CONTINUOUS DEPENDENCE ON MODELING IN THE GAUCHY PROBLEM

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by

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ABSTRACT

The Cauchy problem for various types of second order nonlinear elliptic equations is considered. A substitution \( v = \varepsilon u \) in the equation leads to a perturbed equation whose solution is compared to an appropriate solution of an unperturbed second order linear elliptic equation obtained by formally setting \( \varepsilon = 0 \). In each case a logarithmic convexity argument is used to show that appropriately constrained solutions of the original equation (assumed to exist) are shown to differ from a solution of the associated linear equation in a manner depending continuously on the parameter \( \varepsilon \).
A problem in ordinary or partial differential equations is said to properly posed if it has a unique solution in the class under consideration and if this solution depends continuously on the data in some appropriate measure. Otherwise the problem is said to be improperly posed. Although Hadamard (8) defined the question of proper posedness at the turn of the century and demonstrated that the Cauchy problem for the Laplace equation is improperly posed, relatively little attention was given to improperly posed problems for partial differential equations until the papers of John (9) and Pucci (16) appeared in the 1950's. Up to that time the prevailing attitude seemed to be that only properly posed problems were of interest in applications.

It is realized now, however, that many problems of physical interest are improperly posed. For example, the Dirichlet problem for a second-order linear elliptic equation on a smooth bounded domain in $\mathbb{R}^n$ is properly posed. However, in many physical situations, only a portion of the boundary may be accessible to data measurement. In such cases, one measures additional data--usually the gradient of the unknown function--on that portion of the boundary which is accessible. The resulting problem is an improperly posed Cauchy problem. Payne (14) showed that this problem can be stabilized by imposing an a priori bound on the $L^2$-norm of the solution. (Special cases of this problem had been considered earlier; for example, Lavrentiev (11) showed that imposing a pointwise bound on the solution of a Cauchy problem for the Laplace equation is sufficient to ensure that the solution will depend continuously upon the Cauchy data in some neighborhood of the data surface.) Extensions of the result in (14) to the Cauchy problem for equations of the type
\[ Lu = f(x, u, \text{grad } u) \], \hspace{1cm} (1.1) 

for a uniformly elliptic second-order linear operator \( L \) are found in (6, 18-20). In each case a Hölder continuous dependence result is obtained by restricting the \( L^2 \)-norm of the solution.

This work investigates improperly posed Cauchy problems for some second-order nonlinear elliptic equations which cannot be written in the form (1.1) but which can be written as

\[ Lv = g(x, v, \text{grad } v, Hv), \hspace{1cm} (1.2) \]

where \( Hv \) is the Hessian matrix of \( v \). Examples of such equations are the minimal surface and capillary surface equations. The substitution \( v = \epsilon u \) in (1.2) leads to a perturbed equation of the form

\[ Lu = \epsilon^k g(x, u, \text{grad } u, Hu) \], \hspace{1cm} (1.3) 

for some positive number \( k \). The corresponding unperturbed equation is

\[ Lh = 0 \], \hspace{1cm} (1.4) 

and logarithmic convexity arguments are used to derive stability estimates for \( v - \epsilon h \), where \( h \) is a solution of an appropriate Cauchy problem for (1.4).

The question of the feasibility of approximating a solution of a Cauchy problem for a perturbed equation by a solution of a Cauchy problem for the corresponding unperturbed equation will be referred to as the question of continuous dependence on modeling in the Cauchy problem for the perturbed equation. Adelson’s work (1.2) illustrates this concept. His results apply, for example, to show that an
appropriately constrained solution of the Cauchy problem for the
singularly perturbed equation

\[ \epsilon \Delta^2 v - v = E(x) \]  \hspace{1cm} (1.5)

can be approximated by the solution of a Cauchy problem for

\[ -\Delta w = E(x) \]  \hspace{1cm} (1.6)

which should, in some sense, be the limiting problem as \( \epsilon \) tends to zero.

The question of existence of solutions for perturbed problems for
all values of \( \epsilon \) in some interval \((0, \epsilon_0)\) presents no difficulty in
most reasonable properly posed problems for ordinary or partial
differential equations. Hence, in such problems, one may allow \( \epsilon \) to
tend to zero and prove that the solution of the perturbed equation
converges to the solution of the unperturbed equation in some
appropriate measure.

However, for improperly posed problems, for given data the solution
may fail to exist for some or all values of \( \epsilon \) in the interval. One
can compensate to some extent for this difficulty by allowing for small
variations in the data—not an unreasonable thing to do since the data
usually cannot be measured exactly. This work shall not be concerned
with the complicated questions of existence but shall assume that all
solutions under consideration do indeed exist.
2. Notation

Let $D$ be an $N$-dimensional domain bounded by a closed surface $C$, and let $\Sigma$ be that portion of $C$ on which Cauchy data are prescribed. The complement of $\Sigma$ with respect to $C$ is denoted $\Sigma'$, and no data are given on $\Sigma'$. Assume that the closure $\overline{\Sigma}$ of $\Sigma$ is a $C^{2+\alpha}$ surface for some $\alpha > 0$. Since Cauchy data are given only on the portion $\Sigma$ of $C$, one cannot expect to derive estimates for continuous dependence (on the data) on the entire domain $D$. In particular one might not expect to derive such estimates for subdomains of $D$ whose boundaries contain a portion of $\Sigma'$. Thus a family $(D_\alpha)$ of subdomains of $D$ on which to derive stability estimates is chosen as follows:

Let $(f(x) = \text{constant})$ define a set of (not necessarily closed) surfaces. This set is to be chosen so that for each $\alpha \in (0,1]$ the surface $(f(x) - \alpha)$ intersects $\overline{D}$ and forms a closed region $D_\alpha$ whose boundary consists only of points on $\Sigma$ and points on the surface $(f(x) - \alpha)$. We set $\Sigma_\alpha = \Sigma \cap \overline{D_\alpha}$ and $S_\alpha = \{f(x) = \alpha\} \cap \overline{D_\alpha}$.

Assume that $f(x)$ has continuous second derivatives in $\overline{D_1}$. Furthermore, assume that

\begin{align*}
\text{if } \beta \leq \gamma, \text{ then } D_\beta & \subset D_\gamma; \quad (2.1) \\
|\text{grad } f| & \geq \delta > 0 \text{ in } D_1; \quad (2.2) \\
\Delta f & \leq 0 \text{ in } D_1; \quad (2.3) \\
|\Delta f| & \leq \delta^2 d \text{ in } D_1; \quad (2.4)
\end{align*}

where $\delta$ and $d$ are positive constants. (In Section 5, (2.3) and (2.4) are modified somewhat.) Assume that the surfaces have been chosen so that $D_\alpha$ has positive Lebesgue measure for $0 < \alpha \leq 1$ but that $D_0$
has Lebesgue measure zero. For $N \geq 2$, one can choose a radial harmonic function $f$ which satisfies (2.1) - (2.4).

