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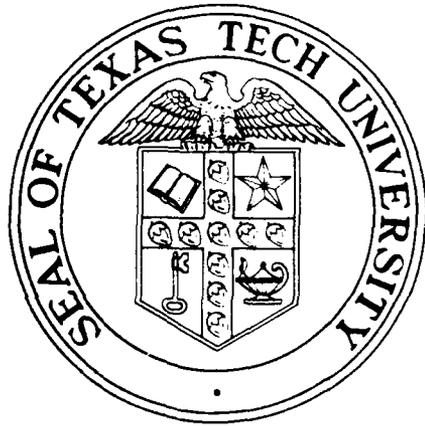
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Second Annual Report

ANALYTICAL AND EXPERIMENTAL RANDOM VIBRATION  
OF NONLINEAR AEROELASTIC STRUCTURES

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| <p>The work accomplished during the second year of this research project is a combination of analytical and experimental investigations:</p> <p>The analytical part deals with the nonlinear response of a three-degree-of-freedom aeroelastic structural model in the neighborhood of combination internal resonance condition. The Fokker-Planck equation approach is used to derive a general differential equation for the response statistical joint moments. This equation is found to constitute a set of infinite coupled first order differential equations. In view of the system complexity an attempt is made to close the infinite hierarchy by using a Gaussian scheme. This scheme leads to 27 differential equations in the first and second response moments. The equations are solved by using numerical integ-</p> <p style="text-align: right;">(continued)</p> |   |  |   |
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ration. The solution shows that the response coordinates are non-stationary random processes and the three normal modes are in complete nonlinear interaction. The interaction is found to be very strong at a region of internal detuning which is shifted from the exact internal resonance condition. This result is under further investigation by using a non-Gaussian closure scheme.

The experimental investigation is conducted out on a two degree-of-freedom model whose analytical solution was obtained during the first year of this project. When the first normal mode is externally excited by a band-limited random excitation, the system mean square response is found to be linearly proportional to the excitation spectral density level up to a certain level above which the two normal modes exhibit discontinuity governed mainly by the internal detuning parameter and the system damping ratios. The results are completely different when the second normal mode is excited. For small levels of excitation spectral density the response is dominated by the second normal mode. For higher levels of excitation spectral density the first normal mode attends and interacts nonlinearly with the second mode in a form of energy exchange.

New directions of this research project have been evolved during this year. These include the influence of random in aerodynamic forces on the nonlinear response of typical aeroelastic structures. Two aeroelastic models, are chosen to carry out this investigation.

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## ABSTRACT

The work accomplished during the second year of this research project is a combination of analytical and experimental investigations:

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The experimental investigation is conducted out on a two degree-of-freedom model whose analytical solution was obtained during the first year of this project. When the first normal mode is externally excited by a band limited random excitation, the system mean square response is found to be linearly proportional to the excitation spectral density level up to a certain level above which the two normal modes exhibit discontinuity governed mainly by the internal detuning parameter and the system damping ratios. The results are completely different when the second normal mode is excited. For small

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New directions of this research project have been evolved during this year. These include the influence of random in aerodynamic forces on the nonlinear response of typical aeroelastic structures. Two aeroelastic models, are chosen to carry out this investigation

## INTRODUCTION

This report presents the main results of the research project "Stochastic Nonlinear Flutter of Aeroelastic Structures" funded by the AFOSR under grant No. AFOSR-85-0008. The report covers only the work performed during the second year of this project. Furthermore, additional new research problems have been evolved during this year. These problems include the influence of the aerodynamic forces on the random response of nonlinear aeroelastic structures. A formal proposal will be submitted next June for new funds to support these new problems.

## SUMMARY OF MAIN RESULTS

### ANALYTICAL INVESTIGATION

The linear and autoparametric modal interactions in a three degree-of-freedom structure subjected to wide band random excitation are examined. For a structure with constant parameter properties the linear response is obtained in a closed form. When the structure stiffness matrix involves random fluctuations, the governing equations of motion, in terms of normal coordinates, are found to be coupled through parametric terms. The structure response is mainly governed by the condition of mean square stability. The boundary of stable-unstable response is obtained as a function of the internal detuning parameter. The results of the linear system with constant parameters are used as a reference to measure the deviation of the system response when the nonlinear inertia coupling is included. In the neighborhood of combination internal resonance the system random response is determined by using the Fokker-Planck equation approach together with the Gaussian closure scheme. This approach results in 27 coupled first order differential equations in the first and second response moments. These

equations are solved by numerical integration. The response is found to deviate significantly from the linear solution when the system internal detuning is close to the exact internal resonance. The autoparametric interaction is found to depend significantly on the system damping ratios and the nonlinear coupling parameter. In the vicinity of combination internal resonance, the second normal mode mean square exhibits an increase associated with a corresponding decrease in the first and third normal modes. The first normal mode shows a very small deviation from the linear solution which implies that the nonlinear interaction takes place between the second and third normal modes. This unexpected feature is currently under further investigation in parallel to a non-Gaussian closure analysis. The results of this work has been accepted for publication in the journal of Probabilistic Engineering Mechanics. A copy of the page proof of this paper is attached.

In order to enhance our understanding to the main results of structural dynamics with parameter uncertainties, the P.I. has conducted an extensive literature survey which has been accepted for publication in the ASME Applied Mechanics Reviews, A preprint of this paper is attached.

#### EXPERIMENTAL INVESTIGATION

A series of experimental tests is conducted on a two degree-of-freedom elastic structural model. The model is subjected to a band-limited random excitation with a central frequency very close to one of two normal mode frequencies. The band width is selected such that only the mode under consideration is excited. The model normal mode frequencies are adjusted to have the ratio 2 to 1. This ratio meets the condition of internal resonance of the analytical model. When the first normal mode is external excited the

system mean square response is found to be linearly proportional to the excitation spectral density up to a certain level above which the two normal modes exhibit discontinuity governed mainly by the internal detuning parameter and the system damping ratio. The results are completely different when the second normal mode is externally excited. For small levels of excitation spectral density the response is dominated by the second normal mode. For higher levels of excitation spectral density the first normal mode attends and interacts with the second normal mode in a form of energy exchange. A number of deviations from theoretical results are observed and discussed in the attached manuscript (28th SDM Conference, Paper No. 87-0079-CP) which will be presented at the AIAA/ASME/ASCE/AHS 28 Structures, Structural Dynamics and Materials Conference.

#### NEW RESEARCH DIRECTIONS

The influence of random aerodynamic forces, in subsonic and supersonic flow regimes, on the nonlinear response of typical aeroelastic structural models has been identified as a potential problem in aeroelastic flutter. Two models, which include a cantilever wing beam and a flat panel, will be adopted. The equations of motion of the first model have been derived by using the Lagrangian formulation. The aerodynamic forces are derived by using the modified strip theory which includes the effect of the span finite length. The linear part of the equations of motion has been considered to derive the flutter boundaries and to identify the system eigenvalues. This preliminary analysis is essential to identify the system parameters which satisfy the condition of internal resonance between bending and torsional motions. The work of this part is in progress.

PUBLICATIONS ACKNOWLEDGING THE AFOSR GRANT

Publications from the first year results:

1. Ibrahim, R. A. and Heo, H., Autoparametric Vibration of Coupled beams Under Random Support Motion," ASME Paper 85-DET-184

Also ASME Journal of Vibration, Acoustics, Stress, and Reliability in Design, Vol. 108(4), pp. 421-426, October 1986.

2. Ibrahim, R. A., "Numerical Solution of Nonlinear Problems in Stochastic Dynamics," Proc. of the International Conference on computational Mechanics, Edited by G. Yagawa and S. N. Atluri, Springer-Verlag, Tokyo, Vol. 2, Section XI, pp. 155-160, May 25-29, 1986.

3. Ibrahim, R. A. and Heo, H., "Stochastic Flutter of Nonlinear Aeroelastic Structures," Proc. of the AIAA/ASME/ASCE/AHS 27th Structures, Structural Dynamics and Materials Conference, Part 2, Paper No. 86-0962, San Antonio, Texas, May 19-21, 1986.

Also AIAA Journal Vol. 25(2), February 1987.

Publications from the Second year results:

4. Ibrahim, R. A. and Hedayati, Z., "Stochastic Modal Interaction in Non-linear Aeroelastic Structures," Proc. of the 5th International Modal Analysis Conference, Imperial College, London, April 6-9, 1987.

Also journal of Probabilistic Engineering Mechanics, in print, 1987.

5. Ibrahim, R. A., "Structural Dynamics with Parameter Uncertainties," ASME Applied Mechanics Reviews, Vol. 40(3), pp. 309-328, 1987.
6. Ibrahim, R. A. and Sullivan, D. G., "Experimental Investigation of Autoparametric Interaction of Structures under Random Excitation," Paper Accepted for Presentation at the AIAA/ASME/ASCE/AHS 28 Structures, Structural Dynamics and Materials Conference, Montrey, California, April 6-8, 1987.

## Professional Personnel

### A. Faculty

1. Raouf A. Ibrahim, Professor of Mechanical Engineering, Principal Investigator.

### B. Graduate Students

1. Zhian Hedayati, M.S. Thesis: Random Modal Interaction of a Nonlinear Aeroelastic Structure. Completed in August 1985.
2. Hun Heo, Ph.D. Dissertation: Nonlinear Stochastic Flutter of Aeroelastic Structural Systems. Completed in December 1985.
3. Douglas G. Sullivan, M.S. Thesis: Experimental Investigation of Random Response of Aeroelastic Structures. Completed in December 1986.
4. Welung Li, Ph.D. Dissertation: Random Vibration of Aeroelastic Structures with Third Order Internal Resonance. in progress.
5. Peter Orono, Ph.D. Dissertation: Effect of Random Aerodynamic Forces on Nonlinear Response of Aeroelastic Cantilever Wings. in progress.

# Stochastic modal interaction in linear and nonlinear aeroelastic structures

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The linear and autoparametric modal interactions in a three degree-of-freedom structure under wide band random excitation are examined. For a structure with constant parameters the linear response is obtained in a closed form. When the structure stiffness matrix involves random fluctuations, the governing equations of motion, in terms of the normal coordinates, are found to be coupled through parametric terms. The structural response is mainly governed by the condition of mean square stability. The boundary of stable-unstable responses is obtained as a function of the internal detuning parameter. The results of the linear system with constant parameters are used as a reference to measure the deviation of the system response when the nonlinear inertia coupling is included. In the neighbourhood of combination internal resonance the system random response is determined by using the Fokker Planck equation approach together with the Gaussian closure scheme. This approach results in 27 coupled first order differential equations in the first and second response moments. These equations are solved numerically. The response is found to deviate significantly from the linear solution when the system internal detuning is close to the exact internal resonance. The autoparametric interaction is found to depend significantly on the system damping ratios and a nonlinear coupling parameter. In the vicinity of combination internal resonance, the second normal mode mean square exhibits an increase associated with a corresponding decrease in the first and third normal modes.

## 1. INTRODUCTION

The modal analysis of aerolastic structures is usually carried out by using one of the available computer codes for eigenvalues and eigenvectors. These computer algorithms are useful in determining the structural dynamic behavior under various types of excitations. The first step usually involves the determination of eigenvalues and eigenvectors. With this information one can determine the linear response to deterministic or random excitations. For systems with constant parameters the mean square response to external white noise is linearly proportional to the excitation spectral density. If the excitation is acting parametrically to the system the equilibrium state could be stable or unstable in a stochastic sense. In certain situations the structure may not behave according to the linear theory of small oscillations and a number of complex response characteristics such as amplitude jump, internal resonance, saturation phenomenon, and chaotic motion<sup>1,2</sup> may be observed. These new characteristics owe their origin to the system inherent nonlinearities which should not be ignored in dynamic analysis.

In aircraft structures several types of nonlinearities have been reported. Breitbach<sup>3</sup> classified structural nonlinearities into distributed and concentrated. Distribution non-linearity is induced by elastic deformation in riveted, screwed and bolted connections

as well as within the structural components themselves. Concentrated nonlinearity acts locally lumped in control mechanisms or in the connecting parts between wing and external stores. This nonlinearity results from back-lash in the linkage elements of the control system, dry friction in control cable and push rod ducts, kinematic limitation of the control surface deflection, and application of spring tab system provided for relieving pilot operation. Breitbach<sup>4</sup> determined the flutter boundaries for three different configurations distinguished by different types of nonlinearities in the rudder and aileron control system of a sailplane. It was shown that the influence of hysteretic damping results in a considerable stabilizing effect and an increase in the flutter speed. However, this special type of non-linearity does not bring the structural response into a bounded limit cycle. Similar effects of nonlinearities due to friction and back-lash were considered by De Ferrari *et al.*<sup>5</sup>, Peloubet *et al.*<sup>6</sup>, Reed *et al.*<sup>7</sup> and Desmarais and Reed<sup>8</sup> examined the effects of control system nonlinearities, such as actuator force or deflection limits, on the performance of an active flutter suppression system. It was shown that a nonlinear system which is stable with respect to small disturbances may be unstable with respect to large ones. Another important feature was that a store on a pylon with low pitch stiffness can provide substantial increase in flutter speed and reduce the dependency of flutter on the mass and inertia of stores relative to that of stiff-mounted stores.

In structural dynamics, the nonlinearity may take one of three classes<sup>9,10</sup>: elastic, inertia, and damping nonlinearities. Elastic nonlinearity stems from nonlinear

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strain-displacement relations which are inevitable. Inertia nonlinearity is derived, in Lagrangian formulation, from the kinetic energy. In multi-degree-of-freedom systems the normal modes may involve nonlinear inertia coupling which may give rise to what are effectively parametric instability phenomena within the system. The parametric action is not due to the external loading, as in the case of parametric vibration, but to the motion of the system itself and, hence, is described as autoparametric<sup>11</sup>. The main feature of autoparametric coupling is that responses of one component of the structure give rise to loading of another component through time-independent coefficients in the corresponding equation of motion. The deterministic autoparametric interactions in two and three freedom systems were examined by Barr and Ashworth<sup>12</sup>, Haddow *et al.*<sup>13</sup>, Ibrahim *et al.*<sup>14</sup>, and Ibrahim and Woodal<sup>15</sup>. These studies have shown that the mode which is externally excited exhibits a saturation phenomenon in which energy is transferred to other modes involved in the nonlinear coupling. The stochastic aspects of parametric and autoparametric vibrations have recently been documented in a recent research monograph by Ibrahim<sup>16</sup>.

To the authors' knowledge the random response of systems with autoparametric coupling has been restricted to two-degree-of-freedom systems. This paper deals with the linear and nonlinear modal interactions of a three degree-of-freedom aeroelastic structure subjected to random excitation. The deterministic responses of this model under various internal resonance conditions  $\sum k_i \omega_i = 0$  (where  $k_i$  are integers and  $\omega_i$  are the system normal mode frequencies) have been determined by Ibrahim *et al.*<sup>14,15</sup>. The system involves quadratic nonlinear inertia which couples the system normal modes. It was shown that under principal internal resonance, the mode which is directly excited is suppressed and energy is transferred to the other mode. When the structure possess combination internal resonance of the summed type the normal mode amplitudes did not achieve a steady state and the response is characterized by energy exchange between the three modes.

The main objectives of this paper are to present the linear, parametric and autoparametric random responses of the same aeroelastic model considered in Refs 14 and 15. The mean square responses will be evaluated for a model with constant parameters and for a model with random variations in its stiffness matrix. The nonlinear random response of the system in the neighbourhood of combination internal resonance of the summed type will be determined by using the Fokker Planck equation approach together with a Gaussian closure scheme. The effects of the system nonlinearity and damping coefficients on the mean square responses will be examined.

## II. BASIC MODEL AND EQUATIONS OF MOTION

Fig. 1 shows a schematic diagram of an analytical model of an aircraft subjected to random excitation  $F(t)$ . The fuselage is represented by the main mass  $m_3$ , linear spring  $K_3$ , and dashpot  $C_3$ . Attached to the main mass on each side are two coupled beams with tip masses  $m_1$  and  $m_2$ , stiffnesses  $K_1$  and  $K_2$ , and lengths  $l_1$  and  $l_2$ . In the analysis

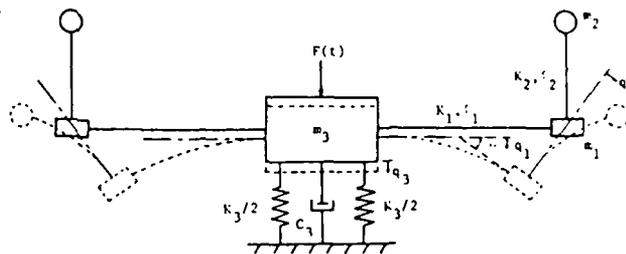


Fig. 1. Schematic diagram of an aeroelastic Structure and Coordinate System

of the shown system only the symmetric motions of the two sides of the model are considered. Under random excitation the system response will be described by the generalized coordinates  $q_1$ ,  $q_2$ , and  $q_3$  as shown in the figure. The equations of motion are derived by applying Lagrange's equation

$$\frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}} \right\} - \left\{ \frac{\partial L}{\partial q} \right\} = 2 \{ Q_{NC} \} \quad (1)$$

where  $L = T - V$ .

The kinetic energy  $T$  is given by the expression<sup>17</sup>

$$\begin{aligned} T = & \frac{1}{2} \left\{ m_1 + m_2 \left[ 1 + \left( \frac{3l_2}{2l_1} \right)^2 \right] \right\} \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 \\ & + \frac{1}{2} (m_1 + m_2 + m_3) \dot{q}_3^2 + \frac{3m_2 l_2}{2l_1} \dot{q}_1 \dot{q}_2 \\ & + (m_1 + m_2) \dot{q}_1 \dot{q}_3 + \frac{9m_2 l_2}{20l_1^2} (q_1 \dot{q}_1^2 + 5q_1 \dot{q}_1 \dot{q}_3) \\ & + \frac{3m_2}{2l_1} \left( \frac{q_1 \dot{q}_1 \dot{q}_2}{5} + q_1 \dot{q}_2 \dot{q}_3 + \dot{q}_1 q_2 \dot{q}_3 + \dot{q}_1^2 q_2 \right) \\ & + \frac{6m_2}{5l_2} (q_2 \dot{q}_2 \dot{q}_3 + \dot{q}_1 q_2 \dot{q}_2) \end{aligned} \quad (2)$$

where a dot denotes differentiation with respect to time  $t$ . Neglecting the gravitational effects, the potential energy  $V$  is given by

$$V = 1/2 (k_1 q_1^2 + k_2 q_2^2 + k_3 q_3^2) \quad (3)$$

Substituting for  $T$  and  $V$  in equation (1), and considering  $F(t)$  as the only nonconservative force (damping forces will be introduced later) results in the equations of motion in terms of the nondimensional coordinates  $\bar{q}_i$ ,

$$\begin{aligned} \omega_3^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & 0 \\ m_{13} & 0 & m_{33} \end{bmatrix} \begin{bmatrix} \bar{q}_1'' \\ \bar{q}_2'' \\ \bar{q}_3'' \end{bmatrix} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \end{bmatrix} \\ = \frac{1}{q_3} \begin{bmatrix} 0 \\ 0 \\ F(\tau/\omega_3) \end{bmatrix} - \frac{m_2 q_3 \omega_3^2}{l_1} \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{bmatrix} \end{aligned} \quad (4)$$

where

$$\bar{q}_i = q_i / q_3, \quad \tau = \omega_3 t$$

$q_3$  is taken as the root-mean-square of the main mass when all other parts are locked under forced excitation.

$\omega_3$  is taken as the third eigenvalue of the system, and

$$m_{11} = m_1 + m_2[1 + 2.25(l_2/l_1)^2]$$

$$m_{22} = m_2$$

$$m_{33} = m_1 + m_2 + m_3$$

$$m_{12} = 1.5m_2(l_2/l_1)$$

$$m_{13} = m_1 + m_2$$

$$\begin{aligned} \ddot{\psi}_1 = & m_2[0.45(l_2/l_1)^2(2q_1\ddot{q}_1 + \dot{q}_1^2 + 5q_1\ddot{q}_3) \\ & + (1.5/l_1)(0.2q_1\ddot{q}_2 + q_2\ddot{q}_3 + 2q_2\dot{q}_1 + 2\dot{q}_1q_2) \\ & + (1.2/l_2)(q_2\ddot{q}_2 + \dot{q}_2^2) \end{aligned} \quad (5)$$

where a prime denotes differentiation with respect to the dimensionless time  $\tau$ .

### III. EIGENVALUES OF THE SYSTEM

The system eigenvalues are determined from the conservative linear part of the equations of motion

$$[m]\{q\} + [k]\{q\} = \{0\} \quad (6)$$

The characteristic equation of (6) is

$$\text{Det}[[k] - \omega^2[m]] = 0 \quad (7)$$

where  $\omega$  is the eigenvalue of the mode in question.

Expanding the determinant gives the cubic equation

$$\begin{aligned} & \left(-1 + \frac{m_{12}^2}{m_{11}m_{22}} + \frac{m_{13}^2}{m_{11}m_{33}}\right)\left(\frac{\omega}{\omega_{33}}\right)^6 + \left[\left(\frac{\omega_{11}}{\omega_{33}}\right)^2\right. \\ & \quad \left. + \left(\frac{\omega_{22}}{\omega_{33}}\right)^2\left(1 - \frac{m_{13}^2}{m_{11}m_{33}}\right) + \left(1 - \frac{m_{12}^2}{m_{11}m_{22}}\right)\right]\left(\frac{\omega}{\omega_{33}}\right)^4 \\ & \quad - \left[\left(\frac{\omega_{11}}{\omega_{33}}\right)^2\left(\frac{\omega_{22}}{\omega_{33}}\right)^2 + \left(\frac{\omega_{11}}{\omega_{22}}\right)^2 + \left(\frac{\omega_{22}}{\omega_{33}}\right)^2\right]\left(\frac{\omega}{\omega_{33}}\right)^2 \\ & \quad \left. + \left(\frac{\omega_{11}}{\omega_{33}}\right)^2\left(\frac{\omega_{22}}{\omega_{33}}\right)^2 = 0 \right. \end{aligned} \quad (8)$$

where the frequency parameters  $\omega_n = K_n/m_n$  are the natural frequencies of the individual components of the structure. The IMSL (International Mathematical and Statistical Library) Subroutine ZPOLR (Zeros of a Polynomial with Real Coefficients) is used to find the roots of equation (8) numerically. Fig. 2 shows a sample of the dependence of the natural frequency ratio  $r = \omega_3/(\omega_1 + \omega_2)$  on the ratios  $\omega_{11}/\omega_{33}$  and  $\omega_{22}/\omega_{33}$  for beams length ratio  $l_2/l_1 = 0.25$ , and mass ratios  $m_2/m_1 = 0.5$ , and  $m_3/m_1 = 5.0$ . Other sets of curves for different system parameters are obtained and reported in Ref. 17. The importance of these curves is to define the critical points where the structure possesses internal combination resonance  $r = 1.0$ . It is seen that the most critical region is located for the curves of  $\omega_{22}/\omega_{33} = 1$  and 2. For the analysis hereafter the following parameters will be used:  $l_2/l_1 = 0.25$ ,  $\omega_{22}/\omega_{33} = 1.4$ .

### IV. TRANSFORMATION INTO NORMAL COORDINATES

Equations (4) include linear and nonlinear dynamic couplings. The linear coupling is eliminated by transforming equations (4) into normalized coordinates

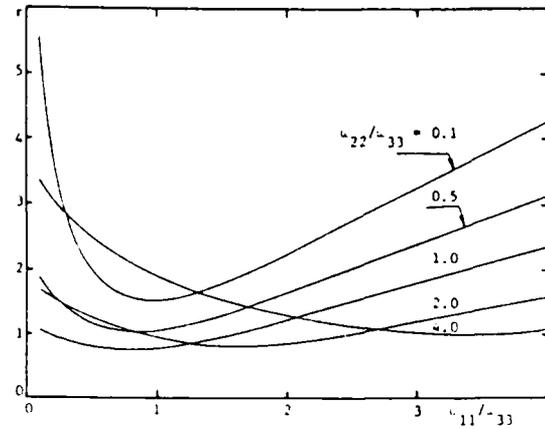


Fig. 2. Dependence of frequency ratio on system parameters for  $l_2/l_1 = 0.25$ ,  $m_2/m_1 = 0.5$ ,  $m_3/m_1 = 5$

$Y_n$  by using the transformation

$$\{q\} = [R]\{Y\} \quad (9)$$

where  $[R]$  is the modal matrix consisting of the normalized eigenvectors,

$$[R] = \begin{bmatrix} 1 & 1 & 1 \\ n_1 & n_2 & n_3 \\ \bar{n}_1 & \bar{n}_2 & \bar{n}_3 \end{bmatrix} \quad (10)$$

the elements of matrix (10) are determined by using the decomposition method<sup>18</sup> and are listed in Ref. 17.

