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IDENTIFICATION OF SPATIALLY VARYING  
STIFFNESS AND DAMPING IN BEAMS

by

H.T. Banks and I.G. Rosen

August 1986

LCDS #86-39

Lefschetz Center for Dynamical Systems

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**COMPUTATIONAL METHODS FOR THE IDENTIFICATION OF SPATIALLY  
VARYING STIFFNESS AND DAMPING IN BEAMS<sup>+</sup>**

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## **ABSTRACT**

A numerical approximation scheme for the estimation of functional parameters in Euler-Bernoulli models for the transverse vibration of flexible beams with tip bodies is developed. The method permits the identification of spatially varying flexural stiffness and Voigt-Kelvin viscoelastic damping coefficients which appear in the hybrid system of ordinary and partial differential equations and boundary conditions describing the dynamics of such structures. An inverse problem is formulated as a least squares fit to data subject to constraints in the form of a vector system of abstract first order evolution equations. Spline-based finite element approximations are used to finite dimensionalize the problem. Theoretical convergence results are given and numerical studies carried out on both conventional (serial) and vector computers are discussed.

## 1. Introduction

We develop here numerical approximation methods for the estimation of functional or more precisely, spatially varying parameters that describe material properties in continuum models for elastic structures. In particular, we consider the identification of the flexural stiffness and Voigt-Kelvin viscoelastic damping coefficients in Euler-Bernoulli models for the transverse vibration of long, slender, flexible beams with tip appendages. The primary motivation for the study we report on here is the modeling and ultimately the control of the dynamics of large flexible spacecraft. The type of structures to which we are referring includes satellites with flexible appendages (solar panels and the like) antennas (reflectors as well as supporting structures) and trussed masts and platforms, both shuttle attached and free flying.

The difficulties involved in the design of efficient and practical control laws and in particular the need for extremely high fidelity models for structures of these types are well documented (see, for example, [1], [8], [21], [22]). Their high flexibility, light damping, construction with new and relatively untested composite materials (usually graphite-epoxy) and overall complexity together with their use in a fuel limited and highly variable environment all contribute to making space structure stabilization and control a formidable task. It is becoming increasingly clear that the use of continuum or distributed models with spatially and / or temporally varying functional parameters has the potential to offer several distinct and significant advantages. Included among them is the ability to, in some sense, capture the physics and inherent infinite dimensionality of the dynamics while at the same time greatly reducing the number of unknown or experimentally indeterminable material parameters which have to be identified (see [15], [18], [23], [28], [35]).

In our study we have considered exclusively Voigt-Kelvin viscoelastic damping which is based on the hypothesis that the damping moment is proportional to strain rate. There exists considerable evidence to suggest that damping mechanisms in composite materials are significantly more complex than the one described by the Voigt-Kelvin model. For example, it has been conjectured by some investigators that an appropriate model might involve hysteretic or hereditary effects. However, since there are a number of materials for which the Voigt-Kelvin assumption is appropriate and moreover, since at present many questions regarding the modeling of structural

damping mechanisms remain open, we feel that the Voigt-Kelvin model leads to a reasonable class of examples and problems on which we can begin to develop, test, and evaluate identification schemes.

Our treatment here is similar in spirit to some of our earlier efforts and the work of others on inverse problems for elastic structures (see [2], [3], [4],[5], [6], [14], [17], [26], [31]). Formulating the identification problem as a least squares fit to data, the scheme we develop involves a spline based finite element approximation to the hybrid system of coupled ordinary and partial differential equations describing the dynamics of the structure together with a spline based discretization of the admissible parameter set.

Our approach here specifically differs from the one taken in [5], [6] in that the present scheme is derived from an alternative state space formulation for the underlying dynamical equations. We consider the higher order analog of the classical conservative formulation for a second order hyperbolic equation as a first order vector system in the natural states of strain  $u_x$  and velocity  $u_t$ . We have considered identification schemes based upon this formulation previously in [31]. However by replacing the semigroup theoretic convergence arguments used there with weak or variational arguments (in the spirit of those commonly found in the finite element literature) as used in [5], we are able to significantly weaken the hypotheses necessary to ensure convergence. We point out below that the weakening of these hypotheses has both theoretical and computational significance.

Along with reporting theoretical convergence results, we discuss numerical findings. Our computational results are based upon extensive numerical studies which involved a variety of examples and two machines. In addition to testing our scheme on a conventional serial computer (an IBM 3081) we vectorized our codes for the Cray 1-S and then benchmarked some of our runs in order to explore the potential of vector architectures in the context of inverse problems for systems described by distributed parameter models.

We provide a brief outline and summary of the remainder of the paper. In Section 2 we specify the ordinary and partial differential equations which govern the underlying dynamics of the structure and precisely formulate the identification problem. We reformulate the initial-boundary value problem as an abstract second order evolution equation and then as a first order vector

system. Existence, uniqueness and regularity results for solutions are summarized. Section 3 contains the abstract approximation theory and convergence results. A spline-based scheme is discussed in detail in Section 4 and our numerical findings are reported and summarized in Section 5.

We use standard notation throughout. For  $X$  and  $Y$  Banach spaces, the Banach space of continuous linear transformations from  $X$  into  $Y$  is denoted by  $\mathfrak{L}(X, Y)$ . When  $X = Y$  we use the shorthand notation  $\mathfrak{L}(X)$ . The spaces of (equivalence classes of) functions  $f$  from an interval  $\mathcal{J}$  into  $X$  which satisfy

$$\int_{\mathcal{J}} |f(\theta)|_X^2 d\theta < \infty \quad \text{or} \quad \text{ess sup}_{\mathcal{J}} |f(\theta)|_X < \infty$$

are denoted respectively by  $L_2(\mathcal{J}; X)$  and  $L_\infty(\mathcal{J}; X)$ . For  $k = 0, 1, 2, \dots$  the space of  $X$ -valued functions with  $k$  continuous derivatives on  $\mathcal{J}$  are denoted by  $C^k(\mathcal{J}; X)$ . When  $k = 0$  we use  $C(\mathcal{J}; X)$ . The completion of the space  $C^k(\mathcal{J}; X)$  with respect to the norm

$$|f|_k = \left( \sum_{j=0}^k \int_{\mathcal{J}} |f^{(j)}(\theta)|_X^2 d\theta \right)^{\frac{1}{2}}$$

is denoted by  $H^k(\mathcal{J}; X)$ . When  $X = \mathbb{R}$  we use simply  $L_2(\mathcal{J})$ ,  $L_\infty(\mathcal{J})$ ,  $C^k(\mathcal{J})$  and  $H^k(\mathcal{J})$ .

## 2. The Identification Problem

We consider the identification, or estimation, of the mass and/or material properties of a long, slender, flexible, viscoelastic beam of length  $\ell$  and spatially varying mass density  $\rho$  which is clamped at one end and free at the other with a body rigidly attached at the free end (see Figure 2.1 below).

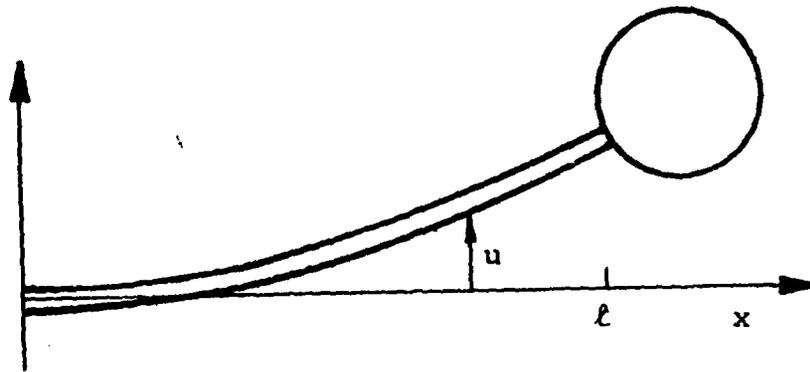


Figure 2.1

We assume that the material behavior of the beam is that of an idealized Voigt-Kelvin solid with modulus of elasticity  $E$  and coefficient of viscosity  $C_D$  (see [30]). We assume further that  $E$ ,  $C_D$  and the cross sectional moment of inertia  $I$  of the beam are in general spatially varying. We take the mass properties of the tip body to be mass  $m$  and moment of inertia  $J$  about the center of mass  $O$  which is assumed to be located at a distance  $c$  from the tip of the beam directed along the beam's tip tangent (see Figure 2.2 below).

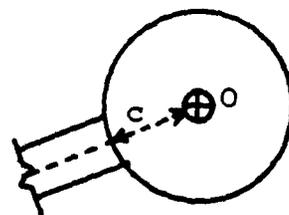


Figure 2.2

We note that there is no essential loss of generality in assuming that the mass center of the tip body is not offset from the tip tangent of the beam. We refer the interested reader to [31] where the more general situation is treated. Also, the problem with non-zero mass center offset can be transformed into a problem of the general form of the one which will be considered here. See [32] for details.

Letting  $u = u(t,x)$  denote the transverse displacement of the beam at position  $x$  at time  $t$  and assuming only small deformations ( $|u(t,x)| \ll \ell$ ,  $|\frac{\partial u}{\partial x}(t,x)| \ll 1$ ), the Euler-Bernoulli theory and elementary Newtonian mechanics yield the hybrid system of ordinary and partial differential equations (see [19], [34])

$$(2.1) \quad \rho \frac{\partial^2 u}{\partial t^2}(t,x) + \frac{\partial^2}{\partial x^2} \left\{ EI \frac{\partial^2 u}{\partial x^2}(t,x) + C_D I \frac{\partial^3 u}{\partial x^2 \partial t}(t,x) \right\} =$$

$$\frac{\partial}{\partial x} \sigma \frac{\partial u}{\partial x}(t,x) + f(t,x), \quad x \in (0, \ell), \quad t > 0$$

$$(2.2) \quad m \frac{\partial^2 u}{\partial t^2}(t, \ell) + mc \frac{\partial^3 u}{\partial t^2 \partial x}(t, \ell) - \frac{\partial}{\partial x} \left( EI \frac{\partial^2 u}{\partial x^2} + C_D I \frac{\partial^3 u}{\partial x^2 \partial t} \right)(t, \ell) = \sigma \frac{\partial u}{\partial x}(t, \ell) + g(t), \quad t > 0$$

$$(2.3) \quad mc \frac{\partial^2 u}{\partial t^2}(t, \ell) + (J + mc^2) \frac{\partial^3 u}{\partial t \partial x}(t, \ell) + EI \frac{\partial^2 u}{\partial x^2}(t, \ell) +$$

$$C_D I \frac{\partial^3 u}{\partial x^2 \partial t}(t, \ell) = -c \sigma \frac{\partial u}{\partial x}(t, \ell) + h(t), \quad t > 0$$

$$(2.4) \quad u(t,0) = 0, \quad \frac{\partial u}{\partial x}(t,0) = 0, \quad t > 0$$

$$(2.5) \quad u(0,x) = \phi(x), \quad \frac{\partial u}{\partial t}(0,x) = \psi(x), \quad x \in [0, \ell].$$

Equation (2.2) and (2.3) are derived from the usual transverse and rotational equilibrium considerations at the free end. The geometric boundary conditions (2.4) are the zero displacement and zero slope constraints at the clamped end. The functions  $f = f(t,x)$ ,  $g = g(t)$ ,  $h = h(t)$  and  $\sigma = \sigma(t,x)$  denote externally applied loads in the form of moments ( $h$ ) and transversally ( $f$  and  $g$ ) or axially ( $\sigma$ ) directed forces exerted on the beam or tip body. (In fact,  $h(t) = \tilde{h}(t) + cg(t)$  where  $\tilde{h}$  is an externally applied torque on the tip body). The temporal boundary conditions (2.5) reflect the initial displacement and velocity distributions which are assumed to be given by the functions  $\phi$  and  $\psi$  respectively.

