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MULTIDIMENSIONAL LEAST
SQUARES FITTING OF FUZZY
MODELS

AIVARS K. R. CELMINS

APRIL 10, 1987

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<p>We describe a new method for the fitting of differentiable fuzzy model functions to crisp data. The model functions can be either scalar or multi-dimensional and need not be linear. The data are n-component vectors. An efficient algorithm is achieved by restricting the fuzzy model functions to sets which depend on a fuzzy parameter vector and assuming that the vector has a conical membership function. A fuzzy model function, equated to zero, defines in the n-space a fuzzy hypersurface. Simple properties of such surfaces are established and a structure in the space of fuzzy manifolds is introduced by defining a discord and collocation between any two fuzzy manifolds. Using these concepts and the restriction to conical membership functions, we derive a simple spread propagation formula for arbitrary functions of fuzzy variables.</p> <p>The model fitting is done in a least squares sense by minimizing the squares of the deviations from one of the membership values of the fitted hypersurface at the observations.</p>			
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19. Abstract (Continued)

Under the outlined restriction, the problem can be reduced to an ordinary least squares formulation for which software is available. ←

Application of the new method is illustrated by two examples. In one example, we are concerned with the hazards caused by enemy fire on armor. An important item of information for the assessment of the involved risks is a predictive model for the hole size in terms of physical properties of the projectile and target plate, respectively. We use a non-linear fuzzy model function for this analysis. The second example is taken from Heshmaty and Kandel.^[4] In this example, the fuzzy model permits one to estimate the risks of economical prediction. The example is of theoretical interest because it allows one to compare the new method with a previously developed method by Tanaka et al.^[5]

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1. INTRODUCTION

We consider model fitting problems in which the data are crisp vectors and the model is a fuzzy function, scalar or multi-dimensional. This type of fuzzified model fitting is attractive in situations where data membership functions are not available, but the model fitting problem nevertheless is considered as fuzzy. The problem is in a sense complementary to the case where the data are fuzzy and the fitting function is assumed to be crisp. Problems of that type were considered by Celmiņš [3]. In both cases the result of the fitting is a fuzzy function.

This paper presents an efficient method for the determination of the model function which is generally assumed to be a non-linear and implicit vector function. The efficiency of the method is achieved by imposing restrictions on the problem formulation. First, membership functions of fuzzy points always are assumed to be conical functions. (These functions are defined in Section 2). Second, the model function is assumed to be a fuzzy set whose elements are defined in terms of a fuzzy parameter vector. Finally, the model fitting is done in a least squares sense by minimizing the squares of deviations from one of the fitted function's membership values at the observed points.

The restriction to model parameter vectors with conical membership functions likely is of minor importance because of the general uncertainty of the particulars of the fuzziness of the fitting function. As far as the least squares minimization is concerned, only experience with fuzzy model fitting may show whether other objective functions than the sum of squares yield more desirable results.

The similarity of problems with fuzzy models and crisp data on one hand and crisp models and fuzzy data on the other hand stems from the fact that in an implicitly formulated model function $F(X,t) = 0$, where X is the observable vector and t is the parameter vector, one does not need to distinguish between "observable" and "parameter" arguments. Therefore, the roles of these arguments formally can be interchanged, and the fitting of a fuzzy model treated as data fitting in the space of parameters. As a consequence, problems of both types can be handled numerically by similar algorithms and software with only minor modifications. The interpretation of the results is of course different in the two cases. Also, in the here considered case with fuzzy model functions, one typically does not a priori know the spread characteristics of the model, which are needed as input for the solution algorithms. The determination of these characteristics therefore

requires an iteration which is not needed in problems with fuzzy data and crisp models.

In Section 2 the basic definitions are provided of fuzzy vector spaces and fuzzy functions with conical membership functions. The least squares model fitting problem is formulated in Section 3. Section 4 discusses the determination of the spread of the fitted model. Examples are presented in Section 5, and Section 6 contains a summary of the results and conclusions.

2. FUZZY VECTORS AND FUZZY FUNCTIONS

We define a fuzzy n -component vector \tilde{A} as a set of vectors with the membership function μ_A . In this paper, we assume that the membership function of a vector always is a conical function which we define as follows.

Let A be a crisp point in R_n , and let P_A be a positive definite $(n \times n)$ -matrix associated with A . Let $\|\cdot\|_A$ be the elliptic norm

$$\|X - A\|_A = [(X - A)^T P_A^{-1} (X - A)]^{1/2} \quad (2.1)$$

of the distance between an arbitrary $X \in R_n$ and A . Using this norm, we define a conical membership function μ_A by

$$\mu_A(X) = 1 - \min\{1, \|X - A\|_A\} \quad (2.2)$$

The fuzzy vector \tilde{A} with the membership function (2.2) is thus specified by A and P_A :

$$\tilde{A} =: \{A, P_A\} \quad (2.3)$$

We call A the *apex* of \tilde{A} and P_A its *panderance matrix*.

In R_1 , the conical membership function is the triangular function

$$\mu_a(x) = 1 - \min\{1, |x - a|/s_a\} \quad (2.4)$$

where $s_a = \sqrt{P_a}$. In R_2 , μ_A is a cone as shown in Figure 1. The membership function of each component of \tilde{A} is a triangular function with a spread which equals the square root of the corresponding diagonal element of P_A . The bases of the triangles are twice the spreads. The boundary of the support of μ_A and the level surfaces $\mu_A = \text{constant}$ are hyperellipsoids in R_n .

Now we consider a space of fuzzy vectors. The distance norm (2.1) is not a convenient measure for the separation of two fuzzy vectors, because in general $\|B - A\|_A \neq \|A - B\|_B$. Therefore we introduce the concept of a *discord* between two fuzzy vectors by the definition

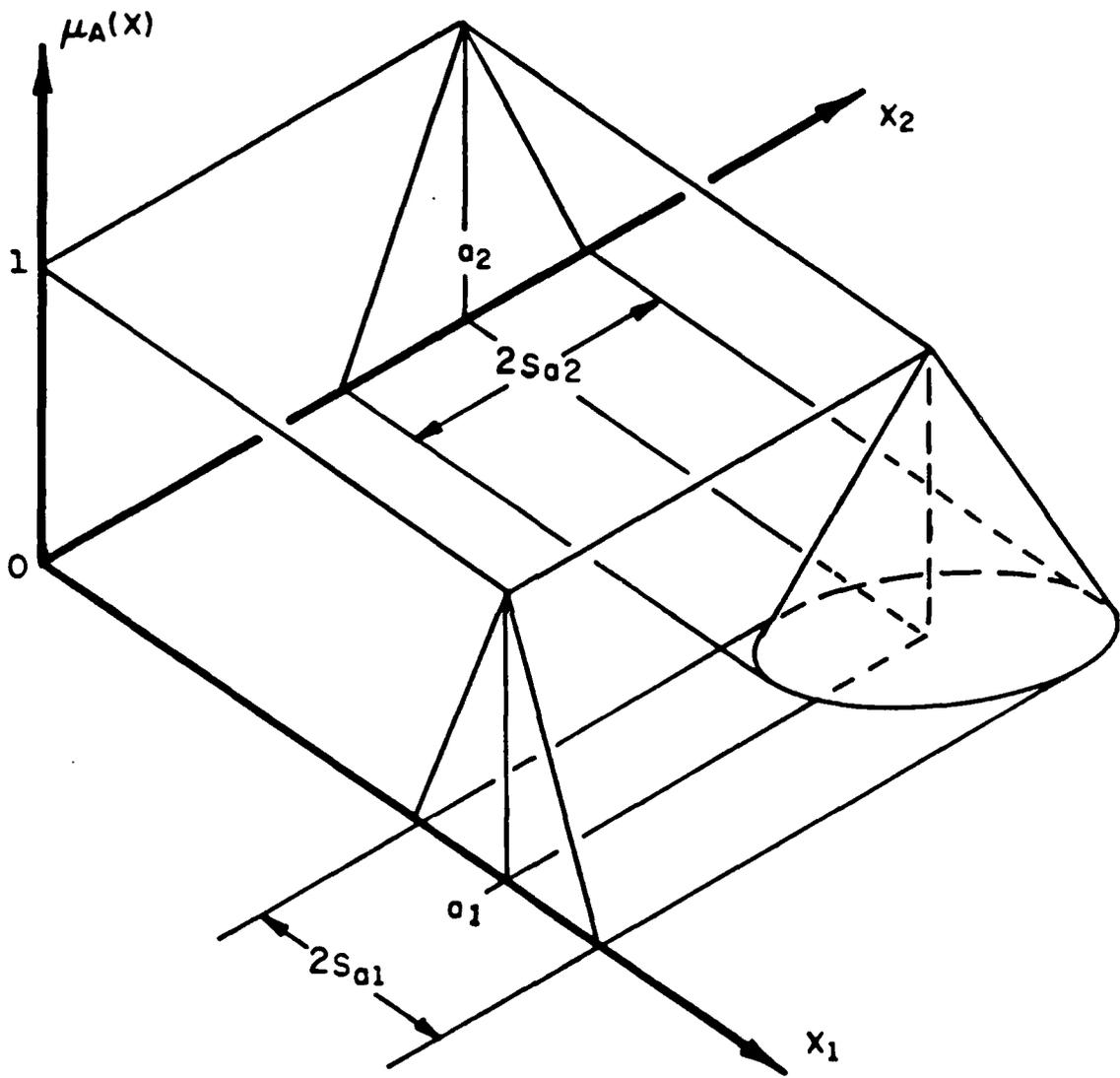


Figure 1. Conical Membership Function in Two Dimensions

$$D(\tilde{A}, \tilde{B}) = \min_{X \in R_n} \max \{ \|X - A\|_A, \|X - B\|_B \} . \quad (2.5)$$

The discord is a symmetric function of \tilde{A} and \tilde{B} , it is positive if $A \neq B$, and it vanishes only if $A = B$. However, it is not a distance in the sense of functional analysis, because it does not satisfy the triangle inequality. We may consider a crisp vector as the limit of a fuzzy vector when its panderance matrix approaches a zero-matrix. The corresponding limit of the discord is the distance (2.1), that is,

$$\lim_{\tilde{A} \rightarrow A} D(\tilde{A}, \tilde{B}) = \|A - B\|_B . \quad (2.6)$$

The discord $D(\tilde{A}, \tilde{B})$ is less than one if and only if the supports of the membership functions of \tilde{A} and \tilde{B} intersect.

Another relation between two fuzzy vectors is the *grade of collocation*. We define it by

$$\gamma(\tilde{A}, \tilde{B}) = \max_{X \in R_n} \min \{ \mu_A(X), \mu_B(X) \} . \quad (2.7)$$

For vectors with conical membership functions one has the following relation between γ and D

$$\gamma(\tilde{A}, \tilde{B}) = 1 - \min \{ 1, D(\tilde{A}, \tilde{B}) \} . \quad (2.8)$$

Now we consider a fuzzy r -component function $\tilde{F}(X)$, and define it as a fuzzy set of functions. In this paper, we only consider fuzzy functions that are defined in terms of a fuzzy p -component parameter vector \tilde{T} , that is,

$$\tilde{F}(X) = F(X, \tilde{T}) \quad (2.9)$$

with

$$\tilde{T} =: \{T, P_T\} . \quad (2.10)$$

Each element t of the fuzzy set \tilde{T} defines an element $F(X, t)$ of the fuzzy function set \tilde{F} . The membership value of t is according to Eq. (2.2)

$$\mu_T(t) = 1 - \min \{ 1, \|t - T\|_T \} , \quad (2.11)$$

where the norm $\|t - T\|_T$ is defined by Eq. (2.1). We assign to the element $F(X, t)$ of \tilde{F} the same membership value:

$$\mu(F(X, t)) = \mu_T(t) . \quad (2.12)$$

The crisp equation $F(X, T) = 0$ defines in R_n a $(n-r)$ -dimensional hypersurface. The fuzzy equation

$$F(X, \tilde{T}) = 0 \quad (2.13)$$

has as a solution in R_n a fuzzy set \tilde{X}_F , which may be geometrically interpreted as a fuzzy hypersurface. The solution of fuzzy equations is discussed in Appendix A where we introduce the concept of a separation $h(X, \tilde{X}_F)$ between a crisp $X \in R_n$ and \tilde{X}_F and define it as (see Eq. (A.13))

$$h(X, \tilde{X}_F) = \min_{t: F(X, t) = 0} \|t - T\|_T. \quad (2.14)$$

If \tilde{X}_F is a fuzzy point \tilde{A} (i.e., a fuzzy vector), then the separation $h(X, \tilde{A})$ equals the norm $\|X - A\|_A$, defined by Eq. (2.1). A generalization to solution sets \tilde{X}_F which are not vectors yields the approximate formula (A.20)

$$h(X, \tilde{X}_F) \approx [F^T (F_t P_T F_t^T)^{-1} F]^{1/2}, \quad (2.15)$$

where $F_t = \partial F / \partial t$, and the functions F and F_t are evaluated at (X, T) . For non-linear F , the approximation may be used if $r=1$ and, with restrictions, if $r>1$. The linearization on which the approximate formula is based is usually not appropriate if $r=n$.