Rewriting equations (4) in the matrix form and using transformation (9) gives

$$[m][R]\{Y''\} + [k][R]\{Y\} = \{F\} - \{\bar{\psi}\} \quad (11)$$

Premultiplying equation (11) by the transpose of the modal matrix results in diagonalizing the mass and stiffness matrices. The resulting equations involve nonlinear coupling and have the form

$$\begin{aligned} & m_1\omega_3^2 \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & 0 & M_{33} \end{bmatrix} \begin{Bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{Bmatrix} + k_1 \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \\ & = \frac{1}{q_3} \begin{Bmatrix} \bar{n}_1 F(\tau, \omega_3) \\ \bar{n}_2 F(\tau, \omega_3) \\ \bar{n}_3 F(\tau, \omega_3) \end{Bmatrix} - \frac{m_2 q_3 \omega_3^2}{l_1} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} \end{aligned} \quad (12)$$

where

$$\begin{aligned} M_{ii} = & 1 + \gamma(1 + 2.25\beta^2 + 3\beta n_i + 2\bar{n}_i + n_i^2 + \bar{n}_i^2) \\ & + [1 + m_{3-1}m]n_i^2 + 2\bar{n}_i \\ k_{ii} = & 1 + (k_2/k_1)n_i^2 + (k_3/k_1)\bar{n}_i^2 \\ \psi_i = & y_1(L_{i11}x_1 + L_{i21}x_2 + L_{i31}x_3) \\ & + y_2^2(L_{i12}x_1 + L_{i22}x_2 + L_{i32}x_3) \\ & + y_3^2(L_{i13}x_1 + L_{i23}x_2 + L_{i33}x_3 + L_{i43}x_4) \\ & + M_{i11}y_1^2 + M_{i22}y_2^2 + M_{i33}y_3^2 \\ & + M_{i12}y_1y_2 + M_{i13}y_1y_3 + M_{i23}y_2y_3 \end{aligned}$$

$$\alpha = m_2/m_1, \quad \beta = l_1/l_1$$

$$\begin{aligned} L_{ijk} &= 0.9\beta + 2.25\beta\bar{n}_k + 0.3n_k + 1.5n_j\bar{n}_k + 3n_j \\ &\quad + (1.2/\beta)n_j n_k + n_i[0.3 + 1.5\bar{n}_i + (1.2/\beta)n_j(1 + \bar{n}_k)] \\ &\quad + \bar{n}_i[2.25\beta + 1.5(n_j + n_k) + (1.2/\beta)n_j n_k] \\ M_{ijk} &= 0.9\beta + 3(n_j + n_k) + (2.4/\beta)n_j n_k \\ &\quad - 2.4n_i + \bar{n}_i[4.5\beta + 3(n_j + n_k) + (2.4/\beta)n_j n_k] \\ &\quad (i \neq k) \\ M_{iii} &= 0.45\beta + 3n_i + (1.2/\beta)n_i^2 \\ &\quad - 1.2n_i + \bar{n}_i[2.25\beta + 3n_i + (1.2/\beta)n_i^2] \\ &\quad (j = k = i) \end{aligned} \quad (13)$$

## V. DYNAMIC MOMENT EQUATIONS

The response coordinates can be approximated as a Markov vector if the random excitation is approximated as a zero mean physical white noise  $W(\tau)$  having the autocorrelation function

$$R_w(\Delta\tau) = E[W(\tau)W(\tau + \Delta\tau)] = 2D\delta(\Delta\tau) \quad (14)$$

where  $2D$  is the spectral density intensity and  $\delta(\cdot)$  is the Dirac delta function. This modelling is justified as long as the relevant Wong-Zakai<sup>16</sup> correction term is introduced. The non-linear functions  $\psi_i$  contain acceleration terms coupled with displacement coordinates such as  $Y_i''Y_j$ . These terms are removed from equations (12) by successive elimination by using MACSYMA software. Equations (12) take the new form

$$Y_i'' + 2\zeta_{i3}r_{i3}Y_i' + r_{i3}^2Y_i = f_iW(\tau) + \varepsilon g_i(Y', Y'') \quad (15)$$

where linear viscous damping terms have been introduced to account for energy dissipation, and

$$\begin{aligned} \omega_i^2 &= (k_{ii}/M_{ii})(k_1/m_1), \quad r_{i3} = \omega_i/\omega_3, \\ f_i &= \bar{n}_i/M_{ii}, \quad \varepsilon = q_3^2/l_1 \end{aligned}$$

$$W(\tau) = \frac{1}{q_3\omega_3^2 m_1} F(\tau/\omega_3)$$

Introducing the transformation into the Markov state vector  $X$

$$\{Y_1, Y_1', Y_2, Y_2', Y_3, Y_3'\} = \{X_1, X_2, \dots, X_6\} \quad (16)$$

equations (15) may be written in the standard form of Stratonovich differential equations

$$dX_i = F_i(X, \tau) d\tau + \sum_{j=1}^6 G_{ij}(X, \tau) dB_j(\tau) \quad (17)$$

where the white noise  $W(\tau)$  has been replaced by the formal derivative of the Brownian motion process  $B(\tau)$ , i.e.,

$$W(\tau) = \sigma dB(\tau) d\tau, \quad \sigma^2 = 2D$$

Alternatively, equations (17) may in turn be transformed into the Ito type equation

$$\begin{aligned} dX_i &= \left[ F_i(X, \tau) + \frac{1}{2} \sum_{k=1}^6 \sum_{j=1}^6 G_{ik}(X, \tau) \frac{\partial G_{kj}(X, \tau)}{\partial X_k} \right] d\tau \\ &\quad + \sum_{j=1}^6 G_{ij}(X, \tau) dB_j(\tau) \end{aligned} \quad (18)$$

where the double summation expression is called the Wong-Zakai correction term<sup>16</sup>.

The system stochastic Ito equations are

$$\begin{aligned} dX_1 &= X_2 d\tau, \quad dX_3 = X_4 d\tau, \quad dX_5 = X_6 d\tau \\ dX_2 &= - \left\{ 2\zeta_{13}r_{13}X_2 + r_{13}^2X_1 + \frac{\varepsilon\alpha}{M_{11}} [(-2\zeta_{13}r_{13}X_2 - r_{13}^2X_1) \right. \\ &\quad \times (L_{111}X_1 + L_{121}X_3 + L_{131}X_5) \\ &\quad - (2\zeta_{23}r_{23}X_4 + r_{23}^2X_3) \\ &\quad \times (L_{112}X_1 + L_{122}X_3 + L_{132}X_5) - (+2\zeta_{33}X_6 + X_5) \\ &\quad \times (L_{113}X_1 + L_{123}X_3 + L_{133}X_5) + M_{111}X_2^2 \\ &\quad + M_{122}X_4^2 + M_{133}X_6^2 \\ &\quad \left. + M_{112}X_2X_4 + M_{113}X_2X_6 + M_{123}X_4X_6 \right\} d\tau \\ &\quad + \left\{ f_1 - \frac{\varepsilon\alpha}{M_{11}} [f_1(L_{111}X_1 + L_{121}X_3 + L_{131}X_5) \right. \\ &\quad + f_2(L_{112}X_1 + L_{122}X_3 + L_{132}X_5) \\ &\quad + f_3(L_{113}X_1 + L_{123}X_3 + L_{133}X_5)] \\ &\quad + \frac{\varepsilon^2\alpha^2}{M_{11}M_{22}} [f_1(L_{211}X_1 + L_{221}X_3 + L_{231}X_5) \\ &\quad + f_2(L_{212}X_1 + L_{222}X_3 + L_{232}X_5) \\ &\quad + f_3(L_{213}X_1 + L_{223}X_3 \\ &\quad + L_{233}X_5)](L_{112}X_1 + L_{122}X_3 + L_{132}X_5) \\ &\quad + \frac{\varepsilon^2\alpha^2}{M_{11}M_{33}} [f_1(L_{311}X_1 + L_{321}X_3 + L_{331}X_5) \\ &\quad + f_2(L_{312}X_1 + L_{322}X_3 \\ &\quad + L_{322}X_5) + f_3(L_{313}X_1 + L_{323}X_3 \\ &\quad + L_{333}X_5)] + L_{113}X_1 \\ &\quad \left. + L_{123}X_3 + L_{133}X_5 \right\} dB \\ dX_4 &= - \left\{ 2\zeta_{23}r_{23}X_4 + r_{23}^2X_3 + \frac{\varepsilon\alpha}{M_{22}} [(-2\zeta_{23}r_{23}X_4 - r_{23}^2X_3) \right. \\ &\quad \times (L_{211}X_1 + L_{122}X_3 + L_{231}X_5) \\ &\quad - (2\zeta_{23}r_{23}X_4 + r_{23}^2X_3) \\ &\quad \times (L_{212}X_1 + L_{222}X_3 + L_{232}X_5) \\ &\quad - (2\zeta_{33}X_6 + X_5) \\ &\quad \times (L_{213}X_1 + L_{223}X_3 + L_{233}X_5) \\ &\quad + M_{211}X_2^2 + M_{222}X_4^2 + M_{233}X_6^2 \\ &\quad \left. + M_{212}X_2X_4 + M_{213}X_2X_6 + M_{223}X_4X_6 \right\} d\tau \\ &\quad + \left\{ f_2 - \frac{\varepsilon\alpha}{M_{22}} [f_1(L_{211}X_1 + L_{221}X_3 + L_{231}X_5) \right. \\ &\quad + f_2(L_{212}X_1 + L_{222}X_3 + L_{232}X_5) \\ &\quad + f_3(L_{213}X_1 + L_{223}X_3 \\ &\quad + L_{233}X_5)] + \frac{\varepsilon^2\alpha^2}{M_{22}M_{11}} \end{aligned}$$

$$\begin{aligned} & \times [f_1(L_{111}X_1 + L_{121}X_3 + L_{131}X_5) \\ & + f_2(L_{112}X_1 + L_{122}X_3 + L_{132}X_5) \\ & + f_3(L_{113}X_1 + L_{123}X_3 + L_{133}X_5)] \\ & \times (L_{211}X_1 + L_{221}X_3 + L_{231}X_5) + \frac{\varepsilon^2 \alpha^2}{M_{22}M_{33}} \end{aligned}$$

$$\begin{aligned} & \times [f_1(L_{311}X_1 + L_{321}X_3 + L_{331}X_5) \\ & + f_2(L_{312}X_1 + L_{322}X_3 + L_{332}X_5) \\ & + f_3(L_{313}X_1 + L_{323}X_3 + L_{333}X_5)] \\ & \times (L_{213}X_1 + L_{223}X_3 + L_{233}X_5) \} dB \end{aligned}$$

$$\begin{aligned} dX_6 = & - \left\{ 2\zeta_3 X_6 + X_5 + \frac{\varepsilon x}{M_{33}} [( - 2\zeta_1 r_{13} X_2 - r_{13}^2 X_1) \right. \\ & \times (L_{311}X_1 + L_{321}X_3 + L_{331}X_5) \\ & - (2\zeta_2 r_{23} X_4 + r_{23}^2 X_3) \\ & \times (L_{312}X_1 + L_{322}X_3 + L_{332}X_5) \\ & - (2\zeta_3 X_6 + X_5)(L_{313}X_1 \\ & \times L_{323}X_3 + L_{333}X_5) \\ & + M_{311}X_2^2 + M_{322}X_4^2 + M_{333}X_6^2 \\ & \left. + M_{312}X_2X_4 + M_{313}X_2X_6 + M_{323}X_4X_6 \right\} d\tau \\ & + f_3 - \frac{\varepsilon x}{M_{33}} [f_1(L_{311}X_1 + L_{321}X_3 + L_{331}X_5) \\ & + f_2(L_{312}X_1 + L_{322}X_3 + L_{332}X_5) \\ & + f_3(L_{313}X_1 + L_{323}X_3 + L_{333}X_5)] \\ & + \frac{\varepsilon^2 \alpha^2}{M_{33}M_{11}} [f_1(L_{111}X_1 + L_{121}X_3 + L_{131}X_5) \\ & + f_2(L_{112}X_1 + L_{122}X_3 + L_{132}X_5) \\ & + f_3(L_{113}X_1 + L_{123}X_3 + L_{133}X_5)] \\ & \times (L_{311}X_1 + L_{321}X_3 + L_{331}X_5) \\ & + \frac{\varepsilon^2 \alpha^2}{M_{33}M_{22}} [f_1(L_{211}X_1 + L_{221}X_3 + L_{231}X_5) \\ & + f_2(L_{212}X_1 + L_{222}X_3 + L_{232}X_5) \\ & + f_3(L_{213}X_1 + L_{223}X_3 + L_{233}X_5)] \\ & \times (L_{312}X_1 + L_{322}X_3 + L_{332}X_5) \} dB \quad (19) \end{aligned}$$

The evolution of the response probability density function is described by the Fokker-Planck equation

$$\begin{aligned} \frac{\partial p(\mathbf{X}, \tau)}{\partial \tau} = & - \sum_{i=1}^6 \frac{\partial}{\partial X_i} [a_i(\mathbf{X}, \tau)p(\mathbf{X}, \tau)] \\ & + \frac{1}{2} \sum_{i=1}^6 \sum_{j=1}^6 \frac{\partial^2}{\partial X_i \partial X_j} [b_{ij}(\mathbf{X}, \tau)p(\mathbf{X}, \tau)] \quad (20) \end{aligned}$$

where  $p(\mathbf{X}, \tau)$  is the response joint probability density function, and  $a_i(\mathbf{X}, \tau)$  and  $b_{ij}(\mathbf{X}, \tau)$  are the first and second

incremental moments evaluated as follows

$$\begin{aligned} a_i(\mathbf{X}, \tau) &= \lim_{\Delta \tau \rightarrow 0} \frac{1}{\Delta \tau} {}^c E[X_i(\tau + \Delta \tau) - X_i(\tau)] \\ b_{ij}(\mathbf{X}, \tau) &= \lim_{\Delta \tau \rightarrow 0} \frac{1}{\Delta \tau} E[\{X_i(\tau + \Delta \tau) - X_i(\tau)\} \\ & \times \{X_j(\tau + \Delta \tau) - X_j(\tau)\}] \quad (21) \end{aligned}$$

The coefficients  $a_i$  and  $b_{ij}$  are evaluated for the present system with the aid of MACSYMA program. It is not possible to solve the resulting Fokker planck equation even for the stationary case. Instead, one may generate a general first order differential equation describing the evolution of response moments of any order. This equation is obtained by multiplying both sides of the system Fokker Planck equation by the scalar function  $\Phi(\mathbf{X})$

$$\Phi(\mathbf{X}) = X_1^{k_1} X_2^{k_2} X_3^{k_3} X_4^{k_4} X_5^{k_5} X_6^{k_6} \quad (22)$$

and integrating by parts over the entire state space  $-\infty < X < \infty$ . The following boundary conditions are used

$$p(\mathbf{X} \rightarrow -\infty) = p(\mathbf{X} \rightarrow \infty) = 0 \quad (23)$$

Due to space limitation the system moment equation will not be listed in this paper. The reader may refer to Ref. 17 for more details. However, the general form of the resulting differential equation is

$$m_N' = F_N(m_1, m_2, \dots, m_N, m_{N+1}) \quad (24)$$

where  $N = \sum_{i=1}^6 k_i$ .

In deriving the system moment differential equation the following notation is adopted

$$m_{k_1, k_2, \dots, k_6} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p(\mathbf{X}, \tau) \Phi(\mathbf{X}) dX_1 dX_2 \dots dX_6 \quad (25)$$

It is found that the differential equation of order  $N$  contains moment terms of order  $N$  and  $N+1$ . The source of this infinite hierarchy is the system nonlinear functions  $\psi_i$  in equations (12). If these nonlinear functions are dropped the system becomes linear and the response moment equations are consistent. In the present study the following three cases will be examined:

- (i) Linear response of constant coefficients structure.
- (ii) Linear response of the structure with random stiffness.
- (iii) Response of the structure with autoparametric interaction involving the internal combination internal resonance  $\omega_3 \cong \omega_1 + \omega_2$ .

## VI. STRUCTURE WITH CONSTANT PARAMETERS

The equations of motion for this case are obtained from equations (11) by excluding the nonlinear functions  $\psi_i$ . The resulting equations of motion are

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} Y_1'' \\ Y_2'' \\ Y_3'' \end{Bmatrix} + \begin{bmatrix} 2\zeta_1 r_{13} & 0 & 0 \\ 2\zeta_2 r_{23} & 0 & 0 \\ 0 & 2\zeta_3 & 0 \end{bmatrix} \begin{Bmatrix} Y_1' \\ Y_2' \\ Y_3' \end{Bmatrix} \\ + \begin{bmatrix} r_{13}^2 & 0 & 0 \\ 0 & r_{23}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{Bmatrix} = W(\tau) \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \quad (26) \end{aligned}$$

For this linear case the response moment differential equations are consistent. The mean squares of the stationary response is obtained in the closed form

$$\begin{aligned} E[Y_1^2] &= Df_1^2 / (2\zeta_1 r_{13}^3), & E[Y_1'^2] &= Df_1^2 / (2\zeta_1 r_{13}), \\ E[Y_2^2] &= Df_2^2 / (2\zeta_2 r_{23}^3), & E[Y_2'^2] &= Df_2^2 / (2\zeta_2 r_{23}), \\ E[Y_3^2] &= Df_3^2 / (2\zeta_3), & E[Y_3'^2] &= Df_3^2 / (2\zeta_3) \end{aligned}$$

Before presenting the linear response graphically, it would be useful to recall that the generalized coordinates  $q_i$  were nondimensionalized with respect to the root-mean square of the main mass response when the coupled system was locked under forced excitation. The value of  $q_3^0$  can be estimated from the single degree of freedom equation of motion

$$(m_1 + m_2 + m_3)\ddot{q}_3^0 + C_3\dot{q}_3^0 + k_3q_3^0 = F(t) \quad (28)$$

which has the stationary response

$$E[q_3^{02}] = E[\dot{q}_3^{02}] = D/2\zeta_3 \quad (29)$$

and therefore

$$q_3^0 = \sqrt{D/2\zeta_3} \quad (30)$$

The excitation parameter level  $D/2\zeta_3$  is chosen so that  $q_3^0$  is chosen so that  $q_3^0$  is unity and as a result any deviation from unity gives a measure of the dynamic interaction (linear or nonlinear) with other modes. For the analysis hereafter the excitation level will be chosen such that

$$D/2\zeta_3 = 1 \quad (31)$$

In this case the mean square response (27) is reduced to the simple form

$$\begin{aligned} E[Y_1^2] &= \frac{\zeta_1 f_1^2}{\zeta_3 r_{13}^3}, & E[Y_1'^2] &= \frac{\zeta_1 f_1^2}{\zeta_3 r_{13}}, \\ E[Y_2^2] &= \frac{\zeta_2 f_2^2}{\zeta_3 r_{23}^3}, & E[Y_2'^2] &= \frac{\zeta_2 f_2^2}{\zeta_3 r_{23}}, \\ E[Y_3^2] &= E[Y_3'^2] = f_3^2 \end{aligned} \quad (32)$$

The linear response for both normalized and generalized coordinates is determined for various damping ratios. Figs 3 and 4 show the mean square responses as a function of the frequency ratio  $r$  for two sets of damping ratios. It is seen that both the first and second normal mode mean square responses decrease faster than the

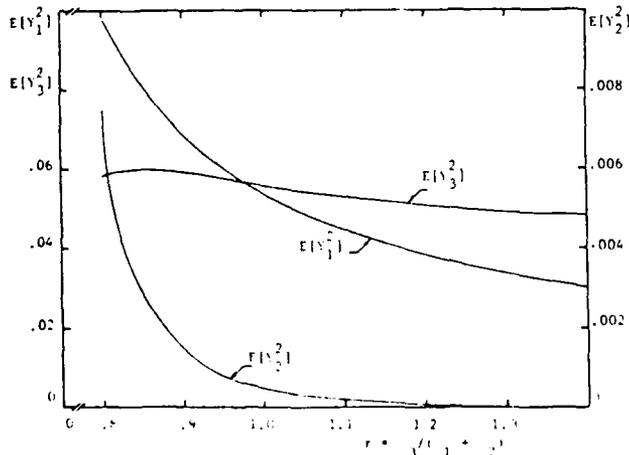


Fig. 3. Mean square response of normal modes for  $\zeta_1 = \zeta_2 = 0.005$ ,  $\zeta_3 = 0.01$

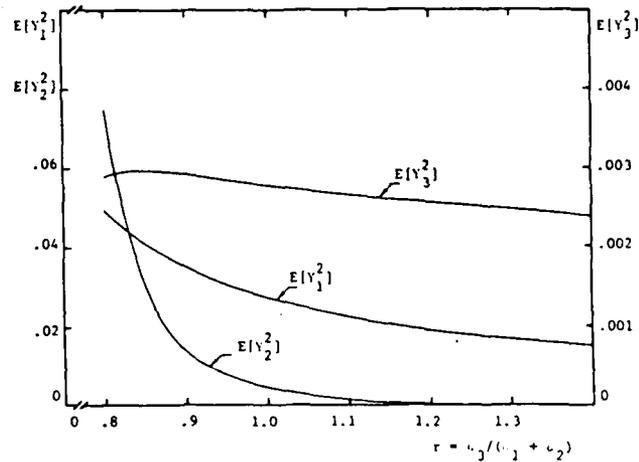


Fig. 4. Mean square response of normal modes for  $\zeta_1 = \zeta_2 = 0.01$ ,  $\zeta_3 = 0.01$

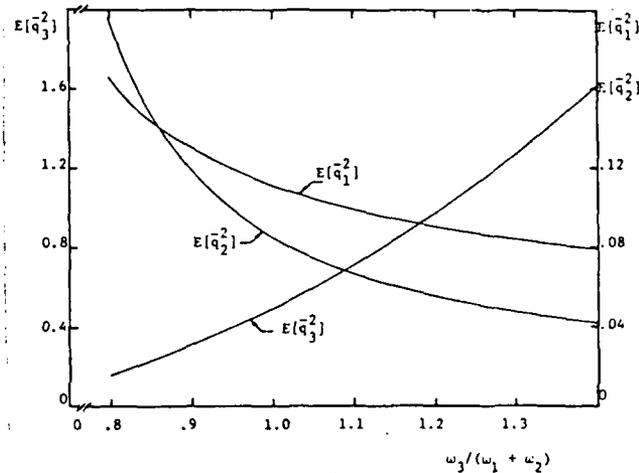


Fig. 5. Mean square response of generalized coordinates for  $\zeta_1 = \zeta_2 = 0.005$ ,  $\zeta_3 = 0.01$

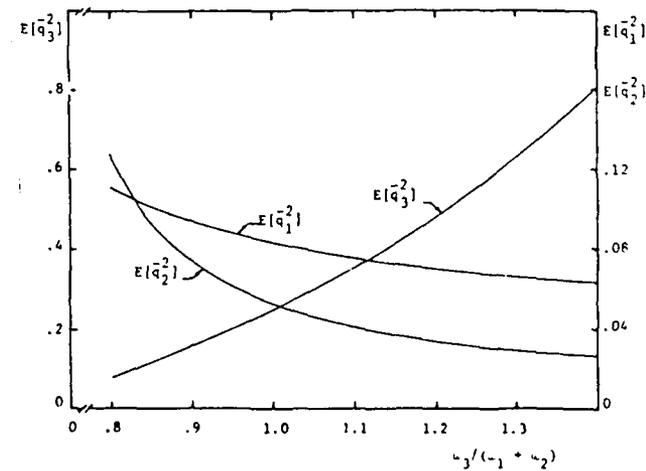


Fig. 6. Mean square response of generalized coordinates for  $\zeta_1 = \zeta_2 = 0.01$ ,  $\zeta_3 = 0.01$

third mode as the frequency ratio increases. In terms of generalized coordinates, Figs 5 and 6 shows that the mean square displacement increases while the two beam displacements decrease with the frequency ratio. The two sets of figures show the well known control damping effect on the mean square responses.

## VII. STRUCTURE WITH RANDOM STIFFNESS

The equations of motion of this case are obtained by including a random component to each stiffness in the original linear equations of motion.

The equations of motion take the

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & 0 \\ m_{13} & 0 & m_{33} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + S_1(t) & 0 & 0 \\ 0 & k_2 + S_2(t) & 0 \\ 0 & 0 & k_3 + S_3(t) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F(t) \end{Bmatrix} \quad (33)$$

Introducing the same dimensionless parameters listed in Sections II and III, the equations of motion in terms of the normal coordinates after introducing linear damping are:

$$\begin{aligned} y_1'' + 2\zeta_1 r_{13} y_1' + r_{13}^2 y_1 + s_{11} W_1(\tau) + s_{12} W_2(\tau) \\ + s_{13} W_3(\tau) y_1 = f_1 W(\tau) \\ y_2'' + 2\zeta_2 r_{23} y_2' + r_{23}^2 y_2 + [s_{21} W_1(\tau) + s_{22} W_2(\tau) \\ + s_{23} W_3(\tau)] y_2 = f_2 W(\tau) \\ y_3'' + 2\zeta_3 y_3' + y_3 + [s_{31} W_1(\tau) + s_{32} W_2(\tau) \\ + s_{33} W_3(\tau)] y_3 = f_3 W(\tau) \end{aligned} \quad (34)$$

where

and  $W_i(\tau)$  are zero mean white noise processes with spectral densities  $2D_i$ . Equations (34) constitute a set of coupled differential equations. The response mean squares are obtained by solving the stationary moment equations. The analytical solution for the stationary response is

$$\begin{aligned} E[Y_1^2] &= Df_1^2 / \{2\zeta_1 r_{13}^3 - D_1 s_{11}^2 - D_2 s_{12}^2 - D_3 s_{13}^2\}, \\ E[Y_1'^2] &= r_{13}^2 E[Y_1^2] \\ E[Y_2^2] &= Df_2^2 / \{2\zeta_2 r_{23}^3 - D_1 s_{21}^2 - D_2 s_{22}^2 - D_3 s_{23}^2\}, \\ E[Y_2'^2] &= r_{23}^2 E[Y_2^2] \\ E[Y_3^2] &= Df_3^2 / \{2\zeta_3 - D_1 s_{31}^2 - D_2 s_{32}^2 - D_3 s_{33}^2\}, \\ E[Y_3'^2] &= E[Y_3^2] \end{aligned} \quad (35)$$

This solution indicates that the system may be unstable depending on the values of  $D_i$ . The fact that the mean square must always be positive provides the stability criteria for mean squares given by (35). These criteria are obtained by keeping the denominators of (35) always positive, i.e.,

$$\begin{aligned} 2\zeta_1 r_{13}^3 &> (D_1 s_{11}^2 + D_2 s_{12}^2 + D_3 s_{13}^2) \\ 2\zeta_2 r_{23}^3 &> (D_1 s_{21}^2 + D_2 s_{22}^2 + D_3 s_{23}^2) \\ 2\zeta_3 &> (D_1 s_{31}^2 + D_2 s_{32}^2 + D_3 s_{33}^2) \end{aligned} \quad (36)$$

The stability boundaries represented by conditions (36) are shown in Fig. 7 as a function of the internal resonance frequency ratio  $r$ . For simplicity the excitation levels  $D_i/2\zeta_i$  of the random stiffness perturbations are assumed to be equal. Samples of the response means squares as function of the excitation level  $D/2\zeta$  are shown in Figs 8

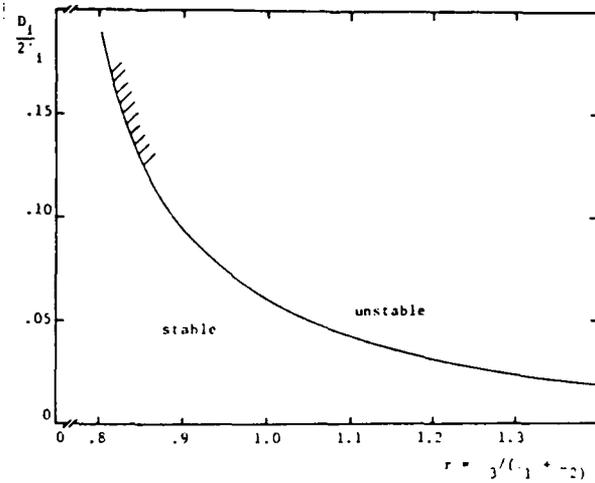


Fig. 7. Mean square stability boundary of the structure with random stiffness, for  $\zeta_i = 0.01$

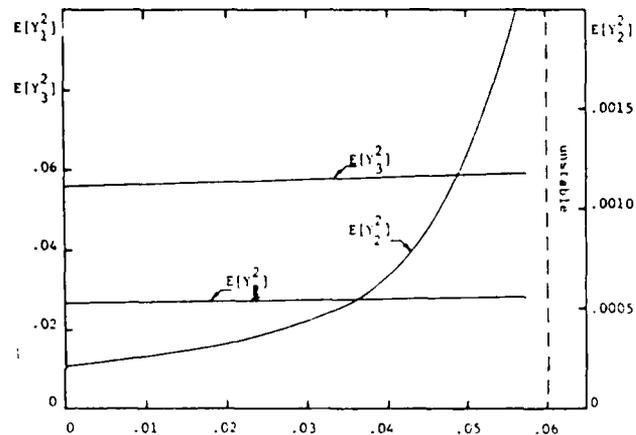


Fig. 8. Mean square response of normal modes with random stiffness for  $r = 1.0$ ,  $\zeta_i = 0.01$

and 9 in terms of normal and generalized coordinates, respectively. It is observed that the response of tip mass of the vertical cantilever is the main source of instability.