We treat the initial-boundary value problem (2.1) - (2.5) in the form of an abstract second order evolution equation which we then rewrite as an equivalent first order vector system. The particular state space formulation we choose forms the basis for the finite dimensional approximation schemes we develop in the next section. It also allows us to easily establish existence, uniqueness and regularity results for solutions to (2.1) - (2.5) using the theory of abstract parabolic systems.

Let  $H$  denote the Hilbert space  $\mathbb{R}^2 \times L_2(0, \ell)$  with inner product

$$\langle (\eta_1, \xi_1, \theta_1), (\eta_2, \xi_2, \theta_2) \rangle_H = \eta_1 \eta_2 + \xi_1 \xi_2 + \langle \theta_1, \theta_2 \rangle_0$$

and let  $V$  denote the Hilbert space

$$V = \{(\eta, \xi, \theta) \in H : \theta \in H^2(0, \ell), \theta(0) = D\theta(0) = 0, \eta = \theta(\ell), \xi = D\theta(\ell)\}$$

with inner product

$$\langle \hat{\theta}_1, \hat{\theta}_2 \rangle_V = \langle EI(D^2\theta_1), D^2\theta_2 \rangle_0$$

for  $\hat{\theta}_i = (\theta_i(\ell), D\theta_i(\ell), \theta_i) \in V$ ,  $i = 1, 2$ . In the above definitions the inner product  $\langle \cdot, \cdot \rangle_0$  is the standard one on  $L_2(0, \ell)$  and  $D$  denotes the spatial differentiation operators  $\frac{d}{dx}$  or  $\frac{\partial}{\partial x}$ . With  $H$  as

the pivot space, we obtain the usual dense embeddings  $V \subset H = H' \subset V'$ .

We consider the system (2.1) - (2.5) in the form of the abstract second order initial value

problem

$$(2.6) \quad \mathfrak{M}_0 \hat{u}_{tt}(t) + \mathfrak{C}_0 \hat{u}_t(t) + \mathfrak{K}_0 \hat{u}(t) = \mathfrak{B}_0(t) \hat{u}(t) + \mathfrak{F}_0(t), \quad t > 0$$

$$(2.7) \quad \hat{u}(0) = \hat{\phi}, \quad \hat{u}_t(0) = \hat{\psi}$$

in the state  $\hat{u}(t) = (u(t, \ell), Du(t, \ell), u(t, \cdot)) \in H$ . The abstract mass, damping and stiffness

operators  $\mathfrak{M}_0$ ,  $\mathfrak{C}_0$  and  $\mathfrak{K}_0$  are given formally by

$$\mathfrak{M}_0(\eta, \xi, \theta) = (m\eta + mc\xi, mc\eta + (J + mc^2)\xi, \rho\theta)$$

$$\mathfrak{C}_0(\eta, \xi, \theta) = (-D(C_D I(D^2\theta))(\ell), C_D I(D^2\theta)(\ell), D^2(C_D I(D^2\theta)))$$

and

$$\mathfrak{K}_0(\eta, \xi, \theta) = (-D(EI(D^2\theta))(\ell), EI(D^2\theta)(\ell), D^2(EI(D^2\theta)))$$

respectively. For each  $t > 0$ , the operator valued function  $\mathfrak{B}_0$  and input or forcing function  $\mathfrak{F}_0$  take on the values

$$\mathfrak{B}_0(t)(\eta, \xi, \theta) = (-\sigma(t, \ell)(D\theta(\ell)), -c\sigma(t, \ell)(D\theta(\ell)), D(\sigma(t, \cdot)(D\theta)))$$

and

$$\mathfrak{F}_0(t) = (g(t), h(t), f(t, \cdot)).$$

The initial conditions  $\hat{\phi}$  and  $\hat{\psi}$  are given by

$$\hat{\phi} = (\phi(\ell), D\phi(\ell), \phi)$$

and

$$\hat{\psi} = (\psi(\ell), D\psi(\ell), \psi).$$

The formal definitions given above can be made precise and the existence and uniqueness of solutions to the initial value problem (2.6), (2.7) can be established if we make the following assumptions.

$A_1$  The functions  $\rho$ ,  $EI$  and  $C_D I$  are elements in  $C[0, \ell]$  and there exists a positive constant  $\alpha$  for which  $\rho(x) \geq \alpha$ ,  $EI(x) \geq \alpha$ ,  $C_D I(x) \geq \alpha$ ,  $x \in [0, \ell]$ .

A<sub>2</sub> The mapping  $t \rightarrow \sigma(t, \cdot)$  is an element in  $L_\infty((0, T); H^1(0, \ell))$  for some  $T > 0$ .

A<sub>3</sub> The mapping  $t \rightarrow f(t, \cdot)$  is an element in  $L_2((0, T); L_2(0, \ell))$  and  $g, h \in L_2(0, T)$ .

A<sub>4</sub> The function  $\phi$  is an element in  $H^2(0, \ell)$  with  $\phi(0) = D\phi(0) = 0$  and  $\psi \in L_2(0, \ell)$  with  $\psi(\ell)$  and  $D\psi(\ell)$  defined.

Under the hypotheses A<sub>1</sub> - A<sub>4</sub> above, the operator  $\mathfrak{M}_0$  is a bounded linear operator from H onto H and  $\mathfrak{C}_0: \text{Dom}(\mathfrak{C}_0) \subset H \rightarrow H$  and  $\mathfrak{K}_0: \text{Dom}(\mathfrak{K}_0) \subset H \rightarrow H$  are densely defined, nonnegative, self -adjoint operators defined on  $\text{Dom}(\mathfrak{C}_0) = \{\theta \in V : C_D I(D^2\theta) \in H^2(0, \ell)\}$  and  $\text{Dom}(\mathfrak{K}_0) = \{\theta \in V : EI(D^2\theta) \in H^2(0, \ell)\}$  respectively (see [32]). For each  $t \in (0, T)$ ,  $\mathfrak{B}_0(t) \in \mathfrak{L}(V, H)$  and  $\mathfrak{F}_0(t) \in H$  while  $\hat{\phi} \in V$  and  $\hat{\psi} \in H$ . It also follows that  $\mathfrak{B}_0 \in L_\infty((0, T); \mathfrak{L}(V, H))$  and  $\mathfrak{F}_0 \in L_2((0, T); H)$ .

We shall call a mapping  $t \rightarrow \hat{u}(t)$  from  $[0, T]$  into H a strong solution to (2.6), (2.7) if

$$\hat{u} \in C([0, T]; V) \cap C^1((0, T); V) \cap C^1([0, T]; H) \cap C^2((0, T); H),$$

$\hat{u}(t) \in \text{Dom}(\mathfrak{K}_0)$ ,  $\hat{u}_t(t) \in \text{Dom}(\mathfrak{C}_0)$ ,  $t \in (0, T]$ , and  $\hat{u}$  satisfies (2.6) and (2.7) where the time derivatives are interpreted in a strong (norm) sense in H. We shall call a mapping  $t \rightarrow \hat{u}(t)$  from  $[0, T]$  into H a weak solution to (2.6), (2.7) if

$$\hat{u} \in C([0, T]; V) \cap H^1((0, T); V) \cap C^1([0, T]; H) \cap H^2((0, T); V)$$

and it satisfies the initial value problem (2.6), (2.7) with the operators  $\mathfrak{C}_0$  and  $\mathfrak{K}_0$  replaced by their natural extensions to operators in  $\mathfrak{L}(V, V')$  and the time derivatives are interpreted in a weak or distributional sense (see [20 ], [27]). A function  $u = u(t, x)$  will be called a strong (weak) solution to the initial-boundary value problem (2.1) - (2.5) if the mapping  $t \rightarrow \hat{u}(t)$  given by  $\hat{u}(t) = (u(t, \ell), Du(t, \ell), u(t, \cdot))$  is a strong (weak) solution to (2.6), (2.7).

Our approximation theory for the estimation problem to be developed below is based upon the reformulation of the initial value problem (2.6), (2.7) as a first order vector system. This reformulation is formally equivalent to rewriting the initial-boundary value problem (2.1) - (2.5)

as a first order system in the states  $D^2u$  (strain) and  $u_1$  (velocity) (see [3], [31]). We note that since the stiffness operator  $\mathcal{K}_0$  is nonnegative and self-adjoint it has a unique nonnegative, selfadjoint square root  $\mathcal{K}_0^{1/2} : V \subset H \rightarrow H$ . It can be written in factored form as

$$\mathcal{K}_0 = L_{EI}^* L$$

where  $L : V \subset H \rightarrow L_2(0, \ell)$  is given by

$$L\theta = D^2\theta,$$

for  $\hat{\theta} = (\theta(\ell), D\theta(\ell))$ ,  $\theta \in V$ , and  $L_{EI}^* : \text{Dom}(L_{EI}^*) \subset L_2(0, \ell) \rightarrow H$  by

$$\text{Dom}(L_{EI}^*) = \{\theta \in L_2(0, \ell) : EI\theta \in H^2(0, \ell)\}$$

$$(2.8) \quad L_{EI}^*\theta = (-D(EI\theta)(\ell), EI\theta(\ell), D^2(EI\theta)).$$

If, for  $\tau \in C[0, \ell]$  with  $\tau(x) \geq \alpha > 0$ ,  $x \in [0, \ell]$ , we let  $L_{2, \tau}$  denote the Hilbert space  $L_2(0, \ell)$  endowed with the inner product

$$\langle \theta_1, \theta_2 \rangle_{0, \tau} = \langle \tau \theta_1, \theta_2 \rangle_0$$

then  $L_{\tau}^*$  given by (2.8) with  $EI$  replaced by  $\tau$  is the Hilbert space adjoint of  $L$  as a mapping from  $V \subset H$  into  $L_{2, \tau}$ .

We note that  $L \in \mathfrak{L}(V, L_{2, EI})$  is a Hilbert space isomorphism with

$$\langle \hat{\theta}_1, \hat{\theta}_2 \rangle_V = \langle \mathcal{K}_0^{1/2} \hat{\theta}_1, \mathcal{K}_0^{1/2} \hat{\theta}_2 \rangle_H = \langle L \hat{\theta}_1, L \hat{\theta}_2 \rangle_{0, EI}$$

and  $L^{-1} : L_2(0, \ell) \rightarrow V$  given by

$$L^{-1}\theta = \left( \int_0^{\ell} \int_0^x \theta(y) dy dx, \int_0^{\ell} \theta(x) dx, \int_0^{\ell} \int_0^x \theta(y) dy dx \right).$$

We also have

$$\mathfrak{C}_0 = L_{CD}^* L.$$

Letting  $\mathfrak{H} = L_2(0, \ell) \times H$  with inner product

$$(2.9) \quad \langle (\theta_1, (\eta_1, \xi_1, \chi_1)), (\theta_2, (\eta_2, \xi_2, \chi_2)) \rangle_{\mathfrak{H}} = \langle \theta_1, \theta_2 \rangle_{0, EI} + \langle \mathfrak{M}_0(\eta_1, \xi_1, \chi_1), (\eta_2, \xi_2, \chi_2) \rangle_H$$

and  $\mathfrak{V} = L_2(0, \ell) \times V$  with inner product

$$\langle (\theta_1, \hat{\chi}_1), (\theta_2, \hat{\chi}_2) \rangle_{\mathfrak{V}} = \langle \theta_1, \theta_2 \rangle_{0, EI} + \langle \hat{\chi}_1, \hat{\chi}_2 \rangle_V$$

we have the dense imbeddings  $\mathfrak{V} \subset \mathfrak{H} \subset \mathfrak{V}'$ . We consider the initial value problem for