The definition of a separation between crisp points and the fuzzy set \tilde{X}_F , and crisp points and a fuzzy point \tilde{A} permits one to define, in generalization of Eq. (2.5), the discord between \tilde{X}_F and \tilde{A} :

$$D(\tilde{A}, \tilde{X}_F) = \min_{X \in R_n} \max \{ h(X, \tilde{A}), h(X, \tilde{X}_F) \}. \quad (2.16)$$

If the solution \tilde{X}_F of Eq. (2.13) is not a point, then the discord $D(\tilde{A}, \tilde{X}_F)$ can be computed by an approximate formula, also derived in Appendix A. For that approximation one first computes the separation between a particular point X_A and \tilde{A} using the formula

$$h(X_A, \tilde{A}) = [F^T (F_X P_A F_X^T)^{-1} F]^{1/2}, \quad (2.17)$$

where $F_X = \partial F / \partial X$ and the functions F and F_X are evaluated at (A, T) . Then

$$D(\tilde{A}, \tilde{X}_F) = \frac{h(X_A, \tilde{A}) \cdot h(A, \tilde{X}_F)}{h(X_A, \tilde{A}) + h(A, \tilde{X}_F)}. \quad (2.18)$$

The separation $h(A, \tilde{X}_F)$ in Eq. (2.18) is given by Eq. (2.15). If either \tilde{A} or \tilde{X}_F approaches a crisp point, then the discord (2.16) approaches $h(A, \tilde{X}_F)$ or $h(\tilde{A}, X_F)$, respectively.

Using the discord we also can compute the grade of collocation between the fuzzy point \tilde{A} and the fuzzy hypersurface \tilde{X}_F :

$$\gamma(\tilde{A}, \tilde{X}_F) = 1 - \min \{ 1, D(\tilde{A}, \tilde{X}_F) \}. \quad (2.19)$$

This relation follows from the definition (2.7) because the membership function $\mu_F(X)$ of \tilde{X}_F is related to $h(X, \tilde{X}_F)$ by

$$\mu_F(X) = 1 - \min \{ 1, h(X, \tilde{X}_F) \}. \quad (2.20)$$

3. FUZZY MODEL FITTING AND SOLUTION ALGORITHM

We consider the following general type of model fitting. Let $X_i \in R_n$, $i = 1, 2, \dots, s$, be a set of observation vectors, $F_i(X_i, t) \in R_r$ be a corresponding set of model functions, and W be an objective function. The goal of the model fitting is to find a model parameter vector t which approximately satisfies the model equations $F_i = 0$ and minimizes the objective function W . The problem is completely defined by providing the sets $\{X_i, F_i, W\}$ and specifying the approximation type of the model equations $F_i = 0$. If at least one of the elements of the sets is fuzzy, then the solution of the problem is a fuzzy model parameter \tilde{T} , and consequently, as described in Section 2, the fitted model functions $F_i(X, \tilde{T})$ are fuzzy functions, even if the a priori formulated model functions $F_i(X, t)$ were crisp. The case with fuzzy X_i and crisp F_i was treated in Celmiņš [3]. In this paper, we consider the dual case where the observations X_i are crisp, but the fitting functions F_i are fuzzy. The objective in this case is to find for each X_i a particular element from the fuzzy set \tilde{F}_i such that the model equation is satisfied, and the element has a high membership value.

Let

$$\tilde{T} = : \{ T, P_T \} \quad (3.1)$$

be the fuzzy parameter of the model functions. The least squares problem is formulated as follows:

$$\left. \begin{array}{l} \text{minimize} \\ W = \sum_{i=1}^s \left[1 - \mu(F_i(X_i, T + c_i)) \right]^2, \\ \text{subject to} \\ F_i(X_i, T + c_i) = 0, \quad i = 1, 2, \dots, s \end{array} \right\} \quad (3.2a)$$

$$\text{and} \quad \mu(F_i(X_i, T + c_i)) > \dot{\gamma}_i, \quad i = 1, 2, \dots, s. \quad (3.2b)$$

The latter condition (3.2b) is a minimum requirement for the quality of the

fitting function, expressed as a required membership level of the solution.

Now we use the definition (2.12) of the membership function $\mu(F_i)$ and reformulate Eq. (3.2) in terms of μ_T :

minimize

$$W = \sum_{i=1}^s [1 - \mu_T(T + c_i)]^2,$$

subject to

$$F_i(X_i, T + c_i) = 0, \quad i = 1, 2, \dots, s$$

and

$$\mu_T(T + c_i) > \hat{\gamma}_i, \quad i = 1, 2, \dots, s. \quad (3.3b)$$

In Eqs. (3.2) and (3.3) $T + c_i$ is a particular element of the fuzzy set \tilde{T} for which the function $F_i \in \tilde{F}_i$ satisfies the constraints at X_i . The minimization of the objective function selects elements $T + c_i$ with high membership values.

The solution of Eq. (3.2) or (3.3) can be found by solving the following related problem:

minimize

$$W = \sum_{i=1}^s c_i^T P_i^{-1} c_i$$

subject to

$$F_i(X_i, T + c_i) = 0, \quad i = 1, 2, \dots, s$$

and

$$c_i^T P_i^{-1} c_i < (1 - \hat{\gamma}_i)^2, \quad i = 1, 2, \dots, s. \quad (3.4b)$$

Because of the definition (2.11) of μ_T , the solutions of (3.3) and (3.4) are identical, but Eq. (3.4) is better suited for numerical algorithms. This is so because the objective function W in Eq. (3.4) is well defined and differentiable for all values of c_i . In Eq. (3.3), the objective function is not differentiable as μ_T approaches zero, requiring special algorithms to handle this singularity.

The unknowns in Eq. (3.4) are the apex T of \tilde{T} and the elements $T + c_i$ of \tilde{T} . A simple procedure for finding the unknowns is to solve Eq. (3.4a), and to check the condition (3.4b) afterwards. If the condition is not satisfied for many observed points X_i (for many i), then this is an indication that the model functions are not compatible with the data. In this case one may either change the model, or dilate or adapt the spread of the solution, as described in Section 4. If the condition is not satisfied at only a few points,

then this usually indicates outliers. The treatment of such points strongly depends on the application of the model and, therefore, will not be discussed here. One can of course try to enforce the condition (3.4b) by seeking a solution for the complete system (3.4a), (3.4b). However, such a solution, if it exists, will have undesirable properties in most cases, because it particularly accommodates outliers. We shall, therefore, restrict our discussion in this section to the solution of Eq. (3.4a).

The constrained minimization problem (3.4a) is very similar to an ordinary least squares problem, or to a model fitting problem with fuzzy data and crisp model. The latter two cases only differ from Eq. (3.4a) in that the constraint equation is

$$F_i(X_i + c_i, T) = 0 \quad (3.5)$$

instead of $F_i(X_i, T + c_i) = 0$, as in Eq. (3.4a). Because of this similarity, the problem can be numerically solved using available software for least squares problems with implicit model equations [2]. The examples in Section 5 were solved using such a program, COLSAC, described in [1]. A special solution algorithm for problems with the constraint (3.4a) is described in Appendix C. The implementation of this algorithm by COLSAC does not require any reprogramming, however, because of the *general formulation* of the computer program.

At the beginning of this section we pointed out that the difference between a crisp and a fuzzy function is in the characteristics of the function parameter T . If T is crisp, then the function $F_i(X, T)$ also is crisp, and if T is a fuzzy \tilde{T} , the $F_i(X, \tilde{T})$ is a fuzzy function $\tilde{F}_i(X)$. Therefore, a "fuzzy model" means that the functions F_i contain a fuzzy unknown parameter. Its fuzziness is given by the panderance matrix P_T . Consequently, we assumed in the problem formulation that P_T is given, and only the apex T of \tilde{T} is unknown.

Such a problem formulation may be adequate in some cases. However, in many other situations, one has not sufficient information on which to base an estimate of P_T and, therefore, needs a process by which P_T is computed concurrently with T . We propose for this purpose the following iteration. We start with an arbitrary initial approximation P_{T0} of P_T and solve Eq. (3.4a) obtaining a T_1 and the corresponding c_1 . We then calculate a panderance matrix P_{T1} from the input P_{T0} by the panderance propagation formula. (See Eq. (C.7) in Appendix C). Because P_{T0} and P_{T1} describe the same fuzzy vector \tilde{T} , one can argue that both should be equal. Therefore, at the next iteration step we start with a panderance matrix that is proportional to P_{T1} , and repeat the process. The proportionality factor is arbitrary in the

sense that such a factor does not influence the apex T of the solution \tilde{T} . A reasonable factor for the iteration process is the dilator Φ_0 , described in Section 4. This factor is determined such that the spread of the fuzzy solution of $F_i(X, \tilde{T}) = 0$ is just large enough to cover the observation X_i if P_T is multiplied by Φ_0^2 . In general, the convergence of the process is slow, but high accuracy of the elements of P_T usually is not needed. Because of this, the convergence end conditions should be formulated in terms of the components of the apex T instead of the elements of P_T . Convergence acceleration techniques, e.g., overrelaxation also can be useful.

4. DILATION AND ADAPTATION OF THE SPREAD OF THE MODEL

The principal result of the model fitting described in Section 3 is the fuzzy model parameter $\tilde{T} = : \{T, P_T\}$, which defines the fuzzy functions $\tilde{F}_i(X) = F_i(X, \tilde{T})$. However, in many applications one is not interested in these functions but in the relations between the components of X which are defined by the equations

$$F_i(X, \tilde{T}) = 0. \quad (4.1)$$

The solution of Eq. (4.1) is a fuzzy set \tilde{X}_{F_i} in R_n which may be geometrically interpreted as a fuzzy $(n-r)$ -dimensional hypersurface in R_n . Some properties of \tilde{X}_{F_i} are discussed in Appendix A, where it is shown that the apex of \tilde{X}_{F_i} is independent of the particular form of Eq. (4.1), whereas the spread of \tilde{X}_{F_i} depends on the formulation of the equation. Therefore, a discussion of the spread always pertains to a particular formulation, for instance to a solution of Eq. (4.1) for certain components of X . Now we assume that a formulation adequate for applications is chosen and represented by $F(X, \tilde{T}) = 0$. For simplicity we omit the index i . The discussion always involves one particular F_i , and it does not matter whether all the $F_i = F$ or are different.

According to the problem formulation in Section 3 we would like the fuzzy relation (4.1) between components of X be such that the observations X_i are compatible with the relations at least to a grade $\dot{\gamma}$. This requirement is expressed by Eqs. (3.3b) or (3.4b) as constraints for the residuals c_i of the parameter vector. In these formulations, the constraints are independent of the model function $F = 0$. Therefore, the following discussion of the properties of the solution \tilde{X}_F of Eq. (4.1) is applicable not only to the model equation, but also to any other such relation between components of X which depends on \tilde{T} . In general one is of course principally interested in the model function.

In terms of X and \tilde{X}_F the condition (3.2b) means that the discord between X_i and \tilde{X}_F should be less than $1 - \dot{\gamma}_i$, or that the grade of collocation between X_i and \tilde{X}_F should be larger than $\dot{\gamma}_i$. Hence one may use the grade of collocation as a measure of compatibility between data and fitted model. We define it most conveniently in terms of the discord, which is according to Eq. (2.15) or Eq. (A.20) approximately given by

$$D(X_i, \tilde{X}_F) = [F^T (F_i P_T F_i^T)^{-1} F]^{1/2}, \quad (4.2)$$

where F and F_i are evaluated at (X_i, T) . Then the *grade of compatibility* between the data vector X_i and the fitted function is

$$\gamma(X_i, \tilde{X}_F) = 1 - \min \{ 1, D(X_i, \tilde{X}_F) \}. \quad (4.3)$$

In terms of the grade of compatibility, the condition (3.2b) simply is $\gamma(X_i, \tilde{X}_F) > \dot{\gamma}_i$. The global grade of compatibility between data and model equation we define as

$$\gamma(\{X_i\}, \tilde{X}_F) = \min_i \{ \gamma(X_i, \tilde{X}_F) \}. \quad (4.4)$$

One can change the grade of compatibility by dilating the spread of the fuzzy hypersurface \tilde{X}_F . The spread is governed by the parameter panderance matrix P_T and by the form of the Eq. (4.1). We only consider the modification of the spread by a modification of P_T . Let P_T be multiplied by a factor Φ_i^2 . Then the spread of \tilde{T} is multiplied by Φ_i , and so is the discord $D(X_i, \tilde{X}_F)$. Hence, if one wants the grade of compatibility, Eq. (4.3), to equal the desired grade $\dot{\gamma}_i$, then the Φ_i must have the value

$$\Phi_i = D(X_i, \tilde{X}_F) / (1 - \dot{\gamma}_i). \quad (4.5)$$

We call Φ_i the *model spread dilator* for the data point X_i . The global model spread dilator we define by

$$\Phi = \max_i \{ \Phi_i \}. \quad (4.6)$$

The dilators depend on the desired grades of compatibility $\dot{\gamma}_i$. For $\dot{\gamma}_i = 0$ in Eq. (4.5) one obtains a dilator which makes the spread of \tilde{X}_F just large enough so that the support of the membership function of \tilde{X}_F includes the observation X_i . We call the corresponding global dilator Φ_0 and have used it in Section 3 in the iteration of the panderance matrix P_T .