## VIII. AUTOPARAMETRIC INTERACTION

In this case the influence of nonlinear modal coupling on the system response will be examined by including the functions in the analysis. These functions are only significant if the structure is tuned internally such that the normal mode frequencies have a linear relationship. For the present system it is found that the following three internal resonance conditions can take place<sup>15</sup>:

$$\begin{aligned} \omega_3 &= \omega_1 + \omega_2 \\ \omega_3 &= 2\omega_1 \quad \text{and} \quad \omega_3 = 2\omega_2 \end{aligned} \quad (37)$$

The random response of the system will be examined under the first internal resonance condition. As mentioned in Section III the response moment equations involve infinite coupling which must be closed in order to solve for the response statistics. It is known that the response of any nonlinear system to a random Gaussian excitation will be non-Gaussian. The deviation of the response from normality depends on the degree of the system nonlinearity. Generally, closure schemes are

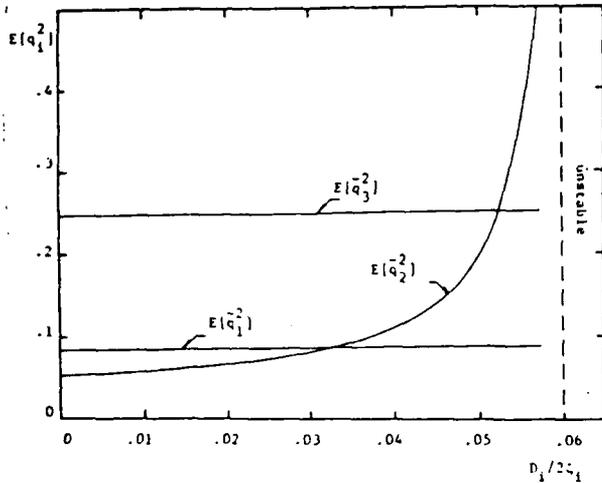


Fig. 9. Mean square response of generalized coordinates for same conditions of Figs 8

classified into Gaussian and non-Gaussian<sup>16</sup>. The Gaussian schemes are useful for dynamic systems with weak nonlinearity. However, in certain situations the application of Gaussian closures may lead to stochastic stability boundaries which are different from those derived by other techniques such as Stratonovich stochastic averaging or non-Gaussian closure approaches. This type of contradiction has been reported for nonlinear systems under parametric random excitations<sup>16</sup>. For two degree-of-freedom systems the Gaussian closure scheme yields nonstationary response while non-Gaussian closure gives strictly stationary response. However, the main response characteristics are found identical as predicted by both methods.

This Section examines the nonlinear response as obtained by using a Gaussian closure scheme which is based on the properties of the cumulants. For the present system 27 equations for the first and second order moments will be generated. The moment equations are closed by setting all third order cumulants to zero, i.e.,

$$\begin{aligned} \dot{z}_3[X_i, X_j, X_k] &= E[X_i, X_j, X_k] - \sum_{i,j,k}^3 E[X_i]E[X_j, X_k] \\ &+ 2E[X_i]E[X_j]E[X_k] = 0 \end{aligned} \quad (38)$$

where the number over summation sign refers to the number of terms generated in the form of the indicated expression without allowing permutation of indices. Relation (38) is used to obtain expressions for the third order moments in terms of first and second order moments.

The solution of the closed 27 coupled moment equations is obtained numerically by using the IMSL DVERK Subroutine (Runge-Kutta-Verner fifth and sixth numerical integration method). Depending on the value of internal detuning parameter  $r$  the system response may be reduced to the same linear response of section VI or may become quasi-stationary which deviates significantly from the linear solution. The response of autoparametric interaction is found to take place in regions of internal resonance ratio slightly deviated from the exact tuning  $r=1$ . The deviation may be attributed to the contribution of nonlinearities incurred during the Gaussian closure procedure. Surprisingly, exact internal resonance yields

linear response characteristics which are displayed in Fig. 10. It is seen that the response fluctuates between two limits during the transient period, then converges to a stationary values which corresponds exactly to the linear solution of section VI. The effect of different initial conditions is examined and it is found that regardless of the initial conditions the solution reaches the same steady state value. For internal resonance ratio  $r=1.175$ , Fig. 11 shows another set of time history responses. In this case the response mean squares do not achieve a stationary state. During the transient period the frequency of the third mode is approximately 1.17 times the sum of the first two mode frequencies. The quasi-stationary behaviour, although present for all three modes, is most prominent for the second mode.

To further illustrate the departure of the nonlinear response from the linear one, Figs 12-15 display the dependence of the normalized mean squares on the internal resonance for various system parameters. The mean squares are normalized by the corresponding linear solution. The subscript G/L refers to the ratio of the nonlinear Gaussian solution to the linear response. In the regions near critical internal resonance the upper and lower envelopes of the quasi-stationary response are plotted. A general trend is observed to exist in all figures. There is a sharp increase in the displacement mean square

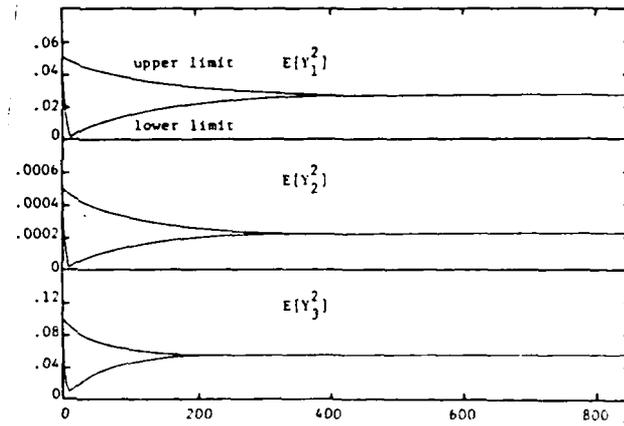


Fig. 10. Time history response of normal coordinates for  $\zeta_1 = 0.01$ ,  $\epsilon = 0.025$ ,  $r = \omega_3 / (\omega_1 + \omega_2) = 1.0$

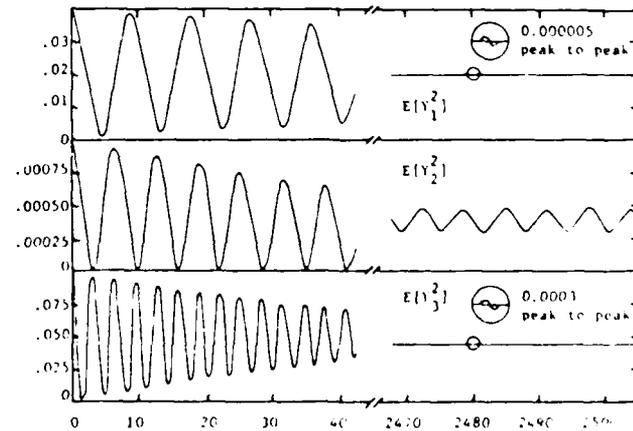


Fig. 11. Time history response of normal coordinates showing autoparametric interaction, for  $\zeta_1 = 0.01$ ,  $\epsilon = 0.025$ ,  $r = \omega_3 / (\omega_1 + \omega_2) = 1.175$

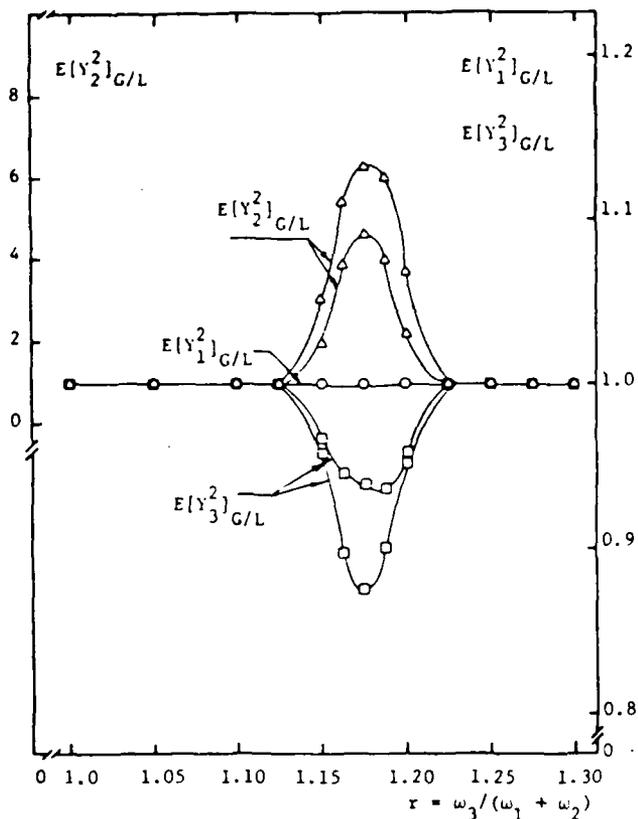


Fig. 12. Mean square response of normalized coordinates as function of internal resonance ratio  $\tau$ , for  $\zeta_1 = \zeta_2 = 0.005$ ,  $\zeta_3 = 0.01$ ,  $\epsilon = 0.025$

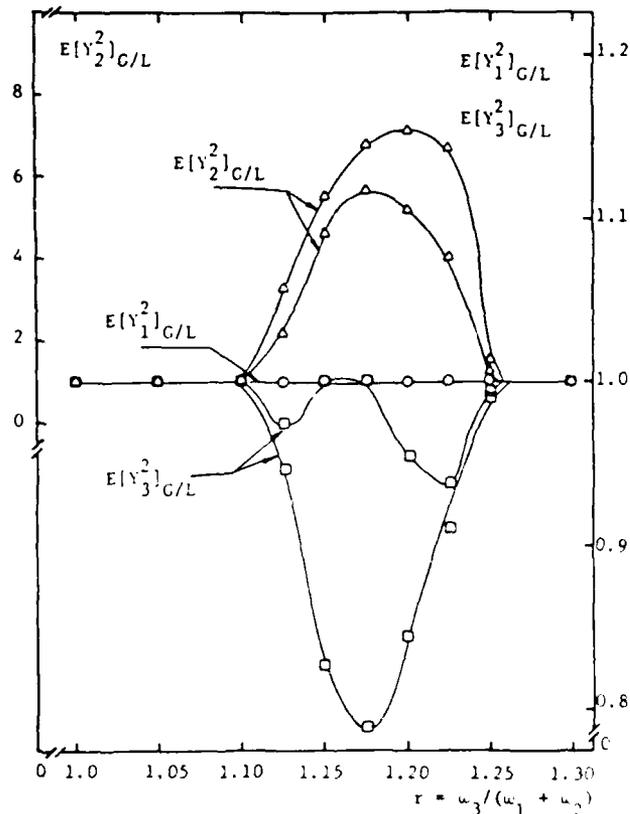


Fig. 14. Mean square response of normalized coordinates as function of internal resonance ratio  $\tau$ , for  $\zeta_1 = \zeta_2 = 0.005$ ,  $\zeta_3 = 0.01$ ,  $\epsilon = 0.05$

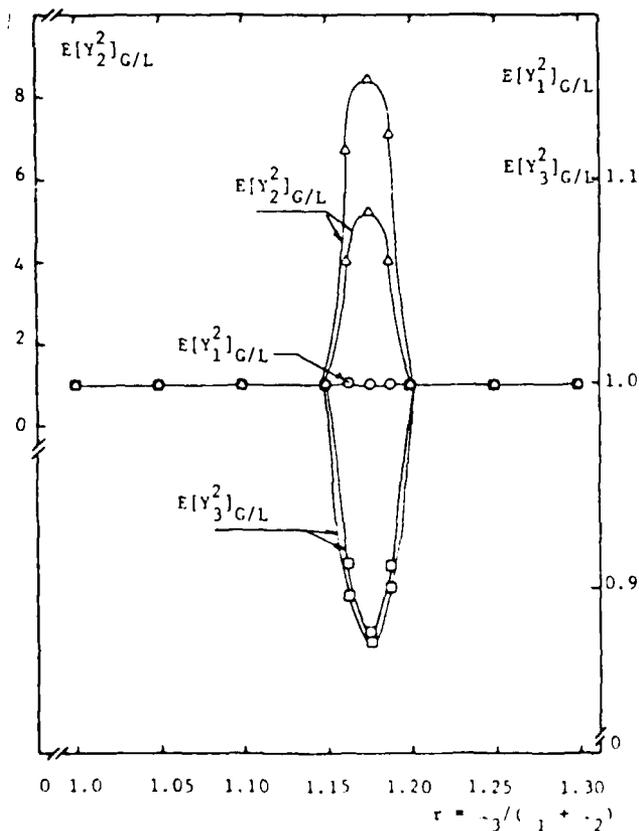


Fig. 13. Mean square response of normalized coordinates as function of internal resonance ratio  $\tau$ , for  $\zeta_1 = 0.01$ ,  $\epsilon = 0.025$

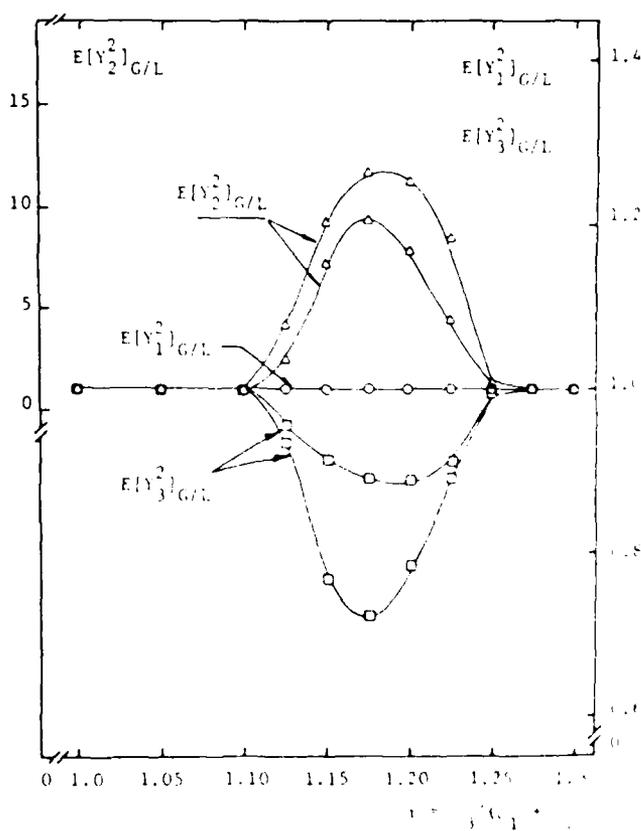


Fig. 15. Mean square response of normalized coordinates as function of internal resonance ratio  $\tau$ , for  $\zeta_1 = 0.01$ ,  $\epsilon = 0.05$

of the second mode associated with a corresponding decrease in the mean square of the third normal mode and very slight drop in the first mode. This feature is similar to a great extent to the deterministic nonlinear absorbing effect reported by Ibrahim and Woodall<sup>15</sup>. Figs 12 and 13 show the effect of damping ratios of the system response. It is seen that nay increase in damping results in narrowing thr region of autoparametric interaction. The nonlinear coupling parameter  $\epsilon$  has a direct influence on the degree of the response deviation from the linear solution as shown in Figs 12–15. As  $\epsilon$  increases from 0.025 to 0.05 the region of autoparametric interaction becomes more wider.

## IX. CONCLUSIONS

The linear and nonlinear modal interactions of a three-degree-of-freedom structure subjected to random excitation is examined. For the linear modelling the response is determined for two cases of structure parameters. The first case is when the parameters are constant coefficients. The mean square response of this case is obtained in terms of the excitation spectral density and the internal detuning parameter. The second case involves random parametric excitations in the stiffness matrix. These excitations result in modal parametric coupling of the normal coordinates. The mean square responses are governed by the spectral densities of parametric excitations which also result in the conditions of mean square stability. The results of the first case are used as a reference to measure the effects of nonlinear inertia coupling of normal modes on the mean square response of the system in the neighbourhood of combination internal resonance. It is found that the critical internal resonance occurs at a value close to  $r = 1.175$  which is deviated from the exact value  $r = 1$ . The nonlinear modal interaction results in an increase of the second normal mode mean square response and in an associated decrease of the first and third normal modes.

## ACKNOWLEDGEMENT

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*Stochastic modal interaction: R. A. Ibrahim and Z. Hedayati*

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1

EXPERIMENTAL INVESTIGATION OF STRUCTURAL AUTOPARAMETRIC  
INTERACTION UNDER RANDOM EXCITATION

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**ABSTRACT**

The paper presents the results of an experimental investigation of random excitation of a nonlinear two-degree-of-freedom structural model. The model normal mode frequencies are adjusted to have the ratio of 2 to 1. This ratio meets the condition of internal resonance of the analytical model. When the first normal mode is externally excited by a band limited random excitation, the system mean square response is found to be linearly proportional to the excitation spectral density up to a certain level above which the two normal modes exhibit discontinuity governed mainly by the internal detuning parameter and the system damping ratio. The results are completely different when the second normal mode is externally excited. For small levels of excitation spectral density the response is dominated by the second normal mode. For higher levels of excitation spectral density the first normal mode attends and interacts with the second normal mode in a form of energy exchange. A number of deviations from theoretical results are observed and discussed.

**1. INTRODUCTION**

The last two decades have witnessed an increasing interest in the study of dynamic behavior of nonlinear systems under deterministic and random excitations. Under certain conditions these systems may experience complex response characteristics such as jump phenomenon, limit cycles, internal resonance, saturation phenomenon, and chaotic motion. These nonlinear phenomena have been predicted theoretically<sup>1-3</sup> and observed experimentally<sup>4-6</sup> under harmonic excitations. However, most of the predicted random response characteristics, including response stochastic stability and statistics,<sup>7,8</sup> have not been verified experimentally. Very few experimental investigations of random vibration of nonlinear systems have been reported in the literature.<sup>7</sup> The lack of experimental verifications may be due to several reasons. These include difficulties in generating the same properties of the random excitation as represented theoretically, and the limitations of experimental equipment. Recently, Bolotin<sup>9</sup> discussed a number of experimental difficulties encountered in experimental measurements of stochastic stability of parametric excited systems.

In deterministic nonlinear vibrations; the amplitude jump, limit cycles, and parametric instability are common features of nonlinear single- and multi-

degree-of-freedom systems. Parametric instability takes place when the external excitation appears as a coefficient in the homogeneous part of the equation of motion. It occurs when the excitation frequency is twice (or multiple) of the system natural frequency. Internal resonance and saturation phenomenon may occur only in nonlinear systems with more than one degree-of-freedom. Internal resonance implies the existence of a linear relationship between the system natural frequencies and causes nonlinear normal mode interaction in the form of energy exchange. Under external harmonic excitation, the mode which is directly excited, exhibits in the beginning, the same features of a single-degree-of freedom system response and all other modes remain dormant. As the excitation amplitude reaches a certain critical level, the other modes become unstable and the originally excited mode reaches an upper bound. In this case, the mode is said to be saturated and energy is transferred into other modes. This interesting phenomenon takes place only in systems with quadratic nonlinear coupling which results in a third order internal resonance.

Under deterministic unsteady aerodynamic forces, most nonlinear characteristics can be predicted by one of the standard techniques of nonlinear differential equations. However, aerospace structures are usually subjected to turbulent air flow, and the aeroelastician is confronted with aerodynamic loads which are random in nature. These loads vary in a highly irregular fashion and can be described in terms of statistical quantities such as means, mean squares, autocorrelation functions and spectral density functions. Ibrahim and Roberts<sup>10,11</sup> and Ibrahim and Heo<sup>12,13</sup> considered nonlinear two degree-of-freedom structural systems and applied Gaussian and non-Gaussian closure techniques to predict the response statistics and response stochastic stability. These studies revealed that a system with internal resonance may experience nonlinear characteristics such as autoparametric in-

teraction. Roberts<sup>14</sup> conducted a series of experimental tests to measure the mean square stability boundaries of a unimodal response of a coupled two-degree-of-freedom system. Roberts reported a number of difficulties in measuring the stability boundaries. Based on the authors experience and other investigators work, it is understood that experimental investigation of nonlinear random vibration is not a simple task and requires careful planning and advanced equipment preparations.

The purpose of the present paper is to report the results of an experimental investigation to measure the response mean squares of a nonlinear two degree-of-freedom structural model under band limited random excitation. The same model was analytically examined by Haddow, et al.<sup>5</sup> under har-

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nic excitation, and by Ibrahim and Ho <sup>12,13</sup> under wide band random excitation. Agreements and disagreements with theoretical predictions will be discussed together with recommendations for future experimental work.

## II. ANALYTICAL BACKGROUND

The random response of a two degree-of-freedom elastic structure has been determined analytically in references [12,13]. The analytical model shown in fig. (1) consists of two beams with end masses. Under vertical support motion  $\xi(t)$  the response of the two beams is mainly governed by linear dynamic and parametric couplings. However, if the system is designed such that the first two normal mode frequencies  $\omega_1$  and  $\omega_2$  satisfy the internal resonance condition  $\omega_2 \approx 2\omega_1$ , the nonlinear inertia forces become dominant and the system dynamic response deviates from the linear response. In terms of the non-dimensional normal coordinates  $\underline{Y}$  the system equations of motion are:

$$[I](Y'') + [C](Y') + [r^2](Y) =$$

$$\xi''(\tau)(a) + \epsilon \xi''(\tau)[b](Y) + \epsilon(\psi) \quad (1)$$

where a prime denotes differentiation with respect to the nondimensional time parameter  $\tau = \omega_1 t$ , and the coordinates  $\underline{Y}$  are related to the dimensional normal coordinates  $\underline{y}$  by the relation  $(Y_1, Y_2) = (y_1, y_2)/q_1^0$ ,  $q_1^0$  is taken as the response root mean square of the system when the length of the vertical beam shrinks to zero, i.e. the response root mean square of the main beam with end mass  $(m_1 + m_2)$ . The elements of the vector (a) and matrix [b] are constants depending on the system properties. The small parameter  $\epsilon = q_1^0/\ell_1$ . The matrix  $[r^2]$  is diagonal with elements 1 and  $(\omega_2/\omega_1)^2$ . The vector ( $\psi$ ) contains all quadratic nonlinear terms which encompasses two groups: nonlinear terms of the same mode and autoparametric terms of the type  $Y_1 Y_1''$ . It is the autoparametric coupling which gives rise to the internal resonance condition  $r = \omega_2/\omega_1 = 2$ .