Throughout this paper commas are used to denote differentiation, and the summation convention is used for repeated indices. For example,

$$u_{,i} u_{,i} = \sum_{i=1}^{N} \left( \frac{\partial u}{\partial x_i} \right)^2 = |\nabla u|^2.$$

The arithmetic-geometric mean (A-G) inequality states that, for positive numbers $a$, $b$, and $c$

$$2ab \leq ca^2 + \frac{1}{c}b^2.$$
3. This section examines the minimal surface equation

\[
\left(1 + |\nabla v|^2\right)^{-1/2} v_{ij} = 0 \text{ in } D. \tag{3.1}
\]

On \( \Sigma \), assume that the Cauchy data satisfy

\[
\int_\Sigma (v^2 + v_{,i} v_{,i}) \, ds \leq \epsilon^2 \tag{3.2}
\]

for some small positive number \( \epsilon \). The substitution \( v = \epsilon u \) in (3.1) yields the perturbed equation

\[
\Delta u = \epsilon^2 \rho^2 u_{,i} u_{,ij} \text{ in } D, \tag{3.3}
\]

where \( \rho = (1 + \epsilon^2 |v|^2)^{-1/2} \). Formally setting \( \epsilon = 0 \) in (3.3) gives Laplace's equation

\[
\Delta h = 0 \text{ in } D. \tag{3.4}
\]

Setting \( w = u - h \) yields

\[
\Delta w = \epsilon^2 \rho^2 u_{,i} u_{,uj} \text{ in } D. \tag{3.5}
\]

Regarding the data on \( \Sigma \), require, for some \( \rho < 6 \), that

\[
\int_\Sigma (u_{,i} u_{,i})^2 \left| (u_{,j} u_{,j} - u_{,j} u_{,j}) n_j \right| ds = O(\epsilon^\rho) \quad . \tag{3.6}
\]
Furthermore assume that

\[
\int_{\Sigma} (w^2 + w, i w, i) ds = 0(\epsilon^{4-\rho}).
\]  

(3.7)

It is shown that if \( u \) and \( h \) are appropriately constrained solutions of (3.3) and (3.4), respectively, which satisfy (3.6) and (3.7), then for \( 0 < \alpha < \alpha_1 < 1 \),

\[
\int_D (u - \epsilon h)^2 dx = 0 \left( \epsilon^{6-\rho} \cdot \nu(\alpha) \right)
\]

(3.8)

where \( \nu \) is a smooth function of \( \alpha \) satisfying

\[
\nu(0) = 1, \quad \nu'(\alpha) < 0, \quad \nu(\alpha_1) = 0.
\]  

(3.9)

The following argument closely resembles the one used by Payne (14) when he computed bounds for solutions of ill-posed Cauchy problems for linear elliptic equations. To begin the derivation of (3.8), set, for \( \alpha \in [0,1] \),

\[
F(\alpha) = Q + \int_0^\alpha (\alpha - \eta) \left\{ \int_D [w, i w, i + \omega \partial w] dx \right\} d\eta.
\]  

(3.10)

where \( Q \) is given by

\[
Q = k_0 \int_{\Sigma} w^2 ds + k_1 \int_{\Sigma} w, i w, i ds + k_2 \epsilon^{4-\rho}.
\]  

(3.11)
and the \( k_i \) are positive constants which will be chosen later. It is
shown that \( F \) satisfies a differential inequality of the form

\[
F'' - (F')^2 \geq -C_1 F' - C_2 F^2
\] 

(3.12)
on the interval \((0, \alpha_1)\) for explicit constants \( C_1 \) and \( C_2 \). The
solution of this inequality then leads to the desired bounds.

The first and second derivatives of \( F \) are

\[
F'(\alpha) = \int_0^\alpha \int_D [w_{,i}w_{,i} + w\Delta w]dx \eta ,
\]

(3.13)

\[
F''(\alpha) = \int_D [w_{,i}w_{,i} + w\Delta w]dx .
\]

(3.14)

Integrating (3.13) by parts, one can write \( F(\alpha) \) and \( F'(\alpha) \) in more
useful forms:

\[
F'(\alpha) = \int_0^\alpha \int_{S_\eta} \left\{ w_{,i}f_{,i} |\nabla f|^{-1} ds + \int_{\Sigma_\eta} w_{,i}n_{,i} ds \right\} d\eta
- \int_D w_{,i}f_{,i} dx + \int_0^\alpha \int_{\Sigma_\eta} w_{,i}n_{,i} d\eta .
\]

(3.15)

Note that on \( S_\eta \) the component \( n_j \) of the unit normal is given by
\( f_{,j} |\nabla f|^{-1} \). Using (3.15),

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\[ F(\alpha) = Q + \int_{0}^{\alpha} F'(\eta) d\eta = Q + \int_{0}^{\alpha} \left\{ \int_{D_\eta} w w_{i} f_{i} dx + \int_{\Sigma_\eta} w w_{i} n_{i} ds \right\} d\eta. \]

Integrating by parts above and using (2.3) and the A-G inequality,

\[ F(\alpha) = Q + \int_{0}^{\alpha} \left\{ \frac{1}{2} \int_{\Sigma_\eta} w^2 |\nabla f|^2 ds + \frac{1}{2} \int_{D_\eta} w^2 f_{i} n_{i} ds \right. \]
\[ - \frac{1}{2} \int_{D_\alpha} w^2 \Delta f dx + \int_{\Sigma_\eta} w w_{i} n_{i} ds \}
\[ \left. d\eta \right\} \geq \frac{1}{2} \int_{D_\alpha} |\nabla f|^2 w^2 dx - \nu_1 \int_{D_\alpha} w^2 dx - \nu_2 \int_{\Sigma_\alpha} w_{i} n_{i} ds + Q \quad (3.16) \]

for computable constants \( \nu_1 \) and \( \nu_2 \). One can now choose the constants \( k_i \) in \( Q \) so that

\[ \frac{1}{2} \int_{D_\alpha} rw^2 dx + Q \leq F(\alpha) \leq \frac{d+1}{2} \left[ \int_{D_\alpha} rw^2 dx + Q \right]. \quad (3.17) \]

where \( r = |\nabla f|^2 \).

The inequality (3.17) enables the use of the solution of (3.12) to estimate the \( L^2 \)-integrals of \( w \) over the domains \( D_\alpha \). To derive (3.12), we need three preliminary estimates.
Lemma 1: Let $\alpha_1 \in (0, 1)$. Then for $\alpha \in (0, \alpha_1)$

$$
\int_{D_\alpha} (\Delta w)^2 dx \leq c \epsilon^4 \left\{ \int_{D_1} [(u,_{i}u,_{i})^3 (1 + \epsilon^2 u,_{j}u,_{j})] dx

\right. 

+ \int_{\Sigma_1} (u,_{i}u,_{i})^2 |(u,_{i}u,_{j,i} - u,_{j}u,_{i})| n_j |ds\right\}

$$

for an explicit constant $c$ independent of $\epsilon$ and $\alpha$.

Proof: Define the function $w$ on $D_1$ by

$$
\omega(x) = \begin{cases} \frac{1}{1-\alpha} & \text{in } D_1 \cup \Sigma_1 \\
\frac{1-f(x)}{1-\alpha} & \text{in } D_\alpha \cup \Sigma_\alpha 
\end{cases}

$$

Note that $\omega = 0$ on $S_1$ and that $|\omega| \leq 1$ in $D_1$. Since $f \in C^2(\overline{D_1})$, the first and second derivatives of $\omega$ are uniformly bounded on $D_1$.