The described dilation of P_T by a constant factor can produce in some parts of the model spreads which are too large to be consistent with the actual discords between observations and model. In such cases the model spread may be adapted by making Φ a function of X , and using the sets

$\{X_i, \Phi_i\}$ to design the function $\Phi(X)$ such that

$$\Phi(X_i) \geq \Phi_i, \quad i = 1, 2, \dots, s. \quad (4.7)$$

One recognizes that the constant global dilator, Eq. (4.6), is an approximation of the function $\Phi(X)$ by a constant Φ_0 .

In model fitting problems where the data are fuzzy one has three different dilators, namely data dilators, minimal model dilators and inclusive model dilators [3]. In the present case with crisp data, the data dilators formally are infinite (to fuzzify the crisp data), whereas both model dilators are equal. Therefore, one only has one dilator type if the data are crisp.

5. EXAMPLES

We present two examples of model fitting with a fuzzy model and crisp data. The first example is a fitting of a non-linear function to data from a terminal ballistics problem. The second example is an economics forecasting problem, and it involves a linear model function. In the latter example we compare our results with those obtained by Heshmaty and Kandel [4] who used a method developed by Tanaka et al. [5], which is based on an optimization principle that is different from ours.

Now we consider the first example. For the assessment of effects of weapons one needs an estimate of the size of perforation that is produced by a projectile or fragment impacting on a metal plate. The size of the perforation depends on the material properties of the projectile and plate, on the geometry of the projectile, and on the velocity and obliquity of the impact. Shear and Dumer [6] have shown that in many cases the ratio of the hole area to projectile's presented area is an increasing function of the dimensionless *ballistic damage indicator*

$$B = K_P / (V_P \sigma_{yT}), \quad (5.1)$$

where K_P is the kinetic energy of the projectile, V_P is its volume and σ_{yT} is the ultimate yield stress of the target material. The relation between the ratio A_H/A_P (crater or hole area by projectile's presented area) and B is fuzzy for two reasons. First, it has no firm theoretical foundations, that is, no particular form of the model function is suggested by a theory of the mechanics of the penetration process. Second, the observations, i.e., the measurements of the involved quantities also are fuzzy. Particularly, the value of σ_{yT} is usually only approximately known, and can vary considerably from specimen to specimen even in laboratory environment. The other quantities, A_H , A_P , K_P and V_P can be more precisely measured, but often the data sources do not contain sufficient information to estimate their distributions in the classical

Table 1. Perforation Data

B	A_H/A_P	B	A_H/A_P
4.060	1.148	21.885	2.599
23.725	2.549	14.010	2.098
60.250	4.774	23.580	2.737
56.405	5.244	19.150	2.518
73.625	6.786	16.865	2.381
57.495	5.148	15.830	2.184
28.770	3.065	15.200	2.064
37.285	3.513	14.755	2.016
8.305	1.209	13.565	1.995
11.540	2.044	8.475	1.571
32.075	4.315	4.390	1.409

sense. Hence, although it is known that the data are fuzzy, one has no means to estimate their spread. This is a typical situation where one would like to fit a fuzzy model to data which, for lack of better information, may be treated as crisp.

Table 1 contains a list of typical experimental data. The list is a subset of the data collected by Shear and Dumer [6], and is presented for illustrative purposes only. The original set contains over 400 observation pairs. We use the data in Table 1 to illustrate the fitting process with the following non-linear fuzzy model function

$$A_H/A_P = \sqrt{B} / (\bar{a}\sqrt{B} + \bar{b}) . \quad (5.2)$$

The numerical solution of the model fitting was obtained as described in Section 3 by using the utility program COLSAC [1]. The model equation (5.2) was formulated in the form

$$F = \bar{a} + \bar{b}/\sqrt{B} - A_P/A_H = 0 , \quad (5.3)$$

where \bar{a} and \bar{b} were treated as data, and B , A_P , and A_H as index variables. In particular, the i -th observation set (B_i, A_{Pi}, A_{Hi}) was used in the form of the constraint

$$F_i = (a + c_{ai}) + (b + c_{bi})/\sqrt{B_i} - A_{Pi}/A_{Hi} = 0 , \quad (5.4)$$

where a and b are the apex values of the model parameters \bar{a} and \bar{b} , respectively, and $a + c_{ai}$ and $b + c_{bi}$ are a set of particular parameter values which satisfy the i -th constraint equation (5.4). The least squares algorithm minimizes the objective function

$$W = \sum_{i=1}^s (c_{ai}, c_{bi}) P_T^{-1} \begin{pmatrix} c_{ai} \\ c_{bi} \end{pmatrix} , \quad (5.5)$$

where P_T is the panderance matrix of the parameter vector (\bar{a}, \bar{b}) . Because P_T was not known, an iteration was carried out as described in Section 3. The iteration end condition was the requirement that the change of the parameter apex values be less than 10^{-6} times their spread. This condition was judged to be sufficient to assure that the inaccuracy of the solution caused by the termination of the iteration was insignificant in comparison to its fuzziness. Convergence under this condition was achieved in 24 iteration steps. The final results were as follows:

$$\left. \begin{aligned} a &= -0.0727, & s_a &= 0.0936, \\ b &= 2.1008, & s_b &= 0.4001, \\ & & c_{ab} &= -0.9766937, \end{aligned} \right\} \quad (5.6)$$

where c_{ab} is the concordance between \bar{a} and \bar{b} . The quoted spreads s_a and s_b and the concordance c_{ab} were used as input for the fitting program, that is, at the final step we assumed that the fuzziness of the model parameters were given by Eq. (5.6), and the apex values a and b were not known. Then the program produced the values of a and b in Eq. (5.6) with panderances computed by the panderance propagation formula from the input. The spreads thus computed were, of course, smaller than the input by the factor Φ_0 , the zero-level dilator. In the present example the value of the dilator was

$$\Phi_0 = 3.317 . \quad (5.7)$$

Because the input panderance matrix was iteratively determined, the concordance c_{ab} between the fitted function parameters is equal to the input concordance between the fuzzy model parameters, given by Eq. (5.6).

Figure 2 displays the results of the model fitting. It shows the observed points, and the fitted curve with its spread. (The spreads in Figure 2 were calculated using the panderance propagation formula (B.17) for the function $A_P/A_H = \bar{a} + \bar{b}/B^{1/2}$. The plots show the inverse A_H/A_P of that function with its spreads. Because the inverse is not a linear function of the parameters, a direct computation of the spread of $A_H/A_P = B^{1/2}/(\bar{a}B^{1/2} + \bar{b})$ would give slightly different results, and be only an approximation. A_P/A_H is, on the other hand, linear with respect to \bar{a} and \bar{b} , and, therefore, the panderance propagation and discord formulas are exact for this function). The solid lines on both sides of the fitted curve indicate the support boundaries of its membership function if the dilator Φ_0 is used. Notice that the magnitude of the dilator is determined by the observation at $B \approx 8.3$. The support boundary passes through that observation. If one specifies instead of zero a compatibility grade of 0.3, then the spread of the fitted function has to be further increased such that the minimum membership value of all observations is 0.3. The corresponding dilator is

$$\Phi_{0.3} = 4.738 . \quad (5.8)$$

The support boundaries corresponding to this dilator are shown in Figure 2 as dotted lines.

Through each observation in Figure 2 we have plotted a short segment of that curve of the fuzzy set of solution curves which passes through the observation and has the highest membership value. (It is given by the parameter values $a + c_{a_i}$ and $b + c_{b_i}$).

One notices in Figure 2 that the spread of the fitted function is quite large for larger values of B , and that this increase of the spread is not consistent with the scatter of the observations. The increasing trend of the

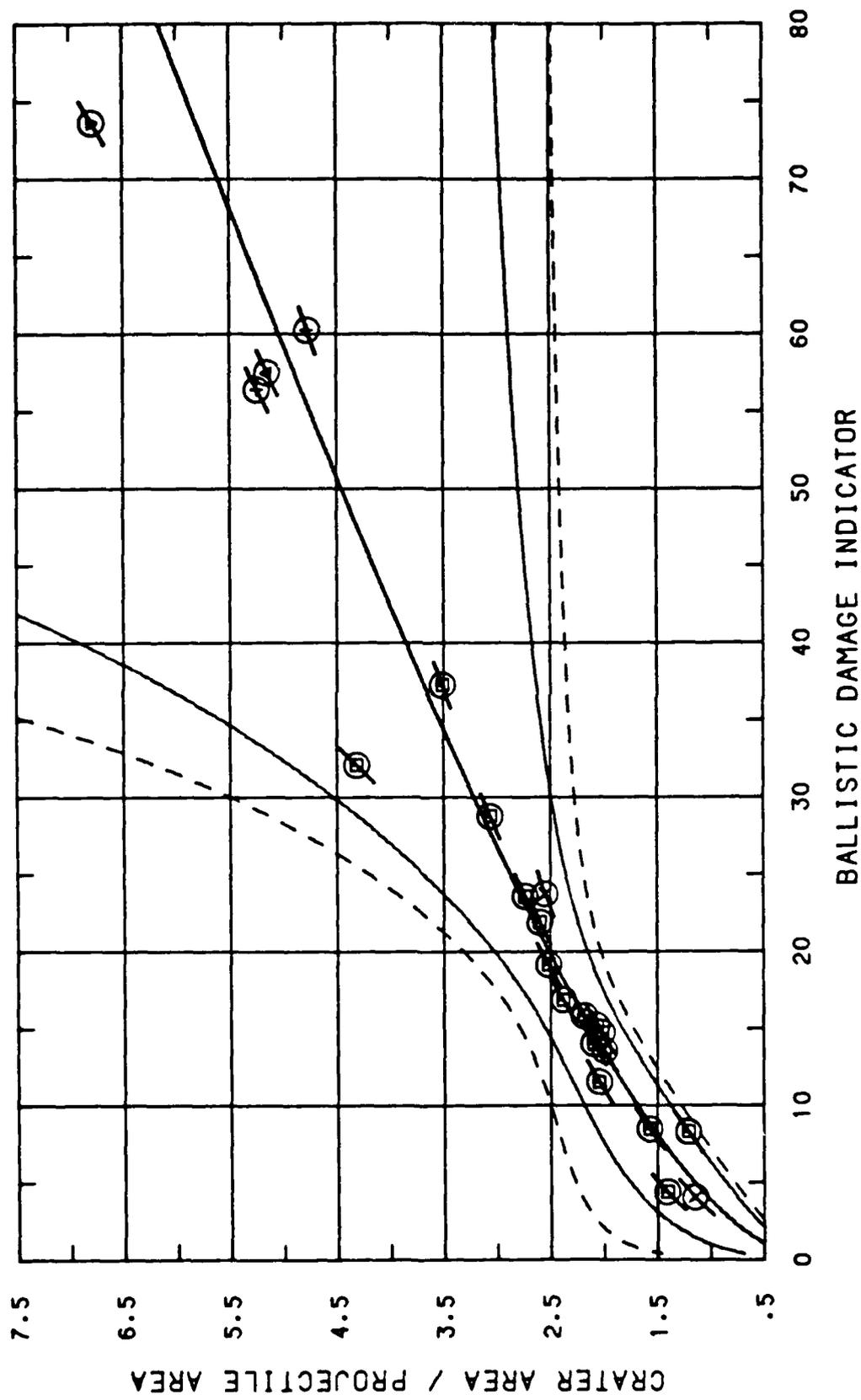


Figure 2. Model of Crater Size

spread of course correctly reflects the uncertainty of the function in the region with few data. The magnitude of the spread, that is, the dilator Φ_0 is, however, generally determined by few outliers. Thus, in the present case an observation at $B \approx 8.3$ is causing the large value of Φ_0 . The fitted function would change little if that observation were discarded, but the spread would be reduced. Therefore, one easily could obtain a less fuzzy fitted model if this observation and other outliers were discarded. Another possibility to reduce the fuzziness of the fitted model is an adaptation of the spread, as described in Section 4. We achieve such an adaptation by constructing a modulator function $\Phi_\gamma(B)$ instead of a constant dilator Φ_γ . Using such a function, the panderance matrix P_{TF} of the parameters of the fitted function is computed by

$$P_{TF}(B) = \Phi_\gamma^2(B) P_T , \quad (5.9)$$

and the spread of the fitted function is computed using $P_{TF}(B)$ instead of the constant $\Phi_0^2 P_T$ or $\Phi_{0.3}^2 P_T$, as in Figure 2. In the present example, different functions $\Phi_{0.3}(B)$ were computed for positive and negative spreads, respectively. The form of both functions was

$$\Phi(B) = \max \{ 1 , \alpha + (B - B_0)^2 \beta \} , \quad (5.10)$$

and the parameters, α , B_0 and β were determined such that the membership value of the fitted function (5.2) was at least 0.3 for each observation.

