The purpose of this note is to give a direct proof of some recent important results of E.B. Dynkin and A. Mandelbaum [2]. This also provides immediately the results in [3] with a very simple proof. This is achieved by avoiding the use of Poisson process.

Let us set up some notation. Let \((L,\Sigma,\mu)\) be a probability space and \((X^k,\Sigma^k,\mu^k)\) be the \(k\)-fold product probability space. Let \(h_k(x_1,\ldots,x_k)\) be a symmetric function of \(k\)-variables. We call it canonical if \(\int h_k(x_1,\ldots,x_{k-1},y)\,d\mu=0\) for all \(x_1,\ldots,x_{k-1}\) and \(x_k\in X^{k-1}\). Let \(X_1,\ldots,X_n\) be i.i.d. \(X\)-valued random variables on a probability space with distribution \(\mu\).
ON A LIMIT THEOREM AND INVARIANCE PRINCIPLE FOR SYMMETRIC STATISTICS

BY

V. Mandrekar

Technical Report No. 142
July 1986
ON A LIMIT THEOREM AND INVARIANCE PRINCIPLE FOR SYMMETRIC STATISTICS

by

V. Mandrekar*
Department of Statistics and Probability
Michigan State University
East Lansing, MI 68824

and

Center for Stochastic Processes
Department of Statistics
University of North Carolina
Chapel Hill, NC 27514

*This research supported by ONR N00014-85-K-0150 and the Air Force Office of Scientific Research Contract No. F4962085C0144.
0. Introduction: The purpose of this note is to give a direct proof of some recent important results of E.B. Dynkin and A. Mandelbaum [2]. This also provides immediately the results in [3] with a very simple proof. This is achieved by avoiding the use of Poisson process. Let us set up some notation. Let \((X, \Sigma, \mu)\) be a probability space and \((X^k, \Sigma^k, \mu^k)\) be the k-fold produce probability space. Let \(h_k(x_1, \ldots, x_k)\) be a symmetric function of \(k\)-variables. We call it canonical if \(\int h_k(x_1, \ldots, x_{k-1}, y) d\mu = 0\) for all \(x_1, \ldots, x_{k-1} \in X^{k-1}\). Let \(X_1, \ldots, X_n\) be a i.i.d. \(X\)-valued random variable on a probability space with distribution \(\mu\). As in [2], define

\[
\sigma^n_k(h_k) = \sum_{1 \leq s_1 < \ldots < s_k \leq n} h_k(X_{s_1}, \ldots, X_{s_k}), \quad \text{for } k \leq n
\]

\[
= 0 \quad \text{for } k > n.
\]

Let \(H = \{ (h_0, h_1, \ldots) : h_k \text{ canonical and } \sum_{k=1}^{\infty} \frac{1}{k!} \| h_k \|_2^2 < \infty \} \) where \(h_0\) is a constant and \(\| \cdot \|_2\) is the norm in \(L^2(X^k, \Sigma^k, \mu^k)\). On \(H\) define

\[
\| h \|_2^2 = \sum_{k=0}^{\infty} \| h_k \|_2^2 / k!.
\]

\(H\) is the so-called exponential (Foch) space of \(L^2(X, \Sigma, \mu)(\phi \in L^2(X, \Sigma, \mu) \text{ with } E\phi(X) = 0)\). It is a Hilbert space under coordinate addition, scalar multiplication and \(\| \cdot \|_2\). For each \(\phi \in L^2_0(X, F, \mu)\), \(h^\phi \in H\) with \(h_k^\phi = \phi(x_1), \ldots, \phi(x_k)\). It can be easily seen that \(sp\{ h^\phi : \phi \in L^2_0(X, F, \mu) \}\) is dense in \(H\). Define for each \(h \in H\),

\[
Y_n(h) = \sum_{k=0}^{\infty} n^{-k/2} \sigma^n_k(h_k).
\]

Since \(\sigma^n_k(h_k) = 0\) for \(k > n\), this is a finite sum. Also, let

\[
Y_n^t(h) = \sum_{k=0}^{\infty} n^{-k/2} \sigma^{[nt]}_k(h_k).
\]

The main purpose is to show directly that \(Y_n(h) \overset{\mathcal{D}}{\rightarrow} \sum_{k=0}^{\infty} \frac{I_k(h_k)}{k!}\) where \(\mathcal{D}\) denotes convergence in distribution and \(I_k(h_k)\) denotes Ito-Wiener multiple
integral of $h_k$ with respect to Gaussian random measure $W$ with
\[ EW(A)W(A') = \mu(A \cap A'). \]