$z(t) = (w(t), \hat{v}(t)) \in \mathfrak{H}$  given by

$$(2.10) \quad w_t(t) = L\hat{v}(t)$$

$$(2.11) \quad \mathfrak{M}_0 \hat{v}_t(t) = -L_{EI}^* w(t) - L_{CD}^* L\hat{v}(t) + \mathfrak{B}_0(t)L^{-1}w(t) + \mathfrak{F}_0(t) \quad 0 < t \leq T$$

$$(2.12) \quad w(0) = L\hat{\phi}, \quad \hat{v}(0) = \hat{\psi}$$

which we rewrite as

$$(2.13) \quad z_t(t) = Q(t)z(t) + \mathfrak{F}(t), \quad 0 < t \leq T,$$

$$(2.14) \quad z(0) = z_0$$

where

$$(2.15) \quad Q(t) = \tilde{Q} + \mathfrak{B}(t)$$

with  $\tilde{Q} : \mathcal{D} \subset \mathfrak{H} \rightarrow \mathfrak{H}$ ,  $\mathfrak{B} \in L_\infty((0, T); \mathfrak{B}(H))$ ,  $\mathfrak{F} \in L_2((0, T); \mathfrak{H})$  and  $z_0 \in \mathfrak{H}$  given by

$$\tilde{Q}(\theta, \hat{\chi}) = (L\hat{\chi}, -\mathfrak{M}_0^{-1}L_{EI}^*\theta - \mathfrak{M}_0^{-1}L_{CD}^*L\hat{\chi})$$

for  $(\theta, \hat{\chi}) \in \mathcal{D} = \text{Dom}(L_{EI}^*) \times \text{Dom}(L_{CD}^*)$ ,

$$\mathfrak{B}(t)(\theta, (\eta, \xi, \chi)) = (0, \mathfrak{M}_0^{-1}\mathfrak{B}_0(t)L^{-1}\theta),$$

for  $(\theta, (\eta, \xi, \chi)) \in \mathfrak{H}$ ,

$$\mathfrak{F}(t) = (0, \mathfrak{M}_0^{-1}\mathfrak{F}_0(t))$$

and

$$z_0 = (L\hat{\phi}, \hat{\psi}).$$

In formulating the inverse problem, we keep technical details to a minimum by considering only

the estimation of the beam's spatially varying flexural stiffness  $EI$  and viscous damping coefficient  $C_{DI}$ . Extending the finite dimensional approximation methods and corresponding convergence theory which are developed below so as to be applicable to the identification of other structural or input parameters, for example mass properties (of the beam and/or tip body), initial conditions or loading, is, at least in principle, routine (see [7] [12] [14] [16] [31]).

Let  $\mathcal{Q} = C[0, \ell] \times C[0, \ell]$  with norm

$$(2.16) \quad \begin{aligned} \|q\|_{\mathcal{Q}} &= \|(q_1, q_2)\|_{\mathcal{Q}} = \|q_1\|_{\infty} + \|q_2\|_{\infty} \\ &= \sup_{x \in [0, \ell]} |q_1(x)| + \sup_{x \in [0, \ell]} |q_2(x)|. \end{aligned}$$

We take the admissible parameter space  $Q$  to be a compact subset of  $\mathcal{Q}$  (compact with respect to the metric topology induced by the norm (2.16)). Recalling assumption (i) we assume further that the set  $Q$  has the property that all  $q = (q_1, q_2) \in Q$  satisfy  $q_1(x) \geq \alpha$  and  $q_2(x) \geq \alpha$ ,  $x \in [0, \ell]$ .

We formulate the identification problem as a least-squares fit-to-data over the admissible parameter space  $Q$ . We assume that the structure has undergone a time varying elastic deformation in response to the initial conditions described by  $\phi$  and  $\psi$  and the input loads represented by  $f, g, h$  and  $\sigma$ . Denoting the observation space by  $\mathcal{Z}$ , we assume that at times  $t_i$ ,  $i = 1, 2, \dots, v$  measurements  $\zeta(t_i) \in \mathcal{Z}$  (e.g. displacement, velocity, slope, strain, etc.) were taken from the structure.

We require that  $\mathcal{Z}$  be a linear space endowed with a norm  $\|\cdot\|_{\mathcal{Z}}$  and let  $\Gamma$  denote an appropriately defined continuous mapping from  $\mathcal{H}$  into  $\mathcal{Z}$ . For example, suppose that displacement measurements have been taken at the points  $x_j$ ,  $j = 1, 2, \dots, \mu$  along the span of the beam. We choose  $\mathcal{Z}$  as Euclidean  $\mu$ -space,  $R^{\mu}$ , and take  $\Gamma$  to be

$$\Gamma(z) = (\theta(x_1), \theta(x_2), \dots, \theta(x_{\mu}))^T$$

where  $z = (w, \hat{v}) \in \mathcal{H}$  and

$$\theta = (\theta(\ell), D\theta(\ell), \theta) = L^{-1}w \in V.$$

With distributed strain or velocity observations, we would take  $\Gamma(z) = w$  or  $\Gamma(z) = \hat{v}$  respectively.

We formulate the identification problem as follows

(ID) Given  $\zeta(t_i) \in \mathcal{Z}$ ,  $i = 1, 2, \dots, v$ , find  $q^* \in Q$  which minimizes

$$J(q) = \sum_{i=1}^v |\Gamma(z(t_i; q)) - \zeta(t_i)|_{\mathcal{Z}}^2$$

where for each  $q = (q_1, q_2) \in Q$ ,  $z(\cdot; q) = (w(\cdot; q), \hat{v}(\cdot; q))$  is the solution to the initial value problem (2.13), (2.14) or (2.10) - (2.12) with EI set equal to  $q_1$ , and  $C_D I$  set equal to  $q_2$ .

It is immediately clear that the optimization problem given above is inherently infinite dimensional. The admissible parameter set  $Q$  is a subset of a function space and the evaluation (and therefore minimization) of the least-squares performance index  $J$  requires the solution of an infinite dimensional evolution equation. The introduction of finite dimensional approximations is essential to the development of practical computational methods. Fundamental to the approach we take here is a weak, distributional, or variational formulation of the initial value problem (2.13), (2.14). We derive the weak form and briefly outline existence, uniqueness and regularity results for solutions.

In the usual manner, we extend the operator  $\mathcal{Q}(t)$  given by (2.15) to an operator in  $\mathfrak{B}(\mathcal{V}, \mathcal{V})$  via

$$(\mathcal{Q}(t)(v))(\tilde{v}) = a(t)(v, \tilde{v}), \quad v, \tilde{v} \in \mathcal{V}$$

where the bilinear form  $a(t)(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$  is given by

$$a(t)((\theta_1, \hat{\chi}_1), (\theta_2, \hat{\chi}_2)) = \langle EI L \hat{\chi}_1, \theta_2 \rangle_0 - \langle EI \theta_1, L \hat{\chi}_2 \rangle_0 - \langle C_D I L \hat{\chi}_1, L \hat{\chi}_2 \rangle_0 - \quad (2.17)$$

$$c\sigma(t, \ell) \int_0^\ell \theta_1(x) dx (D \chi_2(\ell)) - \langle \sigma(t, \cdot) \int_0^\cdot \theta_1(x) dx, \chi_2 \rangle_0.$$

Standard estimates can be used to demonstrate the existence of positive constants  $k$ ,  $\lambda$  and  $\beta$  for which

$$|a(t)(v_1, v_2)| \leq k \|v_1\|_{\mathcal{V}} \|v_2\|_{\mathcal{V}}, \quad v_i \in \mathcal{V}, i = 1, 2,$$

and

$$a(t)(v, v) + \lambda \|v\|_{\mathfrak{B}}^2 \geq \beta \|v\|_{\mathcal{V}}^2, \quad v \in \mathcal{V}, t \in [0, T].$$

Consequently (see [27]) the system (2.13), (2.14) interpreted as an initial value problem in  $\mathcal{V}$  or equivalently, written in weak form as

$$(2.18) \quad \langle z_t(t), v \rangle_{\mathfrak{H}} = a(t)(z(t), v) + \langle \mathcal{F}(t), v \rangle_{\mathfrak{H}} \quad v \in \mathcal{V}, t \in (0, T]$$

$$(2.19) \quad z(0) = z_0$$

admits a unique solution  $z$  with  $z(t) \in \mathcal{V}$ ,  $t \in (0, T]$  and

$$z \in L_2((0, T); \mathcal{V}) \cap C([0, T]; \mathfrak{H}) \cap H^1((0, T); \mathcal{V}').$$

If  $z = (w, \hat{v})$  is the unique solution to (2.18), (2.19) then

$$\hat{u}(t) = L^{-1}w(t), \quad t \in [0, T]$$

is a weak solution to (2.6), (2.7) and it is unique.

Under somewhat stronger hypotheses than those given in  $A_2$  and  $A_3$  above, the existence of strong solutions can be established. Indeed, if in addition to  $A_1$  and  $A_4$ , we assume

$A_2'$  The mapping  $t \rightarrow \sigma(t, \cdot)$  is an element in  $C^1([0, T]; H^1(0, \ell))$  for some  $T > 0$

$A_3'$  The mapping  $t \rightarrow f(t, \cdot)$  is an element in  $C^1([0, T]; L_2(0, \ell))$  and  $g, h \in C^1[0, T]$  (in fact, Holder continuity will suffice, see [29], [37])

then the family of operators  $\{U(t)\}_{t \in [0, T]}$  given by (2.15) generates a unique evolution system  $\{U(t, s) : 0 \leq s \leq t \leq T\}$  on  $\mathfrak{H}$  and  $z$  given by

$$(2.20) \quad z(t) = U(t, 0)z_0 + \int_0^t U(t, s)\mathcal{F}(s)ds, \quad 0 \leq t \leq T,$$

is the unique solution to the initial value problem (2.13), (2.14) and satisfies  $z(t) \in \mathcal{D}$ ,  $t \in (0, T]$

with  $z \in C([0, T]; \mathfrak{H}) \cap C^1((0, T]; \mathfrak{H})$ . Once again, with  $z = (w, \hat{v})$  now given by (2.20),

$\hat{u}(t) = L^{-1}w(t)$ ,  $t \in [0, T]$ , is a strong solution to (2.6), (2.7) and it is unique.

### 3. An Abstract Approximation Framework

We turn next to a discussion of a general approximation framework and convergence theory for the identification problem (ID) formulated above. In the following section we formulate a specific spline-based scheme to which the general theory developed here applies.

Fundamental to our approach is the construction of a sequence of finite dimensional (with regard to both the state dynamics and the admissible parameter set) approximating identification problems each of which, under appropriate hypotheses, can be shown to have a solution that in some sense (specifically, subsequential convergence) approximates a solution to the original infinite dimensional estimation problem (ID).

In the discussion to follow, we exhibit the explicit dependence on  $q = (q_1, q_2) \in \mathcal{Q}$  of the  $\mathcal{H}$  inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  and the bilinear form  $a(t)(\cdot, \cdot)$  given in (2.9) and (2.17) respectively by using the notation  $\langle \cdot, \cdot \rangle_q$  and  $a(t; q)(\cdot, \cdot)$ . For each  $N_1 = 1, 2, \dots$  and each  $N_2 = 1, 2, \dots$  let  $W^{N_1}$  and  $V^{N_2}$  be finite dimensional subspaces of  $L_2(0, \ell)$  and  $V$  respectively. If, for  $N = (N_1, N_2)$  we define  $\mathcal{U}^N = W^{N_1} \times V^{N_2}$ , then  $\mathcal{U}^N$  is a finite dimensional subspace of both  $\mathcal{H}$  and  $\mathcal{V}$ . Let  $\mathcal{P}^N: \mathcal{H} \rightarrow \mathcal{U}^N$  denote the projection map of  $\mathcal{H}$  onto  $\mathcal{U}^N$  given by

$$(3.1) \quad \mathcal{P}^N(w, \hat{v}) = (P_1^N w, P_2^N \hat{v})$$

where  $P_1^N$  is the orthogonal projection of  $L_2(0, \ell)$  onto  $W^{N_1}$  and  $P_2^N$  is the orthogonal projection of  $H$  onto  $V^{N_2}$ , both computed with respect to the standard (unweighted) inner products on the respective spaces  $L_2(0, \ell)$  and  $H$ .