Figure 3 shows the result of the fitted function with the modulated spread. The solid lines indicate the limits of the support of $\mu_F = 0.3$, where μ_F is the membership function of the fitted function. The dotted lines are the limits of the support of μ_F . The spread of the model is more reasonable than in Figure 2, and consistent with the observed scatter of data.

Figure 4 illustrates the described model fitting in the parameter space. The result of the fitting is a fuzzy vector with the components \bar{a} and \bar{b} and a panderance matrix given by Eq. (5.6). The support of the membership function of the fuzzy model parameter is an ellipse. Figure 4 shows the ellipse, dilated by $\Phi_{0.3}$, as a solid line. The level line $\mu_T = 0.3$ is shown in the figure as a dotted ellipse. The figure also contains the locations of the parameter vectors $T + c_i$ which correspond to each of the 22 observations. One notices that all of the latter vectors are inside the dotted ellipse, as required by the calculation of the dilator $\Phi_{0.3}$. (The different symbols in this and the previous figures indicate different combinations of projectile and target materials).

Next, we present the second example with a linear fitting function

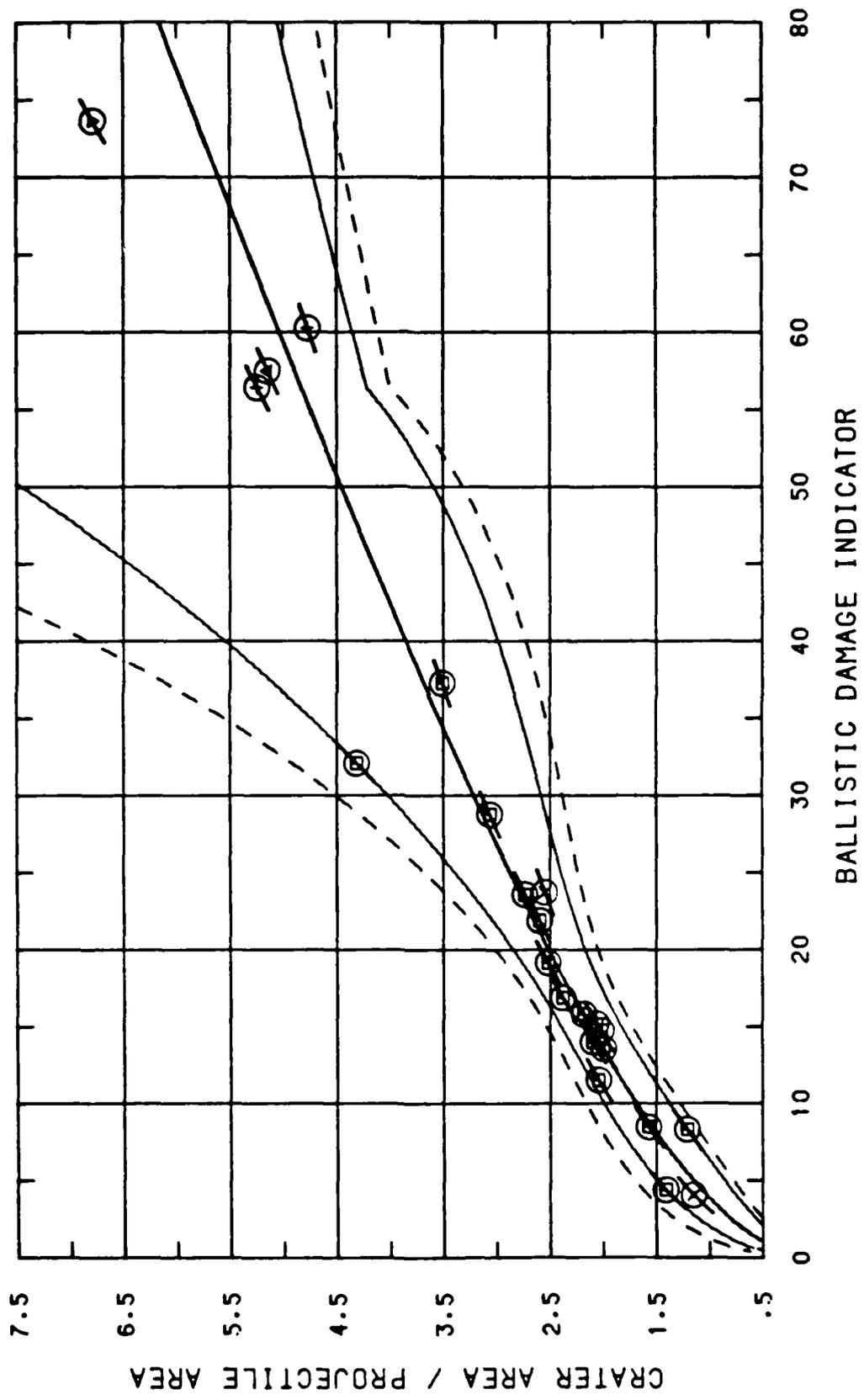


Figure 3. Model of Crater Size with Adapted Spread

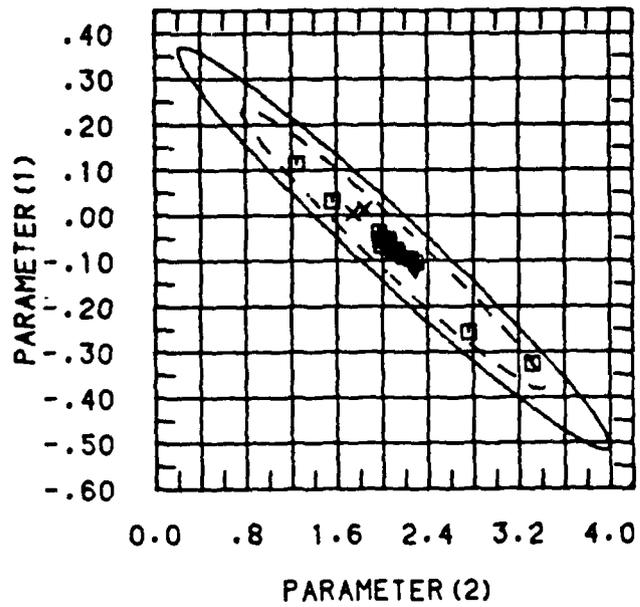


Figure 4. Adjusted Model Parameters of Crater Size Model

$$\tilde{S} = \sum_{i=1}^4 \tilde{A}_i x_i, \quad (5.11)$$

and data taken from Heshmaty and Kandel [4]. Table 2 gives a summary of the data. (A detailed discussion of their sources and significance is given in [4] and will not be repeated here). The purpose of this model fitting is to provide an estimate of \tilde{S} (a total sales value) for the years beyond 1980, for projected values of the x_i . A particular set of projected values for the years 1981-1988 which was used in [4] also is listed in Table 2.

The fuzzy parameter vector $\tilde{A} = \{A_i, P_A\}$ of the model (5.11) was determined as described in Section 3, again using the least squares utility program COLSAC. The panderance matrix of \tilde{A} was determined by iteration. In this example the iteration end criterion was satisfied after 52 iteration steps, whereby an overrelaxation factor 1.7 was used. The resulting model parameter vector \tilde{A} is listed in Table 3 together with the model parameter vector \tilde{A}_{HK} which was found by Heshmaty and Kandel [4] using a different approach. The latter model with the parameter vector \tilde{A}_{HK} is supposed to fit the data for the years 1975 through 1981 with a compatibility grade 0.5. However, a test calculation shows that the model does not fit the data for the year 1975. (See, e.g., Table 4 or Figure 6). Therefore, in order to have a fair comparison between Heshmaty and Kandel results and results by the present approach, we only fitted the 11 data sets for the years 1976 through 1981. Table 4 shows that our model also fits the data for the year 1975 with a compatibility grade 0.77, but this is coincidental. We also note in passing that the two data sets for the year 1975 were found to be influential for the model fitting, that is, their inclusion in the data base substantially changes the fitted model parameter \tilde{A} .

Comparing our parameter vector \tilde{A} with the parameter vector \tilde{A}_{HK} one notices two essential differences. (See Table 3). First, the coefficient \tilde{A}_4 of the price of microcomputers has a negative apex, whereas A_{HK4} is positive. Hence our model predicts an increase albeit small of the total sales volume (in \$) if the price of microcomputers decreases. This might have some economic significance. More important for the mathematical aspect of the model fitting is the fact that \tilde{A}_{HK} is a degenerated fuzzy vector with only one fuzzy component. This means that the estimated spread of the computed sales volume $\tilde{S}_{HK} = \sum \tilde{A}_{HKi} x_i$ only is affected by x_3 . This seems to be a somewhat artificial situation, which can lead to overconfidence in the accuracy of the model if it is used for extrapolation in the variables x_1 , x_2 , or x_4 . In contrast to this situation, all components of a least squares model parameter, presently of \tilde{A} , are fuzzy and, consequently, the estimated model spread increases whenever any of the arguments x_1 through x_4 are used

Table 2. Economic Data

Year	x_1	x_2	x_3	x_4	S_{obs}
1975.0	41.37	3.0	1375.0	4100.00	5855.50
1975.5	46.34	15.5	1622.5	3600.00	6852.40
1976.0	51.30	28.0	1870.0	3100.00	7849.30
1976.5	56.68	45.5	2285.0	2893.50	8727.20
1977.0	62.07	63.0	2700.0	2687.30	9605.10
1977.5	69.52	81.5	3100.0	2522.65	10984.80
1978.0	76.97	100.0	3500.0	2358.00	12364.50
1978.5	83.13	140.0	5208.0	1847.75	14124.00
1979.0	89.29	180.0	6916.0	1337.50	15883.50
1979.5	94.64	407.0	7878.0	1313.15	17845.05
1980.0	100.00	634.4	8840.0	1288.80	19806.60
1980.5	110.06	832.9	9728.5	1191.30	21798.20
1981.0	120.11	1031.4	10617.0	1093.80	23789.80
1982.0	135.00	1390.0	12600.0	900.00	-
1983.0	150.00	1700.0	14600.0	800.00	-
1984.0	165.00	3310.0	16600.0	700.00	-
1985.0	180.00	5260.0	18600.0	600.00	-
1986.0	195.00	7890.0	20600.0	500.00	-
1987.0	210.00	11835.0	22600.0	400.00	-
1988.0	225.00	17752.0	24600.0	300.00	-

The data are taken from Heshmaty and Kandel [4].

- x_1 = User population percent expansion
- x_2 = Microcomputer sales, 10^6 \$
- x_3 = Minicomputer sales, 10^6 \$
- x_4 = Price of microcomputer, \$
- S_{obs} = Sales of computers and peripheral equipment, 10^6 \$

The data (x_1 through x_4 and S_{obs}) for the years 1975 through 1981 are actual observations from different sources. The values of x_1 through x_4 for the years 1982 through 1988 are projected.

Table 3. Economic Model Parameters

Apex Coordinates and Spreads of Model Parameters

i	A_i	s_i	A_{HKi}	s_{HKi}
1	127.95	11.93	130.39	0.0
2	2.177	0.709	4.48	0.0
3	0.616	0.153	0.39	0.15
4	-0.015	0.144	0.14	0.0

Concordance Matrix of \tilde{A}

$$\begin{pmatrix} 1. & 0.498100 & -0.953341 & -0.954782 \\ 0.498100 & 1. & -0.720295 & -0.292541 \\ -0.953341 & -0.720295 & 1. & 0.834829 \\ -0.954782 & -0.292541 & 0.834829 & 1. \end{pmatrix}$$

The apex coordinates A_{HKi} and spreads s_{HKi} are due to Heshmaty and Kandel [4]. The concordance matrix of \tilde{A}_{HK} is the unit matrix. The dilator for the model \tilde{A} with a compatibility level of 0.5 is $\phi_{0.5} = 3.3166$.