Now consider

$$
J(\alpha) = \epsilon^4 \int_{D_\alpha} (\Delta w)^2 dx \leq \int_{D_1} \omega^{2\rho^4 (u,_{i}u,_{i})^2 u,_{j,k}u,_{j,k}} dx

$$
\begin{equation}
\left( u, \partial u, \partial_j u \right)^2 \leq |\nabla u|^2 u, u, k u, j k, j_1 \leq |\nabla u|^4 u, j^k u, j^l .
\end{equation}

Therefore, from (3.18),
\begin{align*}
J(\alpha) & \leq \int_{D_1} \omega_2 \rho^2 (u, \partial u, \partial_1)^2 \left[ u, \partial_j u, \partial_j (1+\epsilon^2 |\nabla u|^2)^2 - \epsilon^2 (u, \partial u, \partial_1)^2 \right] dx \\
& + 4 \int_{D_1} \omega_2 \rho^2 (u, \partial u, \partial_1)(u, k u, j k, j_1 (1+\epsilon^2 |\nabla u|^2) - \epsilon^2 (u, \partial u, \partial_1)^2) dx \\
& - \int_{D_1} \omega_2 (u, \partial u, \partial_1) \left\{ u, \partial_j u, \partial_j - (\Delta u)^2 \right\} dx \\
& + 4 \int_{D_1} \omega_2 (u, \partial u, \partial_1) \left\{ u, k u, j k, j_1 - \Delta u(u, \partial u, \partial_1) \right\} dx .
\end{align*}
To eliminate the second-order terms appearing in the bound for $J(\alpha)$, integrate the first term on the right side of \((3.19)\) by parts. First note that

\[
\int_{D_1} \omega^2 (u, u)^2 u, j \partial_j u, j dx = \int_{\Sigma_1} \omega^2 (u, u)^2 u, j \partial_j n_j ds
\]

\[- \int_{D_1} \omega^2 (u, u)^2 u, j (\Delta u), j dx - 2 \int_{D_1} \omega_1 (u, u)^2 u, j u, j dx
\]

\[- 4 \int_{D_1} \omega^2 (u, u)^2 (u, j u, k) u, j u, k dx . \tag{3.20}
\]

Integrate the first volume integral on the right side of \((3.20)\) by parts to obtain

\[- \int_{D_1} \omega^2 (u, u)^2 u, j (\Delta u), j dx = - \int_{\Sigma_1} \omega^2 (u, u)^2 u, j \partial_j n_j ds
\]

\[+ \int_{D_1} \omega^2 (\Delta u)^2 (u, u)^2 dx + 2 \int_{D_1} \omega_1 (u, u)^2 u, j u, 2u dx
\]

\[+ 4 \int_{D_1} \omega^2 (u, u)^2 u, j u, k u, j k u, 2u dx . \tag{3.21}
\]
Combining (3.20) and (3.21) yields

\[
\int_{D_1} \omega^2(u, u, u)^2 u, j u, j d\mathbf{x} = \int_{\Sigma_1} \omega^2(u, u, u)^2 [u, u, j n_j] - [u, u, j n_j] ds
\]

\[
+ \int_{D_1} \omega^2(\Delta u)^2(u, u, u)^2 d\mathbf{x} + 2 \int_{D_1} \omega u, j (u, u, u)^2 u, j d\mathbf{x}
\]

\[
+ 4 \int_{D_1} \omega^2(u, u, u) u, j u, k u, j d\mathbf{x} - 2 \int_{D_1} \omega u, j (u, u, u)^2 u, j d\mathbf{x}
\]

\[
- 4 \int_{D_1} \omega^2(u, u, u)(u, u, k u, j k) d\mathbf{x}.
\]

(3.22)

Returning to (3.19),

\[
\int_{D_1} \omega^2 \rho^2(u, u, u)^2 [u, j u, j (1+\epsilon^2 |\nabla u|^2)^2 - \epsilon^4 (u, u, j u, j)^2] d\mathbf{x}
\]

\[
+ 4 \int_{D_1} \omega^2 \rho^2(u, u, u)[u, k u, j u, k (1+\epsilon^2 |\nabla u|^2)^2 - \epsilon^2 (u, u, j u, j)^2] d\mathbf{x}
\]

\[
- \int_{\Sigma_1} \omega^2(u, u, u)^2 [u, u, j n_j] - [u, u, j n_j] ds
\]
Using the A-G inequality, the absolute value of the volume integral on the right side of (3.23) is bounded above by

\[ k \int_{D_1} (u_1 u_1 u_1)^3 (1 + \epsilon^2 |\nabla u|^2) dx + \frac{1}{2} \int_{D_1} \omega_2^2 (u_1 u_1 u_1)^2 u_1 u_1 u_1 dx \]  

for a computable constant \( k \). Thus from (3.18), (3.19), (3.23), and (3.24),

\[ \frac{1}{2} J(\alpha) \leq \frac{1}{2} \int_{D_1} \omega_2^2 (u_1 u_1 u_1)^2 u_1 u_1 u_1 dx + 4 \int_{D_1} (u_1 u_1 u_1)^2 u_1 u_1 u_1 + 4 \int_{D_1} \omega_2 (u_1 u_1 u_1)^2 u_1 u_1 u_1 + 4 \int_{D_1} \omega_2 (u_1 u_1 u_1)^2 u_1 u_1 u_1 \]

\[ \leq \int_{\Sigma_1} \omega_2 (u_1 u_1 u_1)^2 [u_1 u_1 u_1 n_1 - u_1 u_1 u_1 n_1] ds \]
\[ + k \int_{D_1} (u, u, u)^3 (1 + \varepsilon^2 |\nabla u|^2) \, dx. \]

The proof of Lemma 1 is now complete.

Before deriving the next two required estimates, it is convenient at this point to place a constraint on the function \( u \). Using the result of Lemma 1 and the definition of \( Q \) as a guide, require that

\[ \int_{D_1} [(u, u, u)^3 (1 + \varepsilon^2 u, u, u)] \, dx = O(\varepsilon^\rho). \] (3.25)

This constraint is used in the proofs of the next two lemmas, which are understood to be valid on the interval \((0, \alpha_1)\).

**Lemma 2:** If \( F(\alpha) \) is given by (3.10), then

\[ |F'| \leq F' + K_1 F \] (3.26)

for an explicit constant \( K_1 \).

**Proof:** From (3.13) it is immediate that

\[ |F'| \leq F' + 2 \left| \int_0^\alpha \int_{\partial D} w \omega \, d\sigma \right|. \] (3.27)

By the A-G inequality,
so that (3.17) and Lemma 1 yield

\[ |F'(\alpha)| \leq F'(\alpha) + 2\varepsilon^{-2}F(\alpha) + O(\varepsilon^4) \]

This completes the proof of Lemma 2.

**Lemma 3:** If \( F(\alpha) \) is given by (3.10), then

\[ \int_{D} \left[ \omega_{i} \omega_{i} - 2r^{1}(\omega_{i}f_{i})^{2} \right] dx \geq -K_{2}F' - K_{3}F, \quad (3.28) \]

for explicit constants \( K_{2} \) and \( K_{3} \).

**Proof:** To establish this lemma, consider the identity (also used in Payne (14))

\[ 2 \int_{D} (\alpha - \eta) f_{i} f_{i} \omega_{i} \omega_{i} dx - \int_{\partial D_{\eta}} \left[ 2f_{i} f_{i} \omega_{i} \omega_{i} \right] \eta_{i} \omega_{i} \eta_{i} dx \]

\[ \cdot \int_{D} (\alpha - \eta) \left[ 2(f_{i} f_{i} \omega_{i} \omega_{i}) f_{i} f_{i} \omega_{i} \omega_{i} - (f_{i} f_{i} \omega_{i} \omega_{i}) \eta_{i} \omega_{i} \eta_{i} \right] dx. \quad (3.29) \]

The integrals over \( S_{\eta} \) may be rewritten as
\[
\int_{D} \left[ 2f_{,1}^{*}w_{,1}^{*} - f_{1} \int_{1}^{n_{1}} |\nabla w|^{2} \right] r^{-1} ds \eta
\]

- \int_{D_{\alpha}} \left[ 2(f_{,1}w_{,1})^{2} r^{-1} - |\nabla w|^{2} \right] dx . \tag{3.30}

Substituting (3.30) into (3.29) and using the A-G inequality,

\[
\int_{D_{\alpha}} \left[ w_{,1}w_{,1} - 2r^{-1} (w_{,1}f_{,1})^{2} \right] dx \geq k_{3} \int_{\Sigma} w_{,1}w_{,1} ds
\]

\[\quad - k_{4} \int_{D_{\alpha}} (\alpha - \eta) w_{,1}w_{,1} dx \quad k_{5} \int_{D_{\alpha}} (\alpha - \eta) f_{,1}w_{,1} dx . \tag{3.31}\]

(The \( k_{i} \) in this proof are all explicit positive constants.) Clearly

\[\quad - k_{3} \int_{\Sigma} w_{,1}w_{,1} ds \geq - k_{6} F . \tag{3.32}\]

Note that

\[
\int_{D_{\alpha}} (\alpha - \eta) w_{,1}w_{,1} dx = \int_{\alpha}^{D_{\eta}} w_{,1}w_{,1} dxd \eta , \tag{3.33}\]

and the proofs of Lemmas 1 and 2 yield the conclusion that
Finally,

\[
\int_{D_\alpha} (a-\eta) f, k, w, k, \Delta w dx d\eta \geq \int_{D_\eta} f, k, w, k, \Delta w dx d\eta
\]

\[
\leq k_9 \int_{D_\eta} w, k, w, k, dx d\eta + k_{10} \int_{D_\eta} (\Delta w)^2 dx d\eta
\]

\[
\leq k_{11} F' + k_{12} F .
\]

Combining (3.32), (3.34), and (3.35), one sees that Lemma 3 holds.