The Galerkin equations in  $\mathcal{U}^N$  corresponding to the system (2.18), (2.19) and  $q \in \mathcal{Q}$  are

$$(3.2) \quad \langle z_1^N(t), v^N \rangle_q = a(t; q)(z^N(t), v^N) + \langle \mathcal{F}(t), v^N \rangle_q, \quad v^N \in \mathcal{U}^N, 0 \leq t \leq T$$

$$(3.3) \quad z^N(0) = \mathcal{P}^N z_0.$$

For each  $M_i \in Z_+ = \{1, 2, \dots\}$ ,  $i = 1, 2$ , let  $S_{M_1}^1$  and  $S_{M_2}^2$  be finite dimensional subspaces of  $C[0, \ell]$  and for  $M = (M_1, M_2)$  define  $Q_M \subset Q$  by  $Q_M = S_{M_1}^1 \times S_{M_2}^2$ . Let  $g_M^1$  and  $g_M^2$  denote mappings from  $C[0, \ell]$  onto  $S_{M_1}^1$  and  $S_{M_2}^2$  respectively and define  $g_M$ , a mapping from  $Q$  onto  $Q_M$ , by

$$g_M(q) = g_M((q_1, q_2)) = (g_M^1(q_1), g_M^2(q_2)), \quad q = (q_1, q_2) \in Q.$$

We define a sequence of approximating admissible parameter spaces  $\{Q_M\}$ ,  $M \in Z_+ \times Z_+$  by

$$(3.4) \quad Q_M = g_M(Q)$$

and formulate the sequence of approximating identification problems as follows:

(ID<sub>M</sub><sup>N</sup>) Given  $\zeta(t_i) \in Z$ ,  $i = 1, 2, \dots, v$ , find  $(q_M^N)^* \in Q_M$  which minimizes

$$J_M^N(q) = \sum_{i=1}^v |\Gamma(z^N(t_i; q)) - \zeta(t_i)|_Z^2$$

over  $Q_M$ , where  $z^N(\cdot; q)$  is the solution to the initial value problem (3.2), (3.3) in  $\mathcal{U}^N$ .

We choose bases  $\{\theta_i^N\}_{i=1}^{K_1^N}$ ,  $\{\chi_i^N\}_{i=1}^{K_2^N}$ ,  $\{\phi_M^i\}_{i=1}^{L_M^1}$  and  $\{\psi_M^j\}_{i=1}^{L_M^2}$  for the finite dimensional spaces  $W^{N_1}$ ,  $V^{N_2}$ ,  $S_{M_1}^1$  and  $S_{M_2}^2$  respectively. Then  $q_M^1 \in S_{M_1}^1$ ,  $q_M^2 \in S_{M_2}^2$  and the solution  $z^N(\cdot; q_M)$  to the initial value problem (3.2), (3.3) with  $q = q_M = (q_M^1, q_M^2)$  can be written as

$$q_M^1 = \sum_{i=1}^{L_M^1} \alpha_M^i \phi_M^i,$$

$$q_M^2 = \sum_{i=1}^{L_M^2} \alpha_M^{i+L_M^1} \psi_M^i$$

and

$$z^N(t; q_M) = \sum_{i=1}^{K_1^N} Z_i^N(t; \alpha_M) \theta_i^N + \sum_{i=1}^{K_2^N} Z_{i+K_1^N}^N(t; \alpha_M) \hat{\chi}_i^N, \quad t \in [0, T],$$

respectively. Moreover,  $Z^N(\cdot; \alpha_M)$  is the solution to the initial value problem in

$\mathbb{R}^{K_1^N + K_2^N}$  given by

$$(3.5) \quad \mathfrak{M}^N(\alpha_M) Z^N(t) = A^N(t; \alpha_M) Z^N(t) + F^N(t), \quad t \in (0, T]$$

$$(3.6) \quad Z^N(0) = Z_0^N.$$

Here the positive definite matrix  $\mathfrak{M}^N(\alpha_M)$  is of the form

$$\mathfrak{M}^N(\alpha_M) = \begin{bmatrix} \mathfrak{M}_1^N(\alpha_M) & 0 \\ 0 & \mathfrak{M}_2^N \end{bmatrix}$$

where  $\mathfrak{M}_1^N(\alpha_M)$  is a  $K_1^N$ -square matrix with components

$$[\mathfrak{M}_1^N(\alpha_M)]_{ij} = \sum_{k=1}^{L_M} \alpha_M^k \langle \phi_M^k \theta_i^N, \theta_j^N \rangle_0,$$

and  $\mathfrak{M}_2^N$  is  $K_2^N$ -square matrix with entries

$$[\mathfrak{M}_2^N]_{ij} = \langle \mathfrak{M}_0 \hat{\chi}_i^N, \hat{\chi}_j^N \rangle_H.$$

For each  $t \geq 0$  the matrix  $A^N(t; \alpha_M)$  is given by  $A^N(t; \alpha_M) = \bar{A}^N(\alpha_M) + B^N(t)$  with

$$\chi^N(\alpha_M) = \begin{bmatrix} 0 & E^N(\alpha_M) \\ -E^N(\alpha_M)^T & -C^N(\alpha_M) \end{bmatrix}$$

and

$$B^N(t) = \begin{bmatrix} 0 & 0 \\ D^N(t) & 0 \end{bmatrix}$$

where  $E^N(\alpha_M)$  is a  $K_1^N \times K_2^N$  matrix with components

$$[E^N(\alpha_M)]_{ij} = \sum_{k=1}^{L_M^1} \alpha_M^k \langle \phi_M^k \theta_i^N, D^2 \chi_j^N \rangle_0,$$

$C^N(\alpha_M)$  is a  $K_2^N$ -square matrix with components

$$[C^N(\alpha_M)]_{ij} = \sum_{k=1}^{L_M^2} \alpha_M^{k+L_M^1} \langle \psi_M^k D^2 \chi_i^N, D^2 \chi_j^N \rangle_0$$

and  $D^N(t)$  is a  $K_2^N \times K_1^N$  matrix with components

$$[D^N(t)]_{ij} = c\sigma(t, \ell) \int_0^\ell \theta_j^N(x) dx D\chi_i^N(\ell) - \langle \sigma(t, \cdot) \int_0^\cdot \theta_j^N(x) dx, D\chi_i^N \rangle_0.$$

The nonhomogeneous term  $F^N(t)$  is given by  $F^N(t) = (0, F_2^N(t))$  where  $F_2^N(t)$  is a  $K_2^N$  vector with entries

$$[F_2^N(t)]_i = \langle \mathcal{F}_0(t), \hat{\chi}_i^N \rangle_H.$$

The initial data is of the form

$$Z_0^N = (G^N)^{-1} (Z_{01}^N, Z_{02}^N)^T$$

where the  $K_1^N$  vector  $Z_{01}^N$  and the  $K_2^N$  vector  $Z_{02}^N$  are given component-wise by

$$[Z_{01}^N]_i = \langle D^2 \phi, \theta_i^N \rangle_0$$

and

$$[Z_{02}^N]_j = \langle \hat{\psi}, \hat{\chi}_j^N \rangle_H$$

respectively, and

$$G^N = \begin{bmatrix} G_1^N & 0 \\ 0 & G_2^N \end{bmatrix}$$

with  $G_1^N$  a  $K_1^N$ -square matrix defined by

$$[G_1^N]_{ij} = \langle \theta_i^N, \theta_j^N \rangle_0,$$

and  $G_2^N$  a  $K_2^N$ -square matrix with components

$$[G_2^N]_{ij} = \langle \hat{\chi}_i^N, \hat{\chi}_j^N \rangle_H.$$

It is now easily seen that the finite dimensional identification problem  $(ID_M^N)$  in fact involves simply the minimization of a least-squares performance index over a subset of

$\mathbb{R}^{L_M^1 + L_M^2}$ . Furthermore, the evaluation of the functional  $J^N$  requires only the solution

of the  $K_1^N + K_2^N$  dimensional, linear, non-autonomous ordinary differential equation (3.5)

with initial conditions specified in (3.6). If the existence of solutions to the finite dimensional optimization problems can be established, it is immediately clear that they can, in principle, be computed using standard techniques. Conditions which guarantee the existence of solutions to problem  $(ID_M^N)$  and the fact that they in some sense approximate solutions to the original infinite dimensional estimation problem (ID) are given in the following theorem.

**Theorem 3.1** Suppose

$H_1$  the mappings  $\mathcal{J}_M$  are continuous from  $Q$  into  $\mathcal{Q}$ ,

$H_2$  for each  $q \in Q$ ,  $\mathcal{J}_M(q) \rightarrow q$  as  $|M| \rightarrow \infty$  with the convergence being uniform in  $q$  for  $q \in Q$ ,

$H_3$  the spaces  $\mathcal{V}^N$  and projections  $\mathcal{P}^N$  are such that if  $\{q^N\}$  is a sequence in  $Q$  with

$q^N \rightarrow \bar{q} = (\bar{q}_1, \bar{q}_2) \in Q$  as  $|N| \rightarrow \infty$  then  $z^N(t; q^N) \rightarrow z(t; \bar{q})$  in  $L_2(0, \ell) \times H$  for each  $t \in [0, T]$  as  $|N| \rightarrow \infty$  where  $z^N(\cdot; q^N)$  is the solution to the initial value problem (3.2), (3.3) with  $q = q^N$  and  $z(\cdot; \bar{q})$  is the solution to the initial value problem (2.18), (2.19) corresponding to  $EI = \bar{q}_1$  and  $C_D I = \bar{q}_2$ .

Then, each of the problems  $(ID_M^N)$  has a solution  $(q_M^N)^*$ . Furthermore, the sequence  $\{(q_M^N)^*\}$  admits a  $\mathcal{Q}$ -convergent subsequence whose limit  $q^*$  is a point in  $Q$  and is a solution to problem (ID).

In the statement of the theorem, for an element  $K = (K_1, K_2) \in Z_+ \times Z_+$  we have adopted the notation  $|K| \rightarrow \infty$  to denote  $K_1, K_2 \rightarrow \infty$ . We remark that it is also true that the limit point of any  $\mathcal{Q}$ -convergent subsequence  $\{(q_M^k)^*\}$  of  $\{(q_M^N)^*\}$  with  $|M|, |N| \rightarrow \infty$  as  $j, k \rightarrow \infty$  is a solution to problem (ID) as well. Moreover, if problem (ID) has a unique solution,  $q^*$ , then the sequence  $\{(q_M^N)^*\}$  itself converges to  $q^*$ . It is also important to note that the hypotheses of the theorem do not require that  $Q_M \subset Q$ .

We have established results analogous to those given in Theorem 3.1 for inverse problems involving parabolic and hyperbolic systems (see, for example [12], [13], [16]) as well as for related methods for higher order equations for elastic structures (see [4], [5], [6]). For the flexible

structure problems treated here, the essential features of the argument remain, for the most part, unchanged. We therefore only briefly sketch them below.

Standard continuous dependence results for linear ordinary differential systems, the continuity assumptions on  $\mathcal{J}_M$  and  $\Gamma$  (and therefore on  $\mathcal{J}^N$  as well) and the fact that  $Q$  is a compact subset of  $\mathcal{Q}$  are sufficient to conclude that there exists a solution  $(q_M^N)^* \in Q_M$  to problem  $(ID_M^N)$ .