Table 4. Economic Forecasts

Observation		Present Model				Heshmaty & Kandel Model [4]			
Year	S _{Obs}	S	s	S-S _{Obs}	γ	S _{HK}	S _{HK}	S _{HK} -S _{Obs}	γ _{HK}
1975.0	5855.50	6085.32	989.84	229.82	.77	6517.92	206.25	662.42	.00
1975.5	6852.40	6908.41	670.22	56.01	.92	7248.49	243.38	396.09	.00
1976.0	7849.30	7730.21	370.91	-119.09	.68	7977.75	280.50	128.45	.54
1976.5	8727.20	8715.42	261.02	-11.78	.95	8890.59	342.75	163.39	.52
1977.0	9605.10	9701.90	194.87	96.80	.50	9804.77	405.00	199.67	.51
1977.5	10984.80	10944.27	265.82	-40.53	.85	10992.00	465.00	7.20	.98
1978.0	12364.50	12186.64	432.27	-177.86	.59	12179.24	525.00	-185.26	.65
1978.5	14124.00	14121.68	350.37	-2.32	.99	13756.33	781.20	-367.67	.53
1979.0	15883.50	16056.71	608.20	173.21	.72	15333.41	1037.40	-550.09	.47
1979.5	17845.05	17828.38	423.23	-16.67	.96	17419.73	1181.70	-425.32	.64
1980.0	19806.60	19602.20	450.19	-204.40	.55	19509.14	1326.00	-297.46	.78
1980.5	21798.20	21870.29	615.44	72.09	.88	22043.01	1459.28	244.81	.83
1981.0	23789.80	24137.10	907.58	347.30	.62	24575.58	1592.55	785.78	.51
1982.0	-	28047.38	1353.35	-	-	28869.85	1890.00	-	-
1983.0	-	31875.00	1695.23	-	-	32980.50	2190.00	-	-
1984.0	-	38532.72	4959.67	-	-	42915.15	2490.00	-	-
1985.0	-	45930.62	9101.17	-	-	54373.00	2790.00	-	-
1986.0	-	54808.88	14851.54	-	-	68877.25	3090.00	-	-
1987.0	-	66549.90	23696.06	-	-	89272.70	3390.00	-	-
1988.0	-	82583.95	37179.41	-	-	118502.71	3690.00	-	-

S_{Obs}, S and S_{HK} are sales of computers and peripheral equipment in millions of dollars. s and s_{HK} are the spreads of the model values S and S_{HK}, respectively. Both are calculated for a desired compatibility grade of 0.50. The compatibility grade γ is calculated by $\gamma = \max\{0, 1 - |S - S_{Obs}|/s\}$ and a corresponding formula is used to calculate the compatibility grade γ_{HK}.

beyond the observation interval. The different behavior of both models is evident from Table 4, which shows that the spread of \tilde{S} increases substantially over that of \tilde{S}_{HK} for the years after 1984, thus warning a potential user of the economic model about the intrinsic inaccuracies of economic forecasts. We notice in passing that the estimated spread of \tilde{S} has been calculated under the assumption that the projected values of x_i are crisp. The formalism of panderance propagation (see Appendix B) permits one also to take into account the spreads (and concordances) of the projected x_i . The resulting spread of \tilde{S} can be easily calculated and would, of course, be larger if finite spreads of the x_i were assumed.

The results of the fitting are shown graphically in Figures 5 through 8. Figures 5 and 6 show the data fitting quality of both models for the years 1975 through 1981. The grade of compatibility is 0.5 for the model \tilde{A} and 0.47 for the model \tilde{A}_{HK} (in the latter case disregarding data from 1975). From a practical view point, both models fit the data equally well although our model (Figure 5) consistently produces smaller differences between computed S and observed S .

Figures 7 and 8 show the predicted sales \tilde{S} and \tilde{S}_{HK} , respectively, up to the year 1988. The curves on both sides of the predicted sales volume curve are the support boundaries of \tilde{S} and \tilde{S}_{HK} , respectively. One notices a divergence of predictions after 1984, and also an increase of the spread of \tilde{S} . The spread of \tilde{S}_{HK} increases only moderately, as discussed above. We also note that the prediction \tilde{S}_{HK} approximately coincides with the upper support boundary of \tilde{S} , that is, \tilde{S}_{HK} is close to a border solution of the fuzzy least squares result. It appears that the increase of the estimated spread of \tilde{S} for values of x_i outside the region of observations is an advantage of the panderance propagation formalism, which is used in this paper.

6. SUMMARY AND CONCLUSIONS

We have presented an efficient and flexible method for the fitting of fuzzy model functions to crisp data. The efficiency of the method is achieved by restricting the membership functions of vector data to a class of conical functions, and by using a least squares objective function. A fuzzy vector with a conical membership function is defined as a set consisting of an apex vector and a panderance matrix which describes the spread of the fuzzy apex point. We have derived for functions of such vectors a panderance propagation formula which is exact for linear vector operations, and gives an approximate estimate of the fuzziness of non-linear differentiable functions of fuzzy vectors.

