THE UNIVERSAL TRANSVERSE MERCATOR GRID FORMULAS FOR CARTOGRAPHIC APPLICATIONS PROGRAMMERS(U) DEFENSE MAPPING AGENCY INTER AMERICAN GEODETIC SURVEY APO MIA.

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The Universal Transverse Mercator
Grid Formulas for Cartographic
Applications Programmers

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The Universal Transverse Mercator Grid formulas for cartographic applications programmers.

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Universal Transverse Mercator grid system is described and the formulas are presented. Sample programs in BASIC are included.

Cartographers frequently need to transform geographic coordinates, via a projection system, to their corresponding rectangular coordinates. The traditional method of using look up tables is no longer efficient. We now can use hand held microcomputers and programmable calculators to do the coordinate transformations. A problem has been the difficulty in locating concise, easy to program transformation formulas.
This paper presents easy to program formulas for the Universal Transverse Mercator Grid (UTM grid). The paper begins with a short history of the development of the UTM grid system and then states its basic characteristics. Following this the forward and inverse transformation formulas are presented. The last section contains two features of aid to the programmer. These are sample programs, written in BASIC and numerical examples.
Cartographers involved in national mapping frequently need to transform geographic coordinates, by way of a specific projection system, to their corresponding rectangular coordinates. They may want to plot control points or construct graticules of meridians and latitude. Until recently this was a time-consuming chore. Now, with the proliferation of hand held microcomputers and programmable calculators, this once tedious job is performed quickly and accurately. In the time it took to transform one set of geographic coordinates into rectangular coordinates using projection tables we now can perform many. But there is one problem that is holding back the greater use of hand held computers. This problem is the difficulty in locating concise, easy to program transformation formulas.

The purpose of this article is to provide Cartographers with the transformation equations for the most widely used grid system, the Universal Transverse Mercator grid system (UTM grid).

The equations perform forward transformations; geographic coordinates to UTM grid coordinates, and inverse transformations; UTM grid coordinates to geographic coordinates. These equations are notable for the ease of which they can be programed.

The article begins with a short history of the development of the UTM grid system and then states its basic characteristics. Following this the forward and inverse transformations are presented. The last section of this article contains two features that will be of aid to the programer. These are sample programs, written in BASIC for a Casio programmable calculator, and numerical examples using the formulas.
The history of the development of the UTM grid system

The Transverse Mercator projection is used in the UTM grid system. The Transverse Mercator projection evolved from the regular Mercator projection, named after the Flemish cartographer Gerhardus Mercator (Gerhard Kremer, 1512 - 1594). The development of the Mercator projection can be represented geometrically by viewing a cylinder that is wrapped around the globe and tangent to the Equator. (see figure 1). A light source approximately 3/4 of the way back along the diameter can be seen as projecting the earth's features out on to the cylinder. The cylinder is then cut and laid out flat to form the map. The feature that is interesting to us here is that there is very little scale distortion close to the standard line, the Equator.

The Transverse Mercator was developed to take advantage of this feature. If you take the cylinder and rotate it 90° you effectively trace a meridian the standard line (see figure 1). This makes the Transverse Mercator useful in mapping narrow, North-South orientated features.

The spherical form of the Transverse Mercator projection was first developed by Johann Heinrich Lambert (1728 - 1777), an Alsatian mathematician, astronomer, physicist and philosopher. His work, Beiträge, published in 1772 contained six new map projections and included the Transverse Mercator and the Lambert Conformal Conic. It is interesting to note that the Lambert Conformal Conic and a modified ellipsoidal form of the Transverse Mercator, the UTM, are the two most widely used projections for large and medium scale mapping in Latin America today.

After Lambert, the Transverse Mercator was worked on by the great German mathematician, Johann Karl Friedrich Gauss (1777 - 1855). It wasn't until 1912 and 1919 that L. Kruger published his studies that for the first time stated the
Transverse Mercator ellipsoidal formulas. Because of the work of Gauss and Kruger, the Transverse Mercator is sometimes referred to as the Gauss-Krüger projection.