The definition of the space  $Q_M$  (see (3.4)) implies the existence of a  $\bar{q}_M^N \in Q$  for which  $(q_M^N)^* = \mathcal{J}_M(\bar{q}_M^N)$ . Since  $Q$  is compact, there exists a subsequence  $\{\bar{q}_{M^j}^{N^k}\}$  of  $\{\bar{q}_M^N\}$  with  $\bar{q}_{M^j}^{N^k} \rightarrow q^* \in Q$  as  $j, k \rightarrow \infty$ . The subsequence  $\{\bar{q}_{M^j}^{N^k}\}$  can always be chosen with  $|M^j|, |N^k| \rightarrow \infty$  as  $j, k \rightarrow \infty$ . It follows that

$$\mathcal{J}^{N^k}((q_{M^j}^{N^k})^*) \leq \mathcal{J}^{N^k}(q), \quad q \in Q_{M^j}$$

and consequently that

$$(3.7) \quad \mathcal{J}^{N^k}((q_{M^j}^{N^k})^*) \leq \mathcal{J}^{N^k}(\mathcal{J}_{M^j}(q)), \quad q \in Q.$$

Assumption  $H_2$  above and

$$|(q_{M^j}^{N^k})^* - q^*|_q \leq |\mathcal{J}_{M^j}(\bar{q}_{M^j}^{N^k}) - \bar{q}_{M^j}^{N^k}|_q + |\bar{q}_{M^j}^{N^k} - q^*|_q$$

imply  $(q_{M^j}^{N^k})^* \rightarrow q^*$  as  $j, k \rightarrow \infty$ . Taking the limit as  $j, k \rightarrow \infty$  in (3.7) above with an

application of assumption  $H_3$ , we find  $\mathcal{J}(q^*) \leq \mathcal{J}(q)$ ,  $q \in Q$ , and hence that  $q^*$  is a solution to problem (ID).

#### 4. A Scheme Using Polynomial Splines

In this section we outline a scheme which uses piecewise polynomial spline functions and show

that it satisfies the conditions and hypotheses of Theorem 3.1. We first treat the discretization of the admissible parameter set  $Q$ .

For each  $M = (M_1, M_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$  let  $\Delta_M^1$  and  $\Delta_M^2$  denote the uniform partitions of the interval  $[0, \ell]$  determined by the meshes  $\{0, \ell/M_1, 2\ell/M_1, \dots, \ell\}$  and  $\{0, \ell/M_2, 2\ell/M_2, \dots, \ell\}$  respectively. For  $m = 1, 2, \dots$  and  $\Delta$  a partition of  $[0, \ell]$  let  $\text{Sp}(m, \Delta)$  denote the usual spline space of functions in  $C^{2m-2}[0, \ell]$  which are polynomials of degree  $2m-1$  on each subinterval of  $\Delta$  (see [36]). We then define  $S_{M_i}^i = \text{Sp}(1, \Delta)$ ,  $i = 1, 2$ . In this case we have  $\dim S_{M_i}^i = L_M^i = M_i + 1$ ,  $i = 1, 2$ , with the usual "hat" functions forming a cardinal basis for each of the spaces  $S_{M_i}^i$ ,  $i = 1, 2$ . For  $i = 1, 2$ , let  $\mathcal{I}_M^i: C[0, \ell] \rightarrow S_{M_i}^i$  be the interpolation operator defined by

$$\left(\mathcal{I}_M^i \gamma\right)\left(\frac{j\ell}{M_i}\right) = \gamma\left(\frac{j\ell}{M_i}\right), \quad j = 0, 1, 2, \dots, M_i$$

for  $\gamma \in C[0, \ell]$ . The theory of interpolatory splines (see [33]) yields the continuous dependence result

$$\left|\mathcal{I}_M^i \gamma_1 - \mathcal{I}_M^i \gamma_2\right|_{\infty} \leq \left|\gamma_1 - \gamma_2\right|_{\infty}, \quad i = 1, 2$$

where  $\gamma_1, \gamma_2 \in C[0, \ell]$  and consequently that hypothesis  $H_1$  of Theorem 3.1 is satisfied. Also, the approximation result (see [36])

$$\left|\mathcal{I}_M^i \gamma - \gamma\right|_{\infty} \leq \omega(\gamma, 1/M_i)$$

where  $\omega(\gamma, \delta)$  is the usual modulus of continuity of  $\gamma \in C[0, \ell]$  with respect to  $\delta$ , together with the assumption that  $Q$  is a compact subset of  $\mathcal{Q} = C[0, \ell] \times C[0, \ell]$  and the Arzela-Ascoli theorem yield that hypothesis  $H_2$  is satisfied as well.

Next we define a state approximation and verify that hypothesis  $H_3$  holds. As above, given  $N = (N_1, N_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ , we define the uniform partitions  $\Delta_i^N$  of the interval  $[0, \ell]$  determined by the meshes  $\{0, \ell/N_i, 2\ell/N_i, \dots, \ell\}$ ,  $i = 1, 2$ . We may then choose either

$$W^{N_1} = \text{Sp}(1, \Delta_1^N)$$

or

$$W^{N_1} = \text{Sp}(2, \Delta_1^N).$$

In the first case, once again the "hat" functions may be chosen as a basis with

$\dim W^{N_1} = K_1^N = N_1 + 1$ . In the second, the standard cubic B-splines (see [33]),

$\{B_j^{N_1}\}_{j=-1}^{N_1+1}$ , corresponding to the partition  $\Delta_1^N$  form an appropriate finite element basis with

$\dim W^{N_1} = N_1 + 3$ . In either case, approximation results for interpolatory splines can be used to obtain

$$(4.1) \quad |P_1^N \theta - \theta|_0 \rightarrow 0 \text{ as } N_1 \rightarrow \infty$$

for  $\theta \in L_2(0, \ell)$ .

We set

$$V^{N_2} = \{(\chi(\ell), D\chi(\ell), \chi) \in H : \chi \in \text{Sp}(2, \Delta_2^N), \chi(0) = D\chi(0) = 0\}.$$

Then  $V^{N_2} \subset V$  and defining

$$\beta_i^{N_2} = B_0^{N_2} - 2B_1^{N_2} - 2B_{-1}^{N_2},$$

$$\beta_i^{N_2} = B_i^{N_2}, \quad i = 2, 3, \dots, N_2 + 1$$

and

$$\hat{\beta}_i^{N_2} = (\beta_i^{N_2}(\ell), D\beta_i^{N_2}(\ell), \beta_i^{N_2}), \quad i = 1, 2, \dots, N_2 + 1,$$

the collection  $\{\hat{\beta}_i^{N_2}\}_{i=1}^{N_2+1}$  forms a basis for  $V^{N_2}$  with  $\dim V^{N_2} = K_2^N = N_2 + 1$ . With  $V^{N_2}$

as defined above, it is not difficult to show (using arguments similar to those in [31])

$$(4.2) \quad |P_2^N(\eta, \xi, \chi) - (\eta, \xi, \chi)|_H \rightarrow 0 \text{ as } N_2 \rightarrow \infty$$

for  $(\eta, \xi, \chi) \in H$  and

$$(4.3) \quad |LP_2^N \hat{\chi} - L\hat{\chi}|_H \rightarrow 0 \quad \text{as } N_2 \rightarrow \infty$$

for  $\hat{\chi} \in V$ .

So as to avoid obscuring the essential features of our argument with technical details, we verify hypothesis  $H_3$  for the spline-based scheme described above in the case  $\sigma \equiv 0$ . The term which results from axial loading is a bounded perturbation and does not involve the unknown parameters. Showing that the desired convergence continues to hold in the presence of a non-zero axially directed acceleration requires only a routine extension of the proof which we give below (see [14]).

Suppose that  $\{q^N\}$  is a sequence in  $Q$  with  $q^N \rightarrow \bar{q} \in Q$  as  $|N| \rightarrow \infty$ . Let  $z^N = (w^N, \hat{v}^N)$  denote the solution to (3.2), (3.3) with  $q = q^N$  and let  $z = (w, \hat{v})$  denote the solution to (2.18), (2.19) corresponding to  $\bar{q}$ . We shall require the assumption that  $z$  is a strong solution.

In the estimates which follow, we simplify our notation by referring to the inner products (norms)  $\langle \cdot, \cdot \rangle_{q^N} (|\cdot|_{q^N})$  and  $\langle \cdot, \cdot \rangle_{\bar{q}} (|\cdot|_{\bar{q}})$  on  $\mathcal{H}$  by  $\langle \cdot, \cdot \rangle_N (|\cdot|_N)$  and  $\langle \cdot, \cdot \rangle (|\cdot|)$  respectively. Also note that with  $\sigma = 0$ , we have  $a(t; q)(\cdot, \cdot) = a(q)(\cdot, \cdot)$ .

Since

$$\|z^N - z\| \leq \|z^N - \mathcal{P}^N z\| + \|(\mathcal{P}^N - I)z\|$$

where  $\|\cdot\|$  denotes the usual (unweighted) product norm on  $\mathcal{H} = L_2(0, \ell) \times H$ , (3.1), (4.1) and (4.2) imply that we need only to consider the term  $\|z^N - \mathcal{P}^N z\|$ . Letting  $y^N(t) = z^N(t) - \mathcal{P}^N z(t)$ , using (2.18), (2.19), (3.2), (3.3) and the fact that  $V^N \subset V$  we find

$$(4.4) \quad \begin{aligned} \langle y_t^N, v^N \rangle_N &= \langle (z - \mathcal{P}^N z)_t, v^N \rangle_N + \langle z_t, v^N \rangle - \langle z_t, v^N \rangle_N \\ &+ a(q^N)(y^N, v^N) - a(q^N)(z - \mathcal{P}^N z, v^N) + a(q^N)(z, v^N) - a(\bar{q})(z, v^N) \\ &+ \langle \mathcal{F}, v^N \rangle_N - \langle \mathcal{F}, v^N \rangle \quad v^N \in \mathcal{V}^N, \quad 0 < t \leq T \end{aligned}$$

$$y^N(0) = 0.$$

Choosing  $v^N = y^N \in \mathcal{V}^N$ , we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} |y^N|_N^2 + \left| \sqrt{q_2^N} L(\hat{v}^N - P_2^N \hat{v}) \right|_0^2 \\
& = \langle (z - \mathcal{P}^N z)_t, y^N \rangle_N + \langle (\bar{q}_1 - q_1^N) w_t, w^N - P_1^N w \rangle_0 \\
& - \langle q_1^N L(\hat{v} - P_2^N \hat{v}), w^N - P_1^N w \rangle_0 + \langle q_1^N (w - P_1^N w), L(\hat{v}^N - P_2^N \hat{v}) \rangle_0 - \\
& \quad \langle q_2^N L(\hat{v} - P_2^N \hat{v}), L(\hat{v}^N - P_2^N \hat{v}) \rangle_0 + \langle (q_1^N - \bar{q}_1) L \hat{v}, w^N - P_1^N w \rangle_0 - \\
& \quad \langle (q_1^N - \bar{q}_1) w, L(\hat{v}^N - P_2^N \hat{v}) \rangle_0 - \langle (q_2^N - \bar{q}_2) L \hat{v}, L(\hat{v}^N - P_2^N \hat{v}) \rangle_0.
\end{aligned}$$