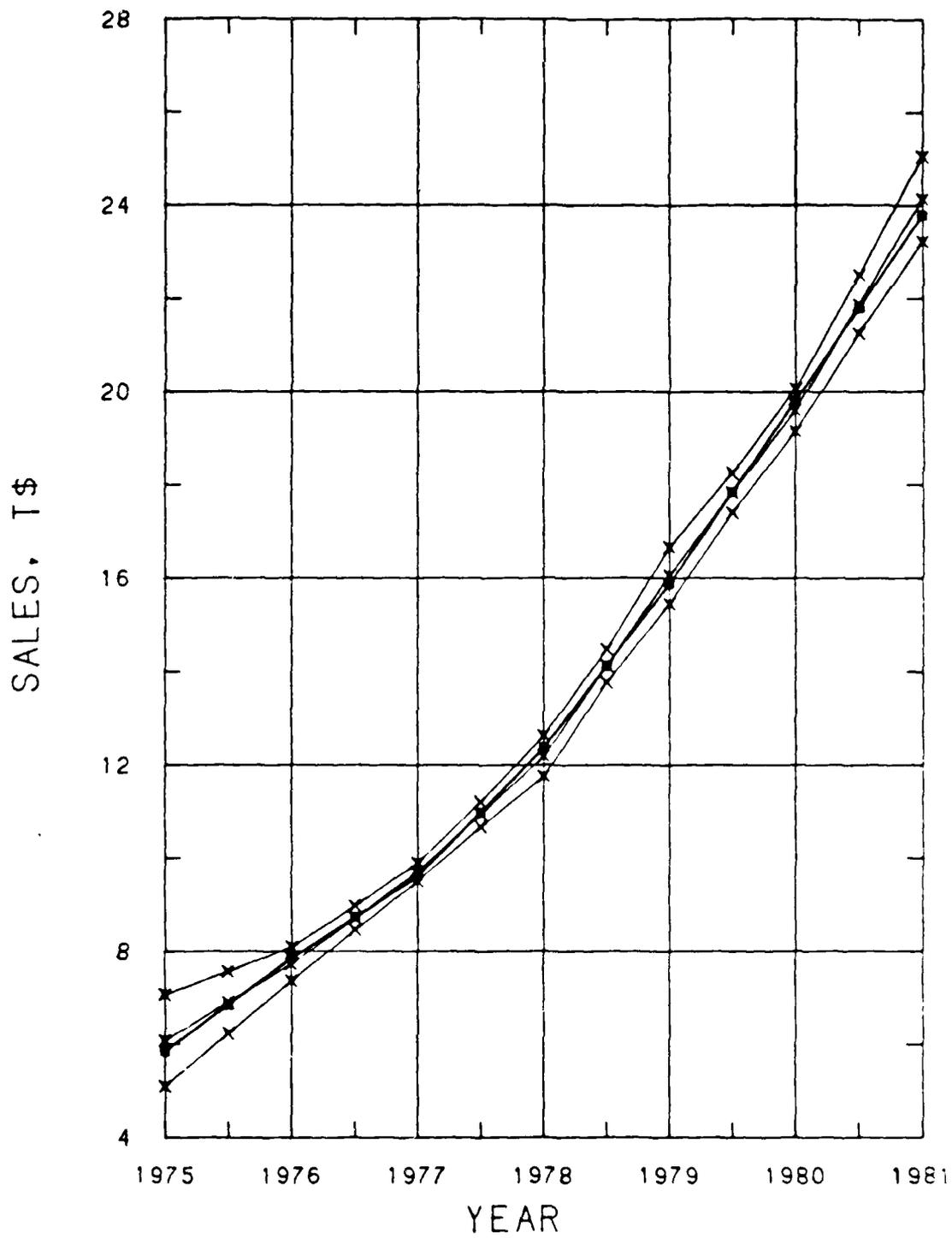


Figure 5. Fitting Result of Economic Model

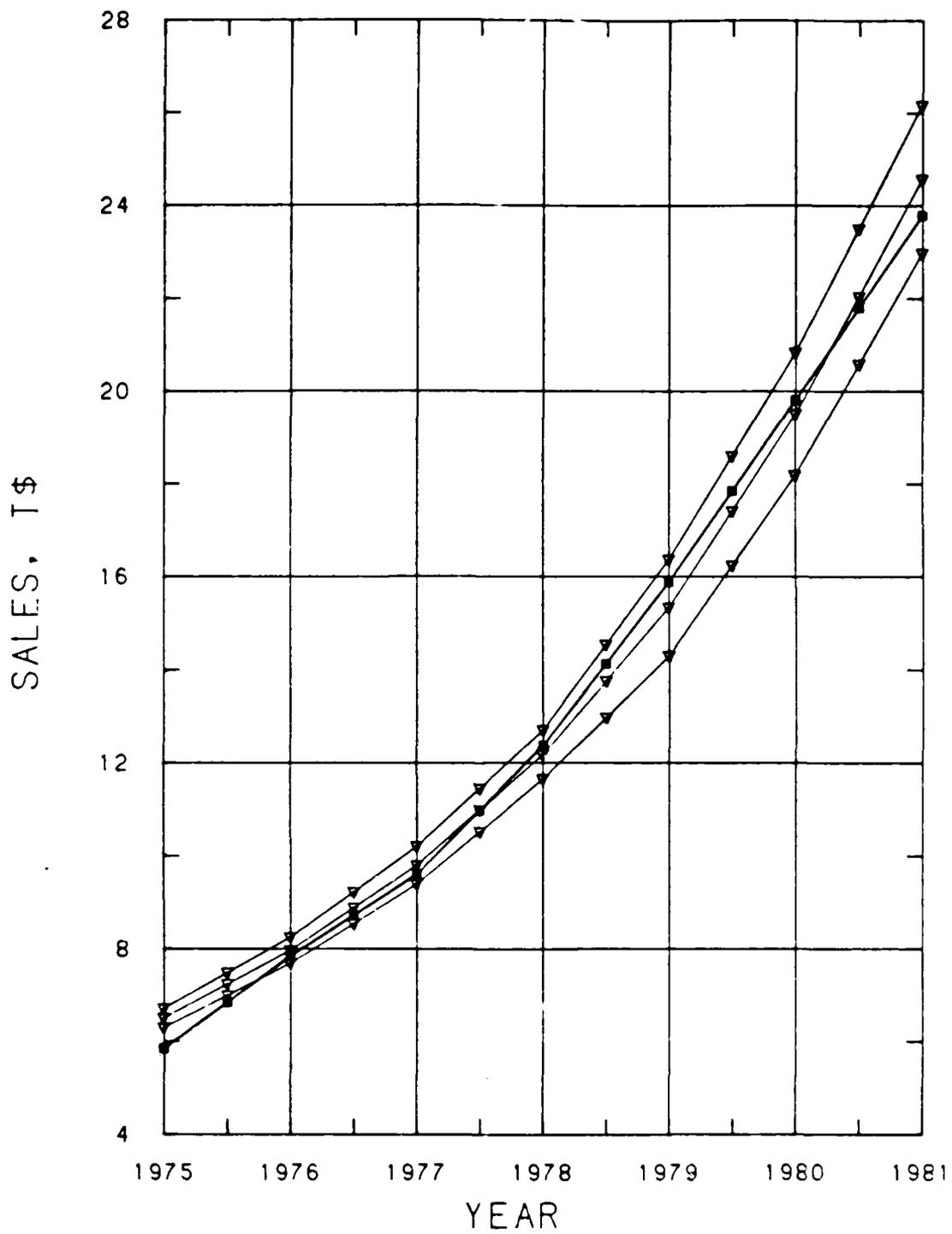


Figure 6. Fitting Result of Economic Model by Heshmaty and Kandel

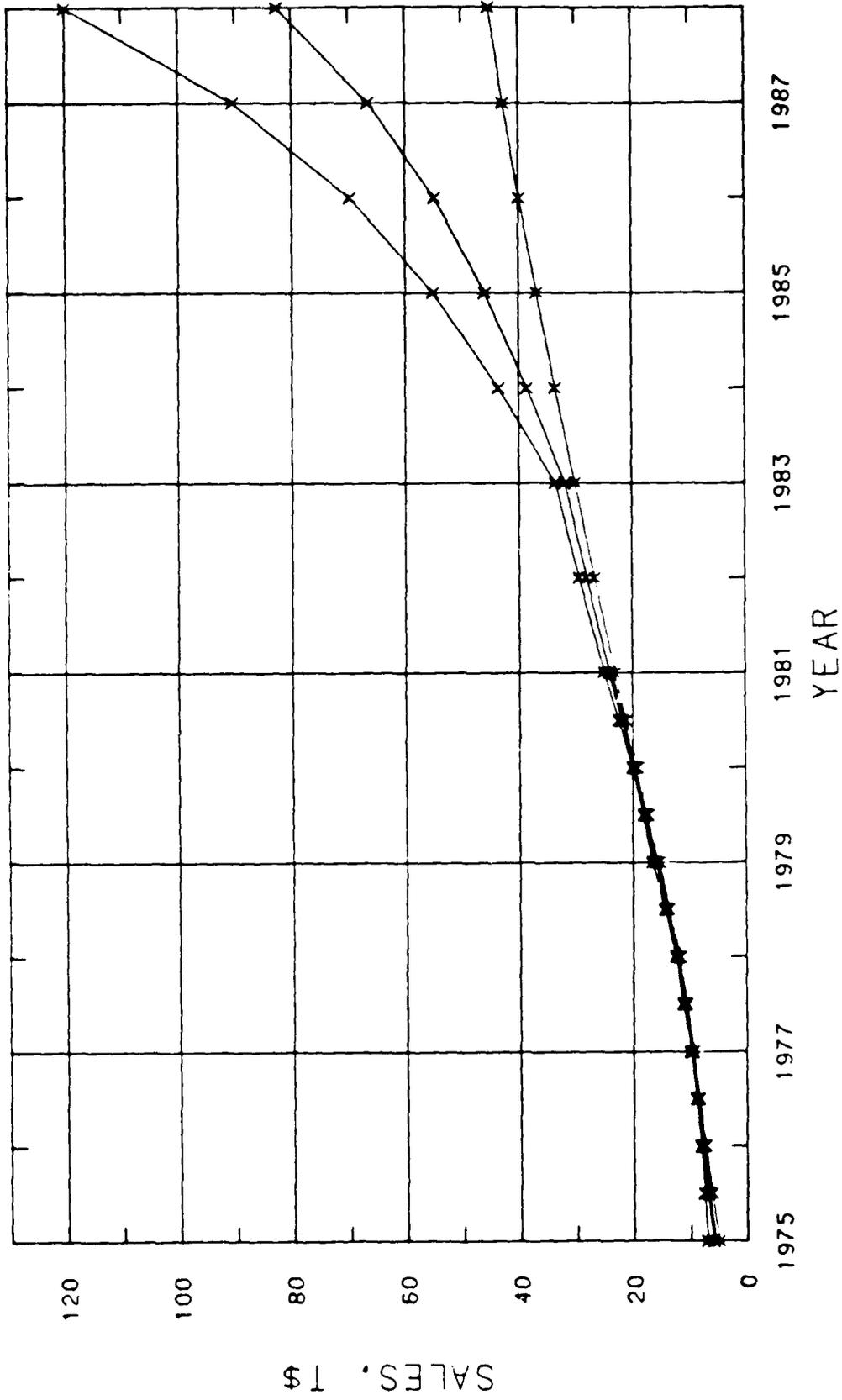


Figure 7. Predicted Sales Volume

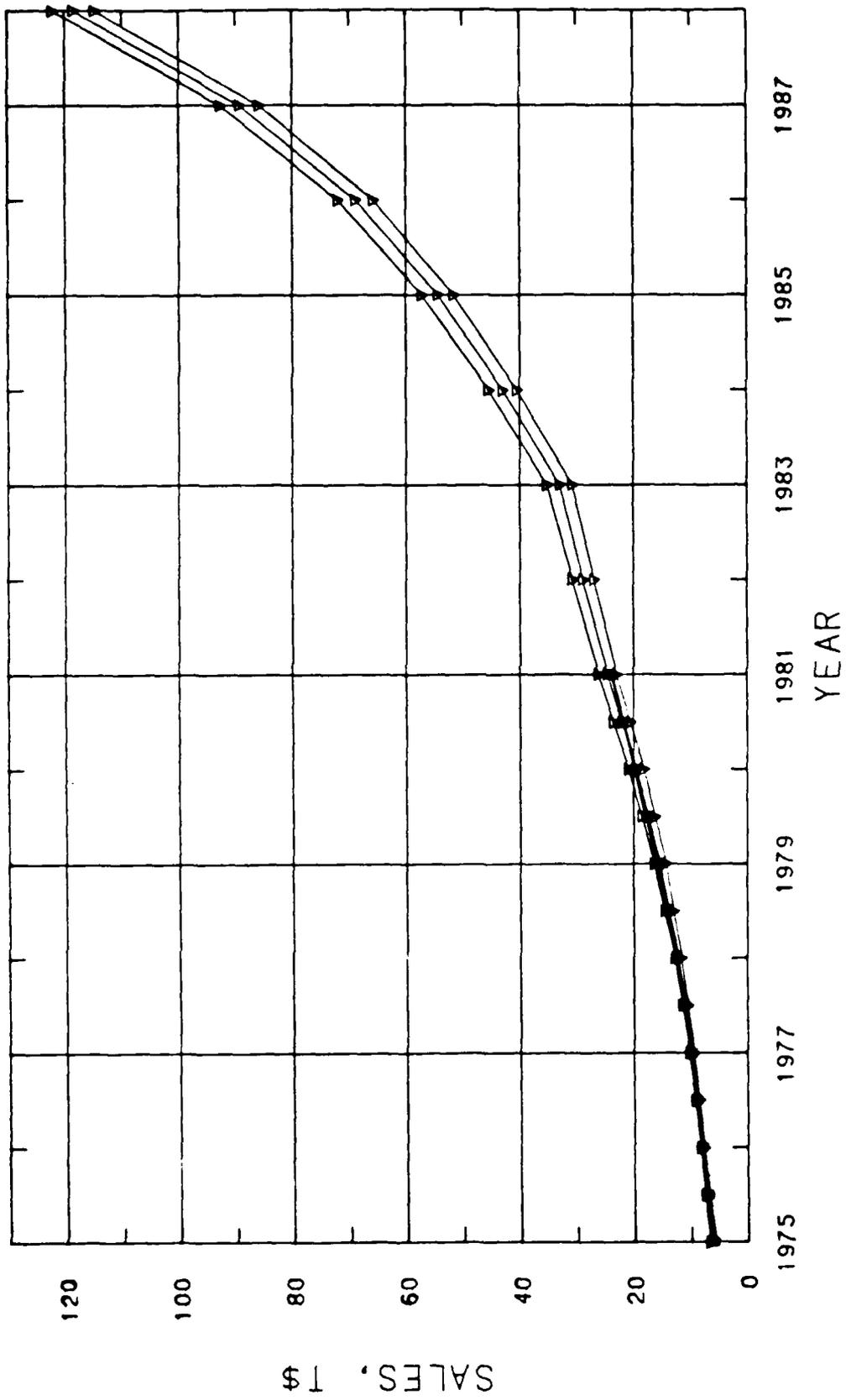


Figure 8. Sales Volume Predicted by Heshmaty and Kandel1 [4]

Fuzzy functions we define in the present context as sets of functions which depend on fuzzy function parameter vectors. These parameter vectors again were assumed to have conical membership functions. Each element of the parameter set defines an element of the function set and we assign to both elements the same membership value. We do not assume that the functions are scalar, thereby achieving general applicability of the presented approach to a large variety of problems. By setting a fuzzy function equal to zero (crisp or fuzzy) one obtains a fuzzy equation. Such an equation defines a fuzzy hypersurface in R_n (n is the dimension of the argument vector of the function). A fuzzy point in R_n is defined by a fuzzy vector and is a special case of a fuzzy hypersurface. Hence, in fuzzy model fitting one deals with fuzzy manifolds, points and surfaces, in R_n .

We introduce a structure in the space of the described fuzzy manifolds by defining a discord between any two elements of the space. It is a measure for the separation of the elements. A concept dual to the discord is the collocation of two elements. We use the concepts of discord and collocation to express a desired grade of compatibility between data and fitted model by a convenient formula. The objective of the model fitting is to find such a fuzzy hypersurface in the space of observables which has high membership values at the observed points. In particular, we minimize the sum of squares of the deviations of these membership values from one. Because we formulate the model function in the form of an implicit equation, we can interpret the problem as the fitting of a crisp model in the parameter space. In this formulation, and because the parameter vector is assumed to have a conical membership function, the numerical treatment can be done using available software for general least squares problems.

We presented two examples: one involves a non-linear function of the observables and the other uses a linear model. The latter case was compared with results published by Heshmaty and Kandel [4], who used a model fitting approach developed by Tanaka et al. [5]. The comparison shows that in this particular example our method produces a model which fits the data slightly better than the Heshmaty and Kandel model, but the difference is not significant. A more important difference is that the method by Tanaka et al. [5] tends to generate model parameters with some crisp and some fuzzy components, and it does not provide estimates of the concordances between the model parameter components. In our method the model parameter only has fuzzy components, and the concordances are explicitly calculated. As a result, the estimated spread of fitted model functions typically increases as any of the observations are extrapolated. In contrast, Tanaka et al. model spread is insensitive to extrapolations of those observations which have crisp parameters as coefficients. It seems that our

estimated spreads are more reasonable, because they would draw the attention of a user of the model to the intrinsic inaccuracies of extrapolation. In comparing our model with Heshmaty and Kandel calculations, one indeed observes that our calculated spread does include the Heshmaty and Kandel solution, albeit with a small membership value, but not vice versa.

We have also discussed a modulation of the spread and shown its application in the first example. Such a modulation might provide more realistic spread estimates than a simple application of the pandurance propagation formula, because the modulation uses more of the available information.

In conclusion, our method has the advantage that it can be easily implemented, using available software, and is not restricted to any special form of the model function. The method produces estimates of model spreads that in the example are more reasonable than those obtained by Tanaka et al. [5]. Restrictions of our method are due to the requirement of conical membership functions. This restriction is essential for the efficiency of the solution algorithms.

The tools and concepts developed for the present problem also can be used to develop solution methods for the more general problem of fitting a fuzzy model to fuzzy data. If estimates of the spreads of model parameters and data are available then such an extension is a straight forward application of the present results. If, however, these spreads are to be estimated concurrently with the model, then more development is needed, because the simple iteration described in this paper cannot be applied to such problems.

Appendix A.

FUZZY EQUATIONS

Let $X \in R_n$ and let $F(X) \in R_r$ be a continuously differentiable function of X . The crisp equation

$$F(X) = 0 \quad (\text{A.1})$$

defines in R_n a $(n-r)$ -dimensional hypersurface X_F . The equation can be fuzzified either by a fuzzification of the function F yielding

$$\tilde{F}(X) = 0, \quad (\text{A.2})$$

or by replacing the crisp right hand side by a fuzzy zero, that is,

$$F(X) = \tilde{0}. \quad (\text{A.3})$$

In the former case one has a set of functions equaling zero, and a corresponding set of solutions. From these solutions one may construct a fuzzy domain in R_n which represents the fuzzy solution \tilde{X}_F of Eq. (A.2). In the case of Eq. (A.3) one has a crisp function $F(X)$ which is set equal to different values close to zero. Each equation produces a solution X_F , and the set of these solutions is the solution of Eq. (A.3). One can formally transform the latter equation into an equation of the same type as Eq. (A.2) by defining a fuzzy function

$$\tilde{G}(X) = F(X) - \tilde{0} \quad (\text{A.4})$$

and setting

$$\tilde{G}(X) = 0. \quad (\text{A.5})$$

We shall first discuss the more general Eq. (A.2), and later demonstrate that the solution of Eq. (A.5) indeed is identical to the solution of Eq. (A.3).

We define the fuzzy function \tilde{F} in terms of a fuzzy parameter vector \tilde{T} . Because we only consider vectors with conical membership functions, \tilde{T} is defined by

$$\tilde{T} =: \{ T, P_T \}, \quad (\text{A.6})$$

where $T \in R_p$ is the apex of \tilde{T} and P_T is its panderance matrix. (P_T is a positive definite $(p \times p)$ -matrix). The membership function μ_T of \tilde{T} is defined as follows

$$\|t-T\|_T = [(t-T)^T P_T^{-1} (t-T)]^{1/2}, \quad (\text{A.7})$$

$$\mu_T(t) = 1 - \min \{ 1, \|t-T\|_T \}. \quad (\text{A.8})$$

The fuzzy function \tilde{F} we define by

$$\left. \begin{aligned} \tilde{F}(X) &= F(X, \tilde{T}) \\ \text{with the membership function} \\ \mu(F(X, t)) &= \mu_T(t). \end{aligned} \right\} \quad (\text{A.9})$$

We also assume that $F(X, t)$ is continuously differentiable with respect to X as well as with respect to t . The Jacobian matrices of the derivatives we denote by subscripts:

$$\partial F / \partial X = F_X, \quad \partial F / \partial t = F_t. \quad (\text{A.10})$$

Each element of the fuzzy set of equations (A.2), i.e., each crisp equation

$$F(X, t) = 0 \quad (\text{A.11})$$

with fixed t defines a crisp relation between components of X . We assign to that relation the membership value $\mu_T(t)$. On the other hand, for any fixed X one has a set (possibly an empty set) of parameters t which satisfy Eq. (A.11). We assign to X the highest membership value of that set:

$$\mu_F(X) = \max_{t: F(X, t) = 0} \mu_T(t). \quad (\text{A.12})$$

The fuzzy solution \tilde{X}_F of Eq. (A.2) then is defined as a set in R_n with the membership function (A.12).