The UTM grid system, which is based on the ellipsoidal form of the Transverse Mercator projection, grew out of the military need for a worldwide plane coordinate system. The ideal universal plane coordinate system was seen as containing the following properties; conformality, continuity over large areas, a minimum number of zones, minimal scale errors, a unique referencing system, uniform transformation formulas and a maximum meridional convergence of five degrees. Based on these criteria, in the late forties the U.S. Army Map Service devised the UTM grid. Today the UTM grid system and its companion system, the Universal Polar Stereographic (UPS), for use in the polar regions, are used extensively in the Americas, Europe and Russia for surveying, mapping and military purposes.

Basic characteristics of the UTM grid system

The UTM grid system is made up of sixty zones each covering six degrees of longitude. The zones are numbered one through sixty starting at 180° west from Greenwich and increasing to the east. The zones extend in latitude from 80°N to 80°S (see figure 2).

Each UTM grid zone is bisected by a central meridian which becomes the y axis origin of the projection. The central meridian is assigned a value of 500,000 meters in order to avoid the use of negative numbers. Along the central meridian a scale factor of 0.9996 is employed. Because of this the central meridian is not a standard line. The effect of using a scale factor of less than one is to create two standard lines, in this case standard meridians, spaced 180 km east and west of the central meridian. By having two standard meridians the scale retention characteristics of the UTM are greatly improved.
(see figures 3 and 4).
The X axis origin is the Equator. When working in the northern hemisphere the Equator is assigned a value of zero. In the southern hemisphere the Equator is given a false value of 10,000,000 meters. The false value is applied to avoid the use of negative numbers.

The UTM Equations
The following formulas are taken from the publication, "Map Projections used by the U.S. Geological Survey" second edition, 1984. They are presented here in a modified form.

Symbols

\( a = \) Semimajor axis of the reference ellipsoid

\( b = \) Semiminor axis of the reference ellipsoid

\( e = \) Eccentricity \( = (1-b^2/a^2)^{1/2} \).

\( x = \) Rectangular coordinate: the distance to the right of the vertical line (the y axis) passing through the origin of the projection, the central meridian.

\( y = \) Rectangular coordinate: the distance above the horizontal line (the x axis) passing through the origin of the projection, the Equator.

\( k = \) The relative scale factor along a parallel of latitude.

\( k_0 = \) The scale factor of the central meridian. \( k_0 = 0.9996 \).
\( \varphi = \) The latitude of the point. For a latitude south of the Equator apply a minus sign.
\( \varphi_0 = \) The latitude of the origin. 0° in the UTM system.
\( \lambda = \) The longitude of the point. For longitude west of Greenwich apply a minus sign.
\( \lambda_0 = \) The longitude of the origin, the central meridian. For longitude west of Greenwich, apply a minus sign.
THE FORWARD EQUATIONS FOR THE UTM GRID

Easting = 500,000 + x
Northing = y, in the northern hemisphere
Northing = 10,000,000 + y, in the southern hemisphere

\[ x = k_0 [A + (1 - T + C) A^3 / 6 + (5 - 18T + T^2 + 72C - 58 \, e^{12}) A^5 / 120] \]
\[ y = k_0 \{ M + N \tan \theta \left[ A^2 / 2 + (5 - T + 9C + 4C^2) A^4 / 24 + (61 - 58T + T^2 + 600C - 330e^{12}) A^6 / 720 \right] \} \]
\[ k = k_0 \left[ 1 + (1 + C) A^2 / 2 + (5 - 4T + 42C + 13C^2 - 28e^{12}) A^4 / 24 + (61 - 148T + 167^2) A^6 / 720 \right] \]