We can now proceed to derive the inequality (3.12). Using (3.17).

\[
F F" - (F')^2 \geq \left\{ \left[ \frac{1}{2} \int_{D_\alpha} r w^2 dx \right] \left[ \int_{D_\alpha} w, i, w, i, dx \right] - \left( \int_{D_\alpha} w, i, f, i, dx \right) \right\}
\]

\[
+ \left[ \int_{D_\alpha} w \Delta w dx - 2 |F'| \right] \cdot \left[ \int_\Sigma \left| w, i, n, i, dx d\eta \right| \right] .
\]
In arriving at (3.36), we have dropped a number of nonnegative terms on
the right. On account of (3.25), one can find a constant $k_{12}$ so that

$$\left| \int_{D_{\alpha}} w \Delta w \, dx \right| \leq k_{12} F , \quad (3.37)$$

and the A-G inequality gives a computable constant $k_{13}$ such that

$$\left| \int_{\Sigma_\eta} \sum w, I n, d s d \eta \right| \leq k_{13} F . \quad (3.38)$$

For the term in braces in (3.36), by the Schwarz inequality, Lemma 3, (3.17), and Lemma 2,

$$\int_{D_{\alpha}} r w^2 \, dx \int_{D_{\alpha}} w, I w, I \, dx - 2 \left( \int_{D_{\alpha}} w, I f, I \, dx \right)^2$$

$$\geq \int_{D_{\alpha}} r w^2 \, dx \left\{ \int_{D_{\alpha}} w, I w, I \, dx - 2 \int_{D_{\alpha}} r^{-1}(w, I f, I) \, dx \right\}$$

$$\geq - 2K_3 F^2 - 2(K_1 K_3 + K_4) F^2 . \quad (3.39)$$
Applying Lemma 2 now to the term $|F'|$ in (3.36) gives (3.12) for explicit constants $C_1$ and $C_2$.

It is well-known (see, e.g., Levine (12)) that a solution $F$ of (3.12) which vanishes for one value of $\alpha$ in the interval $[0, \alpha_1]$ must vanish identically. Thus, assume without loss that $F(\alpha) > 0$ for all $\alpha$ in $[0, \alpha_1]$. Set

$$\sigma = \exp(-c_1 \alpha), \quad G(\sigma) = \log F(\alpha) \cdot \sigma^{-2/C_1^2}, \quad (3.40)$$

to see that

$$G''(\sigma) = (C_1 F\sigma)^{-2} [C_2^2 F^2 + F F'' + C_1 F F' - (F')^2] \geq 0. \quad (3.41)$$

Hence $G$ is a convex function of $\sigma$, so that by Jensen's inequality,

$$F(\alpha) \sigma^{-C_2/C_1^2} \leq \left[ F(\alpha) \sigma_1^{-C_2/C_1^2} \right]^{1-\sigma_1} \left[ F(0) \right]^{\sigma_1-\sigma_1} \quad (3.42)$$

where

$$\sigma_1 = \exp(-C_1 \alpha_1). \quad (3.43)$$

Note that $F(0) = Q$, which is $O(\epsilon^{4-P})$ by assumption.

As has been noted in earlier papers (John (10), Pucci (17)), in order to make $F(\alpha)$ small for $0 < \alpha < \alpha_1$, it does not suffice to make $F(0)$ small. One must ensure that $F(\alpha_1)$ is not excessively large. On account of (3.17), it is necessary to constrain the $L^2$-norms of $u$ and $h$ on $D_1$. Thus, assume that
One can then compute a constant $N_1$ independent of $\epsilon$ so that

$$F(\alpha_1) \sigma_1^2 \leq N_1^2 \epsilon^2.$$  

(3.45)

Insertion of (3.45) into (3.42) gives

$$F(\alpha) = 0\left[\frac{2(\sigma-1)+(4-p)(\sigma-\sigma_1)}{1-\sigma_1}\right].$$  

(3.46)

From (3.17), we now obtain the following.

Theorem 1: If $u$ and $h$ are solutions of (3.3) and (3.4) respectively, which satisfy the boundary conditions (3.6) and (3.7) as well as the constraints (3.25) and (3.44), then the solution $\nu$ of (3.1) satisfies the continuous dependence inequality

$$\int_D (\nu - \epsilon h)^2 dx = 0\left[\epsilon^{(6-p)\nu(\alpha)}\right]$$  

for $0 < \alpha < \alpha_1$

where $\nu(\alpha) = (\sigma - \sigma_1)/(1 - \sigma_1)$.

We close this section with some additional remarks. The reason for deriving (3.12) only on $(0, \alpha_1)$ with $\alpha_1 < 1$ is that the derivatives of the function $\omega$ in the proof of Lemma 1 become unbounded as $\alpha$ approaches 1. By restricting attention to an interval $(0, \alpha_2)$ with $\alpha_2 < \alpha_1$, one can derive bounds for the Dirichlet integral of $\nu - \epsilon h$ as follows:
Let \( \mu(x) = \begin{cases} \frac{\alpha_3 - f(x)}{\alpha_3 - \beta} & \text{in } D_\beta \cup \Sigma \beta \\ 1 & \text{in } D_{\alpha_3} \cap (D_\beta \cup \Sigma \beta) \end{cases} \).

where \( \beta < \alpha_2 \) and \( \alpha_3 \) is a fixed number between \( \alpha_2 \) and \( \alpha_1 \). Then

\[
\int_{D_\beta} (u - h),_i(u - h),_i dx \leq \int_{D_{\alpha_3}} \mu^2 (u - h),_i(u - h),_i dx \\
- \int_{\Sigma_{\alpha_3}} \mu^2 (u - h)(u - h),_i d\Sigma - \int_{D_{\alpha_3}} (u - h)(u - h),_i(\mu^2),_i dx \\
- \int_{D_{\alpha_3}} \mu^2 (u - h)(u - h),_i dx \\
- \frac{1}{2} \int_{D_{\alpha_3}} (u - h)^2 \Delta(\mu^2) dx - \int_{D_{\alpha_3}} \mu^2 (u - h) \Delta(u - h) dx.
\]

Using Theorem 1, one can show that the first volume integral on the right side of (3.47) is \( 0 \left( \frac{6 - p}{\nu(\alpha_3)} \right)^2 \). Using the A-G inequality, Lemma 1, and Theorem 1, we can show that the second volume integral on the right side of (3.47) is also \( 0 \left( \frac{6 - p}{\nu(\alpha_3)} \right)^2 \). Thus,

\[
\int_{D_\beta} |\nabla(u - \epsilon h)|^2 dx = 0 \left( \frac{6 - p}{\nu(\alpha_3)} \right)
\]

for \( \beta < \alpha_2 \).
4. In this section, we consider the capillary surface equation

\[\left(1 + |\nabla v|^2\right)^{-1/2} \frac{\partial v}{\partial n} = c v \text{ in } D, \quad (4.1)\]

where \(c\) is a positive constant. On the surface \(\Sigma\), assume that the Cauchy data satisfy

\[\int_{\Sigma} (v^2 + |\nabla v|^2) ds \leq \epsilon^2 \quad (4.2)\]

for some small positive number \(\epsilon\). The substitution \(v = \epsilon u\) in (4.1) yields the perturbed equation

\[\Delta u - cu = \epsilon^2 \rho^2 u_{i,j} + c \left\{ \frac{1}{\rho} - 1 \right\} u \quad (4.3)\]

where

\[\rho = (1 + \epsilon^2 |u|^2)^{-1/2}.\]