An equivalent but more convenient definition of $\mu_F(X)$ is in terms of a separation measure h of X from \tilde{X}_F :

$$h(X, \tilde{X}_F) = \min_{t: F(X, t) = 0} \|t-T\|_T, \quad (\text{A.13})$$

$$\mu_F(X) = 1 - \min \{ 1, h(X, \tilde{X}_F) \}. \quad (\text{A.14})$$

To be definite, we assign an infinite value to $h(X, \tilde{X}_F)$ if $F(X, t) = 0$ has no solution t for the given X .

Now we derive an approximate expression for the separation $h(X, \tilde{X}_F)$. To that end we reformulate Eq. (A.13) as the following constrained minimization problem:

$$\begin{array}{l}
\text{minimize} \\
\text{subject to}
\end{array}
\left. \begin{array}{l}
W = (t-T)^T P_T^{-1} (t-T), \\
F(X,t) = 0.
\end{array} \right\} \quad (\text{A.15})$$

To solve this problem we use a Lagrange multiplier vector k and obtain the modified objective function

$$\tilde{W} = \frac{1}{2} (t-T)^T P_T^{-1} (t-T) - k^T F(X,t). \quad (\text{A.16})$$

We obtain normal equations for the minimization problem by setting equal to zero the derivatives of \tilde{W} with respect to t and k . The normal equations are

$$\left. \begin{array}{l}
P_T^{-1} \cdot (t-T) - F_t^T(X,t) \cdot k = 0, \\
F(X,t) = 0.
\end{array} \right\} \quad (\text{A.17})$$

Eliminating k from the first equation (A.17) one obtains the equivalent set of equations

$$\left. \begin{array}{l}
(t-T)^T P_T^{-1} (t-T) = (t-T)^T F_t^T (F_t P_T F_t^T)^{-1} F_t (t-T), \\
F(X,t) = 0.
\end{array} \right\} \quad (\text{A.18})$$

(We exclude singular cases by assuming that the matrix $F_t P_T F_t^T$ is positive definite.) The left hand side of the first equation (A.18) is the sought for minimum value of $\|t-T\|_T^2$. On the right hand side, the arguments of F_t are (X,t) , and therefore, it only can be computed after a numerical solution of Eq. (A.17) or (A.18). However, an approximate solution can be obtained by linearizing the second equation (A.18) in the form

$$F(X,t) = F(X,T) + F_t(X,T) \cdot (t-T) = 0. \quad (\text{A.19})$$

Substituting this into the first Eq. (A.18) and also using the arguments (X,T) instead of (X,t) in the product $F_t P_T F_t^T$, one obtains

$$h(X, \tilde{X}_F) = (\|t-T\|_T)_{\min} \approx [F^T (F_t P_T F_t^T)^{-1} F]^{1/2}, \quad (\text{A.20})$$

where the arguments of F and F_t are (X,T) . The expression (A.20) is exact if F is a linear function of the parameter t . (In that case, also Eq. (A.19) is exact with F_t evaluated at (X,T) .) The solution \tilde{X}_F is approximately given by Eqs. (A.20) and (A.14).

Now we consider the fuzzy Eq. (A.3) and show that Eqs. (A.20) and (A.14) also provide a solution to that equation. Let the right hand side of Eq.

(A.3) be the fuzzy vector

$$\tilde{0} =: \{ 0, P_0 \}. \quad (\text{A.21})$$

We construct a solution of Eq. (A.3) as follows. First, we define a separation of $F(X)$ from the fuzzy zero by

$$h(F, \tilde{0}) = [F^T P_0^{-1} F]^{1/2}, \quad (\text{A.22})$$

and a separation of any X from the solution \tilde{X}_0 of Eq. (A.3) by

$$h(X, \tilde{X}_0) = h(F(X), \tilde{0}) = [F(X)^T P_0^{-1} F(X)]^{1/2}. \quad (\text{A.23})$$

The solution of Eq. (A.3) then is the fuzzy set with the membership function

$$\mu_0(X) = 1 - \min \{ 1, h(X, \tilde{X}_0) \}. \quad (\text{A.24})$$

One obtains exactly the same fuzzy set as a solution of Eq. (A.5) by applying the formula (A.20) and Eq. (A.14) to the function $\tilde{G}(X)$, defined by Eq. (A.4). Therefore, general fuzzy equations always can be assumed in the form (A.3), i.e., with a crisp zero on the right hand side. We note that in the case (A.4) the formula (A.20) is exact, because $\tilde{G}(X) = F(X) - \tilde{0}$ is a linear function of the fuzzy parameter $\tilde{0}$.

We have shown that the solution of a fuzzy equation is a fuzzy set (a fuzzy $(n-r)$ -dimensional hypersurface in R_n). Its membership function can be computed by solving Eq. (A.13) pointwise, or by the approximate formula (A.20). We note, however, that the membership function of the solution \tilde{X}_F depends on the formulation of the function $\tilde{F}(X)$. Two algebraically equivalent formulations of $\tilde{F}(X) = 0$, involving the same fuzzy parameter \tilde{T} and having the same crisp solution $X(T)$ of $F(X, T) = 0$, generally produce different fuzzy solutions. These solutions have the same apex, namely the $(n-r)$ -dimensional crisp hypersurface $X(T)$, but their spreads can be different. The fact that the solution depends on the form of the equation is an intrinsic property of fuzzy equations.

Now we discuss the structure of the fuzzy solution \tilde{X}_F of $\tilde{F}(X) = 0$. It is provided by Eq. (A.13) which defines a distance between an X and \tilde{X}_F . That distance is zero if X is a point of the apex $X(T)$ of \tilde{X}_F , i.e., of the solution of $F(X, T) = 0$. The corresponding membership value of X is one. Boundaries of the regions in R_n where $\mu_F(X) \geq \gamma$ are hypersurfaces defined by the equation $h(X, \tilde{X}_F) = 1 - \gamma$. We investigate the surfaces by using the approximate formula (A.20).

Let Y be a point of the hypersurface $X(T)$. Then Y satisfies the equation $F(Y, T) = 0$, which we expand obtaining

$$F(X,T) = F_X(Y,T) \cdot (X-Y) + \dots, \quad (\text{A.25})$$

since $F(Y,T) = 0$. Substituting the linear term of the expansion in Eq. (A.20) for F , one obtains for the distance between any crisp X and a fuzzy $\tilde{Y} \in \tilde{X}_F$

$$h(X, \tilde{Y}) \approx [(X-Y)^T F_X^T (F_t P_T F_t^T)^{-1} F_X (X-Y)]^{1/2}. \quad (\text{A.26})$$

The structure of the hypersurface $h(X, \tilde{Y}) = 1 - \gamma$ is determined by the $(n \times n)$ -matrix

$$P_Y^{-1} = F_X^T (F_t P_T F_t^T)^{-1} F_X. \quad (\text{A.27})$$

We recall that $F \in R_r$, and $r \leq n$. The rank of the matrix P_Y^{-1} is, therefore, at most equal to r . (It equals r if the components of F are linearly independent functions of X . We assume for simplicity that this is the case.) If $r = n$, then the solution \tilde{X}_F is a fuzzy point in R_n , and the surface $h = 1 - \gamma$ is a hyperellipsoid. If $r < n$ then the matrix P_Y^{-1} is semi-definite and the surface is a hypercylinder with rulings that are parallel to the hypersurface $F(X, T) = 0$ at $X = Y$. The support boundaries of $\mu_F(X) - \gamma$ are, therefore, surfaces that are approximately parallel to the apex $X(T)$, of the solution, at least for γ close to one.

Eq. (A.20) provides a measure for a distance between a crisp point X and the fuzzy set \tilde{X}_F . The distance between a fuzzy point \tilde{A} and \tilde{X}_F may be measured by the discord between both sets. We define the discord in analogy to Eq. (2.15) by

$$D(\tilde{A}, \tilde{X}_F) = \min_{X \in R_n} \max \{ h(X, \tilde{A}), h(X, \tilde{X}_F) \}. \quad (\text{A.28})$$

In some special cases, for instance, if the function $F(X)$ is scalar, the discord can be computed by an explicit formula which now will be derived.

Let \tilde{A} be a fuzzy point

$$\tilde{A} =: \{ A, P_A \}. \quad (\text{A.29})$$

The separation of an $X \in R_n$ from \tilde{A} we measure by $h(X, \tilde{A})$, defined by

$$h(X, \tilde{A}) = [(X-A)^T P_A^{-1} (X-A)]^{1/2}. \quad (\text{A.30})$$

The separation of X from the solution \tilde{X}_F of $\tilde{F}(X) = 0$ we measure by

$$h(X, \tilde{X}_F) = [F^T (F_t P_T F_t^T)^{-1} F]^{1/2}. \quad (\text{A.31})$$

Because $h(X, \tilde{A})$ is a convex function and $h(X, \tilde{X}_F)$ is not concave (we assume that the approximation (A.26) is sufficiently accurate), the search for the minimum over X in Eq. (A.28) can be restricted to the locus where $h(X, \tilde{A}) = \text{constant}$ is tangent to $h(X, \tilde{X}_F) = \text{constant}$. In the special case

where that locus is a straight line, $D(\tilde{A}, \tilde{X}_F)$ can be easily calculated. We now consider that special case. First we determine the point B at which the hypersurface $F(X, T) = 0$ is tangent to a hyperellipsoid $h(X, \tilde{A}) = \text{constant}$. Then we shall consider the straight line between B and A , and establish conditions for it being the above mentioned locus.

We find B by solving the problem

$$\left. \begin{array}{l} \text{minimize} \\ (B - A)^T P_A^{-1} (B - A) \\ \text{subject to} \\ F(B) = 0 \end{array} \right\} \quad (\text{A.32})$$

Using a Lagrange multiplier vector k we obtain a modified objective function

$$\hat{W} = \frac{1}{2} (B - A)^T \cdot P_A^{-1} \cdot (B - A) - k^T \cdot F(B) , \quad (\text{A.33})$$

and the normal equations

$$\left. \begin{array}{l} P_A^{-1} \cdot (B - A) - F_X^T \cdot k = 0 , \\ F = 0 \end{array} \right\} \quad (\text{A.34})$$

Eliminating k from the first equation, one finds the relation

$$B - A = P_A F_X^T (F_X P_A F_X^T)^{-1} F_X \cdot (B - A) \quad (\text{A.35})$$

which, using the approximation

$$F(B) = 0 = F(A) + F_X \cdot (B - A) , \quad (\text{A.36})$$

reduces to

$$B - A = - P_A F_X^T (F_X P_A F_X^T)^{-1} F , \quad (\text{A.37})$$

where F and F_X are evaluated at A . The separation of B from \tilde{A} is

$$h(B, \tilde{A}) = F^T (F_X P_A F_X^T)^{-1} F . \quad (\text{A.38})$$

Hence, in order to approximately calculate the separation, one does not have to actually calculate B .

Let C be a point on the straight line through A and B . Then

$$C - A = \alpha (B - A) . \quad (\text{A.39})$$

The tangent plane to $h(X, \tilde{A}) = \text{constant}$ through C is spanned by vectors $X - C$ which satisfy the equation

$$(C - A)^T P_A^{-1} (X - C) = 0 . \quad (\text{A.40})$$

Substituting Eqs. (A.39) and (A.37) in this equation one obtains the equivalent expression

$$\alpha F^T (F_X P_A F_X^T)^{-1} F_X \cdot (X - C) = 0 . \quad (\text{A.41})$$

The tangent plane to $h(X, \tilde{X}_F) = \text{const.}$ through the same point C is spanned by vectors $X - C$ which satisfy the equation

$$F^T (F_t P_T F_t^T)^{-1} F_X \cdot (X - C) = 0 , \quad (\text{A.42})$$

see Eq. (A.26). Comparing Eqs. (A.41) and (A.42) one sees that necessary and sufficient for both equations to define a common tangent plane is that there exists a positive number β such that

$$\beta F_t P_T F_t^T = F_X P_A F_X^T . \quad (\text{A.43})$$

This condition is satisfied if F is a scalar function. Another case in which (A.43) is satisfied is the function $F = X - B$, that is \tilde{X}_F is the fuzzy point $\tilde{B} = \{ B, P_B \}$, and $P_B = P_T = \beta P_A$.