In the next section we discuss the convergence of $Y_n^+(h)$. We observe that for $\phi \in L_0^2(\mathcal{X}, \Sigma, \mu)$

\[
Y_n(h^+) = \sum_{k=0}^{n} \sum_{1 \leq s_1 < \ldots < s_k \leq n} \phi(X_{s_1}) \ldots \phi(X_{s_k}) \frac{\varsigma(X_{s_1}) \ldots \varsigma(X_{s_k})}{\sqrt{n}} \ldots \frac{\varsigma(X_{s_k})}{\sqrt{n}} = \prod_{1 \leq j \leq n} \frac{\phi(X_j)}{1 + \frac{1}{\sqrt{n}}}. 
\]

Let us observe that for any $\varepsilon > 0$,

\[
\sum_{j} \mathbb{P}(\{|\phi(X_j)| > \sqrt{\varepsilon_j}\}) = \sum_{j} \mathbb{P}(\{\phi(X_1)\}^2 > \varepsilon_j) \leq \|\phi\|_2^2 < \infty.
\]

Hence by Borel-Cantelli lemma, a.s. (for $j \leq n$)

\[ |\phi(X_j)| \leq \sqrt{\varepsilon_j} \leq \sqrt{\varepsilon \sqrt{n}} \quad \text{for} \quad j \geq \text{some } N(\omega) \quad (N(\omega) < \infty). \]

But

\[
\prod_{1 \leq j \leq n} \frac{\phi(X_j)}{1 + \frac{1}{\sqrt{n}}} = \frac{\prod_{1 \leq j \leq n} \phi(X_j)}{\prod_{1 \leq j \leq n} (1 + \frac{1}{\sqrt{n}})} \quad \text{giving for a.s. } w, \text{ so}
\]

\[
\lim_{n} Y_n(h^+) = \lim_{n} \frac{\prod_{1 \leq j \leq n} \phi(X_j)}{N(\omega)} = \frac{n \phi(X_j)}{1 + \frac{1}{\sqrt{n}}}. \quad \text{Thus WLOG, we can assume for } n \text{ large}
\]

\[
|\frac{\phi(X_j)}{\sqrt{n}}| < 1 \quad \text{a.s. for all } j \leq n \text{ and } Y_n(h^+) = \frac{n \phi(X_j)}{1 + \frac{1}{\sqrt{n}}}. \quad \text{Taking log on both sides and expanding log}(1+x) \text{ we have}
\]

\[
\log\frac{\prod_{1 \leq j \leq n} \phi(X_j)}{1 + \frac{1}{\sqrt{n}}} = \sum_{1 \leq j \leq n} \frac{\phi(X_j)}{1 + \frac{1}{\sqrt{n}}} - \frac{1}{2} \sum_{1 \leq j \leq n} \frac{\phi(X_j)^2}{1 + \frac{1}{\sqrt{n}}} + \varepsilon_n(\phi)
\]

where $\varepsilon_n(\phi) \overset{p}{\to} 0$ by the WLLN and since $\max|\frac{\phi(X_j)}{\sqrt{n}}| \overset{p}{\to} 0$ by Chebychev's Inequality,
i.e. the \((Y_n(h^\phi))_n\) \(\xrightarrow{D} \exp[I_1(\cdot)] - \frac{1}{2} \|\cdot\|_2^2\). Using Cramér-Wold device and the above argument we get

0.3 Lemma: For any finite subset \(\{\phi_1, \ldots, \phi_k\} \subset L^2(X,\Sigma,\mu)\)

\[
(Y_n(h_1^{\phi_1}), \ldots, Y_n(h_k^{\phi_k})) \overset{D}{=} (\exp(I_1(t_1)) - \frac{1}{2} \|t\|_2^2, \ldots, \exp(I_1(t_k)) - \frac{1}{2} \|t_k\|_2^2).
\]

As a consequence, we get for \(\{\phi_i, i \in I\}\) a finite subset of \(L^2(X,\Sigma,\mu)\) and \(\{c_i, i \in I\} \subset \mathbb{R}\),

\[
(0.3)\ Y_n(\sum_{i \in I} c_i h^{\phi_i}) \overset{D}{=} \sum_{k=0}^{\infty} \frac{I_k([\sum_{i \in I} c_i h^{\phi_i}])}{k!}.
\]