The random acceleration  $\xi''(\tau)$  was assumed to be Gaussian wide band process with zero mean and a smooth spectral density 2D up to some frequency higher than any characteristic frequency of the system. The acceleration terms in the nonlinear functions  $\psi_1$  were removed by successive elimination and the system equations of motion was transformed into a Markov vector via the coordinates transformation

$$(Y_1, Y_2, Y_1', Y_2') = (X_1, X_2, X_3, X_4) \quad (2)$$

A set of first order differential equations of the response statistical moments were generated by using the Fokker-Planck equation approach.<sup>7</sup> These equations were found to be coupled through higher order moments and were closed via two approaches: Gaussian and non-Gaussian closures. These closure techniques are based on the cumulant properties. The Gaussian closure is established by equating all cumulants  $\lambda$  of order greater than two to zero, i.e.

$$\lambda_{N>2} [X_1^{k_1} X_2^{k_2} \dots X_n^{k_n}] = 0, \quad N = \sum_{i=1}^n k_i \quad (3)$$

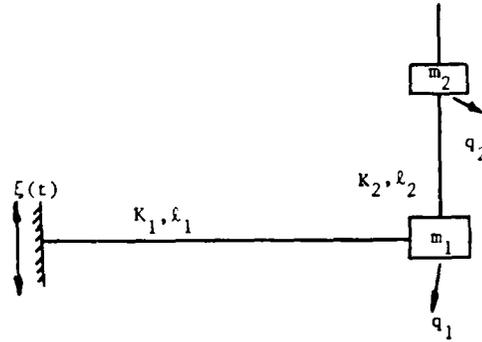


Fig. (1) Model of coupled two beams

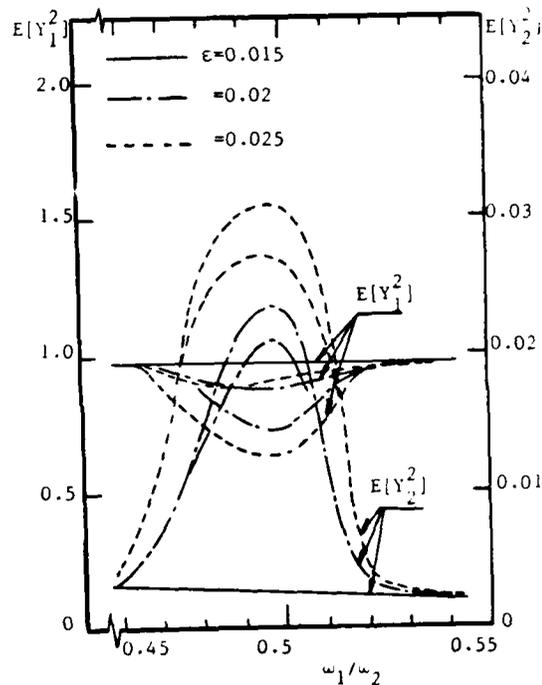


Fig. (2) Gaussian closure solution for various values of nonlinear coupling parameter  $\epsilon$

This approach resulted in fourteen coupled differential equations for first and second order moments of the response coordinates. The numerical integration of these equations revealed that the response mean squares fluctuate between two limits. This fluctuation means that the response does not achieve a stationary state. The autoparametric interaction took place in the neighborhood of internal resonance and was manifested by an energy exchange between the mean squares of the two normal modes. Figure (2), taken from reference 12, shows a sample of the mean square response of the system normal modes against the internal detuning parameter.

The second method takes into account the effect of the response non-normality. As a first order non-Gaussian approximation all cumulants of order greater than four were equated to zero, i.e.

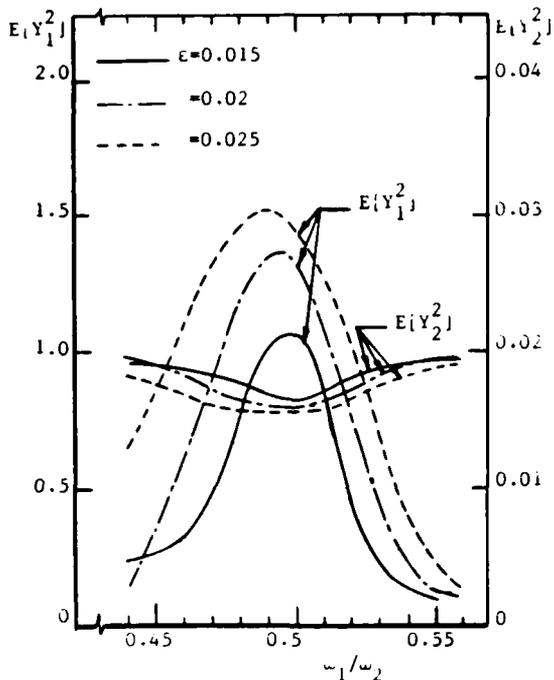


Fig. (3) Non-Gaussian closure solution for various values of nonlinear coupling parameter  $\epsilon$

$$\lambda_{N>4} [X_1^k X_2^k \dots X_n^k] = 0, \quad N = \sum_{i=1}^n k_i \quad (4)$$

This approach resulted in 69 first order differential equations, in the first through the fourth order moments, which were solved numerically. The solution reaches a stationary state after a transient period and exhibits the same nonlinear interaction as predicted by the Gaussian closure solution. Figure (3) shows the stationary mean square response of the normal coordinates against the internal detuning parameter.

Although the two approaches yielded common features to those predicted by deterministic theory<sup>5</sup> of nonlinear vibration such as autoparametric suppression effect, the random analysis did not verify

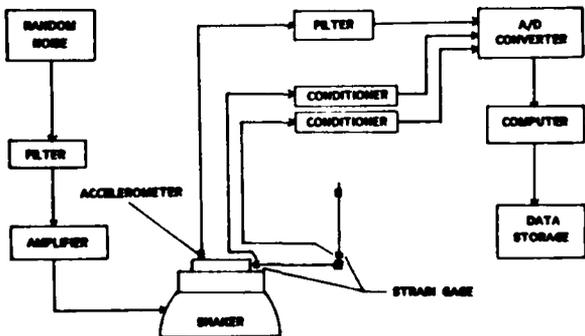


Fig. (4) Arrangement of experimental equipment

the existence of saturation phenomenon. The saturation phenomenon<sup>2</sup> is a well known feature for multi-degree-of-freedom systems involving quadratic nonlinear coupling subjected to harmonic forced excitation.

It is well known that the predicted results are approximate and their validity has not been examined. The next section reports the measured results of a series of experimental tests of the same model under band limited random excitation.

### III. EXPERIMENTAL INVESTIGATION

#### III.1 Experimental Model and Equipment

The model is similar to a great extent to the experimental model used by Haddow, et al.<sup>3</sup> It consists of a horizontal beam of cross section of 0.111"x1.0", length 7.5", and carries a tip mass of 0.015 slug. The tip mass has a provision for clamping the vertical beam which has cross section 0.054"x1.0". The length of the vertical beam can be adjusted by changing the location of its top mass (0.0127 slug). The deflections of the two beams are measured by strain gages fixed at the root of each beam. Two gages are mounted on the horizontal beam in a two arm bridge. Four gages are mounted on the vertical beam in a four arm bridge. The fixed end of the horizontal beam is clamped by a fixture which is bolted on the top of the shaker armature. The shaker is a Caldyne model A88 of thrust 100 lb and provides 1" peak-to-peak stroke. The shaker is powered by a Ling Electronics Model RA-250 power supply and receives a random signal through a GenRad Type 1381 Random Noise generator. The random signal is filtered to a desired band width with a Krohn-Hite Model 3343 Variable Electric Filter. The filtered signal is amplified via a Calex Model 176 Instrument Amplifier. Figure (4) shows a schematic diagram of the instrumentation used in this investigation. The acceleration of the shaker platform is measured by a PCB Piezotronic Model 302A02 shock accelerometer. The accelerometer is powered by a PCB Piezotronic Model 480C06 power unit.

The first two normal mode frequencies of the system are determined theoretically and measured experimentally as a function of the beams length ratio  $l_2/l_1$  as shown in fig. (5). This figure shows that the internal resonance  $\omega_2/\omega_1 = 2$  is obtained in two locations of the length ratio. At these mass locations the normal mode frequencies are:

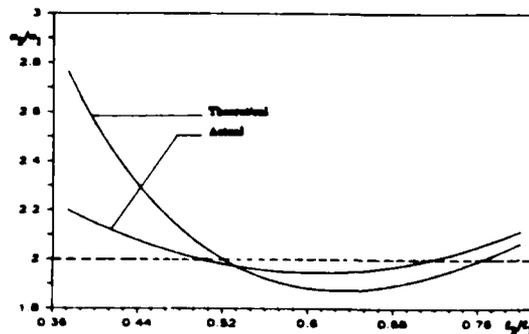


Fig. (5) Measured and theoretical frequency ratio of the first two normal modes.

$$l_2/l_1 = 0.485: f_1 = 9.1 \text{ Hz}, f_2 = 18.2 \text{ Hz} \quad (5a)$$

$$l_2/l_1 = 0.707: f_1 = 7.45 \text{ Hz}, f_2 = 14.9 \text{ Hz} \quad (5b)$$

The analog signals of the excitation and responses are read and converted into binary numbers using a Data Translation Model DT-3752 Intelligent Analog Peripheral (IAP). This IAP is capable of reading either 8 channels (+10v) or 16 channels (0-10v) of input. It can also read and convert analog signals at up to 40k points per second. This unit is mounted in an expansion slot of an IBM System 9001 Benchtop Computer. The control and programming of the Analog/Digital (A/D) system are accomplished through the software controlled registers and field selectable (hardware) options. The software controlled registers are the control registers, status register, and gain/channel register. The control register controls the operation and mode of the A/D system. The modes which are used in this investigation are direct memory transfer and increment mode operation. Direct memory transfer places converted data directly into the memory of the computer. The increment mode allows the A/D to increment the input channel number automatically before each A/D conversion. This allows data to be taken from sequential channels without requiring a program to specify each channel. The status register reports the complete status of the A/D system during the operation. The gain/channel register selects the desired channels from which the data is to be taken and sets a programmable gain for all input signals. This gain is set to one for all tests. The computer controls the DT-3752 through a Fortran program. Analog signals are converted for a specified amount of time or until the computer memory is full. When the computer has completed collecting data, the data is transferred to a floppy disk for future processing.

The data processing is performed at equally spaced intervals. The problem of determining this time interval is well discussed in Bendat and Piersol.<sup>15</sup> Generally, if sampling is prepared at points which are too close together, it will yield correlated and redundant data. This will unnecessarily increase the labor and cost of calculations. Sampling at points which are too far will lead to the problem of aliasing. The aliasing is mainly a confusion between the low and high frequency components in the original data. In order to eliminate the problem of aliasing, a sampling rate should be chosen to be at least two times the maximum frequency that the model will experience. In order to get a good sample data, a sampling rate is chosen which is roughly eight times the maximum frequency. In the present investigation, the sampling rate is chosen to be 80 Hz per channel for the first mode excitation and 160 Hz per channel for the second mode and wide band excitation. Data processing involves another problem known as quantization which is the conversion of data values at the sampling points into digital form. The infinite number of values of the continuous analog signal must be approximated by a fixed set of digital levels. A choice between two consecutive levels will be required because the scale is finite. The accuracy of the approximating process is a function of the available levels which is dependent upon the analog to digital converter resolution. The accuracy of the DT-3752 is the value of the least significant bit which corresponds to a voltage of

$\pm 0.0049v$ . This resolution is analogous to a deflection of the horizontal beam beam of  $\pm 0.00073$ -in and the vertical of  $\pm 0.00097$ -in and an acceleration of  $\pm 0.00044$ -g for the excitation.

The experimental model is tested under various levels of excitation spectral density. This is achieved by keeping the input signal level constant (Master Gain on Ling Amplifier) for the range of internal detuning of the model. The level of amplification is adjusted to five levels for testing of both the first and second normal frequency bandwidths. Another series of tests are conducted for excitation spectral density that covers both normal mode frequencies.

### III.2 Experimental Results

The experimental results include sample records of time history responses and the mean square responses in terms of generalized coordinates and normal coordinates. The mean square response will be represented against the internal detuning parameter  $r$  and the excitation spectral density level. The bandwidth of the random excitation depends on the mode under investigation.

#### III.2.1 First Mode Excitation

The first mode is excited by a limited bandwidth random excitation of bandwidth 5Hz and a central frequency very close to the first normal mode natural frequency. The frequency content of this random process is selected such that it does not excite any higher structural modes. For the five levels of excitation spectral density, the system response is governed mainly by the first mode which does not show any nonlinear coupling. Figure (6) shows a sample of the time history response under excitation spectral level  $S_0 = 0.0142$  ( $g^2/Hz$ ) when the model is internally tuned to the resonance condition  $\omega_r/\omega_1 = 2.0$ . It is seen that the response is characterized by a narrow band random process of frequency close to the first normal mode = 7.5 Hz.

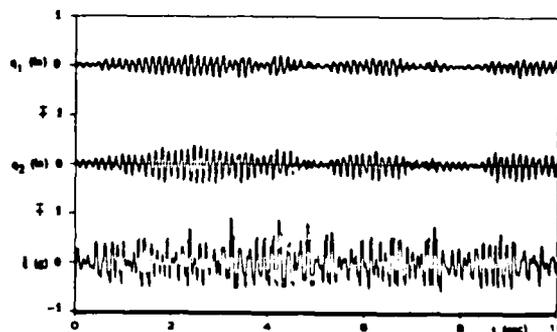


Fig. (6) Time history response of first normal mode excitation, level V,  $S_0 = 0.0142$   $g^2/Hz$ .

Figure (7a) shows the mean square response of the generalized coordinates for the same excitation spectral density level of fig. (6). The empty points are measured when the mass of the vertical beam moves upward while the full points are obtained when the mass moves downward. Both groups

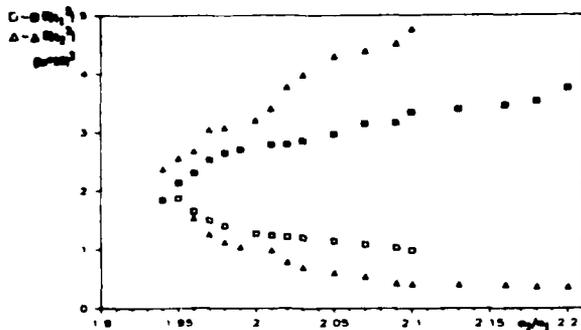


Fig. (7a) Mean square responses of generalized coordinates under first normal mode excitation.

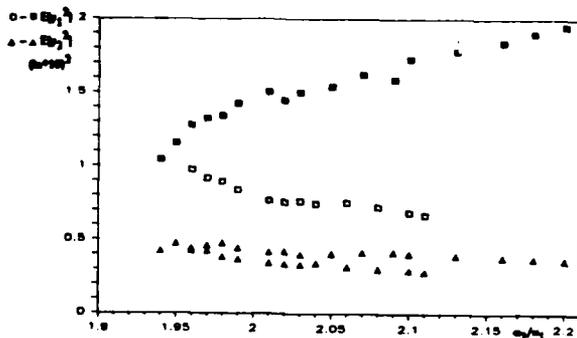


Fig. (7b) Mean square response of normal coordinates under first normal mode excitation.

Excitation level  $V$ :  $S_0 = 0.0142 \text{ g}^2/\text{Hz}$ , empty points correspond higher position of the upper mass while full points correspond lower position.

are measured in the neighborhood of the system internal resonance. The group of full points indicates that the mean square of the horizontal beam increases while the mean square of the vertical beam decreases as the normal mode frequency ratio  $\omega_2/\omega_1$  increases. This implies that the model behaves like a single degree-of-freedom system for  $\omega_2/\omega_1 \gg 2$ . For the second group of results (empty points) the mean square response of the vertical beam increases and the mean square of the horizontal beam decreases. This feature is belonging to the characteristics of linear vibration absorbers due to inertia coupling. The corresponding response curves in normal coordinates are shown in fig. (7b). The square points (empty or full) are belonging to the first normal mode which obviously predominates the response. It is also seen that as the vertical mass moves downward, the model starts to behave like a linear single degree-of-freedom system whose mean square is given by the relationship<sup>17</sup>

$$E[y^2] = D/(\zeta\omega^3 m^2) \tag{6}$$

where  $m$ ,  $\omega$  and  $\zeta$  are the mass, natural frequency, and damping ratio of the system, respectively.  $2D$

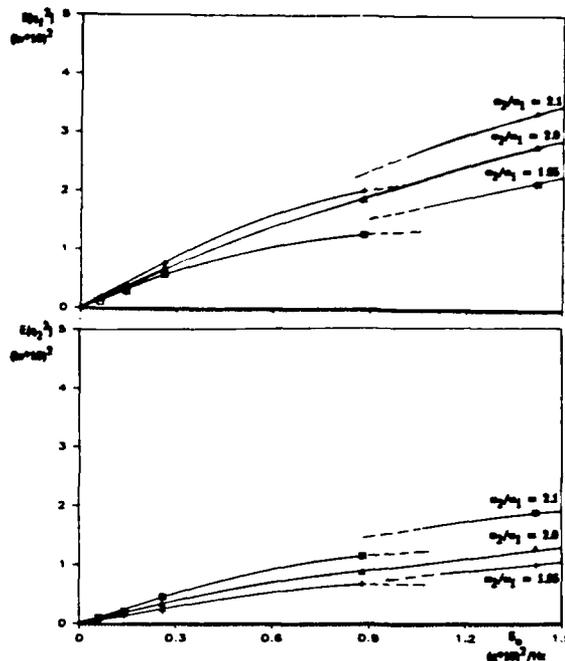


Fig. (8a) Relationship between mean square responses of generalized coordinates and the excitation spectral density for various values of internal detuning.

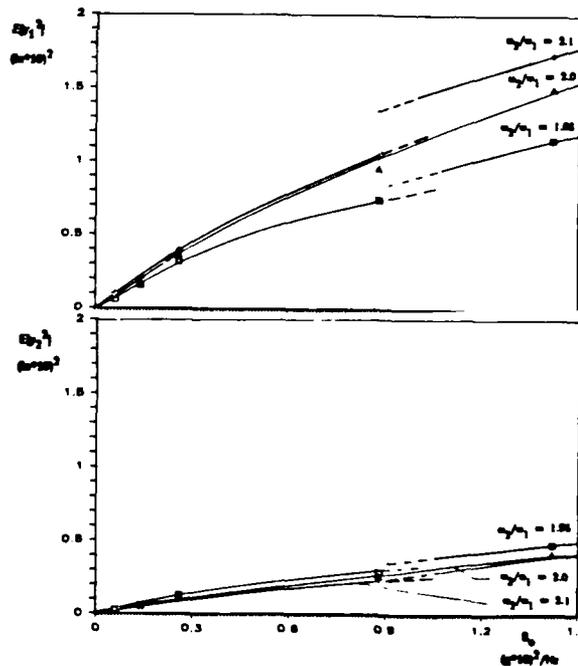


Fig. (8b) Relationship between mean square responses of normal coordinates and the excitation spectral density for various values of internal detuning.

(Measurements are taken for lower position of the upper mass).

is the excitation spectral density of a wide band random excitation.

It is clear that the trend of the full square points agrees with the linear solution (6) that the mean square response is inversely proportional to the cube of the first normal mode frequency.

In order to provide more insight to the system response statistics, the mean square response is plotted against the excitation spectral density level as shown in fig. (8a) for various values of internal detuning. It is seen that the mean squares of the two beams increase with the excitation spectral density up to a certain level above which the curves are discontinuous. The degree of discontinuity depends on the internal detuning. Any deviation from the exact internal detuning results in a strong discontinuity. This discontinuity means that the system is unstable in the mean square sense. Similar features were reported in the deterministic response of the same system by Haddow, et al.<sup>5</sup> The location of discontinuity is strongly dependent on the values of damping ratios and the internal detuning of the structure. Figure (8b) shows the mean square response of the normal coordinates against the excitation spectral density. The curves have the same trend of fig. (8a)

### III.2.2 Second Mode Excitation

The second normal mode is excited by a limited band random excitation of bandwidth 5 Hz and central frequency very close to the second normal mode frequency. Five levels of excitation spectral density ranging from 0.001  $g^2/Hz$  to 0.022  $g^2/Hz$  are selected. A general feature of the time history response records is that both amplitudes  $q_1$  and  $q_2$  increase with the levels of excitation as in the first mode excitation. The records also show that for all selected beam length ratios and for all levels of excitation spectral density, the vertical beam amplitude  $q_1$  is always greater than the horizontal beam amplitude  $q_2$ . Another observation is that when the excitation level is held constant the amplitudes  $q_1$  and  $q_2$  increase slightly as the beam length ratio increases. For small levels of excitation spectral density, the second normal mode is observed to have no interaction with the first mode. However, above a certain level of excitation spectral density, it is found that the first mode appears for a certain period of time and then disappears as the second mode takes over, and so on as shown in fig. (9a). This nonlinear interaction of the two normal modes is more clarified in fig. (9b). Under harmonic excitation, Haddow, et al.<sup>5</sup> reported similar energy exchange between the two modes. Furthermore, it was shown that the directly excited mode becomes saturated and energy is transferred to the first mode. In the present investigation, the energy transfer takes place not only under high levels of excitation spectral density but also when the the internal resonance is approaching the value 2 as vertical beam length is increasing.

The mean square responses of the generalized and normal coordinates are plotted against the internal detuning parameter  $r$  in figs. (10a) and (10b), respectively. The suppression effect of the excited mode takes place only when the vertical mass is moved downward as shown in fig. (10b) by the full triangular points. The second mode mean square (empty triangular points) increases with a corresponding decrease in the first mode mean square (as the vertical mass moves upward).

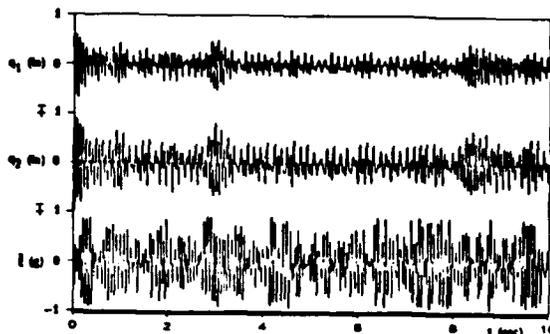


Fig. (9a) Time history response of second mode excitation under excitation spectral density of 0.022  $g^2/Hz$ .

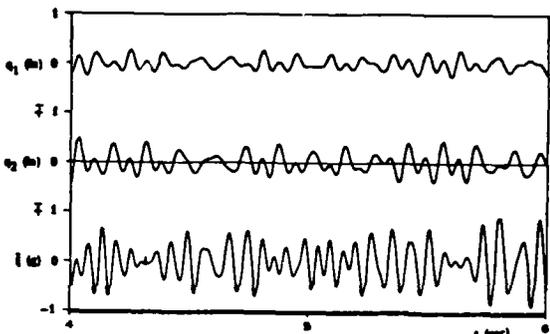


Fig. (9b) Magnification of time history response of second mode excitation showing attendance of first normal mode.

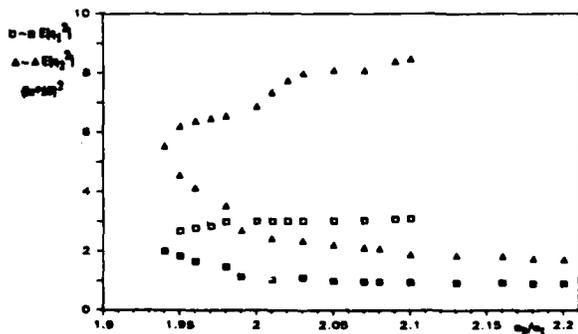


Fig. (10a) Mean square responses of generalized coordinates under second normal mode excitation.

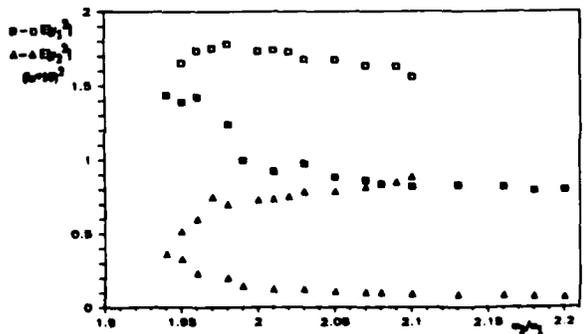


Fig. (10b) Mean square responses of normal coordinates under second normal mode excitation.

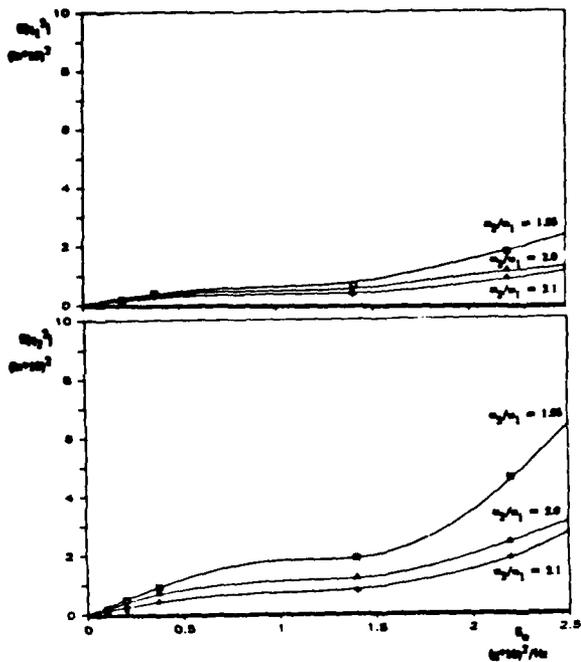


Fig. (11a) Relationship between mean square responses of generalized coordinates and excitation spectral density for various values of internal detuning.

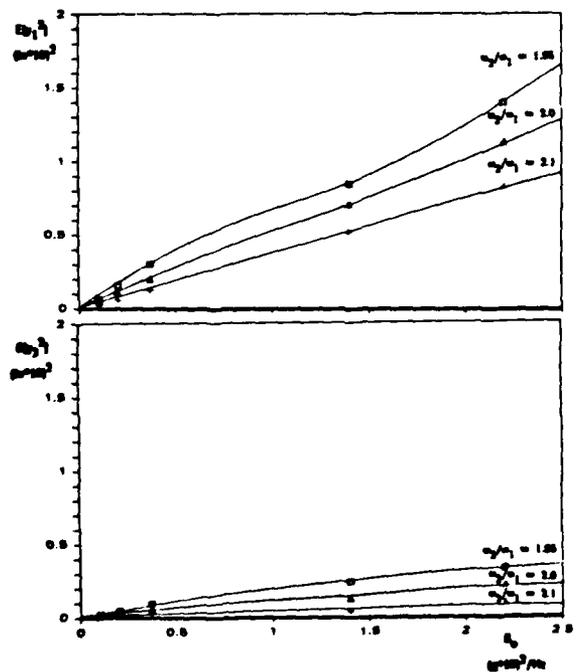


Fig. (11b) Relationship between mean square responses of normal coordinates and excitation spectral density for various values of internal detuning.

(Measurements are taken for lower position of the upper mass.)

Figures (11a) and (11b) show the influence of the excitation spectral density on the normal mode mean square responses of the generalized and normal coordinates, respectively. Figure (11b) indicates that the second normal mode mean square is relatively smaller than the first normal mode mean square response. This suppression effect is due to the nonlinear normal mode interaction. However, the saturation phenomenon, known in deterministic systems with quadratic nonlinearity, is not pronounced in the present results since the excitation is a random process which contains several frequencies each of which may excite the two modes. In deterministic excitation, the external and internal detunings are very important in establishing the saturation phenomenon.