We now assume that (A.43) is satisfied along the straight line. We have then at the point C

$$\left. \begin{aligned} h(C, \tilde{A}) &= |\alpha| \cdot [(B - A)^T P_A^{-1} (B - A)]^{1/2} = \\ &= |\alpha| \cdot h(B, \tilde{A}) , \end{aligned} \right\} \quad (\text{A.44})$$

where $h(B, \tilde{A})$ may be computed by Eq. (A.38). For the same point C one obtains

$$\left. \begin{aligned} h(C, \tilde{X}_A) &= [F(C)^T \cdot (F_t P_T F_t^T)^{-1} \cdot F(C)]^{1/2} \approx \\ &\approx |\alpha - 1| \cdot h(A, \tilde{X}_F) , \end{aligned} \right\} \quad (\text{A.45})$$

where $h(A, \tilde{X}_F)$ may be computed by Eq. (A.31). Neither of the equations actually requires to know the value of B . Restricting the search for a minimum in Eq. (A.28) to the straight line between A and B , that is to α between zero and one, we first determine the value of α for which $h(C, \tilde{A}) = h(C, \tilde{X}_F)$. The corresponding value of h is equal to the discord (A.28). One finds after simple algebra

$$D(\tilde{A}, \tilde{X}_F) = \frac{h(B, \tilde{A}) \cdot h(A, \tilde{X}_F)}{h(B, \tilde{A}) + h(A, \tilde{X}_F)} . \quad (\text{A.46})$$

The grade of collocation between a fuzzy point \tilde{A} and the fuzzy

solution \tilde{X}_F of $\tilde{F}(X) = 0$ we define in terms of the discord by

$$\gamma(\tilde{A}, \tilde{X}_F) = 1 - \min\{1, D(\tilde{A}, \tilde{X}_F)\} . \quad (\text{A.47})$$

Appendix B.

PANDERANCE PROPAGATION FORMULA

Let $X, Z \in R_n$, and let the linear function

$$Z = D \cdot X + Z_0 \quad (B.1)$$

be a coordinate transformation. Let $\tilde{A} =: \{ A, P_A \}$ be a fuzzy point in the X -coordinate system. We seek its representation $\tilde{B} =: \{ B, P_B \}$ in the transformed Z -coordinate system. The transformation formulas are

$$B = D \cdot A + Z_0 \quad (B.2)$$

and

$$P_B = D P_A D^T . \quad (B.3)$$

Eq. (B.3) may be called a panderance propagation formula. It can be readily verified by substitution. By definition one has (see Section 2) the membership functions

$$\mu_A(X) = 1 - \min \{ 1, [(X-A)^T P_A^{-1} (X-A)]^{1/2} \} \quad (B.4)$$

and

$$\mu_B(Z) = 1 - \min \{ 1, [(Z-B)^T P_B^{-1} (Z-B)]^{1/2} \} . \quad (B.5)$$

We have to show that

$$\mu_B(Z(X)) = \mu_A(X) , \quad (B.6)$$

and this is true because

$$\left. \begin{aligned} (Z-B)^T P_B^{-1} (Z-B) &= \\ &= (X-A)^T D^T [(D^T)^{-1} P_A^{-1} D^{-1}] D (X-A) = \\ &= (X-A)^T P_A^{-1} (X-A) . \end{aligned} \right\} \quad (B.7)$$

The panderance propagation formula (B.3) componentwise satisfies the extension principle [7]. In order to show this we consider one component of Eq. (B.1), say,

$$z = d^T X + z_0. \quad (\text{B.8})$$

Let the corresponding component of Eq. (B.2) be

$$b = d^T A + z_0. \quad (\text{B.9})$$

The panderance of the fuzzy number $\tilde{b} = \{ b, P_b \}$ is, according to Eq. (8.3)

$$P_b = d^T P_A d. \quad (\text{B.10})$$

and its membership function is

$$\mu_b(z) = 1 - \min \{ 1, |z - b| / \sqrt{P_b} \}. \quad (\text{B.11})$$

On the other hand, according to the extension principle one should have the relation

$$\mu_b(z) = \max_{X: d^T X + z_0 = z} \mu_A(X). \quad (\text{B.12})$$

We find the maximum on the right hand side of eq. (B.12) by solving the minimization problem

$$\left. \begin{array}{l} \text{minimize} \\ \text{subject to} \end{array} \right\} \begin{array}{l} (X - A)^T P_A^{-1} (X - A) \\ d^T (X - A) - z + b = 0. \end{array} \quad (\text{B.13})$$

Using the Lagrange multiplier k we obtain a modified objective function

$$\tilde{W} = \frac{1}{2} (X - A)^T P_A^{-1} (X - A) - k [d^T (X - A) - z + b] \quad (\text{B.14})$$

and the normal equations

$$\left. \begin{array}{l} P_A^{-1} (X - A) - k d = 0, \\ d^T (X - A) - z + b = 0. \end{array} \right\} \quad (\text{B.15})$$

Eliminating k from the first equation, one obtains

$$P_A^{-1} (X - A) = d \cdot [d^T (X - A)] / (d^T P_A d). \quad (\text{B.16})$$

Multiplying from left by $(X - A)^T$ and using the second Eq. (B.15) one obtains from (B.16)

$$[(X - A)^T P_A^{-1} (X - A)]_{\min} = (z - b)^2 / P_b, \quad (\text{B.17})$$

the square root of which is equal to the expression in Eq. (B.11).

Hence we have shown that the panderance propagation formula (B.3) is consistent with the extension principle for any linear function, even if the matrix D in Eq. (B.1) is not a $(n \times n)$ -matrix, or is singular. If P_B turns out to be singular, then the corresponding support of μ_B is a degenerated hyperellipsoid.

If the function $Z(X)$ is non-linear, then one may obtain an approximate P_Z by linearizing the function. In particular one then obtains

$$\left. \begin{aligned} Z(X - A) &= Z(A) + \frac{\partial Z}{\partial X} (X - A) + \dots, \\ B &= Z(A), \\ P_B &\approx \frac{\partial Z}{\partial X} P_A \left(\frac{\partial Z}{\partial X} \right)^T, \end{aligned} \right\} \quad (\text{B.18})$$

where $\partial Z / \partial X$ is the Jacobian matrix of the function $Z(X)$, evaluated at $X = A$.

As was shown above, the spread of a component of a fuzzy vector equals the square root of the corresponding diagonal element of the panderance matrix. A convenient representation of the panderance matrix in terms of the component spreads is by the formula

$$P_A = S_A C_A S_A, \quad (\text{B.19})$$

where S_A is a diagonal matrix with the spreads of components of A as diagonal elements. The matrix C_A is dimensionless, has ones in the diagonal and, if P_A is positive definite, the off-diagonal elements of C_A have absolute values less than one. We call C_A the *concordance matrix* of \tilde{A} , and its elements c_{jk} the concordances between components a_j and a_k of \tilde{A} . Componentwise the relation (B.19) is for, for $j, k = 1, \dots, n$,

$$P_{jk} = s_{jj} c_{jk} s_{kk}. \quad (\text{B.20})$$

Appendix C.

NORMAL EQUATIONS AND PANDERANCE OF MODEL PARAMETERS

We consider the constrained minimization problem (3.4a) for a fuzzy model and crisp data, viz.,

$$\begin{array}{l}
 \text{minimize} \\
 \\
 \text{subject to}
 \end{array}
 \left. \begin{array}{l}
 W = \sum_{i=1}^s c_i^T P_T^{-1} c_i \\
 \\
 F_i(X_i, T + c_i) = 0, \quad i = 1, 2, \dots, s.
 \end{array} \right\} \quad (C.1)$$

In Eq. (C.1) the c_i and T are p -dimensional vectors, and each F_i is an r_i -dimensional function of the observation vector X_i and the model parameter $t = T + c_i$. However, since the X_i are crisp and not subject to adjustment, they can be included in the definitions of the F_i , which then may be considered as functions of the parameter t only. We assume that these functions are twice differentiable with respect to all components of t . Using Lagrange multiplier vectors k_i , one obtains the following system of normal equations for the problem (C.1):

$$\left. \begin{array}{l}
 P_i^{-1} c_i - \left\{ \frac{\partial}{\partial t} [k_i^T \cdot F_i(T + c_i)] \right\}^T = 0, \quad i = 1, 2, \dots, s, \\
 \\
 \sum_{i=1}^s \frac{\partial}{\partial t} [k_i^T \cdot F_i(T + c_i)] = 0, \\
 \\
 F_i(T + c_i) = 0, \quad i = 1, 2, \dots, s.
 \end{array} \right\} \quad (C.2)$$

A solution of Eq. (C.2) consists of the model parameter vector T , the s correction vectors (residuals) c_i , and the s Lagrange multiplier vectors k_i . The solution of the minimization problem (C.1) is among the solutions of Eq. (C.2). Now we shall discuss the numerical solution of the latter equation.

Eq. (C.2) is non-linear with respect to the unknowns T and c_i . Therefore, its numerical solution in general requires an iteration, and we present iteration formulas based on Newton-Raphson approach. The formulas are obtained by an expansion of Eq. (C.2) at points $\{c_i, k_i, T\}$,

$i = 1, \dots, s$, which represent an approximation to the solution. Let corrections to the approximate residuals be ϵ_i , corrections to the approximate Lagrange multipliers be κ_i , and corrections to the approximate model parameter be τ . Then the linear terms of the expansion yield the following system of linear equations for the corrections

$$\left. \begin{aligned}
 [I - P_i (k_i^T F_i)_{tt}] \epsilon_i - P_i (F_i)_t^T (k_i + \kappa_i) - P_i (k_i^T F_i)_{tt} \tau &= -c_i, \\
 i &= 1, 2, \dots, s, \\
 \sum_{i=1}^s (k_i^T F_i)_{tt} \epsilon_i + \sum_{i=1}^s (F_i)_t^T (k_i + \kappa_i) + \sum_{i=1}^s (k_i^T F_i)_{tt} \tau &= 0, \\
 (F_i)_t \epsilon_i + (F_i)_t \tau &= -F_i, \\
 i &= 1, 2, \dots, s.
 \end{aligned} \right\} \quad (C.3)$$

In Eq. (C.3) we have used the subscript t to indicate the derivative $\partial/\partial t$ and the subscript tt to indicate the second derivative $\partial^2/\partial t^2$. The equation system may be rearranged by algebraic manipulations to obtain more convenient iteration equations. Now we formulate such a set of equations, and define to that end the following matrices for each $i = 1, 2, \dots, s$

$$\left. \begin{aligned}
 G &= (F_i P F_i^T)^{-1}, \\
 A &= P F_i^T G F_i - I, \\
 D &= P^{-1} A P (k_i^T F_i)_{tt}, \\
 \Gamma &= [I + P D]^{-1}, \\
 E &= \Gamma [A c - P F_i^T G F_i], \\
 N &= F_i^T G F_i - D \Gamma A.
 \end{aligned} \right\} \quad (C.4)$$

The rearranged set of equations is

$$\left. \begin{aligned}
\sum_{i=1}^s N_i \tau &= \sum_{i=1}^s \{ (F_i)_t^T G_i [(F_i)_t c_i - F_i] - D_i E_i \} , \\
k_i + \kappa_i &= G_i [(F_i)_t c_i - F_i] + G_i (F_i)_t [I + P_i (k_i^T F_i)_{tt}] \tau - \\
&\quad - G_i (F_i)_t P_i (k_i^T F_i)_{tt} \epsilon_i , \\
\epsilon_i &= E_i - (\Gamma_i A_i + I) \tau , \quad i = 1, 2, \dots, s .
\end{aligned} \right\} \quad (C.5)$$

Numerical experiments show that the convergence of the iteration is enhanced if Eq. (C.5) is used in a sub-iteration mode, iterating alternately τ and $k + \kappa$ (with fixed c_i), and $k + \kappa$ and ϵ (with fixed T), respectively. One obtains a Gauss-Newton iteration scheme from Eq. (C.5) by dropping all second order derivative terms $(k^T F)_{tt}$.

The panderance matrix P_T of the solution vector \tilde{T} can be obtained in terms of the input panderances P_i by using the linearized panderance propagation formula (Appendix B). We obtain the formula by first expanding Eq. (C.2) at the solution in terms of τ , κ_i and ϵ_i , and eliminating the κ_i . The result is

$$\sum_{i=1}^s N_i \tau = \sum_{i=1}^s N_i \epsilon_i , \quad (C.6)$$

giving a linear relation between changes of c_i and T . Applying the panderance propagation formula (B.3) or (B.18) to this relation one obtains

$$P_T = (\sum N_i)^{-1} (\sum N_i P_i N_i^T) [(\sum N_i)^{-1}]^T . \quad (C.7)$$

One notices that the linearized formula (C.7) contains the second order derivatives $(k^T F_i)_{tt}$ of the model functions F_i . This is due to the fact that the normal equations (C.2) already contain first order derivatives of F_i . Their linear expansion, therefore, includes second order derivatives.

The formulas (C.3) through (C.7) describing the solution algorithm of the problem (C.2) are somewhat simpler than the corresponding formulas for a general least squares model fitting problem [2], because the F_i do not depend on the X_i in the present case. However, the simplifications are not of such an extent that a special computer program would be much more efficient than a general program. One may, therefore, solve the problem (C.1) with any computer program for least squares model fitting with implicit constraint equations, for instance the COLSAC program described in [1].

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