We now observe that for \(h, h' \in H\),

\[
(0.4)\ E[Y_n(h) - Y_n(h')]^2 = \sum_{k=0}^{n} (-1)^k \|h_k - h'_k\|^2 \leq E\|h - h'\|^2,
\]

since \(E_0^n(h_k - h'_k)^2 = (\|h_k - h'_k\|^2)^2 \leq E\|h - h'\|^2\) by ([2], p. 744). Also,

\[
(0.5)\ E\left(\sum_{k=0}^{\infty} \frac{I_k(h_k)}{k!} - \sum_{k=0}^{\infty} \frac{I_k(h'_k)}{k!}\right)^2 = \|h - h'\|^2.
\]

Thus we get

\[
(0.6)\ \text{Theorem: For any } h \in H,\]

\[
Y_n(h) \overset{D}{=} W(h) = \sum_{k=0}^{\infty} \frac{I_k(h_k)}{k!}
\]

Proof: Let \(h \in H\) and \(\epsilon > 0\). Choose \(h' = \sum_{i \in I} c_i h^{\phi_i}\) such that \(\|h - h'\|^2 < \epsilon/2\).

Now consider for \(t \in \mathbb{R}\)

\[
|E(e^{itY_n(h)} - e^{itW(h)})| \leq E|e^{itY_n(h)} - e^{itY_n(h')}| + E|e^{itY_n(h')} - e^{itW(h')}| + E|e^{itW(h')} - e^{itW(h')}|.
\]

Using Schwartz's Inequality and the fact \(|e^{ix} - 1| \leq |x|\) we get that the first
and third term of the above inequality are dominated by \( t^2E\|h-h'\|^2 \) using (0.4) and (0.5). Hence by (0.3)',

\[
\lim_{\|Ee\| \to \infty} \frac{it\gamma_n(h)}{|Ee|} - Ee\frac{it\gamma(h)}{|Ee|} \leq \varepsilon/2.
\]

As \( \varepsilon \) is arbitrary we get the result.

Finally, we make some observations to be used later.

\[
(0.7) \quad Y^t_n(h, \phi) = \sum_{k=0}^{[nt]} n^{-k/2} \sum_{1 \leq s_1, \ldots, s_k \leq [nt]} \phi(x_{s_1}) \ldots \phi(x_{s_k}) = \frac{1}{n^2} (1 + \frac{1}{\sqrt{n}}).
\]

Also, \( \min(t,s)\cup(A\cap A') \) is a covariance on \( [0,\infty) \times \Sigma \) giving that there exists a centered Gaussian process \( W(t,a) \) with \( EW(t,a)W(s,a') = \min(t,s)\cup(A\cap A') \). Let for \( T < \infty \)

\[
\mathcal{H}_T = \{(h_0, h_1, \ldots) \in \mathcal{H} : \sum_{k=0}^{T} \frac{\|h_k\|^2}{k!} < \infty\}.
\]

1. **Invariance Principle:** Let \( D[0,T], (T < \infty) \) be the space of right continuous functions on \( [0,T] ([0,\infty)) \) with left limits at each \( t \leq T \). The space \( D[0,T] \) is endowed with Skorohod topology [1]. The topology in \( D[0,\infty) \) is the one described in Whitt [4]. We note that

\[
X_{[nt]} = \sum_{k=0}^{[nt]} \phi^2(x_{s_k}) - \frac{1}{n} \phi^2
\]

has stationary independent increments. So for \( \varepsilon > 0 \)

\[
P(\sup_{0 \leq t \leq T} |X_{[nt]}| > \varepsilon) \leq C.P(|X_{[nt]}| \geq \varepsilon) \to 0
\]

by the weak law of large numbers. Using this, the arguments preceding Lemma 0.3, invariance principle and Cramér-Wold device we get the following analogue of Lemma 0.3.

**Lemma 1.1:** \( (Y^t_n(h_1), \ldots, Y^t_n(h_k)) \overset{D_{k,T}}{\to} (\exp(\frac{1}{2}t\phi_j^2) - \frac{1}{2}t\phi_j^2), j=1, \ldots, k) \)
where $I^t(\cdot) = \int [0,t] \phi(x) \mathbb{W}_k(du, dx)$. Here $D_k[T]$ denotes convergence in $D^k[0,T]$ with respect to product topology.