Recalling that  $Q$  is a compact subset of  $\mathcal{Q}$  and that for  $q = (q_1, q_2) \in Q$ ,  $EI = q_1$  and  $C_D I = q_2$  are assumed to satisfy assumption  $A_1$  of Section 2, we find

$$\begin{aligned}
& \frac{d}{dt} \|y^N\|^2 + |L(\hat{v}^N - P_2^N \hat{v})|_0^2 \leq \kappa_0 \|(I - \mathcal{P}^N) z_t\|^2 \\
& + \|y^N\|^2 + |\bar{q}_1 - q_1^N|_\infty^2 |w_t|_0^2 + |w^N - P_1^N w|_0^2 + \\
& |L(I - P_2^N) \hat{v}|_0^2 + |w^N - P_1^N w|_0^2 + \frac{1}{4\epsilon} |(I - P_1^N) w|_0^2 \\
& + \epsilon |L(\hat{v}^N - P_2^N \hat{v})|_0^2 + \frac{1}{4\epsilon} |L(I - P_2^N) \hat{v}|_0^2 + \\
& \epsilon |L(\hat{v}^N - P_2^N \hat{v})|_0^2 + |q_1^N - \bar{q}_1|_\infty^2 |L \hat{v}|_0^2 + \\
& |w^N - P_1^N w|_0^2 + \frac{1}{4\epsilon} |q_1^N - \bar{q}_1|_\infty^2 |w|_0^2 + \\
& \epsilon |L(\hat{v}^N - P_2^N \hat{v})|_0^2 + \frac{1}{4\epsilon} |q_2^N - \bar{q}_2|_\infty^2 |L \hat{v}|_0^2 \\
& + \epsilon |L(\hat{v}^N - P_2^N \hat{v})|_0^2 \}
\end{aligned}$$

where  $\kappa_0$  is a positive constant and  $\epsilon$  is an arbitrary positive constant. Gathering up like terms

and choosing  $\epsilon < \frac{1}{4}$  we obtain

$$\frac{d}{dt} \|y^N\|^2 \leq \kappa_1 \{ \|(I - \mathcal{P}^N)z_1\|^2 + |q_1^N - \bar{q}_1|_{\underline{}}^2 |w_1|_0^2 + |L(I - P_2^N)\hat{v}|_0^2 +$$

$$|(I - P_1^N)w|_0^2 + |q_1^N - \bar{q}_1|_{\underline{}}^2 |L\hat{v}|_0^2 + |q_1^N - \bar{q}_1|_{\underline{}}^2 |w|_0^2 + |q_2^N - \bar{q}_2|_{\underline{}}^2 |L\hat{v}|_0^2 \} + \kappa_2 \|y^N\|^2$$

where  $\kappa_1$  and  $\kappa_2$  are positive constants. Integrating both sides of the above inequality from 0 to  $t$  and recalling (4.4) we obtain

$$(4.5) \quad \|y^N(t)\|^2 \leq \delta + \kappa_2 \int_0^t \|y^N(s)\|^2 ds$$

where

$$\delta = \kappa_1 \int_0^T \{ \|(I - \mathcal{P}^N)z_s(s)\|^2 + |q_1^N - \bar{q}_1|_{\underline{}}^2 |w_s(s)|_0^2 + |L(I - P_2^N)\hat{v}(s)|_0^2 + |(I - P_1^N)w(s)|_0^2 +$$

$$|q_1^N - \bar{q}_1|_{\underline{}}^2 |L\hat{v}(s)|_0^2 + |q_1^N - \bar{q}_1|_{\underline{}}^2 |w(s)|_0^2 + |q_2^N - \bar{q}_2|_{\underline{}}^2 |L\hat{v}(s)|_0^2 \} ds .$$

Since  $q^N \rightarrow \bar{q}$  as  $|N| \rightarrow \infty$  and  $z = (w, \hat{v})^T$  was assumed to be a strong solution, (4.1), (4.2), (4.3) and (4.5) together with an application of the Gronwall inequality yield the desired result,  $\|y^N(t)\| \rightarrow 0$ .

A close inspection of the estimates above reveals that they depend, to a large extent, on the presence of the viscous damping term  $\langle C_D I L \hat{\chi}_1, L \hat{\chi}_2 \rangle_0$  in the bilinear form  $a(t)(\cdot, \cdot)$  given in (2.17). That is, we require that  $q_2 \geq \alpha > 0$  for some  $\alpha > 0$  for all  $q = (q_1, q_2) \in Q$ . In the absence of the Voigt-Kelvin damping we can still argue the convergence of  $z^N$  to  $z$ ; however, we must assume that  $Q$  is  $H^2$ -compact. If one is to enforce the compactness constraint on  $Q$  when solving the finite dimensional optimization problems (a desirable implementation feature in many cases - see [10],[11]), this stronger assumption becomes especially unappealing. On the other hand, by employing a somewhat different (but closely related) factorization of the stiffness operator  $\mathcal{K}_0$  than the one which was used here (one which is formally equivalent to rewriting the initial-boundary value problem (2.1) - (2.5) as a first order system in the states  $EI D^2u$  and  $u_t$  as opposed to  $D^2u$  and  $u_t$ ) hypothesis  $H_3$  of Theorem 3.1 can be verified for the resulting spline-based

scheme under the present assumptions on  $Q$ . Unfortunately this scheme is also difficult to implement and from a numerical standpoint, has not performed as satisfactorily as the one based on the formulation given in this paper. The present scheme performed well whether or not damping was present in the equation and hence the assumption that  $C_D I \geq \alpha > 0$  may be an artifact of our proof of convergence (see Example 5.3 below).

## 5. Numerical Findings

We present and discuss some of the results which we obtained from our numerical studies of the scheme that was described in Section 4. All codes were written in FORTRAN, and tested and run on the IBM 3081 at either Brown University or the University of Southern California. The same codes were, with only minor modification, run on the Cray 1-S at Boeing Computer Services in Seattle with support made available to us through the National Science Foundation's Super Computer Initiative program. Examples were benchmarked so that the potential benefits of vectorization to our research program could be accurately and effectively assessed. Our findings are described below. This information will become especially important to us when we begin to consider the extension of our general approach to inverse problems involving the vibration of two dimensional structures, such as flexible plates or platforms, or vibrations of structures in which nonlinearities play a significant role. The finite dimensional optimization problems  $(ID_M^N)$  were solved using the IMSL routine ZXSSQ, an implementation of the iterative Levenberg-Marquardt quasi-Newton algorithm. The finite dimensional initial value problems (3.5), (3.6) were solved in each iteration of the minimization procedure (for the evaluation of the least-squares performance index  $J$  and its gradient with respect to the parameters) using Gear's method for stiff systems (IMSL routine DGEAR).

Our codes were written to take full advantage of the banded structure of the generalized mass, stiffness and damping matrices afforded by the use of polynomial B-spline elements. All necessary inner products were computed using a two point composite Gauss-Legendre quadrature scheme.

All of the examples presented here involve fits based upon displacement measurements obtained

through simulation. "True" values (which, in the examples below will be denoted with an asterisk, for example  $EI^*$ ,  $C_D I^*$ , etc.) for the unknown parameters were chosen. The resulting initial boundary value problem (2.1) - (2.5) was then solved using an independent integration scheme. (We used a seven element, quintic spline based Galerkin method applied directly to the second order system (2.6), (2.7)). This procedure produced sufficient noise in the data so that the use of a random noise generator was not required.

In addition to the test example numerical studies we report on here we have successfully used methods similar to those developed above with experimental data. These results are presented in detail in [9].

In the examples which follow we took the axial loading to be induced by an acceleration of the base or root of the structure in the positive  $x$ -direction. In this case we have (see [34])

$$\sigma(t,x) = -a_0(t) \left( m + \int_0^x \rho(y) dy \right)$$

where  $m$  is the mass of the tip body,  $\rho$  is the linear mass density of the beam and  $a_0$  is the time dependent base acceleration.

In Examples 5.1 thru 5.4 below we took  $\ell = 1$ ,  $\rho(x) = 3 - x$  for  $0 \leq x \leq 1$ ,  $f(t,x) = e^x \sin 2\pi t$ ,  $g(t) = 2e^{-t}$ ,  $h(t) = e^{-2t}$ ,  $a_0(t) = 1$  for  $0 \leq t \leq 1.5$ ,  $a_0(t) = 0$  for  $t > 1.5$ ,  $m = 1.5$ ,  $c = .1$  and  $J = .52$  and considered the estimation of the flexural stiffness coefficient  $EI$  and/or the viscoelastic damping coefficient  $C_D I$  only. In Example 5.1, 5.2 and 5.4, the fits we describe are based upon observations at times  $t_i = .2i$ ,  $i = 1, 2, \dots, 5$  at locations  $x_j = .5, .75$  and  $1$ . In Example 5.3 observations at times  $t_i = .5i$ ,  $i = 1, 2, \dots, 10$  at locations  $x_j = .75$  and  $1$  were used. In all of the examples we discuss here the space  $W^{N_1}$  was generated by cubic splines (i.e. as  $Sp(2, \Delta_1^N)$ ) with  $N_1 = N_2 = N$ . This corresponds to the approximation of the first and second components of  $z$  with respectively  $N + 3$  and  $N + 1$  piecewise cubic  $C^2$  elements.

The compactness constraints on the spaces  $Q_M$  were not enforced when the finite dimensional optimization problems ( $ID_M^N$ ) were solved. When  $M_1$  and  $M_2$  became large, the inherent ill-posedness of the inverse problem became apparent as the performance of our schemes deteriorated. There is evidence strongly suggesting that this situation can be remedied either by

imposing the compactness constraints on the admissible parameter space and then solving the minimization problem using a constrained optimization procedure (see [10], [11]) or by regularizing the least squares performance index (see [24], [25]). We intend to direct our attention to these ideas in the near future.

### Example 5.1

In this example we consider the simultaneous estimation of a constant flexural stiffness coefficient,  $EI^* = .15$ , and a damping coefficient given by  $C_D I^*(x) = \gamma(1.5 - \tanh(3x - 1.5))$ ,  $x \in [0,1]$ , with  $\gamma = .01$ . With  $N = 4$ ,  $M_1 = 1$  and  $M_2 = 3$  and taking start up values (for the least squares minimization algorithm)  $EI^0 = .1$  and  $C_D I^0(x) = .015$ ,  $x \in [0,1]$  we obtained the results shown in Figure 5.1 below. This particular run required approximately 30 seconds of CPU time on the IBM 3081.

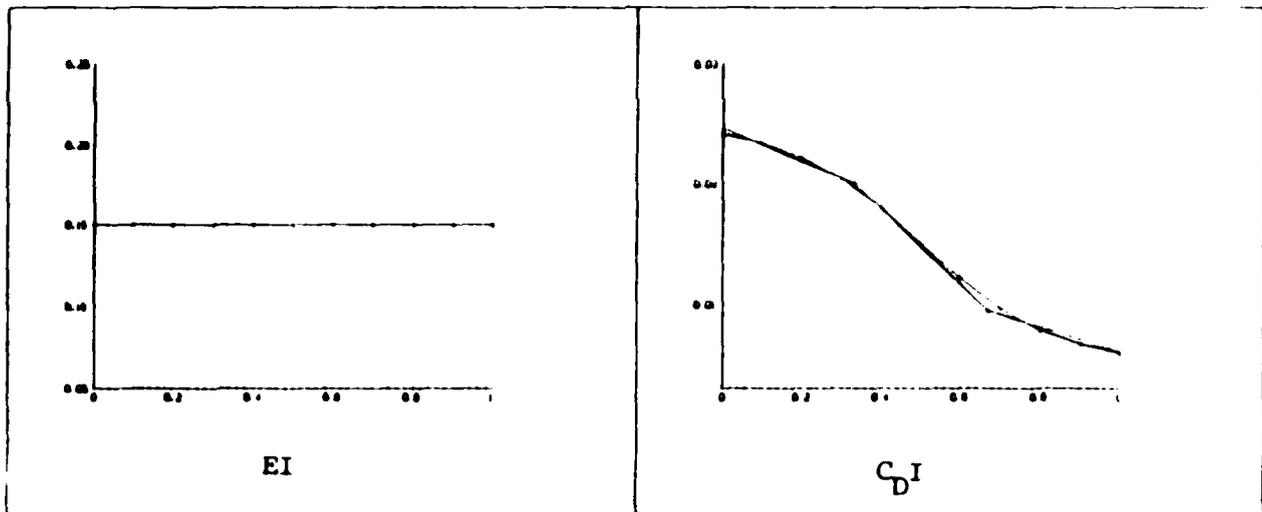


Figure 5.1

We observed that how well the scheme performed depended upon the magnitude of the scaling factor  $\gamma$ . As  $\gamma$  was decreased, so too did the "sensitivity" of the least squares performance index to the damping coefficient. Results similar to those shown in the figures above were obtained with  $\gamma = .005$ . With  $\gamma = .001$ , on the other hand, we were unable to simultaneously identify both of the unknown parameters. However, again with  $\gamma = .001$ , but this time fixing  $EI$  at the true value, we

were able to identify  $C_D I$  alone.

When we replaced the constant  $EI^*$  with the linear function  $EI^*(x) = 1 - \frac{1}{2}x$  and took  $\gamma = 1$ , the performance of the scheme, from a qualitative point of view, remained unchanged.