III.2.3 TWO MODE EXCITATION

The purpose of these tests is to explore the behavior of the system under random excitation which covers both normal mode frequencies. Due to the shaker limitation the tests are conducted under single excitation spectral density level  $S_0 = 0.0026 \text{ g}^2/\text{Hz}$ . A sample of the time history response record is shown in fig. (12) which reveals the presence of the two modes. The amplitude of oscillation of each beam depends on the vertical mass location which yields the same internal resonance condition. Figures (13a) and (13b) show the dependence of the mean square response on the internal detuning in terms of generalized and normal coordinates, respectively. The full points reveal linear response characteristics while the empty points show a nonlinear interaction between the two modes.

IV. CONCLUSIONS AND DISCUSSION

The results of an experimental investigation of nonlinear modal interaction in a two-degree-of-freedom structural model under random excitation are reported. The model equations of motion include linear and nonlinear inertia couplings of the generalized coordinates. The normal mode frequencies  $\omega_1$  and  $\omega_2$  of the model are adjusted to meet the internal resonance condition  $r = 2.0$ . This frequency ratio is found to exist at two beam length ratios  $l_1/l_2 = 0.49$  and  $0.71$ . At these locations the system response characteristics are completely different when the model is excited by a band limited random excitation. Three main series of tests are conducted to examine the system response behavior when the first and second modes are excited separately and when both modes are excited simultaneously.

When the first normal mode is externally excited it is found that the mean squares of the two modes are increasing monotonically with excitation spectral density. The response-excitation relationship is almost linear for small excitation levels. When the two beams are tuned to the exact internal resonance, the response-excitation relationship follows a continuous curve. For different internal detuning, the response curves exhibit a discontinuity. This feature is similar to the deterministic characteristics of the same model.

When the second normal mode is externally excited, the system response is dominated by the second normal mode up to an excitation spectral density level above which the first normal mode attends and

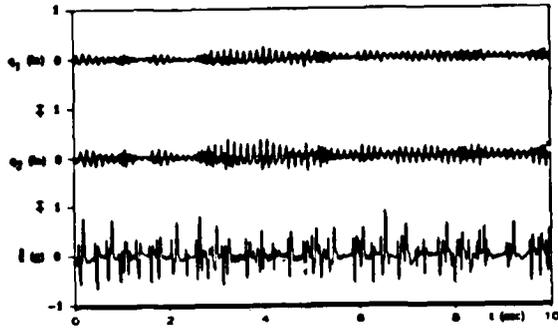


Fig. (12) Time history response of two normal modes excitation.

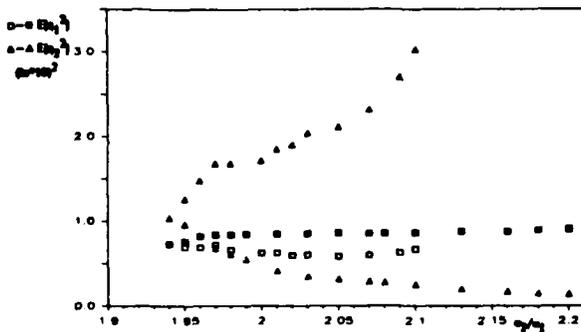


Fig. (13a) Mean square responses of generalized coordinates under band-limited random excitation of the two normal modes.

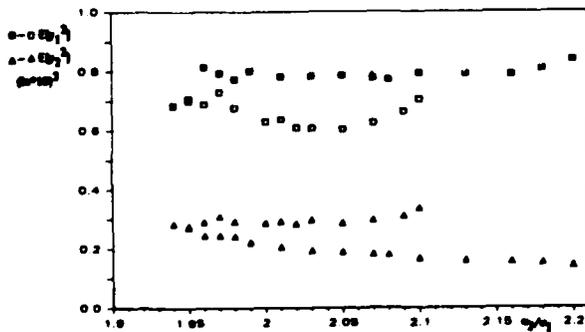


Fig. (13b) Mean square responses of normal coordinates under band-limited random excitation of the two normal modes.

[ $S_0 = 0.0026 \text{ g}^2/\text{Hz}$ , points notation follows fig. (7)].

the two normal modes exhibit nonlinear interaction. Above this excitation level, the first normal mode shows large random motion which results in a suppression of the second mode. The results do not display any evidence for the existence of saturation phenomenon. The main features of the vibration autoparametric absorber effect reported theoretically by Ibrahim and Heo<sup>12</sup> are not exactly confirmed in the measured results. It is believed that the

deviation from theory is attributed to the fact that the experimental excitation is a band limited random process, while in theory it is represented by a wide band random process. Another source of deviation is that the transformation into normal coordinates is not exact since it does not take into account the effect of structural damping. To eliminate this problem, it is convenient to adopt other models whose generalized and normal coordinates are the same. With new equipment and more powerful shakers the first author is currently undertaking an experimental research program supported by the NSF.

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# Structural dynamics with parameter uncertainties

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The treatment of structural parameters as random variables has been the subject of structural dynamicists and designers for many years. Several problems have been involved during the last few decades and resulted in new theorems and interesting phenomena. This paper reviews a number of topics pertaining to structural dynamics with parameter uncertainties. These include direct problems such as random eigenvalues and random responses of discrete and continuous systems. The impact of these problems on related areas of interest such as sensitivity of structural performance to parameter variations, design optimization, and reliability analysis is also addressed. The paper includes the results of experimental investigations, the phenomenon of normal modes localization, and the effect of mistuning of turbomachinery blades on their flutter and forced response characteristics.

## 1. INTRODUCTION

The concept of uncertainty plays an important role in the investigation of various engineering and physical chemistry problems. In fluid mechanics, for example, the inaccuracy of measurements is called "uncertainty" which differs from the concept of error (Kline, 1985). An error in measurement is the difference between the true value and the measured value. On the other hand, an uncertainty is a possible value that the error might take on in a given measurement. Because the uncertainty can take on various values over a range, it is inherently random. In control theory, the differential equations of control systems often involve uncertain bounded state variables. The parameters of transfer functions of certain models usually vary with a certain degree of uncertainty (Ashworth, 1982). Thus a probabilistic transfer function can be defined with uncertain parameters and can lie anywhere within the ranges which are determined from simulation tests. The identification of uncertain parameters has recently been examined by Skowronski (1981, 1984).

Another class of problems involving parameter uncertainties is the random heterogeneity of real media which possess properties that are described in a probabilistic sense. More specifically, these properties vary randomly with respect to time and position, and thus constitute a random field. The theory of wave propagation in random media is very complicated and involves partial differential equations whose coefficients are random functions of space and time. The difficulty of random wave propagation problems stems from the fact that the solution of a linear partial differential equation depends nonlinearly upon the coefficients (Chernov, 1960; Frisch, 1968; Sobczyk, 1985).

In physical chemistry the problem of determining the vibrational properties of randomly disordered crystal lattices involves the calculations of the frequency spectrum, electronic energy levels of binary alloys, thermodynamic properties of alloys, isotropic mixtures, and other solid state phenomena. Of particular importance is the "normal localization" or "confinement" phenomenon which was first reported by Anderson

(1958). Anderson showed that the electron eigenstates in a disordered solid may become localized and results in a reduction of metallic conductivity. In structural dynamics with parameter uncertainties, irregularities may inhibit the propagation of vibration within the structure and the vibration modes become localized. The similarities between the propagation of vibration in an elastic system and the conduction of electrons in a solid is discussed by Hodges (1982), Hodges and Woodhouse (1983), and Pierre et al (1986). Several problems in physics and physical chemistry pertaining to crystal lattice dynamics were reviewed by Elliot et al (1974) and recently documented in a monograph by Bottger (1983).

In structural dynamics, uncertainties arise from two main sources (Prasthofer and Beadle, 1975). The first is a statistical one and is due, for example, to the stiffness or damping fluctuations caused by random variations in material properties, randomness in boundary conditions, and variations caused by manufacturing and assembly techniques. The second is nonstatistical and is due, for example, to the inaccuracies and assumptions introduced in the mathematical modeling of the structure. In the first class the mechanical properties of dynamic systems are subject to a certain degree of uncertainty because the physical properties of their elements are not measured exactly. In addition, the physical properties can experience variations with the passage of time as a result of wear and tear or just inherent deterioration. These properties should be modeled as random variables with a probability distribution representing the distribution of the measured values. This modeling results in random eigenvalues, eigenvectors, and random responses of the system in question. The analysis of random eigenvalues and eigenvectors has been a subject of several studies by mathematicians and engineers and will be reviewed in section 3.

Figure 1 shows five examples of structural systems involving parameter and load uncertainties. They include "almost" periodic structures, similar component subsystems, multi-span beams, rocket fins, and turbomachinery rotors. The rocket fins

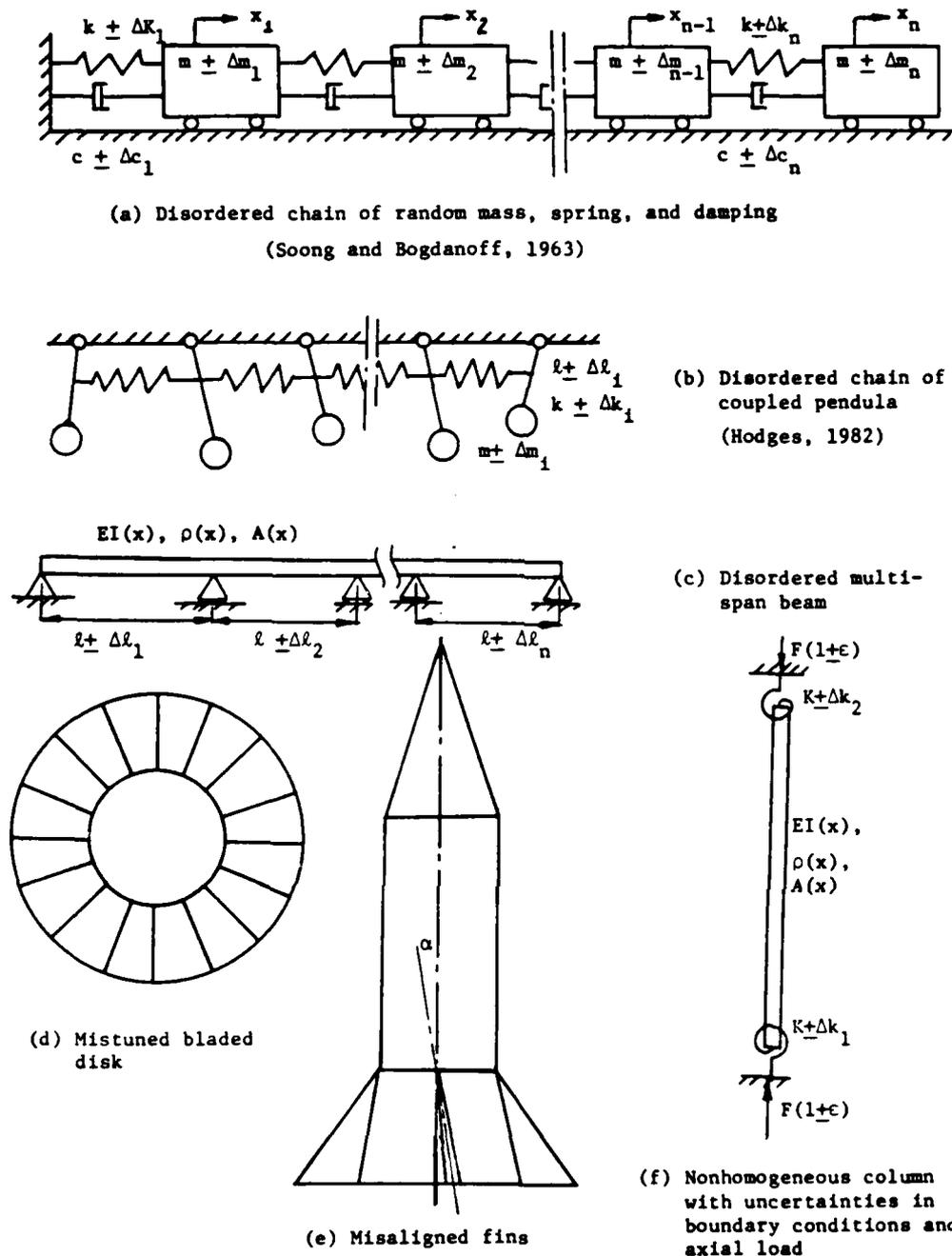


FIG. 1. Examples of disordered systems.

are not usually identical in their areas and each fin has some misalignment with the rocket longitudinal axis. For the case of turbomachinery rotors, there is always some mass and stiffness eccentricity in the disks. Parameter variations exist in disk blades and result in corresponding variations in the individual natural frequencies of the blades. This problem is known as mistuning (Srinivasan, 1984) which may have a significant effect on the forced response amplitude of the blades and also in the value of the flow speed at which flutter of the blades occurs. Other examples include buried pipelines, railroad tracks, and interconnected girders. The uncertainties in these systems affect to a large extent their design and operating performance.

It should be noted that parameter irregularities may cause significant changes in the dynamic characteristics of structural

systems. In particular, they may cause the occurrence of mode localization which can be used as a means of passive control of vibrations. In civil engineering the mechanical and strength properties of the material vary from one point to another point and are seldom prone to certain *in situ* measurements but only to indirect estimates (Augusti et al, 1984). The uncertainty of these properties has a direct relationship to the reliability of such structures. These uncertainties are usually manifested in the applied loads, stiffness, and theoretical models that are used to describe and relate loading and resistance. The design of structures under conditions of uncertainty implies a balancing decision between risk of failure and cost or weight (Ang and Tang, 1984; Frangopol, 1986). The risk is an unavoidable consideration for structural optimization problems. It has been

customary in most reliability studies to measure the risk by the probability of failure (ie, the likelihood of occurrence of some specified limit state). On the other hand, when restrictions and constraints of the design are imprecisely described, the design objective functions become fuzzy (Zadeh, 1965, 1973; Brown, 1980; Brown and Yao, 1983). Recently, the fuzzy set theory has been applied in multi-objective fuzzy optimization design of ship grillage structures (Gangwu and Suming, 1986).

The degree of sensitivity of structures to either deterministic design changes, or stochastic parameter variations is of great importance to the structural dynamicist. In particular, it is essential to determine if small perturbations can result in significant changes of the free or forced response amplitudes. This sensitivity analysis is of great concern to those who are involved in the control of large flexible space structures (Meirovitch et al, 1983; Nurre et al, 1984). These structures possess several modes densely packed at low frequencies. When they are discretized, model errors occur and the free modes of vibration cannot be determined accurately. Thus when a control system is designed for natural frequencies whose values are assumed to be exact, the model errors and structural uncertainties may deteriorate the performance of the control loop, and may even make the system unstable. This problem results in what is known as robustness, ie, a control system is termed robust if it is relatively insensitive to model errors and structural uncertainties.

This paper provides a review of the recent theorems and results pertaining to structural dynamics with parameter uncertainties. An early account of the subject was provided by Soong and Cozzarelli (1976). Three main problems will be addressed. These are:

1. Random eigenvalues,
2. Random response characteristics, and
3. Design optimization and reliability.

Before reviewing these three problems the differences between parametric random vibration and structural dynamics with parameter uncertainties will be discussed first.

## 2. BETWEEN PARAMETRICALLY EXCITED AND DISORDERED SYSTEMS

It is very important to distinguish between two types of parameter variations encountered in structural dynamics. The first type arises due to random parametric excitation of systems with essentially fixed properties while the second class is internal and is associated with the system when its parameters are represented in a probabilistic sense. In the former case the system equations of motion are stochastic differential equations with random coefficients represented by random processes (Ibrahim, 1985), while in the latter case the equations of motion are differential equations with random parameters represented by random variables (Soong, 1973). The methods of treating dynamic systems under parametric random excitations are different from those used in solving differential equations with random variable coefficients. Parametric random vibration is basically a combination of the theory of stochastic processes, stochastic differential equations, and applied dynamics. Systems with parameter uncertainties (referred to in the literature as "disordered systems"), on the other hand, involve boundary-value problem and random field theory (Vanmarcke, 1984). The term "disorder" has been extensively used in the literature to distinguish between the case of random perturbation of the system parameters (described by a probabilistic law) and the case when these parameters are perturbed in a deterministic sense.

## 3. RANDOM EIGENVALUES

### 3.1. Basic concept of random eigenvalue

The value of the natural frequency of simple single degree-of-freedom systems is given by the square root of the stiffness to mass ratio. This value is assumed by constant for identical systems. However, experiments have shown that this value varies randomly (Mok and Murray, 1965) because in reality the physical properties of the elements can neither be measured exactly nor manufactured exactly. Thus, the eigenvalues are random variables whose statistical properties are determined by the random coefficients of the inertia and stiffness terms of the equations of motion. Consider for example the natural frequency of a simple mass-spring system

$$\lambda = \omega^2 = k/m,$$

the variation of  $\lambda$  due to variations in stiffness  $k = \bar{k} + \delta k$  and mass  $m = \bar{m} + \delta m$ , may be expressed as a Taylor series

$$\delta\lambda = \lambda - \bar{\lambda} = -\frac{\partial\lambda}{\partial k}\delta k + \frac{\partial\lambda}{\partial m}\delta m + \frac{1}{2}\frac{\partial^2\lambda}{\partial k^2}(\delta k)^2 + \frac{1}{2}\frac{\partial^2\lambda}{\partial m^2}(\delta m)^2 + \dots \quad (1)$$

where overbar quantities refer to mean values and  $\bar{\lambda} = \bar{k}/\bar{m}$ .

When the variations  $\delta m$  and  $\delta k$  are random variables the natural frequency will be a random variable. The mean and variance of  $\lambda$  can be evaluated as follows

$$E[\lambda] = \bar{\lambda} + \frac{1}{2}\frac{\partial^2\lambda}{\partial k^2}E[\delta k^2] + \frac{1}{2}\frac{\partial^2\lambda}{\partial m^2}E[\delta m^2] + \dots \quad (2)$$

and

$$\begin{aligned} E[(\lambda - \bar{\lambda})^2] &= \left(\frac{\partial\lambda}{\partial m}\right)^2 E[\delta m^2] + \left(\frac{\partial\lambda}{\partial k}\right)^2 E[\delta k^2] \\ &+ 2\frac{\partial\lambda}{\partial k}\frac{\partial\lambda}{\partial m} E[\delta k \cdot \delta m] \\ &+ \frac{1}{4}\left(\frac{\partial^2\lambda}{\partial k^2}\right)^2 E[\delta k^4] + \frac{1}{4}\left(\frac{\partial^2\lambda}{\partial m^2}\right)^2 E[\delta m^4] \\ &+ \frac{1}{2}\left(\frac{\partial^2\lambda}{\partial k^2}\right)\left(\frac{\partial^2\lambda}{\partial m^2}\right) E[\delta k^2 \cdot \delta m^2] + \dots \quad (3) \end{aligned}$$

The same is applied when the mass moment of inertia is included in the equations of motion. Collins and Thomson (1967) derived the statistical characteristics of principal moments of inertia and principal axes directions.

Generally, the structural dynamicist is interested in determining the probability that one or more eigenvalues lie in a given range or less than a certain value (Boyce, 1968). However, the probabilistic description of the eigenvalues and the eigenvectors has been examined for a limited and simple class of problems. In most cases, it is possible to calculate the statistical functions (such as expectations, variances, and covariance functions) of the eigenvalues and eigenvectors.

The random eigenvalue problem has been examined for a limited number of linear discrete and continuous systems. The treatment of these systems is based on the analysis of random matrices and random differential operators (Scheidt and Purkert, 1983). The next subsections will review the methods and main results reported in the literature

### 3.2. Random eigenvalues of discrete systems

The statistics of random eigenvalues and eigenvectors of discrete systems may be determined by using one of three main approaches. These are the transfer matrix method, the random perturbation method, and the Monte Carlo numerical simulation algorithm. The transfer matrix method (Kerner 1954, 1956; Soong, 1962) utilizes a perturbational expansion of the random eigenvalues in terms of the random perturbations of the system parameters. The perturbation method is based on an asymptotic expansion and combines the ordinary perturbation and multivariate statistical analysis. The multivariate establishes the probability distributions of random eigenvalues in terms of the distributions of the matrix coefficients in the equations of motion. The Monte Carlo method, on the other hand, generates a random sample of the system random parameters which are used for computing numerically the eigenvalues and eigenvectors for each set of parameters in the sample. Monte Carlo simulations are expensive since they require a large number of numerical solutions to define the probability level at the tails of the distribution. This disadvantage becomes evident when one deals with large or medium size systems where numerical simulations become unrealistic on conventional digital computers. The first two methods will be outlined in the next two sections.

#### 3.2.1. Transfer matrix method

This method was first developed for disordered periodic lattice systems by Kerner (1954, 1956). It was adopted by Soong and Bogdanoff (1963) to examine the statistics of the random eigenvalues of disordered spring-mass chain of  $N$  degrees of freedom of the type shown in Figure 1(a). Basically the method is an extension of the transfer matrix developed originally for free vibration of deterministic discrete systems (Thomson, 1981). The method transfers the displacement vector  $\{X\}_j$  of the  $j$ th mass into next mass displacement vector  $\{X\}_{j+1}$ , i.e.

$$\{X\}_j = [I + T]_{j-1} \{X\}_{j-1}, \quad (4)$$

where  $I$  is the unit matrix and  $[I + T]$  is the transfer matrix. The first displacement vector  $\{X\}_1$  is related to the last displacement vector  $\{X\}_N$  by the relationship

$$\{X\}_1 = \left[ \prod_{j=1}^N [I + T]_j \right] \{X\}_N. \quad (5)$$

In order to demonstrate the method, a periodic disordered chain with random masses and constant equal springs of stiffness  $K$  will be considered. Let the random mass be defined by the expression

$$m_j = \bar{m}(1 + \epsilon_j), \quad (6)$$

where  $\bar{m}$  is the mean value of the mass and  $\epsilon_j$  is a small random variable with zero mean.

The transfer matrix can be written in the form

$$[I + T]_j = [I + T]_j + [E]_j, \quad (7)$$

where  $[E]_j$  is a perturbational transfer matrix which results from the random perturbations  $\epsilon_j$ .

The characteristic equation can be established from eq. (5). The roots of this equation are the system eigenvalues  $\omega_n$ . In order to determine the statistical properties of the eigenvalues it is necessary to express  $\omega_n$  in terms of the random variables  $\epsilon_j$ . It will be assumed that the range over which the values of  $\epsilon_j$  are distributed is small and  $\omega_n$  can be explained in powers of the

random variables  $\epsilon_j$ :

$$\begin{aligned} \omega_n &= \bar{\omega}_n + \sum_{j=1}^N \omega_{1j} \epsilon_j + \sum_{j=1}^N \omega_{2j} \epsilon_j^2 \\ &+ \sum_{j=1}^{N-1} \sum_{k=2}^N \omega_{1jk} \epsilon_j \epsilon_k + \dots \end{aligned} \quad (8a)$$

$$\approx \bar{\omega}_n + \sum_{j=1}^N \omega_{1j} \epsilon_j \quad \text{for small } \epsilon_j, \quad n = 1, 2, \dots, N. \quad (8b)$$

Let the random variables  $\epsilon_j$  be statistically independent, identically and normally distributed with zero mean. This means that the probability density function of each is

$$p(\epsilon) = \frac{1}{\sigma_j \sqrt{2\pi}} \exp\{-\epsilon^2/2\sigma_j^2\}, \quad (9)$$

where  $\sigma_j^2$  is the variance of the random variable  $\epsilon_j$ .