Since

\[\frac{1}{\rho} - 1 = O(\epsilon^2 |u|^2),\]

compare \(u\) to a function \(h\) satisfying

\[\Delta h = ch \text{ in } D. \quad (4.4)\]

Setting \(w = u - h\),

\[\Delta w - cw = \epsilon^2 \rho^2 u_{i,j} + c \left\{ \frac{1}{\rho} - 1 \right\} u. \quad (4.5)\]

Under appropriate constraints on \(u\) and \(h\), we obtain a continuous dependence inequality for \(v - \epsilon h = \epsilon(u - h)\) which is similar to that of the previous section. In order to choose appropriate constraints as
well as to prove analogs of Lemmas 2 and 3 of Section 3, we need the following analog of Lemma 1.

**Lemma 4:** Let \( q_1 \in (0.1) \). Then there are constants \( \ell_1 \) and \( \ell_2 \) independent of \( \epsilon \) so that for each \( \alpha \in (0.1) \)

\[
\int_{D_{\alpha}} (\Delta w)^2 \, dx \leq \ell_1 \int_{D_{\alpha}} w^2 \, dx
\]

\[
+ \ell_2 \epsilon \left\{ \int_{\Sigma} (u, u, i) \epsilon^2 |\mathbf{u}, j_{x}^{\alpha} n_{i} - u, j_{x}^{\alpha} n_{j}| \, ds
\right.
\]

\[
+ \int_{D_{1}} \left[ (u, u, i)^3 (1 + \epsilon^2 |\nabla u|^2) + u^2 (u, u, i)^2 (1 + \epsilon^2 |\nabla u|^2)^2
\right.
\]

\[
\left. + |u|^2 (u, u, i) \right) \, dx \right\}.
\]

**Proof:** From (4.5) and the A-G inequality

\[
(\Delta w)^2 \leq c_1 w^2 + c_2 \epsilon^2 |\nabla u|^4 + c_3 \epsilon^4 (u, i u, j u, i j)^2
\]

(4.5)

for computable constants \( c_1, c_2, \) and \( c_3 \). As in section 3, consider

\[
J(\alpha) = \int_{D_{\alpha}} \rho^4 (u, i u, j u, i j)^2 \, dx
\]

(4.7)
and note that

\[ J(\alpha) \leq \int_{D_1} \omega \rho^2 (u, u, u) J^2 u, jk u, jk dx \]

\[ \leq \int_{D_1} \omega \rho^2 \left[ (u, u, u) J^2 u, jk u, jk + 4(u, u, u) u, k u, jk u, jk \right] dx \]

\[ \leq \int_{D_1} \omega \rho^2 (u, u, u) \left[ u, jk u, jk (1 + \epsilon^2 |v|^2)^2 - \epsilon^2 (u, u, u, u, u, u) \right] dx \]

\[ + 4 \int_{D_1} \omega \rho^2 (u, u, u) \left[ (u, k u, jk u, jk) (1 + \epsilon^2 |v|^2) - \epsilon^2 (u, u, u, u, u, u) \right] dx \]

\[ = A. \quad (4.8) \]

where \( \omega(x) \) is defined as in section 3. As in the previous section, note that

\[ \int_{D_1} \omega^2 (u, u, u) u, jk u, jk dx = \int_{\Sigma_1} \omega^2 (u, u, u) u, jk u, jk \rho^2 u, n_j ds \]

\[ + \int_{D_1} \omega^2 (\Delta u)^2 (u, u, u) dx + 2 \int_{D_1} \omega (u, u, u) u, jk u, jk dx \]
Use the equation (4.3) to write

\[ -\int_{D_1} \omega^2(u_i u_i)^2 (\partial^2 u_{ij} u_{ij})^2 \, dx \]

\[ -\int_{D_1} \omega^2(u_i u_i)^2 \left( \Delta u - \frac{1}{\rho} u \right)^2 \, dx \]

\[ -\int_{D_1} \omega^2(u_i u_i)^2 (\Delta u) \, dx - c^2 \int_{D_1} \omega^2(u_i u_i)^2 \rho^{-2} u^2 \, dx \]

\[ + 2c \int_{D_1} \omega^2(u_i u_i)^2 \omega \Delta u \rho^{-1} \, dx \quad (4.10) \]

Combining (4.9) and (4.10),

\[ \int_{D_1} \omega^2(u_i u_i)^2 \rho^4 \left[ u_{ij} u_{ij} (1 + \epsilon^2 |\nabla u|^2)^2 - \epsilon^4 (u_{ij} u_{ij})^2 \right] \, dx \]
\[- \int_{\Sigma_1} \omega^2 (u, j u, j u) \left[ (u, j u, j u, j u, j u) - (u, j u, j u, j u) \right] d_s \]

\[ + 2 \int_{D_1} \omega (u, j u, j u) \left[ \omega (u, j u, j u, j u) - \omega (u, j u, j u, j u) \right] d_x \]

\[ + 4 \int_{D_1} \omega^2 (u, j u, j u, j u, j u) \left[ (u, j u, j u, j u, j u) - (u, j u, j u, j u) \right] d_x - 4 \int_{D_1} \omega^2 (u, j u, j u, j u, j u) d_x \]

\[- c^2 \int_{D_1} \omega^2 (u, j u, j u, j u, j u) (\Delta u)^2 d_x + 2c \int_{D_1} \omega^2 (u, j u, j u) (\Delta u)^2 d_x \]

\[ (4.11) \]

Returning to (4.8), we use (4.3) to write

\[ 4 \int_{D_1} \omega^2 \rho^2 (u, j u, j u, j u, j u) \left[ (u, j u, j u, j u, j u) - (u, j u, j u, j u) \right] d_x \]

\[ - 4 \int_{D_1} \omega^2 (u, j u, j u, j u, j u) d_x \]

\[- 4 \int_{D_1} \omega^2 (u, j u, j u, j u, j u) \left( \Delta u - \frac{1}{\rho} u \right) d_x \]

\[ (4.12) \]
Adding (4.11) to (4.12),

\[
A = \int_{\Sigma_1} \omega^2(u, u, )^2 [u, j u, j n_2 - u, j u, j n_j] ds
\]

\[+ 2 \int_{D_1} \omega(u, u, )^2 [\omega, j u, j u, - \omega, j u, j n_j] dx
\]

\[- c^2 \int_{D_1} \omega^2(u, u, )^2 \rho^{-2} u^2 dx
\]

\[+ 2c \int_{D_1} \omega^2(u, u, )^2 u(\Delta u) \rho^{-1} dx
\]

\[+ 4c \int_{D_1} \omega^2(u, u, ) \rho^{-1} u dx \]

(4.13)

Using the A-G inequality, one can find a constant \( k \) so that

\[
A \leq \int_{\Sigma_1} \omega^2(u, u, )^2 [u, j u, j n_2 - u, j u, j n_j] ds
\]

\[+ k \int_{D_1} (u, u, )^2 (1 + \epsilon^2 |\nabla u|^2) dx
\]