### Example 5.2

We again consider the simultaneous estimation of the stiffness and damping coefficients. We again set  $EI^* = .15$  but this time choose  $C_D I^*(x) = .01 (1.5 - \tanh(20x - 10))$ ,  $x \in [0,1]$ . The identification of this steeper hyperbolic tangent function has, in past test examples, proven to be a somewhat stiffer challenge for our methods (see [5],[6]). With  $N = 4$ ,  $M_1 = 1$ ,  $M_2 = 3$ ,  $EI^0 = .1$  and  $C_D I^0(x) = .015$  for  $0 \leq x \leq 1$ , we obtained the estimates which are plotted along with the true parameters in Figure 5.2.

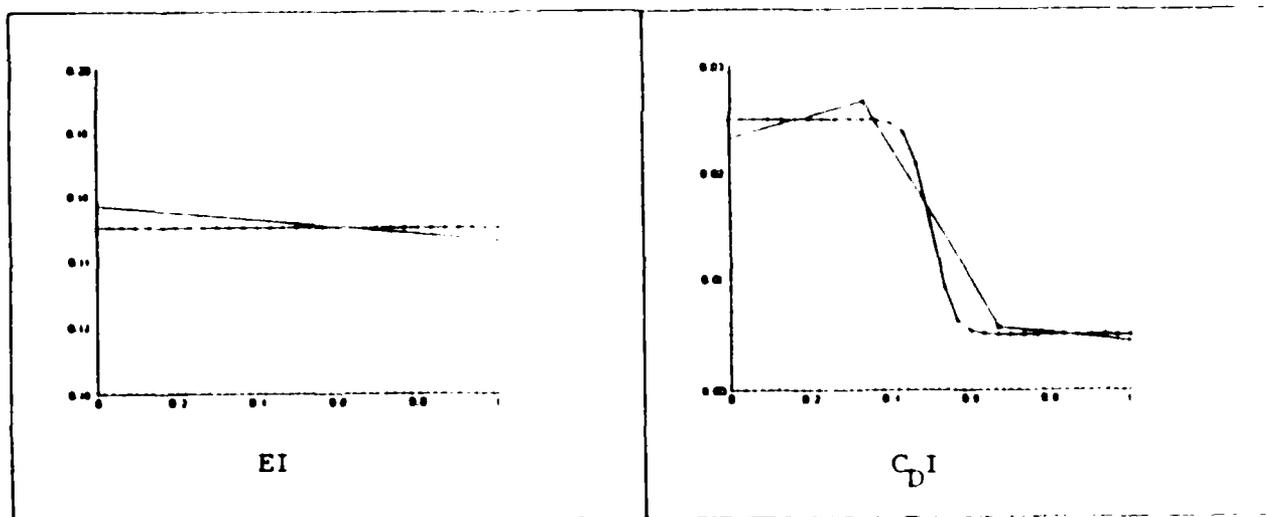


Figure 5.2

Also, although the theory was not explicitly treated here, we note that elements other than linear splines can be used to discretize the admissible parameter space. Our investigations have included numerical studies with 0-order splines (i.e. piecewise constant functions) and cubic spline elements. Using two linear elements to approximate  $EI$  (i.e.  $M_1 = 1$ ) and nine cubic elements to discretize  $C_D I$  we obtained the estimates shown in Figure 5.3. We have obtained an acceptable

estimate for  $C_D I$  with as few as six cubic elements.

In the tests reported on for the present example, residuals were typically in the range  $10^{-6}$  to  $10^{-8}$  with CPU times from 25 to 40 seconds.

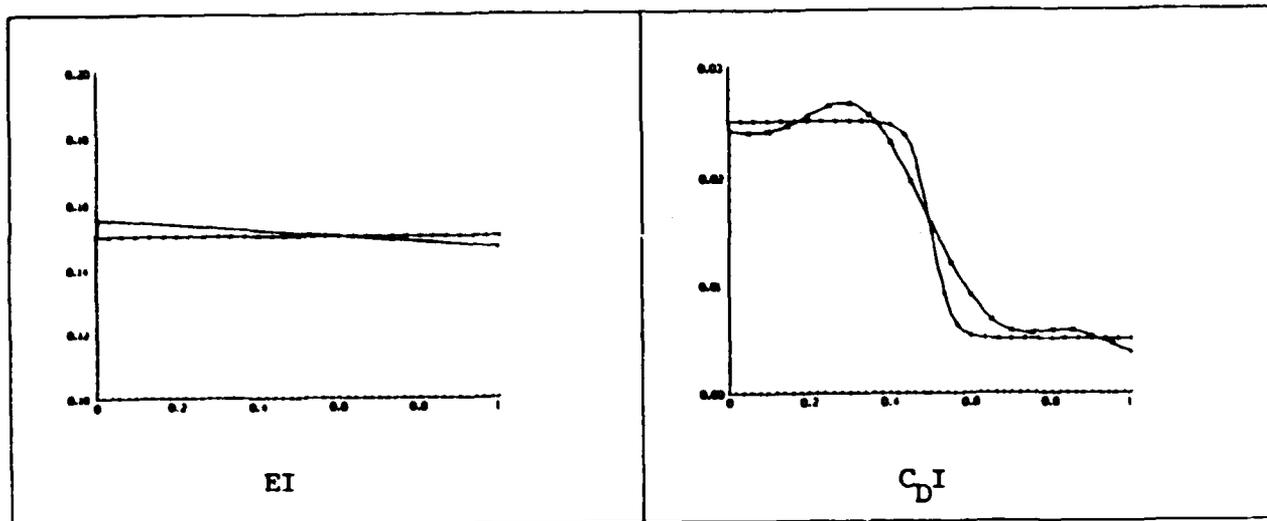


Figure 5.3

### Example 5.3

In this example we identify only the spatially varying flexural stiffness coefficient  $EI^*(x) = 1.5 - \tanh(3x - 1.5)$ ,  $x \in [0, 1]$ , in a model with no viscoelastic damping ( $C_D I = 0$ ). In Section 4 we remarked that our convergence arguments required either the presence of viscoelastic damping in the model or that the admissible parameter set  $Q$  be compact in the stronger  $H^2$  topology. The results shown in the figure below suggest that this is only an artifact of our proof and not a fundamental requirement for the convergence of our approximation (i.e. the absence of damping does not appear to effect the overall performance of our scheme).

Taking  $N = 4$  and  $M_1$  equal to 1 thru 8 we produced the results shown in the series of graphs in Figure 5.4. The initial estimate or start-up value for  $EI^*$  was taken to be the constant function  $EI^0(x) = 1$  for  $0 \leq x \leq 1$ .

Recalling our earlier remarks, the oscillations which appear in the graphs corresponding to

$M_1 = 6, 7$  and  $8$  due to the inherent ill posedness of the estimation problem are not unexpected. In fact, as  $M_1$  or  $M_2 \rightarrow \infty$ , the appearance of the undesirable oscillations in our final estimates occurred in virtually every test we ran. As we have noted earlier however, preliminary findings in related studies [10] and [11] regarding the enforcing of the compactness constraints and the subsequent use of constrained optimization techniques to solve the approximating finite dimensional identification problems suggest that this difficulty can be overcome. Our investigations in these directions are continuing.

In addition, the series of tests corresponding to the graphs in Figure 5.4 were benchmarked on the IBM 3081 and the Cray 1-S. The same estimates were obtained on both machines. However, we were able to achieve a speed-up factor  $f$ , of 7 - 10 on the vector machine. The CPU times are reported in Table 5.1. In comparing the CPU times on the 3081 for this example with the times reported for the previous examples it is important to note that the results here were based upon observations taken over the longer time interval,  $[0,5]$ , versus the interval  $[0,1]$  for examples 5.1 and 5.2.

#### Example 5.4

In Figure 5.5 below we plot the final estimates obtained when we attempted to use our scheme to simultaneously identify the spatially varying flexural stiffness coefficient  $EI^*(x) = .5 + 4x(1 - x)$ ,  $x \in [0,1]$ , and viscoelastic damping coefficient,  $C_D I^*(x) = .1 (1.5 - \tanh(3x - 1.5))$ ,  $x \in [0,1]$ . The start-up values for the iterative least squares minimization routine were taken to be the constant functions  $EI^0(x) = 1$  and  $C_D I^0(x) = .15$  for  $0 \leq x \leq 1$ . The graphs in the figure were obtained with  $N = 4$  and a linear spline discretization of the admissible parameter set  $Q$  with  $M_1 = M_2 = 3$ . In all of our tests with this example the minimum sum of the squares of the residuals was in the range  $10^{-7} - 10^{-8}$  with the optimization typically requiring 50 - 70 seconds of CPU time.

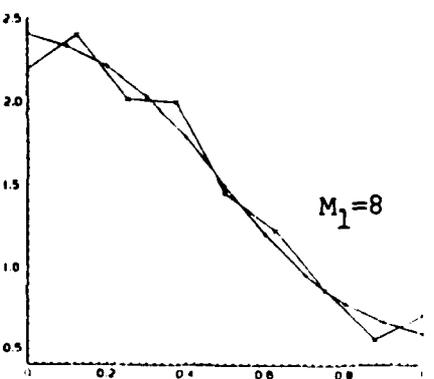
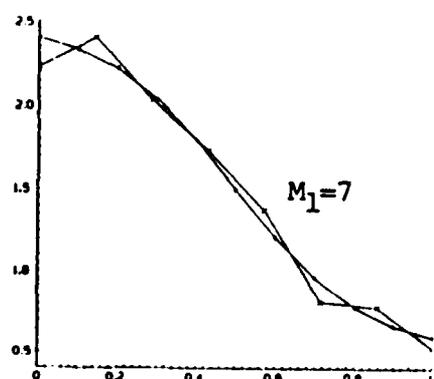
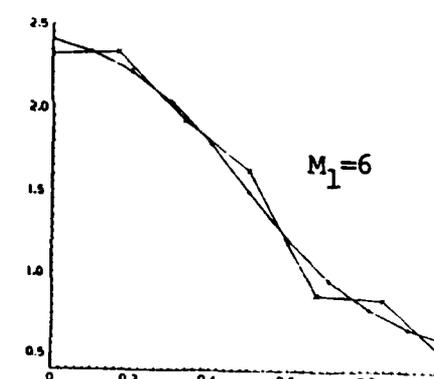
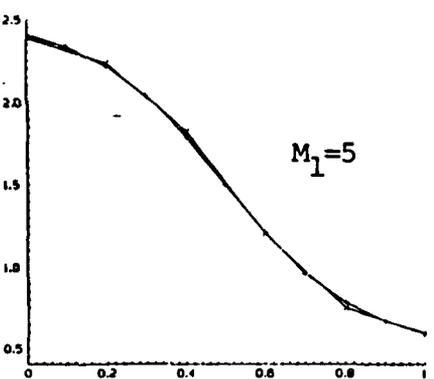
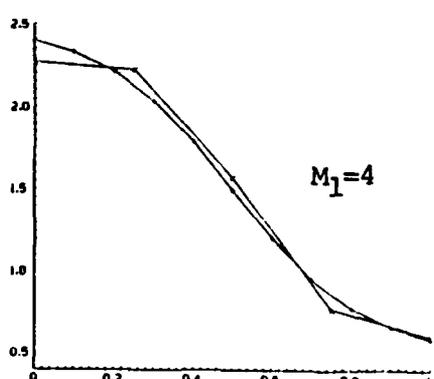
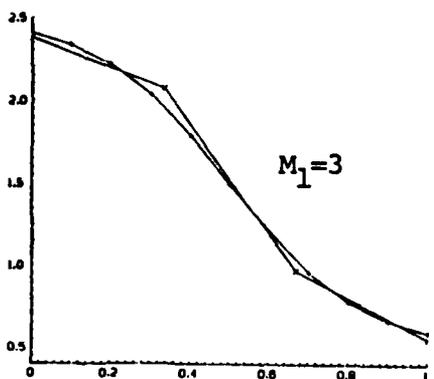
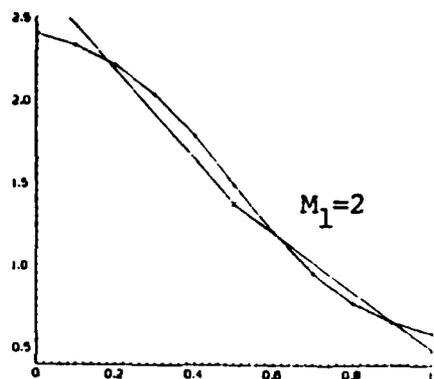
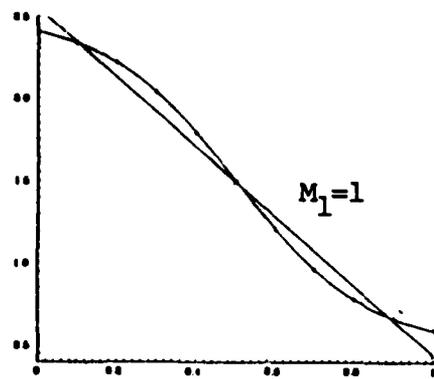