From the theory of random processes (Laning and Battin, 1956), it is known that if the random variables  $\epsilon_j$  are independent and normally distributed the random eigenvalues will be normally distributed with mean value  $\bar{\omega}_n$  and variance  $\sigma_n^2 = \sum_{j=1}^N \omega_{1j}^2 \sigma_j^2$ . These two statistical parameters provide the elements of the probability density function of  $\omega_n^2$ , i.e.

$$p(\omega_n) = \frac{1}{\sigma_n \sqrt{2\pi}} \exp\{-(\omega_n - \bar{\omega}_n)^2/2\sigma_n^2\}. \quad (10)$$

Figure 2 shows  $p(\omega_n)$  and the standard deviation  $\sigma_n$  for a spring-mass chain of 10 degrees of freedom with  $\sigma_j = 0.05$ . It is seen that the randomness of the masses results in a considerable dispersion in the high frequency region. The standard deviation of the random eigenvalues increases with the standard deviation of the mass perturbations  $\epsilon_j$  according to the formula (Soong, 1962)

$$\sigma_n/\sigma_j = \left\{ \sum_{j=1}^N \omega_{1j}^2 \right\}^{1/2}. \quad (11)$$

#### 3.2.2. Random perturbation method

The perturbation method for the deterministic eigenvalue problem is well documented (Cole, 1968; Meirovitch, 1980). The method has recently been extended for random eigenvalues by Scheidt and Purkert (1983). The eigenvalues of discrete systems are usually determined from the conservative part of the system equations of motion whose eigenvalue equation is given in the form

$$[\mathbf{K}(s) - \lambda \mathbf{M}(s)] \{x\} = \{0\}, \quad (12)$$

where  $\mathbf{K}(s)$  and  $\mathbf{M}(s)$  are symmetric stiffness and mass matrices, respectively. The elements of these matrices are taken from the entire sample space  $S$ , i.e.  $s \in S$ .  $\lambda_j$  and  $\{x\}_j$  are the  $j$ th eigenvalue and eigenvector, respectively. The random matrices  $\mathbf{K}(s)$  and  $\mathbf{M}(s)$  can be written as the sum of deterministic and random matrices

$$\begin{aligned} \mathbf{K}(s) &= \bar{\mathbf{K}} + \bar{\mathbf{k}}(s), \\ \mathbf{M}(s) &= \bar{\mathbf{M}} + \bar{\mathbf{m}}(s), \end{aligned} \quad (13)$$

where  $\bar{\mathbf{k}}(s)$  and  $\bar{\mathbf{m}}(s)$  represent random fluctuations in the stiffness and mass matrices, respectively, with zero means such that

$$\begin{aligned} |\bar{\mathbf{k}}(s)| &= \left\{ \sum_{i,j=1}^n \bar{k}_{ij}^2(s) \right\}^{1/2} < \epsilon_k, \\ \text{and} \\ |\bar{\mathbf{m}}(s)| &= \left\{ \sum_{i,j=1}^n \bar{m}_{ij}^2(s) \right\}^{1/2} < \epsilon_m \end{aligned} \quad (14)$$

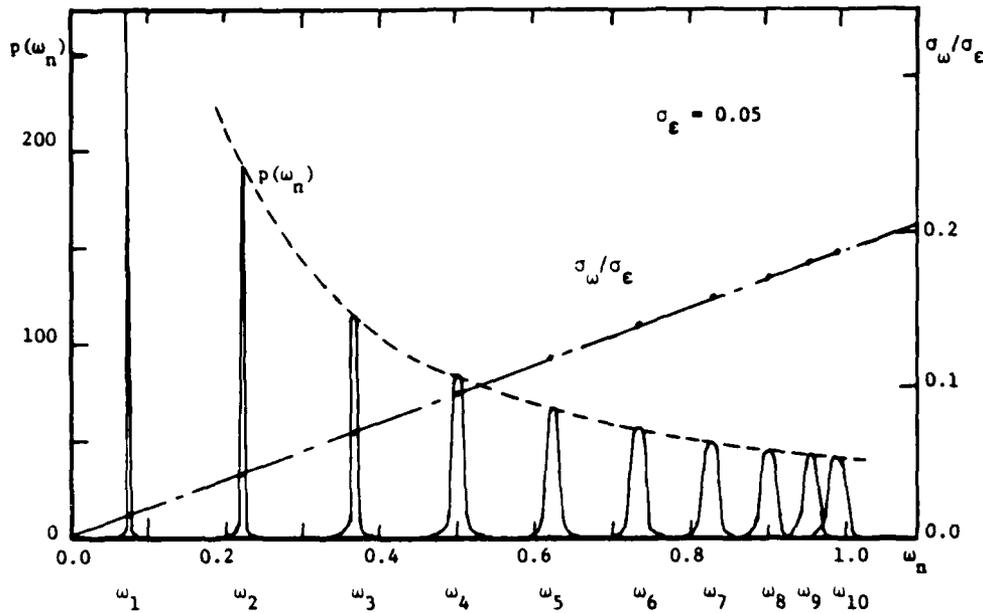


FIG. 2. Probability density functions and standard deviation of  $\omega_n$ .

Alternatively, the problem can be stated by transforming eq. (12) into the standard form

$$[\mathbf{A} - \lambda \mathbf{I}]\{\mathbf{x}\} = \{0\}, \quad (15)$$

where  $\mathbf{A}$  is the system dynamic matrix which is symmetric positive and has the random perturbational form

$$\mathbf{A}(s) = \bar{\mathbf{A}} + \mathbf{a}(s). \quad (16)$$

The deterministic matrix  $\bar{\mathbf{A}}$  has the simple eigenvalues

$$\bar{\lambda}_1 < \bar{\lambda}_2 < \dots < \bar{\lambda}_n. \quad (17)$$

while the random matrix  $\mathbf{A}(s)$  has the random eigenvalues

$$\lambda_1(s) < \lambda_2(s) < \dots < \lambda_n(s). \quad (18)$$

it is clear that the existence of the first two moments of the eigenvalues  $\lambda_j(s)$  is implied by the existence of the first two moments of the elements of  $\mathbf{A}(s)$ .

The eigenvectors  $\{\mathbf{x}_j\}$  are normalized by the relation

$$(\mathbf{x}_j, \mathbf{x}_j) = 1. \quad (19)$$

where  $(\mathbf{x}_j, \mathbf{x}_j)$  denotes the scalar (or inner) product of the same vector  $\mathbf{x}_j$ , i.e.  $\{\mathbf{x}_j\}_i^T \{\mathbf{x}_j\}_i$ . Introducing the two expansions

$$\lambda_j(s) = \bar{\lambda}_j + \sum_{k=1}^{\infty} \bar{\lambda}_{j,k}(s) \quad (20)$$

$$\{\mathbf{x}(s)\}_j = \{\bar{\mathbf{x}}\}_j + \sum_{k=1}^{\infty} \{\bar{\mathbf{x}}(s)\}_{j,k} \quad (21)$$

where  $\{\bar{\mathbf{x}}\}_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$  is the normalized eigenvector associated with  $\bar{\lambda}_j$ ,  $\bar{\lambda}_{j,k}(s)$  and  $\{\bar{\mathbf{x}}(s)\}_{j,k}$  are the contributions due to the perturbed elements of  $\mathbf{a}(s)$ . From the analytical dependence of  $\lambda_j$  and  $\{\mathbf{x}\}_j$  on the elements of  $\mathbf{a}(s)$ , Scheidt and Purkert (1973) showed that expansions (20) and (21) converge at least for sufficiently small values of the elements of  $\mathbf{a}(s)$ . The homogeneous terms  $\bar{\lambda}_{j,k}(s)$  and  $\{\bar{\mathbf{x}}(s)\}_{j,k}$  up to fourth order are given by Scheidt and Purkert (1983). These terms can then be used to determine the expectations and correlation relations of the random eigenvalues and eigenvectors. If the correlation between the elements of  $\mathbf{a}(s) = [a_{ij}]$  are only given, then up to first-order perturbation the means of the eigenvalues and eigen-

vectors are

$$E = [\lambda_j(s)] = \bar{\lambda}_j + \sum_{i=1}^n E[a_{ij} a_{ij}] / \bar{\lambda}_{ij} + \dots \quad (22)$$

$$E[\mathbf{x}(s)_j] = \left(1 + \frac{1}{2} E[(\bar{\mathbf{x}}_{j,1}, \bar{\mathbf{x}}_{j,1})] + E[(Z_{j,2})_{j,2}] + \dots\right) \bar{\mathbf{x}}_{j,0} \quad (23)$$

where  $\bar{\lambda}_{ij} = \bar{\lambda}_i - \bar{\lambda}_j$ ,  $\{Z_{j,2}\}_{j,2} = \{Z_{21}, Z_{22}, \dots, Z_{2n}\}_j^T$ , and the elements of  $\{Z_{j,2}\}_{j,2}$  are given by the expression

$$(Z_{j,2})_i = \frac{1}{\bar{\lambda}_{ij}} \left\{ \sum_{k=1}^n \frac{1}{\bar{\lambda}_{ik}} a_{ik} a_{kj} - \frac{1}{\bar{\lambda}_{ij}} a_{ij} a_{ij} \right\} \quad \text{for } i \neq j \text{ and } (Z_{j,2})_j = 0.$$

On the other hand, the correlation relations of the eigenvalues and eigenvectors up to the  $(k+1)$ -th order in the perturbations  $a_{ij}$  are

$$\begin{aligned} R_{\lambda}(j, k) &= E[\lambda_j, \lambda_k] = E[a_{jj} a_{kk}] \\ &+ \sum_{i=1}^n \frac{1}{\bar{\lambda}_{ij}} E \left[ a_{ij} a_{ij} a_{kk} \left(1 + \frac{a_{jj}}{\bar{\lambda}_{ij}}\right) \right] \\ &+ \sum_{i=j}^n \frac{1}{\bar{\lambda}_{ik}} E \left[ a_{ik} a_{ik} a_{jj} \left(1 + \frac{a_{kk}}{\bar{\lambda}_{ik}}\right) \right] \\ &+ \sum_{i=1}^n \frac{1}{\bar{\lambda}_{ij} \bar{\lambda}_{ij}} E[a_{ij} a_{ij} a_{ij} a_{ij}] \quad (24) \\ &+ \sum_{i=1}^n \frac{1}{\bar{\lambda}_{ik} \bar{\lambda}_{ik}} E[a_{ik} a_{ik} a_{ik} a_{ik}] \\ &+ \sum_{i=1}^n \frac{1}{\bar{\lambda}_{ij} \bar{\lambda}_{ik}} \{ E[a_{ij} a_{ij} a_{ik} a_{ki}] \\ &- E[a_{ij} a_{ij}] E[a_{ik} a_{ki}] \}. \end{aligned}$$

The analysis is called first order perturbation if first-order terms in expansions (20) and (21) are retained and higher-order terms are excluded. It is second order if terms up to second order are kept. However, second-order perturbation is tedious and involves multivariate statistical analysis. Most of the analyses reported in the literature deal with the first-order perturbation.

Problems involving a random symmetric matrix with multiple eigenvalues of the unperturbed matrix have been treated by Scheidt and Purkert (1983). The analysis consists in the formulation of a convergence condition for the perturbation expansions.

Collins (1967) and Collins and Thomson (1969) considered first-order perturbation and derived the eigenvalue and eigenvector statistics of a multi-degree-of-freedom system in terms of the covariance matrix of the system elements. With reference to the eigenvalue eq. (12) they showed that the variations in the mass and stiffness matrices result in the following first order variations in the eigenvalue and eigenvector, respectively:

$$\lambda_i - \bar{\lambda}_i = \sum_{j=1}^{n^2} \frac{\partial \lambda_i}{\partial k_j} (k_j - \bar{k}_j) + \sum_{l=1}^{n^2} \frac{\partial \lambda_i}{\partial m_l} (m_l - \bar{m}_l) + \dots, \quad (25)$$

$$x_{r,i} - \bar{x}_{r,i} = \sum_{j=1}^{n^2} \frac{\partial x_{r,i}}{\partial k_j} (k_j - \bar{k}_j) + \sum_{l=1}^{n^2} \frac{\partial x_{r,i}}{\partial m_l} (m_l - \bar{m}_l) + \dots \quad (26)$$

If the elements of the mass and stiffness matrices of eq. (12) are random variables with means  $\bar{k}_j$  and  $\bar{m}_l$  and variances  $\sigma_{k_j}^2$  and  $\sigma_{m_l}^2$ , then the expected eigenvalues and eigenvectors are  $\bar{\lambda}_i$  and  $\bar{x}_{r,i}$ , respectively, and the variance of the eigenvalue is

$$\sigma_{\lambda_i}^2 = \text{Var}(\lambda_i) = \sum_{j=1}^{n^2} \sum_{l=1}^{n^2} \frac{\partial \lambda_i}{\partial k_j} \cdot \frac{\partial \lambda_i}{\partial k_l} \text{cov}(k_j, k_l) + 2 \sum_{j=1}^{n^2} \sum_{l=1}^{n^2} \frac{\partial \lambda_i}{\partial k_j} \cdot \frac{\partial \lambda_i}{\partial m_l} \text{cov}(k_j, m_l) + \sum_{j=1}^{n^2} \sum_{l=1}^{n^2} \frac{\partial \lambda_i}{\partial m_j} \cdot \frac{\partial \lambda_i}{\partial m_l} \text{cov}(m_j, m_l) \quad (27)$$

where

$$\text{cov}(k_j, k_l) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_j - \bar{k}_j)(k_l - \bar{k}_l) p(k_j, k_l) dk_j dk_l = \rho_{jl} \sigma_{k_j} \sigma_{k_l} \quad (28)$$

and  $p(k_j, k_l)$  is the joint probability density function for  $K_j$  and  $K_l$ , and  $\rho_{jl}$  is the correlation coefficient for  $k_j$  and  $k_l$ . Expressions for  $\text{cov}(k_j, m_l)$  and  $\text{cov}(m_j, m_l)$  follow the same format of relation (8).

For a simple chain of equal springs and masses with uncorrelated random masses or with random uncorrelated stiffnesses, Collins and Thomson showed that the standard deviation of the frequency is governed linearly with the standard deviations of the masses and stiffnesses. The results were confirmed by an independent Monte Carlo simulation and were very close to those obtained earlier by Soong and Bogdanoff (1963). However, these linear relationships disappear when correlation exists in the masses or stiffnesses and the eigenvalues are not closely spaced. Recently Pierre (1985) considered two different discrete systems and employed a first-order perturbation to solve for the statistics of their eigenvalues. The first system is a mass-spring chain with random mass and the second is a chain of coupled pendula with random lengths. His results were found identical to those obtained by Soong and Bogdanoff.

Schiff and Bogdanoff (1972a, b) derived an estimator for the standard deviation of a natural frequency in terms of second-order statistical properties of the system parameters. The derivation was based on the mean square approximation developed by Bogdanoff (1965, 1966).

It may be noticed that the statistical properties of random eigenvalues are usually based on the assumption of normal distribution of the system random parameters. However, for correlated non-Gaussian parameters the analysis can be performed in terms of another set of Gaussian random parameters which are evaluated by using the Rosenblatt (1952) transformation. This transformation has extensively been used in reliability analysis when the performance function is nonlinear. This issue will be addressed in detail in section 5.1.

### 3.3. Random eigenvalues of continuous systems

#### 3.3.1. Methods of analysis

Continuous systems may involve uncertainties from two main sources. These are (Boyce and Goodwin 1964):

- (i) Uncertainties in the geometry and the material properties. The random variation in space dependent parameters results in variations of the differential operators governing the free vibrations of the structure.
- (ii) Uncertainties in the support mechanism of the system (or the boundary conditions).

The uncertainties of the first class constitute a random field. According to Vanmarcke (1984) the behavior of disordered systems is governed by two general laws. The first is a statement of the "conservation of uncertainty" as measured by the product of the variance by the scale of fluctuation of the property in the random field. The scale of fluctuation is taken as the area under the correlation function. This product remains invariant under linear transformation that preserves the mean. The second law states that the degree of disorder of a homogeneous random field, as measured by the direction-dependent bandwidth measure, tends to increase when a random field is subjected to local aggregation.

For the two classes of uncertainties the random eigenvalue has been determined for a limited class of dynamical systems. These include elastic strings and bars (Boyce, 1962; Goodwin and Boyce, 1964) and elastic beams (Boyce and Goodwin 1964; Bliven and Soong, 1969; Hoshiya and Shah, 1971; Shinozuka and Astill, 1972; Vaicatis 1974). Boyce (1968) outlined a number of techniques for determining the statistics of the eigenvalues of systems described by partial differential equations and boundary conditions involving uncertainty in their parameters. These differential equations are of order  $2n$  and usually written in the form

$$\mathcal{L}w(x) = \mathcal{N}w(x), \quad (29)$$

subject to the boundary conditions

$$\mathcal{U}_i(w) = 0, \quad i = 1, 2, \dots, 2n. \quad (30)$$

where  $\mathcal{L}$ ,  $\mathcal{N}$ , and  $\mathcal{U}_i$  are differential operators (with respect to the spatial coordinate  $x$ ) whose coefficients are random variables.  $w(x)$  is the displacement of the system at  $x$ . Equation (29) involves values of  $w$  and its first  $2n - 1$  derivatives at the end points of the interval in which solutions are sought. The eigenvalue problem defined by eqs. (29) and (30) is assumed to be self-adjoint and positive definite. The investigation of random eigenvalues has been carried out via analytical or numerical approaches. The numerical methods include the Monte Carlo simulation and stochastic finite element methods. The

analytical treatment of the random eigenvalue problem of systems described by eqs. (29) and (30) is outlined by Boyce (1968) and Scheidt and Purkert (1983). The mathematical methods which have been used to determine the statistical moments of eigenvalues are classified according to whether the statistical or nonstatistical part of the analysis is performed first. One class consists of first expressing the solution in terms of the system parameters, without regard to whether these parameters are random or deterministic. Having obtained such a solution, the statistical properties are then determined. According to Keller (1962, 1964) this approach is referred to as "honest" and the solution can be determined by using one of the following techniques (Boyce, 1968; Scheidt and Purkert, 1983):

- (i) Perturbation methods.
- (ii) Variational methods.
- (iii) Asymptotic estimate methods.
- (iv) Integral equation methods.

The "honest" approach does not provide an exact solution and the above four methods are not suitable for every problem. For example, the variational methods are not suitable for structures with random boundary conditions. Variational methods and integral equation methods are limited because they only lead to statements for the first eigenvalue of the system. Moreover, in order to apply the integral equation methods, very strong conditions for the calculation of the mean of the eigenvalues are required. Under certain conditions pertaining to the spatial correlation function, the asymptotic methods and perturbation techniques lead to the same results. The perturbation methods have less restrictions and are extensively used in the literature.

The approach, on the other hand, is called "dishonest" (Boyce, 1967) if the statistics of the eigenvalue problem are directly determined by performing averaging analysis to the system's partial differential equation and its associated boundary conditions. The statistics can be evaluated by using one of the following methods:

- (i) Iteration methods.
- (ii) Hierarchy methods (Haines, 1965, 1967; Adomian, 1983).

The iteration methods are based on some assumptions for the correlation relations in order to solve the averaged integral equations of the random eigenvalue. The hierarchy methods take into consideration further equations so that all statistical functions in question can be calculated.

In a series of papers by Purkert and Scheidt (1977, 1979a, b), a number of theorems pertaining to functionals of weakly correlated processes encountered in the eigenvalue problems, boundary value problems, and initial value problems were established. They treated the stochastic eigenvalue problem for ordinary differential equations with deterministic boundary conditions. The coefficients of the differential operator were independently weakly correlated processes of small correlation spatial length. They showed that as the correlation length becomes very small, the eigenvalues and eigenvectors possess Gaussian distributions. This result has recently been confirmed by Boyce and Xia (1983). When the random terms are not small the perturbation method is no longer valid and the second term in the Hermite-Chebyshev expansion (Ibrahim, 1985) of the distribution function will not vanish. This implies that the distribution of the eigenvalue will not be normal. Boyce and Xia (1985) obtained the upper bounds for the mean of eigenvalues through a variational characterization of the eigenvalues. For stochastic boundary value problems Linde (1969) and

Boyce (1966, 1980) considered a Sturm-Liouville problem with a stochastic nonhomogeneous term. In their recent monograph Scheidt and Purkert (1983) analyzed the moments of the eigenvalues and mode shapes of random matrices and random ordinary differential operators. The calculations of these moments were based on perturbation expansions, and so required the random terms to be appropriately small. Day (1980) developed a number of asymptotic expansions for the random eigenvalues and eigenvectors of continuous systems.

The concept of the Wiener field, which is obtained by replacing the time variable of the Wiener process by a space coordinate, was adopted by Wedig (1976, 1977) as a basic model for randomly distributed loadings or imperfections of continuous structural systems. The solution of such boundary value problems may thus be described by integral equations defined on the Wiener field and thus possesses the Markov properties. Wedig showed that these integral equations may be interpreted in the mean square sense via the boundary and eigenvalue problems of elastic structures with random distributed imperfections or loadings.

### 3.3.2. Applications

The random eigenvalue of a column under axial force  $F$ , shown in Figure 1(f), is described by the second order partial differential equation

$$\frac{\partial^2}{\partial x^2} EI(x)^2 \frac{\partial^2 w(x, t)}{\partial x^2} + F \frac{\partial^2 w(x, t)}{\partial x^2} + A(x) \frac{\partial^2 w(x, t)}{\partial t^2} = 0 \quad (31)$$

and the boundary conditions:

$$\begin{aligned} EI(0) \frac{\partial^2 w(x, t)}{\partial x^2} \Big|_{x=0} - K_1 \frac{\partial w(x, t)}{\partial x} \Big|_{x=0} &= 0, \\ w(0, t) &= 0; \\ EI(L) \frac{\partial^2 w(x, t)}{\partial x^2} \Big|_{x=L} + K_2 \frac{\partial w(x, t)}{\partial x} \Big|_{x=L} &= 0, \\ w(L, t) &= 0. \end{aligned} \quad (32)$$

where  $w(x, t)$  is the lateral displacement at distance  $x$  and time  $t$ ,  $L$  is the length of the column,  $EI(x)$  is the flexural stiffness, and  $\rho A(x)$  is the column mass per unit length.  $K_1$  and  $K_2$  are the stiffnesses of the end springs. For simple supports  $K_1 = K_2 = 0$ , and for fixed supports  $K_1 = K_2 = \infty$ .

The solution of eq. (31) may be expressed in the form

$$w(x, t) = \sum U_j(X) \exp(i\omega t). \quad (33)$$

Introducing the following substitutions

$$\begin{aligned} X &= x/L, \\ I(X) &= I[1 + \alpha(X)], \\ A(X) &= \bar{A}[1 + a(X)], \\ \mu &= FL^2/EI, \\ \lambda &= \rho \bar{A} L^4 \omega^2 / EI, \end{aligned}$$

where  $a(X)$  and  $\alpha(X)$  are random variables, eq. (31) and the boundary conditions (32) for mode  $j$  become:

$$\{[1 + \alpha(X)]U''(X)\}'' + \mu U''(X) - \lambda[1 + a(X)]U(X) = 0, \quad (34)$$

$$\begin{aligned} [1 + \alpha(0)]U''(0) - (K_1 L/EI)U'(0) &= 0 & U(0) &= 0; \\ [1 + \alpha(1)]U''(1) + (K_2 L/EI)U'(1) &= 0 & U(1) &= 0. \end{aligned} \quad (35)$$

where a prime denotes differentiation with respect to  $X$ , and subscript  $j$ , indicating the mode number in expansion (33), is removed.

Hoshiya and Shah (1971) employed the standard perturbation analysis to determine the expected value and variance of the eigenvalue of the  $n$ th mode by using a linearized perturbation technique. They found that the variance of the  $n$ th natural frequency is proportional to the variances of the stiffness coefficients at the boundaries and the axial load. This linear relationship implies that the principle of superposition can be applied in a modified form. For the buckling case, i.e. when  $\lambda = 0$ , the eigenvalue problem is reduced to determine the statistics of the buckling eigenvalue (Augusti et al, 1981, 1984). Shinozuka and Astill (1972) considered the case when both  $K_1$  and  $K_2$  are random variables.

The natural frequencies of transverse vibration of elastic beams were analyzed by Boyce and Goodwin (1964). They considered the geometry of the cross-section of the beam and its support mechanism as random variables. The statistics of the eigenvalues were determined by using three different techniques. These were the perturbation method, the method of integral equations, and numerical solution. Bliven and Soong (1969) determined the statistics of the natural frequencies of a simply supported elastic beam with random imperfections in the beam stiffness. The beam was modeled as a lumped-parameter model and the properties of the frequencies were derived by using a perturbation method. The stiffness random variation was represented by the relation

$$EI = \bar{EI}[1 + \alpha(x)],$$

where  $\bar{EI}$  is the mean value of the beam stiffness and  $\alpha(x)$  is a stationary random field process with zero mean and autocorrelation function given by the relation

$$E[\alpha(x_1), \alpha(x_2)] = \sigma^2 \exp(-|x_1 - x_2|/d), \quad (36)$$

where  $d$  is a non-negative constant known as the correlation distance.

The standard deviation of the natural frequency of the beam was obtained in the closed form

$$\sigma_{\omega}(n) = \bar{\omega}(n) \sigma \sqrt{g(n)}, \quad (37)$$

where  $\bar{\omega}(n)$  is the  $n$ th mode natural frequency of the uniform beam  $= n^2 \pi^2 \bar{EI}/mL^4$ ,

$$g(n) = \int_0^1 \int_0^1 \sin^2(n\pi x_1) \sin^2(n\pi x_2) \\ \times \exp[-|x_1 - x_2|/d] dx_1 dx_2,$$

and  $\bar{m}$  is the beam mass per unit length.

Bliven and Soong found that when the stiffness fluctuation has zero correlation distance  $d = 0$ , the natural frequency standard deviation vanishes. The standard deviation was found to reach the value of  $\sigma_{\omega}(n) = 0.5\bar{\omega}(n)\sigma$  when the stiffness variation is perfectly correlated ( $d \rightarrow \infty$ ).

The random eigenvalue of a beam-column supported at its ends by a rotary springs was examined by Shinozuka and Astill (1972). The spring supports and axial applied force were treated as random variables. The distribution of material and geometric properties were considered correlated homogeneous random functions. The distributions of these properties were generated by using a Monte Carlo simulation for multivariate and multidimensional random processes developed originally by Shinozuka (1971). The mean and variance of the eigenvalues were determined by using the perturbation analysis and Monte Carlo simulation. It was found that the application of approximate methods, such as the perturbation technique based on exact or an assumed mode shape, causes a considerably

greater error for the buckling case than in the vibration case. Furthermore, the perturbation solution for the eigenvalue variance can be approximated reasonably well by using an assumed mode shape in place of the unperturbed mode shape. Vaicaitis (1974) employed a two-variable perturbation expansion procedure to determine the eigenvalues and normal modes of beams with random and/or nonuniform characteristics which do not deviate considerably from the beam mean properties. A Monte Carlo simulation was used to determine the statistical averages of beam eigenvalues and mode shapes. Two cases of random fluctuations of beam cross section were considered. For one particular case there was significant deviation attributed mainly to the fact that gradual change in the beam stiffness was permitted. In this case the beam is "soft" at one end and "hard" at the other end.

Hart and Collins (1970), Collins et al (1971), and Hasselman and Hart (1971, 1972) developed a numerical method for computing the variance of structural dynamic mode properties by using component mode synthesis which was formulated originally by Hurty (1964, 1965). Numerical solution provided reasonable results for lower modes even when a relatively small percentage of available component modes is used. Hart (1973) developed a general algorithm for calculating the statistics of the natural frequencies and mode shapes of structures acted upon by an external static loading. This type of problems involve considerable calculations due to the fact that the proportionate axial load in each member of the structure is dependent upon the structural parameters which are random variables. For the two-bar truss shown in Figure 3 Hart determined the first natural frequency's mean and standard deviation. The influence of the static load on the statistics of the first natural frequency is shown in Figure 4. It is seen that the standard deviation of the natural frequency increases with the axial load. The implication of this increase was further demonstrated in Figure 5 by using normal probability density function. The observed flattening shape of the probability density function with increased compressive loading shows a marked decrease in confidence with the magnitude of loading.

