \leq i_1, \ldots, i_m \leq N} c_{i_1} \ldots c_{i_m} \varphi_{i_1}(t) \ldots \varphi_{i_m}(t) \]

\[ = T_N(\varphi)(t), \quad \text{where } \varphi = (\varphi_1, \ldots, \varphi_N) \in H_N, \]

and also that, for any given \( \varphi = (\varphi_1, \ldots, \varphi_N) \in H_N \), if we put \( f(u, t) \)

\[ N \]

\[ = \sum_{i=1}^{\infty} e_i(u)\varphi_i(t), \text{ then } f \in L^2_{[0,1]} \otimes H_1, \left\| f \right\| = \left\| \varphi \right\|_H \text{ and } Af = T_N(\varphi). \]

From the above we immediately obtain the following

**Lemma 4** (1) For any \( g = Af \in \mathcal{G} \) with \( f \in L^2_{[0,1]} \otimes H_1 \) and for any \( \delta > 0 \), there is
an element \( g_N \) of the form \( g_N = T_N(\psi) \) with \( \psi \in H \), \( ||\psi||_H \leq ||f|| \) such that
\[ ||g - g_N||_C < \delta \]
for all sufficiently large \( N \).

(i1) \( T_N(K_r) \subset C_r \) for all \( r > 0 \), \( N \geq 1 \).

Proof of Theorem 1 (i) Let \( g \in C, \delta, \delta' > 0 \) be given, and assume that \( g \) is of the form \( g = A \varphi, \varphi \in \mathcal{L}^2[0, 1] @ H \). Choose \( N \) large enough that \[ ||g - g_N||_C < \delta/4 \] with \( g_N = T_N(\psi), \psi \in H \) (Lemma 4 (i)), and \( M_N > (4/\delta)^{2/m} (||f||^2 + \delta') \) (cf. Lemma 1).

Then, by Lemmas 1, 2 and 3(i),
\[
P(||X - g||_C < \delta) \geq P(||T_N(B)/\alpha - T_N(\psi)||_C < \delta/4)
- P(||X - X_N||_C > \alpha \delta/4)
- P(||X_N - T_N(B)||_C > \alpha \delta/4)
\]
\[
\geq \exp\left[-(\alpha^{2/m}/2)(||\psi||_H^2 + \delta'/3)\right]
- C \cdot \exp\left[-M_N(\alpha \delta/4)^{2/m}\right]
- C' \cdot \exp[-C'' \cdot \alpha^{2/(m-2)}]
\]
for all sufficiently large \( \alpha \). Noting that \( ||\psi||_H \leq ||f|| \), we thus obtain
\[
P(||X - g||_C < \delta) \geq \exp\left[-(\alpha^{2/m}/2)(||f||^2 + 2\delta'/3)\right]
\]
for all sufficiently large \( \alpha \), and the assertion follows from the definition of \( D(g) \).

Proof of Theorem 1 (ii) Given \( \delta, \delta', r > 0 \), choose \( N \) large enough that
\[ M_N > (3/\delta)^{2/m}(r^2 - \delta') \] (cf. Lemma 1). Then, by Lemmas 1, 2, 3(ii) and 4(ii),
\[
P(d(X/\alpha, C_r) > \delta) \leq P(d(X/\alpha, T_N(K_r)) > \delta)
\leq P(d(T_N(B)/\alpha, T_N(K_r)) > \delta/3)
+ P(||X - X_N||_C > \alpha \delta/3) + P(||X_N - T_N(B)||_C > \alpha \delta/3)
\leq \exp\left[-(\alpha^{2/m}/2)(r^2 - \delta'/2)\right]
+ C \cdot \exp\left[-M_N(\alpha \delta/3)^{2/m}\right] + C' \cdot \exp[-C'' \cdot \alpha^{2/(m-2)}]
\leq \exp\left[-(\alpha^{2/m}/2)(r^2 - \delta')\right]
\]
for all sufficiently large \( a \). This completes the proof of Theorem 1.

3. Proof of Theorem 2

It follows from Lemmas 1 and 2 that
\[
P(\|X-\widehat{T}_N(B)\|_C > z) \leq \exp[-Mz^{2/m}]
\]
for all sufficiently large \( z \) and \( N \), where \( M \) is any given positive constant.

Note also that both processes \( X(\cdot) \) and \( \widehat{T}_N(B)(\cdot) \) are self-similar with parameter \( m/2 \), i.e., \( X(c\cdot) \) and \( c^{m/2}X(\cdot) \), and also \( T_N(B)(c\cdot) \) and \( c^{m/2}T_N(B)(\cdot) \), have the same finite dimensional distributions for any \( c>0 \). Using these facts and the first Borel-Cantelli lemma, we get by the standard argument.

**Lemma 5** For any \( \varepsilon > 0 \), with probability 1
\[
\limsup_{n \rightarrow \infty} \|Z_N - T_N(B)(nt)/(2n \log_2n)^{m/2}\|_C \leq \varepsilon
\]
for sufficiently large \( N \).

Note that
\[
T_N(B)(nt)/(2n \log_2n)^{m/2} = T_N(B_n)(t).
\]
where \( B_n(t) = B(nt)/(2n \log_2n)^{1/2} \). Thus, by Strassen's law of the iterated logarithm for \( B \) and the continuous mapping theorem, we have

**Lemma 6** For any \( N \geq 1 \), with probability 1 the random sequence \( \{T_N(B_n), n \geq 3\} \) in \( C_1 \) is relatively compact and the set of its limit points is \( T_N(K_1) = \{T_N(\varphi); \varphi \in K_1\} \).

Theorem 2 follows from Lemmas 4, 5 and 6.
References


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