Figure 5.4

$M_1$	IBM 3081 (CPU sec.)	CRAY 1-S (CPU sec.)	$f$
1	110	12.5	8.8
2	164.1	20.8	7.9
3	207	23	9
4	249	32	7.8
5	245.5	36	6.8
6	404.4	41.6	9.7
7	346.5	45.6	7.6
8	437.9	49.1	8.9

Table 5.1

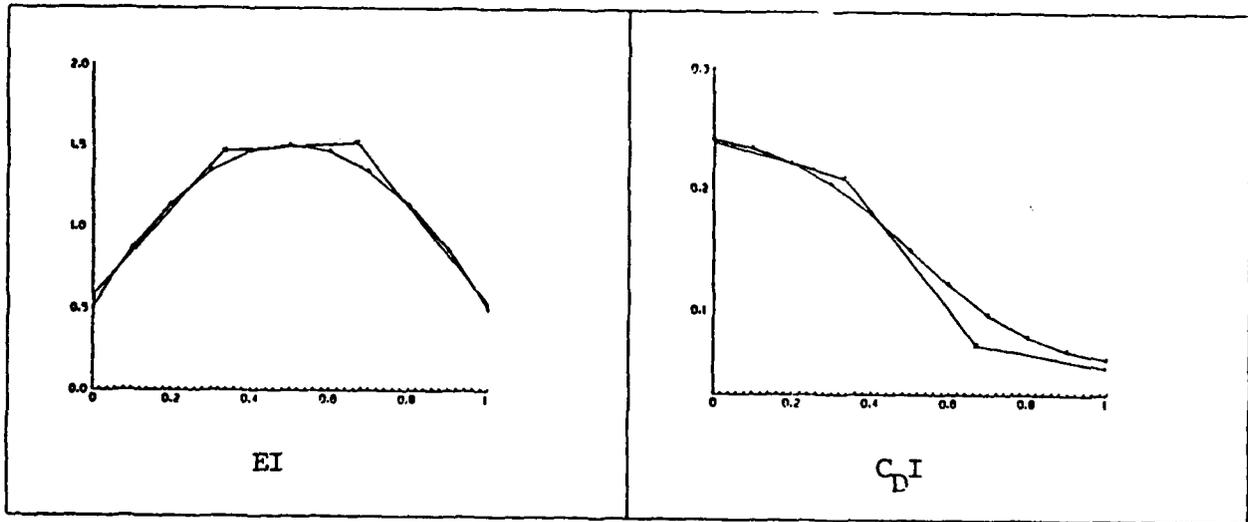


Figure 5.5

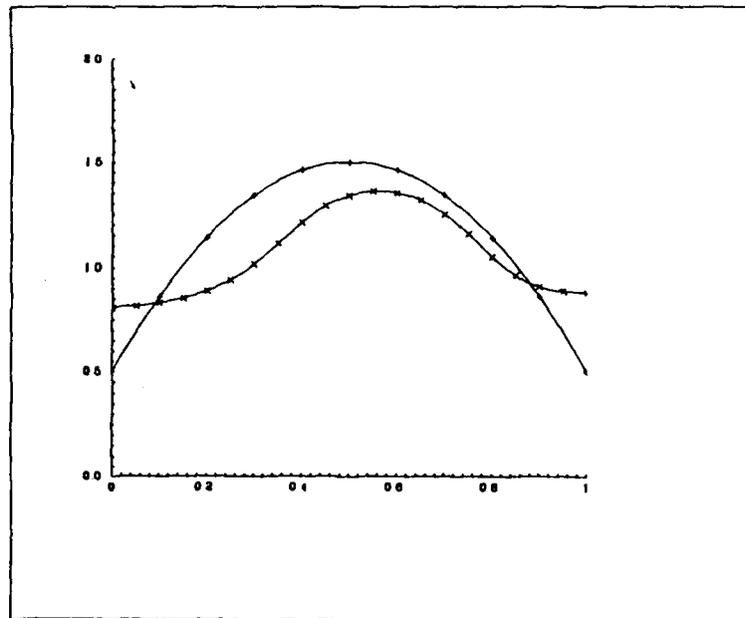


Figure 5.6

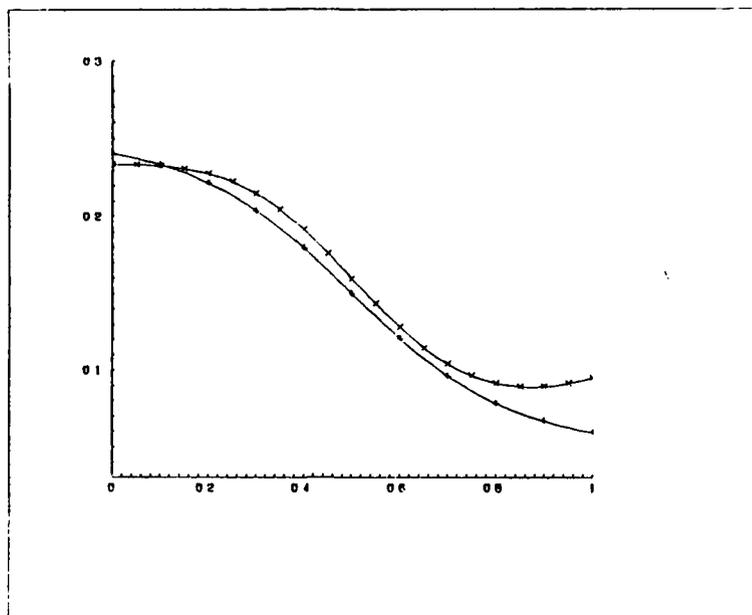


Figure 5.7

For this example we also tried a cubic-spline based discretization for  $Q$ . We considered all possible combinations, linear splines for  $EI^*$  together with cubic splines for  $C_D I^*$ , cubic splines for  $EI^*$  together with linear splines for  $C_D I^*$ , etc. Although small values for the sum of the squares of the residuals were obtained in each instance, our by far best approximation to the true parameters is the one shown in Figure 5.5 which corresponds to a linear spline based discretization for both components of the admissible parameter set.

Holding  $C_D I$  fixed at the true value and using cubic splines to identify  $EI$  and then holding  $EI$  fixed at the true value and using cubic splines to identify  $C_D I^*$  we were able to obtain the estimates plotted in Figures 5.6 and 5.7 respectively. The estimate for  $EI^*$  graphed in Figure 5.6 was obtained with 10 cubic elements while the estimate for  $C_D I^*$  in Figure 5.7 is a linear combination of 6 cubic elements. An inspection of these figures reveals that while the approximations obtained are at least marginally acceptable, it is also not surprising that our scheme had some difficulty when we attempted to identify both parameters simultaneously with a cubic spline-based discretization for either one or both components of  $Q$ .

For this example we also looked at the robustness of our iterative scheme with respect to the initial values chosen (i.e.,  $EI^0$  and  $C_D I^0$ ). In Figure 5.8 we plot those points in the  $C_D I^0 - EI^0$

plane which correspond to the start up values we tried. The point marked with " \* " corresponds to the startup values which produced the approximations shown in Figure 5.5. The points marked with " x " correspond to start-up values which led to essentially the same estimates as those shown in the figures. The points marked with an " ⊕ " correspond to start-up values for which the scheme did not converge. The region whose boundary is denoted with dashed lines corresponds to a "convergence envelope" for the vector valued function  $(C_D I^*, EI^*)$ . An analogous study was carried out for Example 5.2, for which similar robustness results were obtained.

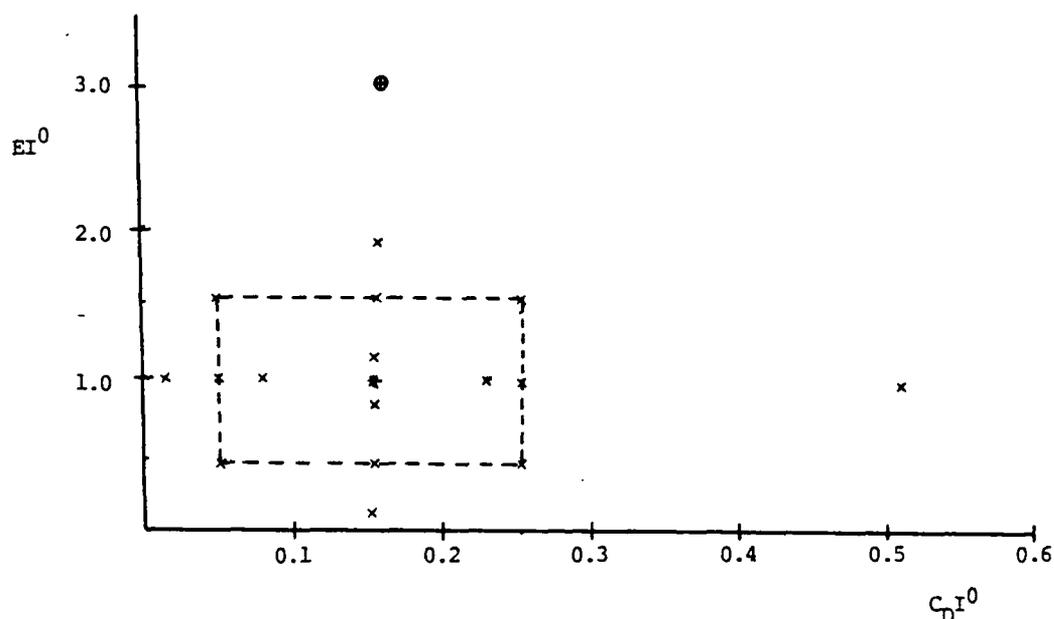


Figure 5.8

Finally we offer several summary comments on some of our other numerical findings. In virtually all examples we tried, we found that the estimates yielded by the scheme which we develop here based on state space coordinates  $(D^2u, u_t)$  and the ones yielded by the scheme based on a state space formulation in coordinates  $(u, u_t)$  described in [5] and [6] were comparable. Although in any given example one scheme or the other may produce a somewhat better approximation to the true parameters, we found it impossible to designate or identify a clear favorite

among the two methods.

We also ran a series of tests in which we varied the boundary conditions at the free end of the beam. That is, in addition to the tip body end condition we considered a beam which is clamped at one end and free at the other with either a point mass ( $c = J = 0$ ) or no mass ( $m = c = J = 0$ ) rigidly attached at the tip. We also studied the effect that the presence or absence of external forces and/or moments at the tip of the beam (i.e.  $g$  and  $h$ ) has on the performance of our scheme. Based upon these tests, we found it difficult to make definitive statements regarding "best" experimental procedures for identification of structural parameters with our schemes. However, we are able to offer several observations. For example, with a point mass at the tip, the schemes performance was enhanced when an external moment was applied at the tip (i.e.  $h \neq 0$ ). On the other hand, the presence of an externally applied force in the transverse direction (i.e.  $g \neq 0$ ) did not appear to have any effect at all. Also, with no mass at the tip, the scheme was most effective when  $g = h = 0$ . In general we found the scheme to be most dependable with tip body end conditions.

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