The random eigenvalue problem of disordered periodic beam was considered by Lin and Yang (1974). They used a first-order perturbation procedure to derive expressions for the variances of natural frequencies and normal modes for different cases of random bending stiffness and span lengths. The natural frequencies were found to be more sensitive to span variations than to bending stiffness fluctuation. It was shown that if the random variations in bending stiffness for different spans are uncorrelated then there is no effect on the statistics of the eigenvalues. The effect exists only when there is a correlation in the random variation in the individual spans. For a random variation in the span lengths it was shown that the variance of the natural frequency is inversely proportional to the number of

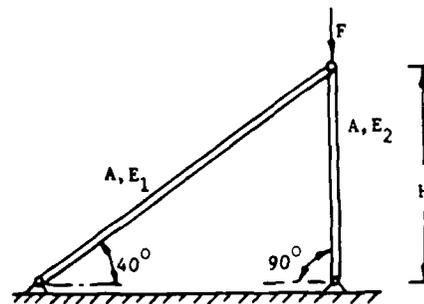


FIG. 3. Two bar truss (Hart, 1973).

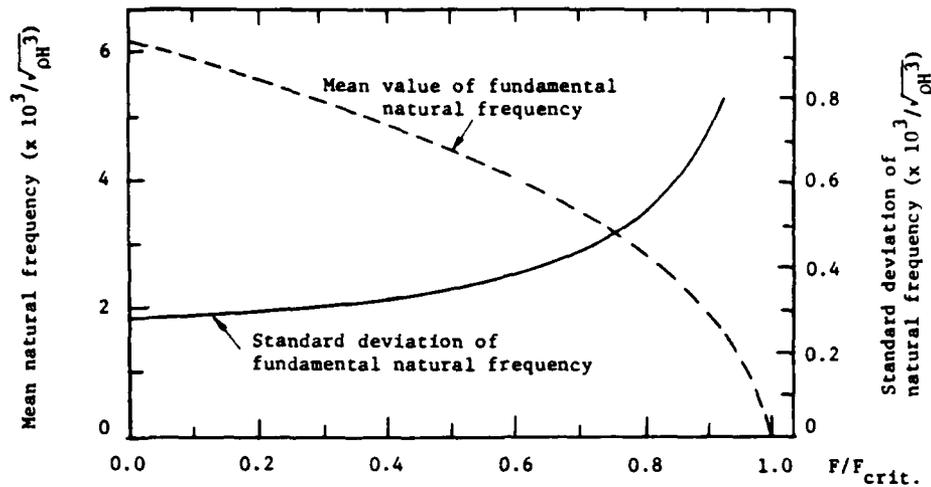


FIG. 4. Variation in fundamental natural frequency statistics with applied load (Hart, 1973).

spans. The random imperfections in spatial periodicity also resulted in variability in the normal modes. However, due to the arbitrary choice of modal amplitude the variance of the normal mode was not a unique function of space.

The statistics of natural frequencies of mistuned blades of a circumferentially closed packet of turbomachinery were examined by Ewins (1973) and Huang (1982). When the bladed disk assembly is tuned and all the blades are identical the natural frequencies and mode shapes are quite regular. Each mode may be described as having a particular number of nodal diameters, just as for an unbladed disk. However, when the blades are mistuned to a degree which might well exist in service, the mode shapes and frequencies becomes irregular. In this case the natural frequencies of the individual blades can be randomly different from one another. This problem is belonging to systems with periodic random parameters and such systems are modeled by a stiff ring supported by transverse springs with randomly distributed stiffness and mass parameters. Huang adopted an exponential form for the auto- and cross-correlation function of the random structural parameters. This form was originally assumed by Hoshiya and Shah (1971). The analysis of Huang was based on a spectral analysis method. He found that the mean of the natural frequency of the structure with random parameters is identical to the natural frequency of the structure

with homogeneous parameters. The standard deviation of the natural frequency of  $m$ th mode was expressed in terms of the  $(2m)$ th Fourier coefficients of the random parameters and was represented as a vector sum of their standard deviations. While the normal modes of a homogeneous structure have a shape of harmonic waves with symmetrically located nodal diameters, for a structure with random parameters the mode shapes are complicated and the nodal diameters are located unsymmetrically. It was shown that these modes have a shape involving not only the main harmonic, but also an infinite number of harmonics. In addition, these random normal modes are orthogonal despite their complicated form. Another important feature was that the phase angles of random normal modes are not arbitrary (as in the case of a homogeneous structure) but are random variables independent of the initial conditions.

Recently, the stochastic finite element method has been used by Nakagiri et al (1985) to determine the uncertain eigenvalue of fiber reinforced plastic (FRB) laminated plates. These composite materials usually exhibit anisotropy and heterogeneity. The elastic constants may fluctuate around the mean values due to some slackness during the manufacturing process which causes spatial distribution of the volume fraction. In addition, another parameter known as the stacking sequence is usually used as a major design parameter of the FRB laminated plates

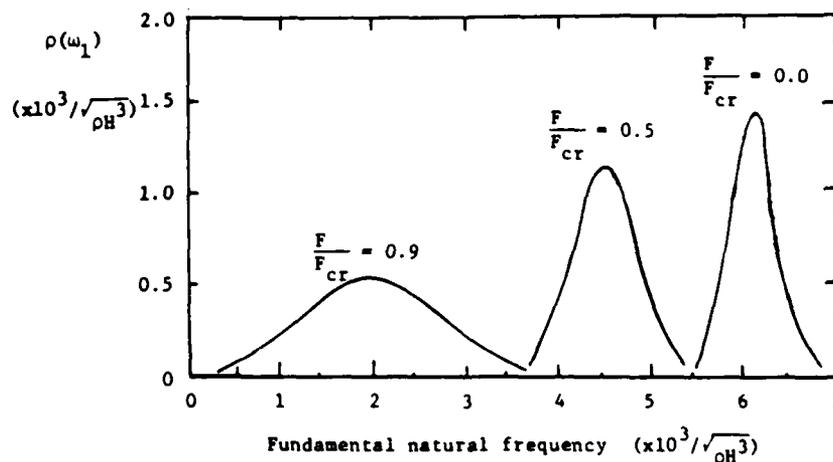


FIG. 5. Probability density function variation with applied load (Hart, 1973).

The stacking sequence (Vinson and Chou, 1975) implies a group of parameters such as elastic constants, layer number, fiber orientation, and layer thickness. Nakagiri et al considered the effect of the fluctuation of the overall stiffness due to uncertain variation of the stacking sequence. The uncertain stacking sequence was treated as a set of random variables for the case of simply-supported graphite/epoxy plates. It was found that the eigenvalue is more sensitive to the standard deviation of the fiber orientation, and the effect of the stacking sequence is more pronounced for the rectangular plate than for the square one.

### 3.3.3. Normal mode localization

Periodic structures with slight variations in their periodicity can exhibit a phenomenon known as normal mode localization. This phenomenon takes place in a manner that vibrational energy injected into the structure by an external source cannot propagate to arbitrarily large distances, but is instead substantially confined to a region close to the source. Hodges (1982) called this phenomenon as "Anderson localization" due to Anderson (1958) who discovered mode localization in solid state physics in an attempt to understand electrical conduction processes in disordered solids. The effect of irregularities has a similar effect to damping in that it limits the propagation of vibrations at large distances from the excitation source. This effect is mainly caused by confinement of the energy close to the source, not by dissipation of the energy as it propagates out.

The phenomenon of mode localization can be well understood by using the coupled pendula example [Fig. 1(b)] which was adopted by Hodges (1982). Hodges provided an excellent explanation of mode localization: If all pendula are identical so that their individual natural frequencies are precisely equal, then the normal modes of oscillation when these pendula are coupled together extend throughout the system, the amplitude of oscillation of each pendulum varies sinusoidally with its position in space. On the other hand, if the natural frequency of oscillation varies from pendulum to pendulum in some kind of random fashion, then in the limit of zero coupling, normal modes consist of oscillation of individual pendula at frequencies equal to their natural frequencies. For small coupling the normal modes remain localized close to individual pendula and the normal mode frequencies approximate the natural frequencies of the pendula. Thus for a particular mode one pendulum is oscillating close to its natural frequency with a large motion. Its nearest neighbors, unlike the ordered system, are driven off resonance, and since the coupling is weak they respond with much smaller amplitudes. These neighbors in turn drive pendula further out and so on, but at each step the driving force and response tend to diminish in magnitude. A typical mode shape diagram is shown in Figure 6. In terms of forced oscillations, mode localization implies localization of the response in the vicinity of the excitation point.

The effect of mode localization was examined by Bendiksen (1984a, b) and Valero and Bendiksen (1985) who showed that

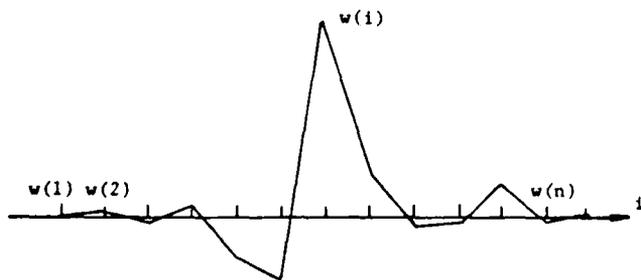


FIG. 6. Schematic diagram of the site amplitude  $w(t)$  for a localized normal mode (Hodges, 1982).

irregularities in shrouded blades of jet engine rotors can result in a stabilizing mechanism which is closely connected with the phenomenon of mode localization. In the framework of localization theory, the stabilizing mechanism is explained based on the fact that the original monochromatic flutter wave is scattered into waves of different and more stable wavelengths and inter-blades phase angles. While the effect of mistuning between turbomachinery blades is favorable in flutter (see also Kaza and Kielb, 1982) it can lead to an increase in amplitude on at least one blade in forced vibration situation as will be shown in section 4.1.2.

For periodic multispan beams Miles (1956) showed that the natural frequencies are clustered in an infinite number of groups, or bands, with  $n$  frequencies in each band, where  $n$  is the number of spans. If a torsional spring is placed at the  $n-1$  intermediate support location, then the width of the frequency bands diminishes as the spring constant  $k$  increases. In the limit as the spring constant goes to infinity, the beam becomes clamped at the constraint locations and the width of the frequency bands is reduced to zero. Pierre et al (1986) established an internal coupling parameter which is equivalent to the inverse of the torsional spring constant  $1/k_\theta$ . For  $k_\theta = 0$  the spans are fully coupled. For large values of the spring constant and irregular spacing between supports, a multispan beam can be regarded as a disordered chain of weakly coupled subsystems. Pierre (1985) and Pierre and Dowell (1986) developed a theoretical analysis for the mode localization phenomenon and indicated that the free modes of vibration are susceptible to becoming localized and the natural frequencies of the multispan beam are in bands of small width if the spring constant is large. They proposed a general criterion stating that localization may occur if the width of the frequency band of the ordered system is of the order of, or smaller than, the spread in individual natural frequencies of the disordered component systems.

Pierre et al (1986) determined the free modes of transverse vibration of a disordered two-span beam by using a Rayleigh-Ritz formulation with the constraint conditions enforced by means of Lagrange multipliers. They developed a modified perturbation method to analyze the localized modes. Figure 7 shows the mode shapes for tuned and mistuned beam for torsional spring parameter  $c = 1000$ , where  $c = 2Ik_\theta/EI$ ,  $l$  is the length of the beam and  $E$  and  $I$  are the Young's modulus and area moment of inertia of the beam, respectively. For a mistuned beam it is seen that mode localization is manifested in that the peak deflection is much larger in one span than in the other one.

## 4. RANDOM RESPONSE

The response of linear structural components with uncertain parameters can be determined by using standard techniques such as the impulse and frequency response functions and perturbation methods, or numerical approaches such as stochastic finite methods and Monte Carlo simulation. The results reported in the literature will be reviewed in the next two subsections.

### 4.1. Standard techniques

#### 4.1.1. Simple structural components

In an attempt to examine certain aspects of the dynamical response of statistically defined systems, Chenea and Bogdanoff (1958) and Bogdanoff and Chenea (1961) considered a linear single degree-of-freedom system with independent discrete distributions in the mass, damping, and stiffness coefficients. The analysis of Bogdanoff and Chenea was based on a partial

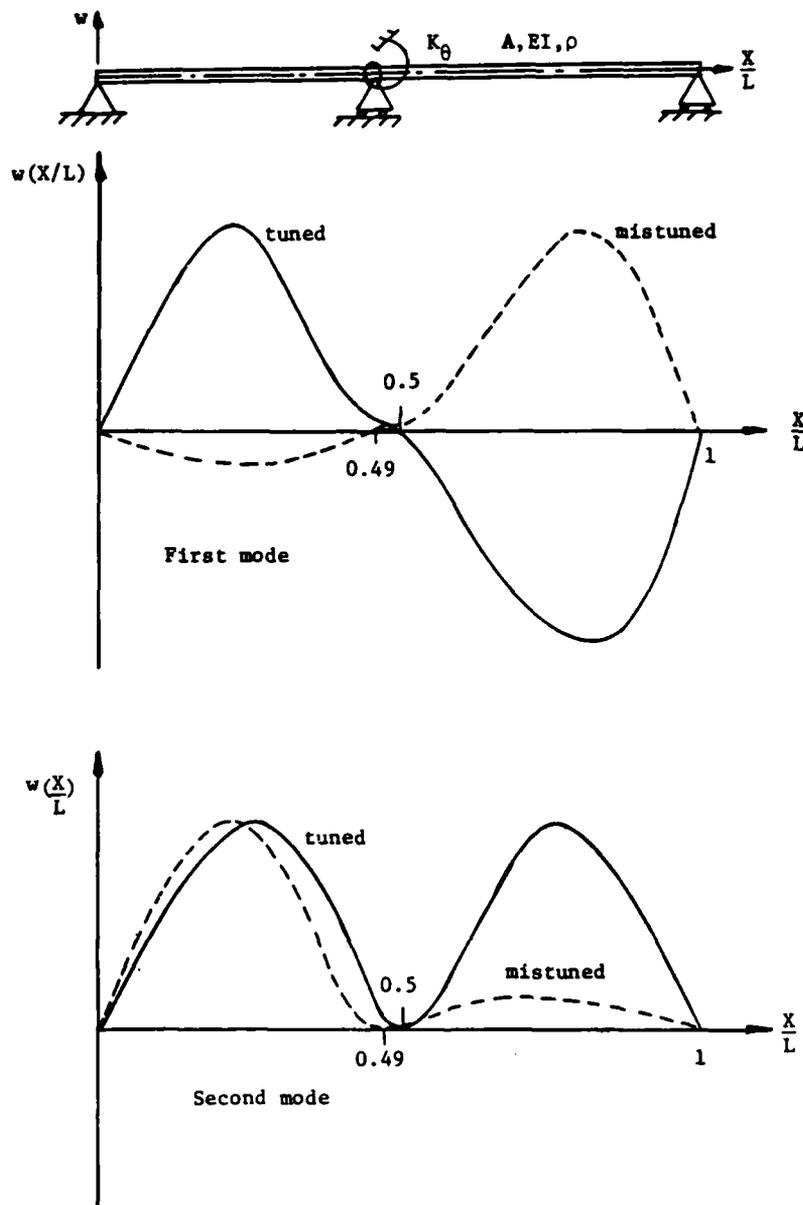


FIG. 7. First two mode shapes for tuned (—) and mistuned (---) two-span beam:  $\Delta l = 0.01$ ,  $c = 1000$  (Pierre et al, 1986).

differential equation for the response joint density function (Kozin, 1961). This equation is known as the Liouville equation (Soong, 1973) and is identical to the Fokker-Planck equation with zero diffusion coefficient. Small dispersions in the system parameters were found to result in a considerable dispersion in the frequency response. The impulse response of the same system was determined by using the perturbation method by Chen and Soroka (1973). They considered a linear system described by the differential equation

$$\ddot{X} + 2\zeta\omega_n\dot{X} + \omega_n^2 X = f(t), \quad (38)$$

where the natural frequency is considered random  $\omega_n = \bar{\omega}_n + \epsilon\bar{\omega}_n$ ,  $\bar{\omega}_n$  is a constant and the perturbation  $\bar{\omega}_n$  is a random variable with zero mean.  $\epsilon$  is a small perturbational parameter and  $f(t)$  is an impulse excitation. Chen and Soroka derived the solution of equation (38) by using a perturbational technique. Figure 8 shows a sample of the time history response curves for damping ratio  $\zeta = 0.05$ . It is seen that both the mean and

standard deviation of the response amplitude are nonstationary and the standard deviation is 90 degrees out of phase from the mean. The amplitude of the response standard deviation increases with time, and gradually dampens out after it reaches a certain level. For systems with a very high natural frequency, the uncertainty in the natural frequency was found to have very small effect on the response statistics. However, the effect is significant if the natural frequency is low. As the damping factor decreases, the dispersion from the mean became substantial.

The response of multi-degree-of-freedom systems with random parameters was examined by Soong and Bogdanoff (1963, 1964) and Chen and Soroka (1974). Soong and Bogdanoff determined the statistics of the impulse admittance and frequency response of a linear chain with random masses distributed in a small range. Chen and Soroka developed a method which relates the statistics of response parameters to the statistics of the system eigenvalues and eigenvectors. They showed

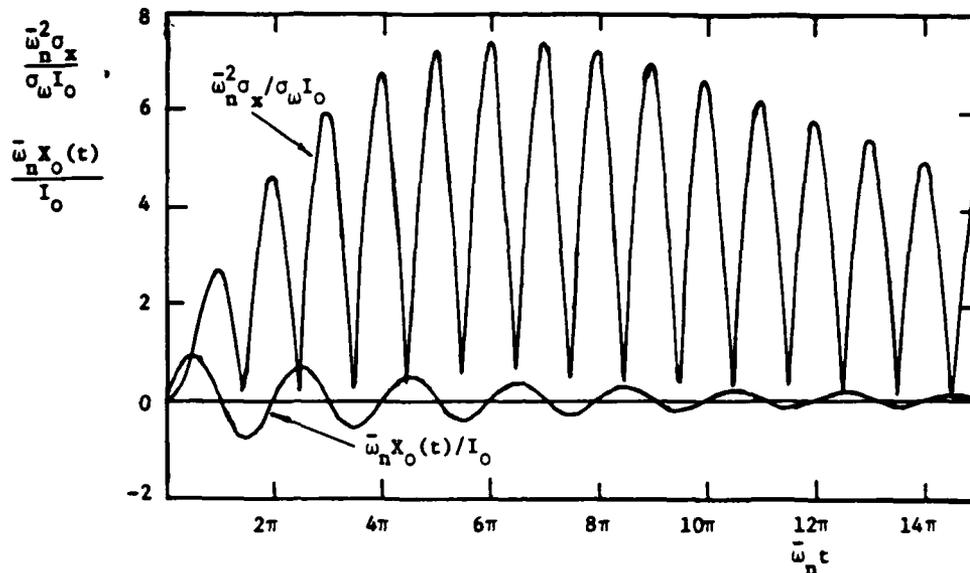


FIG. 8. Mean and standard deviation of impulse function  $I_0 \delta(t)$  for  $\zeta = 0.05$  (Chen and Soroka, 1973).

that the response statistics of disordered systems are higher than those of purely deterministic systems. The instantaneous transient response statistics of an undamped linear multi-degree-of-freedom system, with random stiffness, subjected to arbitrary but deterministic forcing functions was investigated by Prasthofer and Beadle (1975). For the case of an impulsive excitation, they found that the growth of the response uncertainty is exponential. As the standard deviation of the stiffness increases the response mean square increases rapidly with time. For a multi-degree-of-freedom system the response decay rate decreases as the correlation coefficient between the stiffness elements increases. The influence of damping uncertainty on the frequency response of a linear multi-degree-of-freedom system was examined by Caravani and Thomson (1973). They determined the mean and standard deviation of the response by using a linearization technique and a Monte Carlo simulation. They pointed out that an accurate estimate of the damping coefficients for lightly damped systems, in the neighborhood of a natural frequency, is very important in determining the mean and standard deviation of the system response.

The means and variances of the frequency response functions of a disordered periodic beam were studied by Yang and Lin (1975). Two types of excitation were considered. These were a concentrated force (or moment) and a distributed force connected at a constant velocity. It was shown that the magnitude of the statistical average of the frequency response function can be considerably greater than the value computed without taking into account the random variation in the span lengths. In the neighborhood of resonance frequencies the standard deviation of the frequency response function becomes quite large, indicating greater uncertainty in such regions. In the case of connected loading the use of a perfect periodic model cannot account for the response in certain vibration modes while these modes can be induced in a disordered periodic beam.

#### 4.1.2. Mistuned bladed disks

It has been indicated in section 3.3.3 that the mistuning of turbomachinery bladed disks could have beneficial effect in the case of blade flutter. However, the effect is reversed in the case of forced vibration (Whitehead, 1966; Ewins, 1969; Stange and MacBain, 1983). It is believed that Tobias and Arnold (1957) have made the first attempt to understand the effect of blade mistuning on the response of stationary waves (modes traveling

opposite to the direction of disk rotation so as to appear stationary to a fixed observer). An interesting and important structural phenomenon resulting from mistuning is the splitting of a bladed disk's diametral modes of vibration (modes having  $1, 2, \dots, n$  nodal diameters) into "twin" or "dual" modes. The presence of dual modes characteristics in a bladed disk can significantly affect either or both of its aeroelastic stability and resonant response characteristics. Whitehead (1966) showed that there is an upper limit to the effect of mistuning and is given approximately by the factor of  $(1 + N)/2$ , where  $N$  is the number of blades in the row. This upper limit was obtained under the assumption that the damping forces are substantially less than the aerodynamic coupling forces. Jay and Burns (1984) conducted a series of rotating and unrotating test to identify mistuning, damping, split factors for various diametral patterns and dynamic strains signatures from resonant tests of a shrouded fan blade/disk. System mode responses to various distortion patterns were found to involve standing waves and traveling waves.

A number of lumped mass models of bladed disk assemblies have also been used to study the effects of various blade mistune distributions on the maximum resonant response of the blades (Wagner, 1967; Dye and Henry, 1969; El-Bayoumy and Srinivasan, 1975; MacBain and Whaley, 1984). The nature of the lumped parameter models used in these studies is such that individual blade response was studied in terms of single- or two-degree-of-freedom blade modes whose vibratory response was altered by mechanical coupling via the disk portion of the models. Hence, the basis or starting point for these lumped mass models was the individual blade resonant frequencies. The results showed how much greater or smaller the individual blade response would be for a set of mistuned blades compared to the response of a tuned set of blades. For a given mistuning distribution and excitation, the response of the mistuned set of blades was found to be many times greater or smaller (depending upon the disk circumferential location) than the response of tuned blades. Ewins and Han (1984) conducted a series of case studies to examine the influence of various parameters on the resonant response levels of individual blades on a disk. They found, for the case of a 33-bladed disk, that mistuning always increases the highest resonant response level from that experienced by a tuned system but while some blades are

more highly stressed, others suffered a lower level and the mean value is roughly constant. It was also concluded that the highest response is always experienced by a blade of extreme mistune.

Analytical investigations of mistuning fall into three categories (Griffin and Hoosac, 1984): *deterministic* (Dye and Henery, 1969; Ewins, 1973; El-Bayoumy and Srinivasan, 1975), *statistical* (Huang, 1982), and *combined and statistical approaches* (Sogliero and Srinivasan, 1980; Kazan and Kielb, 1982; Muszynska et al, 1981). Basu and Griffin (1986) used a deterministic/statistical approach and developed a model involving aerodynamic and structural interaction for studying the effect of mistuning on bladed disk vibration. They found that the mistuning effect significantly decreases as the density of the gas flowing through the turbine is decreased. On the other hand the effect was found to increase linearly with the number of blades on the disk.

#### 4.2. Stochastic finite element methods

Recent developments of stochastic finite element methods have promoted the analysis of structural dynamics with uncertain parameters. These techniques could be broadly classified into statistical and nonstatistical (Liu et al, 1985b). The statistical approach is based on numerical simulation via Monte Carlo, stratified sampling, and Latin Hypercube sampling. A comparative discussion of these techniques is provided by McKay et al (1979). All simulation methods require that the joint probability distributions of the excitation and random parameters be available. However, these distributions are seldom to be available. Instead, one usually may assume that the input random variables are mutually independent and Gaussian. If these random inputs are non-Gaussian distributed, one may use the Rosenblatt (1952) transformation to transform non-Gaussian correlated variables to Gaussian uncorrelated ones. Nonstatistical approaches include numerical integration (Liu et al, 1985a, 1986), second moment analysis (Cornell 1972) and stochastic finite element methods (Nakagiri et al, 1984; Liu et al, 1985a, b; Hisada and Nakagiri, 1982; Hisada et al, 1983). A major advantage of these methods is that the multivariate distribution functions need not to be known but only the first two moments. Recently several stochastic finite element approaches have been developed by Vanmarcke and Grigoriu (1983), Liu et al (1985a, b), Dias and Nagtegaal (1985), and Mori and Ukai (1986). Linear problems in structural mechanics with uncertain parameters have been solved by second-moment analysis (Contreras, 1980; Nakagiri et al, 1984).

Astill et al (1972) examined the problem of impact loading of structures with random geometric and material properties. Their approach is a combination of finite element method and a Monte Carlo simulation. For the case of an axisymmetric concrete cylinder they assumed spatial distributions of Young's modulus and density for each realization of the test cylinder. Each test cylinder was subjected to the same axial impact loading. The algorithm gave a sample of 100 maximum stress intensities from which the sample mean and standard deviation were computed. For a certain intermediate location of the test cylinder it was found that the axial stress is always different from the corresponding stress in a uniform cylinder.

Vanmarcke and Grigoriu (1983) developed a stochastic finite element analysis for solving first- and second-order statistics of the deflection of structural members whose properties vary randomly along their axis. The covariance matrix of these element averages was obtained by simple algebraic operations on the variance function which in turn depends primarily on the scale fluctuation. Although this approach was used to determine the free end deflection of elastic members, Vanmarcke and

Grigoriu claimed that it can be applied to determine the response statistics to external dynamic excitations even when the statistical information about spatial variation of material properties is limited. Recently Liu et al (1985a, b, 1986) developed a number of probabilistic finite elements methods for nonlinear structural dynamics. These methods are applicable for correlated and uncorrelated discrete random variables. For elastic-plastic bar with end load, they (Liu et al, 1985b) computed the mean and variance of the displacement at the free end by using the probabilistic finite element and Monte Carlo simulation. The solutions of the two methods compared very well, however, the probabilistic finite element approach required much less computer time than the Monte Carlo simulation. Unfortunately these results did not reflect the influence of parameter uncertainties on the random response.