We note that $W(t,A)$ is a Brownian motion for each $A \in \Sigma$. Thus we can choose $I^t(\cdot)$ continuous for each $\phi$ and a martingale in $t$ as $I^t(\cdot) = \int [0,t] \phi(x) \mathbb{W}(du, dx)$. We get for $\{c_1, \ldots, c_k\} \subseteq \mathbb{R}$, (k finite),

$$Y^t(\sum_{j=1}^{k} c_j \phi_j) = \sum_{j=1}^{k} c_j \exp(I^t(\phi_j) - \frac{1}{2} \|\phi_j\|^2).$$

Let $\phi \in L^2_0(X,\Sigma,\mu)$, $\|\phi\| = 1$, and denote

$$(\phi^k)^t = \phi(x_1) \ldots \phi(x_k)I(0,t)(u_1) \ldots I(0,t)(u_k).$$

Define $I_k(\cdot)^t = k!H_k(t, I(\phi))$ where $H_k$ is Hermite polynomial, i.e.

$$\sum_{k=0}^{\infty} k! H_k(t, x) = \exp(yx - \frac{1}{2}y^2t).$$

For $\phi \in L^2_0(X,\Sigma,\mu)$, $\|\phi\| = 1$, we define for $h^t = (1, \phi^t, \phi^2, \ldots)$,

$$W(h^t) = \sum_{k=0}^{\infty} \frac{I_k(\phi^k)^t}{k!},$$

and extend it linearly to $(\sum_{j=1}^{k} \phi_j^j)^t$. It is a martingale. Let $h \in H_T \{h(n)\}$ a sequence in $\text{sp}(\phi^t)$, $\phi$ in CONS in $L^2_0(X,\Sigma,\mu)$, $\|\phi\| = 1$, then

$$P(\sup_{t \leq T} |W^t(h(n) - h(m))| > \varepsilon) \leq E|W^t(h(m) - h(n))|^2$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\|h_k(m) - h_k(n)\|^2}{k!}$$

using Doob's inequality and argument as in (0.5). Define for $h \in H_T^t$,

$W_t^t(h) = -\lim_{n} W_t^t(h_n)$ where the limit is uniform on compact for $h_n \to h$. Then $W_t^t(h)$ is right continuous martingale and has the same distribution as $\sum_{k=0}^{t} I_k^t(h_k)/k!$. Now we derive the main theorem of [3].

Theorem 1.2: $\mathbb{Y}_n^t(h) \rightarrow W_t^t(h)$ in $D(0,T)$ for $h \in H_T$ for each $T < \infty$. 

Proof: Let \( h \in H \) and \( \varepsilon > 0 \), choose \( h'_k \in \text{sp}\{h' : \phi \in L^2_0(X,\mathbb{F}) \cap h_k + h \} \). Now define

\[
\begin{align*}
X_{nk}^* &= Y_n(h'_k), \\
Z_n &= Y_n(h), \\
X_k &= W_n(h'_k) \quad \text{and} \quad X = W(h).
\end{align*}
\]

Then \( X_{nk}^* \overset{D}{\to} X_k \) as \( n \to \infty \) in \( D[0,T] \) for each \( T < \infty \) by Lemma 1.1. Also \( X_k \overset{D}{\to} X \) as \( n \to \infty \) in \( D[0,T] \) for each \( T < \infty \). In addition,

\[
P(\sup_{0 \leq t \leq T} |X_{nk}^* - Z_n| \geq \varepsilon) \leq E|Y_n(h - h'_k)|^2 \leq T||h - h'_k||
\]

giving \( \lim\lim_{k \to \infty} P(\sup_{n} |X_{nk}^* - Z_n| \geq \varepsilon) = 0 \) with \( \rho \) being the Skorohod metric on \( D[0,T] \).

This implies by ([1], Thm 4.2, p. 25) that \( Z_n \overset{D}{\to} W(h) \) in \( D[0,T] \) \( (T < \infty) \) giving the result.

Remark: In the above arguments we may use an interpolated version of \( Y_n(h) \) from the beginning and use appropriate versions of Donsker's Invariance Principle to conclude above convergence occurs in \( D[0,T] \) in sup norm giving \( W(h) \) continuous.

References


END
4-8-87
DTIC