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\[
+ \frac{1}{2} \int_{D_1} \omega^2 (u_i u_i)^2 u_{,j} u_{,j} dx \\
+ k \int_{D_1} \omega^2 (u_i u_i)^2 u^2 (1 + \epsilon^2 |\nabla u|^2)^2 dx \\
+ 4c \int_{D_1} \omega^2 (u_i u_i)^2 \rho^{-1} u \ dx ,
\]

We can now conclude that

\[
J(\alpha) \leq \ell \left\{ \int_{\Sigma_1} \omega^2 (u_i u_i)^2 (u_{,j} u_{,j} n_j - u_{,j} u_{,j} n_j) ds \\
+ \int_{D_1} (u_i u_i)^3 (1 + \epsilon^2 |\nabla u|^2) dx \\
+ \int_{D_1} u^2 (u_i u_i)^2 (1 + \epsilon^2 |\nabla u|^2)^2 dx \\
+ \int_{D_1} \rho^{-1} (u_i u_i) dx \right\} ,
\]

for an explicit constant \( \ell \). Thus Lemma 4 holds.
Having obtained an estimate for

$$
\int_{D_a} (\Delta w)^2 \, dx ,
$$

one can argue in a manner similar to that of the preceding section. For \( a \in [0,1] \) set

$$
F(a) = Q + \int_0^a (a - \eta) \left\{ \int_{D_\eta} [w, w, w, w + w \Delta w] \, dx \right\} \, d\eta
$$

(4.16)

where \( Q \) is given by

$$
Q = k_0 \int_\Sigma w^2 \, ds + k_1 \int_\Sigma w, w, w, w \, ds + k_2 \epsilon^{4-p}
$$

(4.17)

with \( 2 \leq p < 6 \). The \( k_i \) are positive constants chosen so that (3.17) holds. Assume that

$$
\int_\Sigma (w^2 + w, w, w, w) \, ds = O(\epsilon^{4-p}) ,
$$

(4.18)

and also impose the constraint
\[
\int (u, u, u)^2 |u, j u, j f n_j| \, ds
\]

\[
+ \int_{D_1} [(u, u, u)^3 (1 + \epsilon^2 \|u\|^2) + u^2 (u, u, u)^2 (1 + \epsilon^2 \|u\|^2)^2
\]

\[
+ |u|^\rho - 1 (u, u, u) \, dx = O(\epsilon^p). \tag{4.19}
\]

We now state the remaining estimates, which are understood to hold on the interval \((0, \alpha_1)\). The proofs are similar to those of the previous section.

**Lemma 5:** If \( F(\alpha) \) is given by (4.16), then

\[
|F'| \leq F' + K_1 F \tag{4.20}
\]

for a computable constant \( K_1 \).

**Lemma 6:** If \( F(\alpha) \) is given by (4.16), then

\[
\int_{D_2} [w, u, w, u - 2r^{-1} (w, f, f)^2] \, dx \geq - K_2 F' - K_3 F. \tag{4.21}
\]

for explicit constants \( K_2 \) and \( K_3 \) (recall that \( r = |\nabla f|^2 \)).

One may now conclude as in the last section that on the interval \((0, \alpha_1)\).
for explicit constants $C_1$ and $C_2$. Assuming that

$$\int_{D_1} u^2 \, dx = O(\epsilon^{-2}) \quad \text{and} \quad \int_{D_1} h^2 \, dx = O(\epsilon^{-2}),$$

we have

**Theorem 2:** If $u$ and $h$ are solutions of (4.3) and (4.4), respectively, which satisfy (4.18), (4.19), and (4.22), then for $0 < \alpha < \alpha_1$

$$\int_{D_\alpha} (v - \epsilon h)^2 \, dx = O(\epsilon \gamma(\alpha))$$

with $\gamma(\alpha) = (6 - p)(\sigma - \sigma_1)/(1 - \sigma_1)$, $\sigma = \exp(-C_1 \alpha)$, and $\sigma_1 = \exp(-C_1 \alpha_1)$.

As in section 3, one can find a continuous dependence inequality for the Dirichlet integral of $v - \epsilon h$. Introduce the function $\mu$ as in the last section with $\beta < \alpha_2$ and a fixed number $\alpha_3$ between $\alpha_2$ and $\alpha_1$. Then

$$\int_{D_\beta} (u-h)_i (u-h)_i \, dx \leq \int_{D_{\alpha_3}} \mu_2 (u-h)_i (u-h)_i \, dx$$

$$- \int_{\Sigma_{\alpha_3}} \mu (u-h)(u-h)_i n_i \, ds - \int_{D_{\alpha_3}} (u-h)(u-h)_i (\mu^2)_i \, dx$$

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\[ - \int_{D_{\alpha_3}} \mu^2(u-h)\Delta(u-h) \, dx \]
\[ - 0(\varepsilon^{4-p}) + \frac{1}{2} \int_{D_{\alpha_3}} (u-h)^2 \Delta(\mu^2) \, dx \]
\[ - \int_{D_{\alpha_3}} \mu^2(u-h)\Delta(u-h) \, dx \]  
\[ (4.23) \]

Using the A-G inequality and the estimates of this section, we conclude that

\[ \int_{D_{\beta}} (v - \varepsilon h)_+ (v - \varepsilon h)_- \, dx = O\left(\varepsilon^{\gamma_{(\alpha_3)}}\right) \]

with \( \gamma(\alpha) \) given as in Theorem 2.
5. In this section, we examine a general second-order nonlinear elliptic equation of the form

\[ (a^{ij}(x, v(x))v_i, v_j) = g(x, v, \nabla v) \text{ in } D. \quad (5.1) \]

Assume that \( g(x, p, \xi) = O(|p| + |\xi|) \) for \( x \in \bar{D}, p \in \mathbb{R}, \) and \( \xi \in \mathbb{R}^N. \) Also assume that the \( a^{ij} \) are \( C^1 \) functions in the \( \bar{D} \times \mathbb{R} \) with \( a^{ij} = A^{ij} \) and that for \( 1 \leq i, j, k \leq N \)

the functions \( a^{ij}(x,p) \) and their first derivatives with respect to the variables \( x_k \) are uniformly (5.2)

Lipschitz continuous in \( \bar{D} \times \mathbb{R} \).

To ensure that equation (5.1) is uniformly elliptic, we require that there be positive constants \( a_0 \) and \( a_1 \) so that

\[ a_0 |\xi|^2 \leq a^{ij}(x,p)\xi_i \xi_j \leq a_1 |\xi|^2 \quad (5.3) \]

for each \( x \in \bar{D}, p \in \mathbb{R}, \) and \( \xi \in \mathbb{R}^N. \) We use the notation

\[ b^{ij}(x) = a^{ij}(x,v(x)) \quad (5.4) \]

for a fixed solution \( v \) of (5.1) to be considered here. Thus,

\[ (b^{ij})_k(x) = (a^{ij})_k(x,v(x)) + (a^{ij})_p(x,v(x))v_k \quad (5.5) \]

Consider the Cauchy problem for (5.1) with \( v \) and \( \nabla v \) measured on \( \Sigma. \) The continuous dependence estimate derived for \( v \) is less sharp than those of the previous sections.