The dynamic response of random parametered structures under random excitation has been examined in a number of studies by Paez and his group (Chang, 1985; Bennett, 1985; Branstetter and Paez, 1986). These studies provide computer programs in a finite element framework to establish response moments on a step-by-step basis. These numerical algorithms evaluate the system response characteristics at an advance time by using the statistical information about response structural characteristics, and excitation at a previous time. Branstetter and Paez (1986) examined their computer programs for several damped single degree of freedom systems and several undamped two degree-of-freedom systems. The responses of these systems to white noise excitations were obtained for random stiffness parameters while all other system parameters were fixed. It was shown that single-degree-of-freedom systems display greater response variance than systems with deterministic stiffness. The difference in response variance is found to be small when the structure initial conditions are zero. The difference increases and assumes an oscillatory character when the initial conditions depart from zero. The mean response is non-zero for structures with nonzero initial conditions and/or non-zero mean load.

Bennett (1985) considered uncertainties in the stiffness and damping of single- and multi-degree-of-freedom structural systems. The random variables of the system parameters were replaced by a deterministic component (equal to the mean of the original random variable) and a random component with zero mean and with variance equal to that of the original random variable. For a single-degree-of-freedom system Bennett found that the value of the peak response increases monotonically with the standard deviation of the stiffness. For lightly damped systems which do not have zero mean, the effects of the damping randomness on the response are less pronounced than those obtained when the stiffness was random. The standard deviation of the response at the time of peak response was found to increase with the correlation between the stiffness and damping.

## 5. DESIGN OPTIMIZATION AND RELIABILITY

### 5.1. Reliability-based design

The study of response of disordered systems is very important for design purposes. These responses can help the designer to establish acceptable tolerances on system components. The main problem which concerns the designer is how to govern the fluctuations of the system parameters for safe operations. For example when the values of the elastic displacement of a structure are significant, the problem is to set up an optimum standard of manufacturing the structure components.

Here the permissible fluctuation in the structure parameters becomes a restrictive condition. Generally, design optimization of structures subject to reliability requirements is regarded as the ultimate goal of any design procedure. The basic approach in most reliability-based structural optimization is to impose a set of constraints on overall system reliability or probability of failure (Ang and Tang, 1975, 1984; Moses, 1973; Parmi and Cohn, 1978). Another approach suggests to minimize the total cost or weight for a specified allowable overall failure probability (Frangopol, 1984a; Hilton, 1960; Moses and Kinser, 1967).

One of the main objectives of the designer is to establish an acceptable probability of failure. Several procedures for the analysis of probability of failure of structures have been developed (Frangopol, 1984a, b, 1985a, b; Frangopol and Nakib, 1986; Kam, 1986; Moses, 1974; Moses and Kinser, 1967; Moses and Stevenson, 1970). In order to establish a probability of failure consider a structural system subjected to a number of external loads. The structure is said to survive if the applied stress  $\sigma_d$  in the built-in section due to all external loads is smaller than an ultimate limit stress  $\sigma_y$ .

$$\sigma_d \leq \sigma_y \quad (39)$$

The equality sign in eq. (39) corresponds to the state of the collapse threshold of the structure. In general, for each limit state, it is possible to establish a critical inequality similar to eq. (39) and identify, in the space of the relevant parameters, a "safe region  $\mathcal{S}$  (or success region)", where the critical inequality holds, and unsafe region  $\mathcal{F}$  (or failure region), where it does not hold. These regions are shown in Figure 9(a) according to Augusti et al (1984), where

$$S = \sigma_d \quad \text{and} \quad R = \sigma_y \quad (40)$$

In most cases the applied load  $S = S(t)$  is a random process, while the resistance  $R$ , which is calculated or measured, is a random variable. For each actual structure, the resistance takes up a constant value  $R_0$ , although uncertain, and the representative point  $(R, S)$  moves in time up and down the solid line in Figure 9(a). Figure 9(b) shows a possible realization of the random loading process  $S'(t)$ . The limit state is attained when  $S(t)$  violates the threshold  $R_0$ . The time to failure  $t_{fail}$  can be used as a measure of the structure reliability. Alternatively, one can consider a time interval  $(0, t)$  and then check the critical inequality in the worst possible condition. This can be formulated in probabilistic terms by stating that the probability of failure  $P_{fail}$  and the complementary probability of success (reliability)  $r = P_{suc}$  coincide respectively with the probability that the critical inequality is violated at least once in the interval  $(0, t)$ . In space random variables, the probability that a point  $Q$ , which represents the significant input and system parameters, falls either in the failure region  $\mathcal{F}$  or in the success region

$\mathcal{S}$ . Symbolically, these states are

$$P_{fail} = \text{Prob}[Q \in \mathcal{F}], \quad \text{and} \quad P_{suc} = \text{Prob}[Q \in \mathcal{S}] \quad (41)$$

Among the basic formulations of reliability calculations are the level 1 and level 2 approaches. In level 1 one simply applies the characteristic safety factor  $\gamma = R/S$ . In level 2 one needs to determine a reliability index  $\beta$  which measures, in units of standard deviation, the distance between the average point and the boundary of failure region. This means that larger values of  $\beta$  imply smaller probability of failure. The probability of failure is found (August et al, 1984) to be less dependent on the coefficient of variation  $\delta_s = \sigma(S)/E[S]$  of external excitation if the corresponding coefficient of resistance  $\delta_R = \sigma(R)/E[R]$  is relatively large, where  $\delta(S)$  and  $\delta(R)$  are the standard deviations of the applied stress and the resisting stress, respectively. Level 2 reliability methods include the estimation of the minimum distance  $\beta$  which is regarded as a safety measure of the smallest distance of the surface separating the safe and unsafe regions from the origin in the space of random variables  $Q$ .

Generally the level of performance of any structural system depends on the properties of the system. Thus, it is possible to characterize a function  $g(Q)$  known as the performance function such that

$$\begin{aligned} g(Q) > 0 &= \text{the safe state, and} \\ g(Q) < 0 &= \text{the failure state.} \end{aligned} \quad (42)$$

Geometrically the limit-state equation  $g(Q) = 0$  is an  $n$ -dimensional surface that is referred to as the "failure surface." The performance function could be linear or nonlinear. The evaluation of the exact probability of safety for nonlinear performance function is generally involved and the determination of the required reliability index would not be as simple as in the linear performance function (Ang and Tang, 1984). For correlated non-Gaussian random variables, the safety index may be evaluated in terms of another set of independent Gaussian variables through the Rosenblatt transformation (1952). Hohenbichler and Rackwitz (1981) developed an algorithm to determine the safety index by using the Rosenblatt transformation.

Tanaka and Onishi (1980) developed a method of regulating the deviations of random parameters and derived a restrictive conditional formula in terms of the permissible displacement (or natural frequency) fluctuation. The method is based on the linear deviation analysis with partial differential analysis together with sequential linear programming (SLP) for a number of restrictive conditions. Tanaka et al (1982) treated the optimization problem of the allowable variance of random parameters by using a perturbation method and Monte Carlo simulation. They computed the deviation of the steady state response of

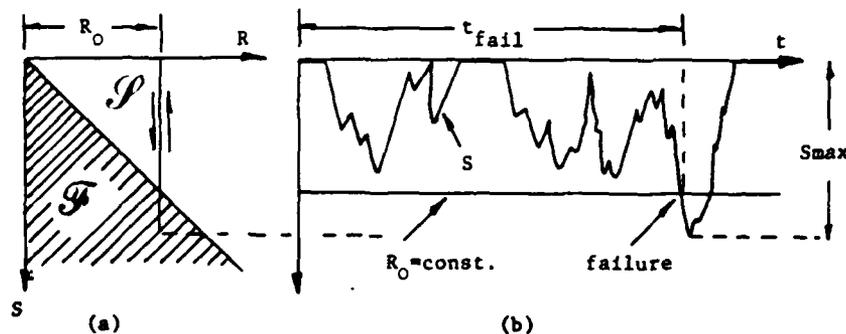


FIG. 9. (a) Safe and unsafe regions for (b) realization of  $S(t)$  (Augusti et al, 1984).

structural systems involving random parameters with the purpose of regulating the deviation of the random parameters when the deviation of the response is specified.

The techniques of mathematical programming have been extensively used to minimum-weight design of deterministic structures subject to constraints on stresses, displacements, dynamic response, and stiffness (Moses and Kinser, 1967); Moses and Stevenson, 1970; Moses 1973, 1974). The stochastic programming of dynamically loaded structures was developed originally by Charnes and Cooper (1959) and is well documented by Rao (1979). The basic idea of this method is to convert the probabilistic problem into an equivalent deterministic one by minimizing the expected value of the objective function subject to certain constraints. Davidson et al (1977) applied the mathematical programming techniques for optimization of structures subject to reliability requirements. Their work resulted in a general formulation of the minimum-weight optimization for indeterminate structures with random parameters. Jozwiak (1985, 1986) applied the stochastic programming based on expected values in the problem of optimization of dynamically loaded structures with random parameters. The mean values of joint displacements and their derivatives were determined by solving the equations of motion of the structure under the constraints of minimum weight.

Other techniques such as multi-objective optimization methods (Rao, 1982, 1984; and Schy and Giesy, 1981) and fuzzy sets (Zadeh 1965, 1973; Brown, 1980; Brown et al, 1983) have been employed to the design of simple structural elements and aeroplane control systems involving uncertain parameters and stochastic processes. The basic idea in multi-objective design is to include all important objectives in a vector objective function. The problem of optimizing structural systems involving dynamic restrictions, random parameters, stochastic processes, and multi-objectives has been outlined by Rao (1982). By considering the imprecision of the restrictions such as use, design, construction, one may assume that, some of the constraints and goals for each of the objective functions are fuzzy or imprecise in multi-objective fuzzy optimization design. If the corresponding expectation functions for objective and admission for constraint are introduced it is possible to quantify the fuzzy objectives and constraints. Guangwu and Suming (1986) employed the concept of multi-objective fuzzy design optimization for ship grillage structures.

## 5.2. Design sensitivity to parameter variations

### 5.2.1. Basic concept of sensitivity analysis

The sensitivity of a structural system to variations of its parameters is one of the basic aspects in the design of structures. The sensitivity theory is a mathematical problem which investigates the change in the system behavior due to parameter variations. The basic concepts of sensitivity theory are well documented in several books, see, eg, Frank (1978). The sensitivity problem can be stated by defining the actual system parameters represented by the vector  $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}^T$  which differ from the nominal value  $\alpha_0$  by a deviation  $\Delta\alpha$ . These parameters are related to a certain vector  $x$  which characterizes the dynamic behavior of the system. In structural dynamics the vector  $x$  can be taken as the system response vector. The mathematical model of the system response can be written in terms of the first order differential equations

$$\{\dot{x}\} = \{f(x, \alpha, t, F)\}, \quad \{x(t_0)\} = \{x^0\}, \quad (43)$$

where  $F$  represents the input vector.

Generally a unique relationship between the parameter vector and the response vector is assumed. However, this is not

possible in real problems because they cannot be identified exactly. It is a common practice in sensitivity theory to define a sensitivity function  $S$  which relates the elements of the set of the parameter deviations  $\Delta\alpha$  to the elements of the set of the parameter-induced errors of the system function  $\Delta x$  by the linear relationship

$$\Delta x = S(\alpha_0) \Delta\alpha. \quad (44)$$

This relation is a linear approximation of eq. (43) and is valid only for small parameter variations, ie,  $\|\Delta\alpha\| \ll \|\alpha_0\|$ .  $S$  is a matrix function known as the trajectory sensitivity matrix which can be established either by a Taylor series expansion or by partial differentiation of the state equation with respect to the system nominal parameters.

When the system is random, the function  $S$  is referred to as stochastic sensitivity function. Szopa (1984) developed equations for stochastic sensitivity functions to determine the influence of changes in the initial conditions on the response. These functions were applied to a stochastic nonlinear oscillator with a limit cycle. It was found that the mean values and the variances of the stochastic sensitivity functions converge to zero. Szopa (1986) used the sensitivity theory to investigate the influence of changes in system parameters on solutions of dynamical systems. The statistics of the stochastic sensitivity functions were found to have finite values when the response exhibit chaotic characteristics.

### 5.2.2. Design derivatives

Consider the eigenvalue problem given by eq. (15). It will be assumed that the eigenvalues  $\lambda_i$  of the system matrix  $A$  are distinct. The elements of  $A$  are function of the system parameters  $\alpha$ . The sensitivity of the free vibration of the structure as well as the sensitivity of its relative stability with respect to any parameter of  $A$  can be characterized by the sensitivity of the eigenvalues  $\lambda_i$  with respect to the parameters.

The partial derivative

$$S_{\alpha_i}^{\lambda_j} = \partial \lambda_j / \partial \alpha_i |_{\alpha_0} \quad (45)$$

is known as the eigenvalue sensitivity or the eigenvalue derivative.

The eigenvector sensitivity (or derivative) of the system matrix is also given by the partial differentiation

$$S_{\alpha_i}^{\lambda_j} = \partial x_j / \partial \alpha_i |_{\alpha_0}. \quad (46)$$

The eigenvalue sensitivity has been examined mathematically by McCalley (1960), Mantey (1968), and Reddy (1969). Frank (1978) developed a number of formulae to determine the eigenvalue sensitivity. The derivatives of the eigenvalues and eigenvectors are very helpful in design optimization of structures under dynamic response restrictions. They have been extensively used in studying vibratory systems with symmetric mass, damping, and stiffness properties (Fox and Kapoor, 1968; Kiefling, 1970) and in nonself-adjoint systems (Rogers, 1970; Plaut and Huseyin, 1973; Rudisill, 1974). For distributed parameter systems, design derivatives of eigenvalues were first encountered in optimization studies by Haug and Rousselet (1980) and Reiss (1986). Reiss used a relatively simple method to determine explicit results for the design derivatives of eigenvalues and eigenvectors. He expressed self-adjoint operator equations in terms of integral form by using Green's function (Reiss, 1983). Recently Kuo and Wada (1986) developed the nonlinear sensitivity coefficients and correction terms, usually eliminated during the linearization process in the Taylor expansion. The nonlinear correction terms were found significant in problems involving many finite element analyses where the size of the eigenmatrix is of order 10E06 and the difference in the eigenvectors may be of order 0.01.

Lund (1979) developed a method to calculate the sensitivity of critical speeds of a conservative rotor to changes in the design using a state vector-transfer matrix formulation. Fritzen and Nordman (1983) have developed the eigenvalue and eigenvector derivatives for general vibratory system (with nonsymmetric system matrices) and used them in evaluating stability behavior due to parameter changes in rotor dynamics. Palazzolo et al (1983) presented a generalized receptance approach for eigensolution reanalysis of rotor dynamic systems. Their method has the advantage of accommodating system modification of arbitrary magnitude and treats the modifications simultaneously. Rajan et al (1986) developed the eigenvalue derivatives for the damped natural frequencies of whirl of general linear rotor systems modeled by finite element discretization. For under-damped modes, the eigenvalue derivative is complex. The real part represents the damping sensitivity coefficient while the imaginary part gives the whirl speed sensitivity. Rajan et al showed that the combination of design parameter and whirl frequency sensitivity coefficients may be used to evaluate the damped critical speed sensitivity coefficients.

In reliability-based design optimization it is useful to examine the results to sensitivity analysis in order to determine the influence of the statistical parameters on the optimum solutions. The essential objectives of sensitivity analysis of any system is to establish a measure of the way each response quantity varies with changes in the parameters that define the system (Grierson, 1983). Recently, Arora and Haug (1979) and Frangopol (1985a) have developed a technique for determining the reliability-based optimum design sensitivity of redundant ductile structures. Frangopol investigated the sensitivity of an optimum design to changes in the statistical parameters that define the loading and resistance strength of the structure.

## 6. EXPERIMENTAL RESULTS

The first attempt to measure the statistics of structural modal frequencies is believed to be made by Mok and Murray (1965). They carried out a series of free flexural vibration tests of a bar with a stepped profile and a maximum variation in the cross section of 50%. The predicted and measured results were found very close. Twenty years later, Paez et al (1985, 1986) conducted a series of experimental investigations to measure the random variation of the natural frequency of a cantilever beam. One end of the beam was mounted on a fixture through a screw and two washers, and the other end carries a concentrated mass. The torque in the screw establishes a preload which governs the stiffness of the beam at the fixture. Paez et al conducted 19 experiments each with different values of base torque and stiffness. The variation of the fundamental frequency with the base stiffness was obtained experimentally and numerically (by using a finite element program). It was shown that the standard deviation of modal frequency increases with the mean modal frequency. Another interesting feature observed by Paez et al was that the magnitude of random variation in modal frequency can become greater than the spacing between modal frequencies as the frequency order increases.

The phenomenon of normal mode localization was first examined experimentally by Hodges and Woodhouse (1983). Their model was a thin high-tensile steel wire stretched between two supports. Seven small lead weights were securely attached initially at equal lengths and then were shifted slightly to give a controlled amount of irregularity. Under a step function force with repeatable amplitude the string motion was observed and measurements were taken for the energy transmission from end to end of the string. Levels of energy attenuation in the dis-

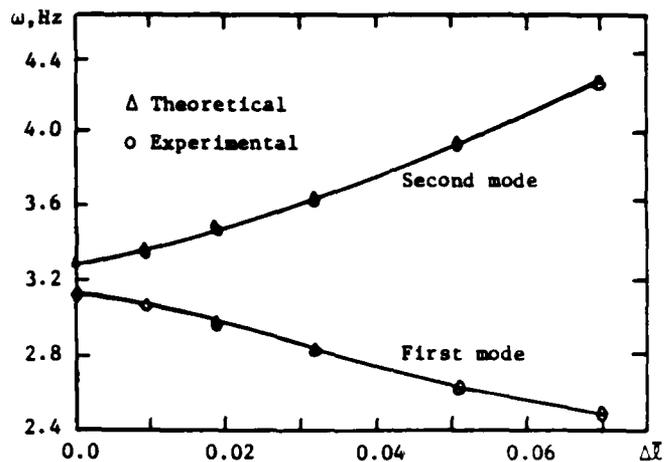


FIG. 10. Comparison of experimental and theoretical natural frequencies of the first two modes for  $c = 281.8$  (Pierre et al, 1986).

ordered case were found in some cases quite large (99%) with only 2.4% standard deviation in the mass positions.

Pierre et al (1986) conducted an experimental investigation to verify the existence of localized modes for two disordered two-span beams shown in Figure 7. The beam was pinned at both ends while the third support with variable torsional stiffness was located near the mid-span. This middle support can be moved to various locations. A pure excitation torque was applied to the specimen beam near its intermediate support. Figure 10 shows the comparison between theoretical and experimental natural frequencies of the first two modes versus mistuning parameter  $\delta l = \Delta l/l$ , where  $l$  is the length of the beam, and  $\Delta l$  is the variation from the middle of the beam. The coupling parameter  $c = 2k_\theta l/EI$ , where  $k_\theta$  is the stiffness of the torsional spring,  $E$  and  $I$  are the Young's modulus and the area moment of inertia of the beam, respectively. The degree of localization of a mode is expressed by the ratio  $A = A_1/A_2$ , which represents the peak deflection in one span to the peak deflection in the other span, such that the numerator of this ratio corresponds to the span with smaller peak deflection. This peak ratio is shown in Figure 11 for the two modes for two different values of torsional spring constant  $c$ . The mode shapes of tuned and mistuned beams are shown in Figure 12. It was reported that for  $\delta l = 2\%$  and  $c = 281.8$ , the first mode of the mistuned beam is strongly localized in the second span, whereas the one of the tuned beam is collective, that is the peak deflection is the same in both spans.

A comprehensive experimental and theoretical investigations were conducted by Ewins (1976) to determine the effects of turbomachinery blades mistuning. His bladed disk testpiece model consists of 24 blades. A provision for adjusting the tune of each blade individually was accomplished by adding a number of washers to a nut and bolt attached near the tip of each blade. The test piece was excited by placing an electromagnet close to its surface and passing an alternating current through the magnet. The response of the bladed disk was detected by a set of strain gages fixed near the root of each blade. The natural frequencies were then measured by adjusting the frequency of the magnet so as to produce a large response in the strain gage outputs. The shape of each mode was determined by examination of the relative amplitudes of all the blades. It was observed that there was a distinct, though complex, pattern linking the basic (tuned) mode shape with the mistuned mode shape and the mistuned pattern, particularly for the lower diametral modes. Jay and Burns (1986) conducted a series of rotating and non-

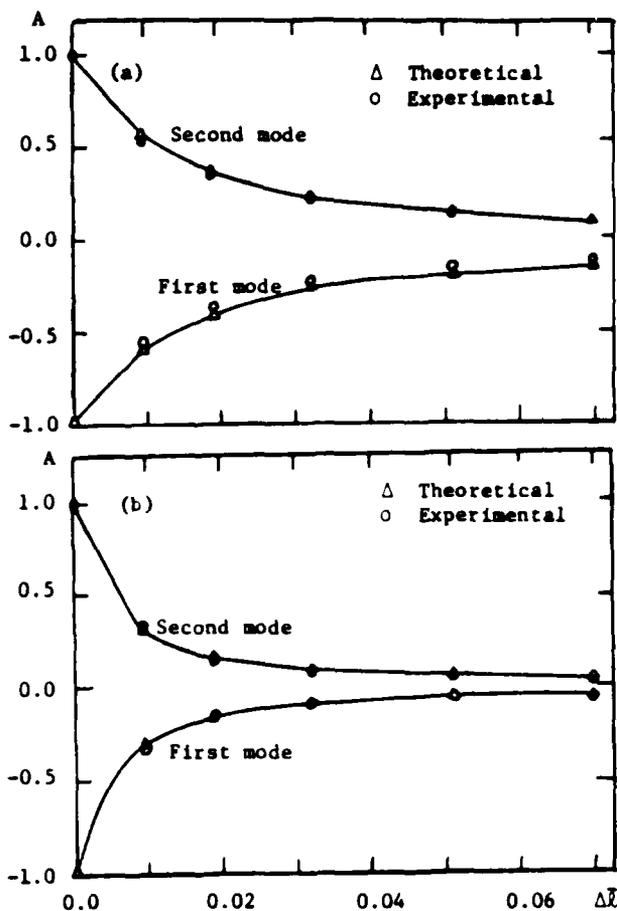


FIG. 11. Comparison of experimental and theoretical peak ratio  $A$  of the first two modes for (a)  $c = 90.4$ , (b)  $c = 281.8$  (Pierre et al. 1986).

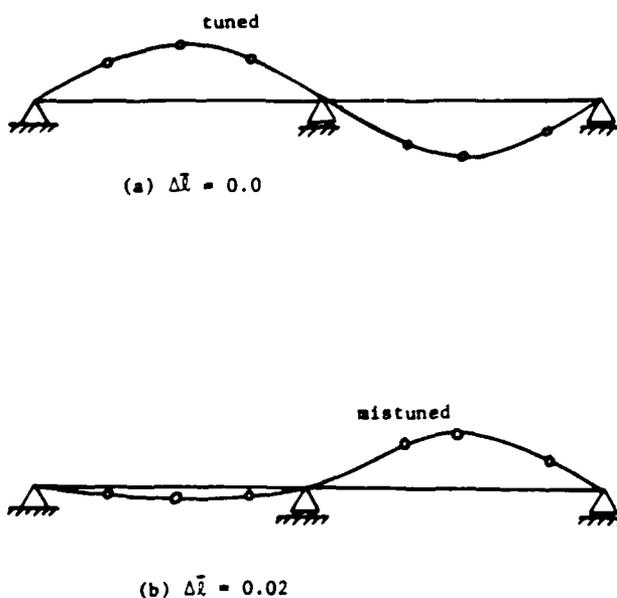


FIG. 12. Measures first mode shape for tuned (a) and mistuned (b) two-span beam for  $c = 281.8$  (Pierre et al. 1986).

rotating tests to identify individual blade frequencies, mode shapes, mistuning, damping, and split factors for diametral patterns of the 3, 4, 5, and 6 diametral mode families. The first harmonic of the normalized axial velocity deficit at the proper mass flow rate was used to construct a gust perturbation velocity. These spanwise gust perturbation velocities multiplied by the product of the density and the relative velocity squared results in the normalized force parameter. It was found that any increase in the perturbation force parameter results in an increase in the dynamic stress in the bladed disk. In addition the perturbation parameter does account for the interaction between the wake and modal response of the system as they are changed by aerodynamic loading.

## 7. CONCLUSIONS

Several problems in structural dynamics involving parameter uncertainties have been treated in the literature. These problems include the random eigenvalue of disordered systems, normal mode localization, random response, design optimization, and reliability. The mathematical theory of the random eigenvalue has reached the maturity stage, however, this theory has not been fully implemented to treat real engineering problems. It is observed that some progress has been made towards the development of numerical algorithms such as stochastic finite element methods and Monte Carlo simulations to determine the response of structural elements. These developments have promoted the investigation of several problems including mistuned turbomachinery bladed disks, reliability-based design and derivatives of eigenvalues in design optimization. Few attempts have been made to employ new approaches such as multi-objective optimization and fuzzy sets in design optimization problems. It is believed that these techniques will have new research avenues in many design problems. Another area of potential future research is the optimum design sensitivity in reliability-based design under multilevel reliability constraints to evaluate the significance of various uncertainties and approximations on the optimum solutions.

The problems treated in the literature have been restricted within the framework of the linear theory. The limitations of the linear formulation need to be defined to provide the structural dynamicist the influence of nonlinearities as a source of uncertainty. Future studies should include the influence of geometric and material nonlinearities. Experimental investigations are also very important to examine the influence of parameter uncertainties of composite structures on their dynamic performance.

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