As in the preceding sections, assume that
\[ \int_{\Sigma} (v^2 + v_i v_{,i}) \, ds \leq \epsilon^2 \tag{5.6} \]

for some small positive number \( \epsilon \). Making the formal substitution \( v = \epsilon u \) leads to the following perturbed equation for a function \( u \): \[
a^{ij}(x, \epsilon u) u_{,i,j} + (a^{ij}, j)(x, \epsilon u) u_{,i} + \epsilon (a^{ij}, p)(x, \epsilon u) u_{,i} u_{,j} = 0(\epsilon) (|u| + |\nabla u|). \tag{5.7} \]

Compare \( u \) to a solution \( h \) of the linear equation \[
(A^{ij}(x) h, j) = 0 \tag{5.8} \]

where \( A^{ij}(x) = a^{ij}(x, 0) \). Subtracting (5.8) from (5.7), \[
(A^{ij}(x) w, j) + [a^{ij}(x, \epsilon u) - a^{ij}(x, 0)] u_{,i,j} + [(a^{ij}, j)(x, \epsilon u) - (a^{ij}, j)(x, 0)] u_{,i} + \epsilon (a^{ij}, p) u_{,i} u_{,j} = 0(\epsilon) (|u| + |\nabla u|), \tag{5.9} \]

where \( w = u - h \). Setting \( Lw = (A^{ij} w, j) \), note that \( L \) is a uniformly elliptic operator in \( \hat{\Omega} \). To choose the domains \( D_{\alpha} \) on which to obtain \( L^2 \) estimates for \( w \), replace the conditions (2.3) and (2.4) on the auxiliary function \( f \) by \[
L f \leq 0, \quad |L f| \leq \delta^2 d \tag{5.10} \]
where $\delta$ and $d$ are positive constants.

To find appropriate constraints under which one can derive $L^2$ bounds for $w$, we need to examine the $L^2$ integral of $Lw$. From (5.9),

$$(Lw)^2 \leq r\varepsilon^2 [u^2 u,_{ij} u,_{ij} + u^2 u,_{k} u,_{k} +$$

$$+ (a^{ij}), p (a^{ij}), p (u,_{i} u,_{j})^2 + u^2 + u,_{i} u,_{i}]$$

(5.11)

for an explicit constant $r$. One can estimate

$$\int \int_{D_{\alpha}} (Lw)^2 dx$$

in terms of data and volume integrals involving only $u$ and its first derivatives by means of

**Lemma 7:** Let $\alpha \in (0, 1)$. Then for $\alpha \in (0, \alpha_1)$

$$\int \int_{D_{\alpha}} (Lw)^2 dx \leq \kappa \varepsilon^2 \left\{ \int \int_{\Sigma} u^2 \left[ b^{ik} \left( \frac{\partial}{\partial x_i} n_k - \frac{\partial}{\partial x_k} n_i \right) \right] u,_{j} +$$

$$+(b^{jk}),, u,_{k} u,_{i} n_i + (b^{jk}),, u,_{i} u,_{j} n_k ds + \int_{D_1} \int |u^2| ds$$

$$+(1 + \varepsilon|u|)^2 (u^2 u,_{i} u,_{i} + (u,_{j} u,_{j})^2) + (u,_{i} u,_{i}) (1 + u^2)$$

$$+ \varepsilon^2 u^2 (u^2 + u,_{i} u,_{i}) + a^{ij}, p (a^{ij}), p (u,_{i} u,_{i})^2 + \varepsilon^2 u^2 u,_{j} u,_{j}) dx \right\}$$

(5.12)
for an explicit constant $k$ independent of $\epsilon$ and $\alpha$.

**Proof:** Using the cutoff function $\omega$ of the last two sections and the ellipticity condition (5.3),

\[
\int_{D_\alpha} u^2 u_{ij} u_{ij} \, dx \leq \frac{1}{a_0} \int_{D_1} \omega^2 b^j k u_{ij} u_{ik} \, dx .
\] (5.13)

Integration by parts gives

\[
\int_{D_1} \omega^2 u^2 b^j k u_{ij} u_{ik} \, dx = \int_{\Sigma_1} \omega^2 u^2 b^j k u_{ij} u_{ik} \, ds
\]

\[
- 2 \int_{D_1} \omega x, u^2 b^j k u_{ij} u_{,i} \, dx - 2 \int_{D_1} \omega u x, b^j k u_{ij} u_{,i} \, dx
\]

\[
- \int_{D_1} \omega^2 u^2 (b^j u_{ij})_{,k} u_{,i} \, dx .
\] (5.14)

Rewrite the boundary term in (5.14) as

\[
\int_{\Sigma_1} \omega^2 u^2 b^j k u_{ij} u_{,i} n_k \, ds
\]

\[
- \int_{\Sigma_1} \omega^2 u^2 b^j k \left[ n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k} \right] \cdot u_{,i} \, ds
\]
\[ + \int_{\Sigma_1} \omega^2 u^2 u_{i,j} n_j b_{j,k} u_{j,k} \, ds \]

\[ - \int_{\Sigma_1} \omega^2 u^2 u_{i,j} b_{j,k} \left( n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k} \right) u_{j,k} \, ds \]

\[ + \int_{\Sigma_1} \omega^2 u^2 u_{i,j} n_k g \, ds - \int_{\Sigma_1} \omega^2 u^2 u_{i,j} n_i (b_{j,k})_{,j,k} \, ds \]  \hspace{1cm} (5.15)

Since

\[ n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k} \]

is a tangential derivative on \( \Sigma \), the three terms on the right side of (5.15) involve only Cauchy data.

Examine now the volume integrals on the right side of (5.14). Using the A-G inequality and (5.3),

\[ -2 \int_{D_1} \omega \omega_{k,u} b_{j,k} u_{,i,j} u_{,i} \, dx - 2 \int_{D_1} \omega u u_{,k} b_{j,k} u_{,i,j} u_{,i} \, dx \]

\[ \leq \delta \int_{D_1} \omega^2 b_{j,k} u_{,i,j} u_{,i,k} \, dx + \]

\[ K_1 \int_{D_1} b_{j,k} b_{j,k} [(u_{,i} u_{,i}) u^2 + (u_{,i} u_{,i})^2] \, dx \]  \hspace{1cm} (5.16)

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where $\delta$ is a positive constant which we may choose to be as small as we like and $K_1$ is a computable constant depending on $\delta$. For the last integral on the right side of (5.14), note that

\[
- \int_{D_1} \omega^2 u^2 (b^j k_{i,j}),_i k_u, i dx = - \int_{D_1} \omega^2 u^2 (b^j k_{i,j}),_i k_i u, i dx
\]

\[
+ \int_{D_1} \omega^2 u^2 (b^j k_{i,j}),_i u, i k_i u, i dx.
\] (5.17)

The first integral on the right side of (5.17) can be written as

\[
- \int_{D_1} \omega^2 u^2 (b^j k_{i,j}),_i k_i u, i dx = - \int_{\Sigma_1} \omega^2 u, i g(x, v, \text{grad } v) n_i ds
\]

\[
+ \int_{D_1} \omega^2 u^2 (\Delta u) g(x, v, \text{grad } v) dx
\]

\[
+ \int_{D_1} \left[ (\omega^2 u^2),_i u, i \right] g(x, v, \text{grad } v) dx
\] (5.18)

where we have integrated by parts and used (5.1). Using the A-G inequality on the right side of (5.18),
\[- \int_{D_1} \omega u^2 (b^j k u^i)_i,_{j_{\cdot k_{, i}}} \, dx \leq - \int_{\Sigma_1} \omega u^2 g(x, \nabla v, \nabla u) u,_{i_{\cdot n_{, i}}} \, ds \]

\[+ \delta \int_{D_1} \omega u^2 (b^j k u^i)_i,_{j_{\cdot k_{, i}}} \, dx + O(1) \int_{D_1} (u,_{i_{\cdot u_{, i}}})(u,_{j_{\cdot u_{, j}}} + u^2) \, dx \]

\[+ O(\epsilon^2) \int_{D_1} u^2 (u^2 + u,_{j_{\cdot u_{, j}}} + u^2) \, dx. \quad (5.19)\]

Integrating the second integral on the right side of (5.17) by parts,

\[\int_{D_1} \omega u^2 [(b^j k),_{i_{\cdot u_{, j}}}],_{, i_{\cdot k_{, i}}} \, dx = \int_{\Sigma_1} \omega u^2 (b^j k),_{i_{\cdot u_{, i}}},_{j_{\cdot n_{, i}}} \, ds \]

\[\quad - \int_{D_1} \omega u^2 (b^j k),_{i_{\cdot u_{, i}}},_{k_{, j_{\cdot i}}} \, dx - \int_{D_1} (\omega^2),_{k_{\cdot u^2 (b^j k),_{i_{\cdot u_{, i}}},_{j_{\cdot k}}} \, dx \]

\[\quad - 2 \int_{D_1} \omega u^2 u,_{i_{\cdot u_{, i}}},_{j_{\cdot k_{, i}}} \, dx. \quad (5.20)\]
Using the A-G inequality and (5.3),

\[- \int_{D_1} \omega^2 u^2 (b^j k)_{i_1 u_1 k u_1} \, dx \leq \delta \int_{D_1} \omega^2 u^2 b^j k u_{i_1 j} u_{i_1 k} \, dx \]

\[+ K_2 \int_{D_1} u^2 (b^j k b^j k)_{i_1 u_1 u_2} \, dx \]

\[+ K_4 \int_{D_1} u^2 (u_{i_1 u_1}) \, dx \]

(5.21)

where \( K_2 \) is a computable constant depending on \( \delta \). Another use of the A-G inequality gives

\[- \int_{D_1} (\omega^2)_{i_1 u_1} u^2 (b^j k)_{i_1 u_1} \, dx \leq K_3 \int_{D_1} u^2 (b^j k b^j k)_{i_1 u_1 u_2} \, dx \]

\[+ K_4 \int_{D_1} u^2 (u_{i_1 u_1}) \, dx \]

(5.22)

for computable constants \( K_3 \) and \( K_4 \). Finally, use the A-G inequality to bound the last integral in (5.20) by

\[K_5 \int_{D_1} u^2 (b^j k b^j k)_{i_1 u_1} \, dx + K_6 \int_{D_1} (u_{i_1 u_1})^2 \, dx \]

(5.23)

for computable constants \( K_5 \) and \( K_6 \).
Combining (5.13) through (5.23), we find that for $0 < \alpha < \alpha_1$,

\[
\int_{D_{\alpha}} u^2 u_{,i} u_{,j} \, dx \leq K_7 \epsilon^2 \left\{ \int_{\Sigma_1} v^2 u^2 \left[ n_k \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_k} \right] u,_{j} 

- (b^j k),_{j} u,_{k} u,_{i} n,_{j} + (b^j k),_{i} u,_{j} u,_{j} n,_{k} \right\} ds

+ \int_{D_1} \left\{ (1 + \epsilon |u|)^2 (u^2 u_{,i} u_{,i} + (u_{,i} u_{,i})^2) + \epsilon^2 u^2 (u^2 + u,_{i} u,_{i})

+ \epsilon^2 (a^j k,_{j} p^j k,_{j} u^2 u,_{j} u,_{j}) \right\} \right\}

\]

for an explicit constant $K_7$. This, combined with (5.11), gives the result of Lemma 7.

Now impose the constraint that the term in braces in the statement of Lemma 7 is $0(\epsilon^{-q})$ for some $q < 4$. Thus,

\[
\int_{D_{\alpha}} (Lw)^2 \, dx = O(\epsilon^{2-q}). \quad (5.24)
\]
We proceed to derive a differential inequality for the functional

\[ F(\alpha) = Q + \int_0^\alpha (\alpha - \eta) \left\{ \int_D \left[ w^2 + A^{ij} w_{,i} w_{,j} \right] dx \right\} d\eta \]  

(5.25)

where \( Q \) is given by

\[ Q = k_0 \int_\Sigma w^2 ds + k_1 \int_\Sigma w_{,i} w_{,i} ds + k_2 \varepsilon^{2-q} \]  

(5.26)

As in section 3, one can choose the constants \( k_i \) in (5.26) so that an analog of (3.17) holds, i.e.,

\[ \frac{1}{2} \left[ \int_D r w^2 dx + Q \right] \leq F(\alpha) \leq \frac{d + 1}{2} \left[ \int_D r w^2 dx + Q \right] \]  

(5.27)

where \( r = A^{ij} f_{,i} f_{,j} \). Assuming that

\[ \int_\Sigma (w^2 + w_{,i} w_{,i}) ds = O(\varepsilon^{2-q}) \]  

(5.28)

we are prepared to state the remaining estimates, which are understood to hold on the interval \((0, \alpha_1)\). Here, the proofs are very similar to those found in Payne (14).
Lemma 8: If \( F(a) \) is given by (5.25), then

\[
|F'| \leq F' + K_1 F
\]  
(5.29)

for a computable constant \( K_1 \).

Lemma 9: If \( F(a) \) is given by (5.25), then

\[
\int_{D_\alpha} A^{ij} w_i w_j dx - 2 \int_{D_\alpha} r^{-1} [A^{ij} w_i f_j]^2 dx \geq - K_2 F' - K_3 F
\]

for computable constants \( K_2 \) and \( K_3 \).

One may now conclude as in section 2.1, that on the interval \((0, \alpha_1)\),

\[
F F'' - (F')^2 \geq - C_1 FF' - C_2 F^2
\]

for explicit constants \( C_1 \) and \( C_2 \). Assuming that

\[
\int_{D_1} u^2 dx = O(\epsilon^{-2}), \quad \int_{D_1} h^2 dx = O(\epsilon^{-2}), \quad (5.30)
\]

we have

Theorem 3: If \( u \) and \( h \) are solutions of (5.7) and (5.8), respectively, which satisfy (5.24), (5.28), and (5.30), then for \( 0 < \alpha, \alpha_1 \)
\[ \int_{D_\alpha} (v - \epsilon h)^2 \, dx = O(\epsilon \gamma(\alpha)) \]

with \( \gamma(\alpha) = (4 - q)(\sigma - \sigma_1)/(1 - \sigma_1) \), \( \sigma = \exp(-C_1 \alpha) \), and \( \sigma_1 = \exp(-C_1 \sigma_1) \).

As in the other sections of this chapter, one can find a continuous dependence inequality for the Dirichlet integral of \( v - \epsilon h \). Introduce the function \( \mu \) as in section 3 with \( \beta < \alpha_2 \) and \( \alpha_3 \) a fixed number between \( \alpha_2 \) and \( \alpha_1 \). Then

\[
\begin{align*}
\alpha_0 \int_{D_\beta} (u-h),_i (u-h),_j \, dx & \leq \int_{D_\alpha_3} \mu_{A}^{i\bar{j}} w,\bar{w},_j \, dx \\
- \int_{\Sigma_\alpha_3} \mu_{A}^{i\bar{j}} w,\bar{w}_n \, ds & - \int_{D_\alpha_3} \mu^2 w L \, dx \\
- \int_{D_\alpha_3} (\mu^2),_{A}^{i\bar{j}} w,\bar{w} \, dx & = \int_{\Sigma_\alpha_3} (\mu_{A}^{i\bar{j}} w,_{i} \bar{n}_j) \\
- \frac{1}{2} (\mu^2),_{A}^{i\bar{j}} w^2 n_i \, ds & - \int_{D_\alpha_3} \mu^2 w L \, dx \\
+ \frac{1}{2} \int_{D_\alpha_3} w^2 [A_{i\bar{j}} (\mu^2),_j] \, dx. 
\end{align*}
\]  

(5.31)
Use the A-C inequality and the estimates of this section to conclude that

\[ \int_{D_\beta} (v - \epsilon h)_+ (v - \epsilon h)_- dx = o(\epsilon^{(a_3)}) \]
6. Concluding Remarks

The arguments in this work do not yield continuous dependence results if $p = 6$ in the constraints (3.25) and (4.19), or $q = 4$ in (5.24). Such constraints would be desirable since they would not impose any apriori "smallness" conditions on volume integrals of the solutions or their derivatives. It is not clear, however, that continuous dependence results could be obtained from such constraints.

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