Where \( k_0 = 0.9996 \)

\[ e^{12} = e^2 / (1 - e^2) \]
\[ N = a / (1 - e^2 \sin^2 \theta)^{1/2} \]
\[ T = \tan^2 \theta \]
\[ C = e^{12} \cos^2 \theta \]

\[ A = \cos \theta \left( \lambda - \lambda_0 \right), \text{ with } \lambda \text{ and } \lambda_0 \text{ in radians.} \]
\[ M = a \left[ (1 - e^2 / 4 - 3e^4 / 64 - 5e^6 / 256 - \ldots) \theta - (3e^2 / 8 + 3e^4 / 32 + 45e^6 / 1024 - \ldots) \right] \]
\[ \sin 2\theta + (15e^4 / 256 + 45e^6 / 1024 - \ldots) \sin 4\theta - (35e^6 / 3072 + \ldots) \sin 6\theta + \ldots \]}

With \( \theta \) in radians.
THE INVERSE EQUATIONS FOR THE UTM GRID

\[ \rho = \rho_1 - (\kappa_1 \tan \rho_1 / R_1) [D^2/2 - (5 + 3T_1 + 10C_1 - 4C_1^2 - 9e'^2) D^6/24 + (6 + 90T_1 + 298C_1 + 45T_1^2 - 252e'^2 - 3C_1^2) D^8/720], \]
where \( \rho \) is in radians. To convert to degrees multiply \( \rho \) by 180/\( \pi \).

\[ \lambda = \lambda_0 + \left[ D_0 \left( 1 + 2T_1 + C_1 \right) D^3/6 + \left( 5 - 2C_1 + 28T_1 - 3C_1^2 + 8e'^2 + 24T_1^2 \right) D^5/120 \right] / \cos \rho_0, \]
where \( \lambda \) is in radians. To convert to degrees multiply \( \lambda \) by 180/\( \pi \).

Where \( \rho_1 \) is the foot point latitude of the perpendicular from the central meridian.

\[ \rho_1 = \mu \left( 3e_1/2 - 27e_1^3/32 + ... \right) \sin 2u + \left( 21e_1^2/16 - 55e_1^4/32 - ... \right) \sin 4u + \left( 15e_1^3/96 + ... \right) \sin 6u + ..., \]
where \( \rho_1 \) is computed in radians and then converted to degrees by multiplying by 180/\( \pi \).

Where \( e_1 = [1 - (1-e^2)^{1/2}] / [1 + (1-e^2)^{1/2}] \).

\[ \mu = \eta / [a(1-e^2/4 - 3e^4/64 - 5e^6/256 - ...)], \]
where \( \mu \) is in radians.

\[ N = y / k_0, \]
\[ k_0 = 0.9996 \]

With \( \rho \) known, the other terms are calculated for use in the equations \( \rho \) and \( \lambda \).

\[ e'^2 = e^2 / (1 - e^2) \]
\[ C_1 = e'^2 \cos^2 \rho_1 \]
\[ T_1 = \tan^2 \rho_1 \]
\[ N_1 = a(1-e^2) / (1-e^2 \sin^2 \rho_1)^{1/2} \]
\[ R_1 = a(1-e^2) / (1-e^2 \sin^2 \rho_1)^{3/2} \]
\[ D = x / (k_1 k_0) \]

\( y \) = The northing in the northern hemisphere
\( y \) = The northing - 10,000,000 in the southern hemisphere
\( x \) = Easting - 500,000
The UTM coordinate transformation programs

The two programs presented are written in BASIC for the Casio FX-801P Programmable Calculator. They perform the forward and inverse transformations.

The programs operate interactively. The forward transformation begins by inquiring of the user which reference ellipsoid he wishes to use. This is accomplished by the interrogative "A=" and "E CUADRADO=". A is the semimajor axis of the reference ellipsoid and E CUADRADO is the eccentricity squared. These values for four reference ellipsoids common to Latin America are given in table 1. A and e² need to be entered only once. The program will skip these inquiries on the second point transformation. After entering the ellipsoid constants the program will ask for the "LAT DEL PUNTO" and the "LON DEL PUNTO" - the latitude and longitude are entered in the following manner. To enter 35° 22' 14.22" N you press 35 then execute, 22 then execute and 14.22 and execute. To enter a point located in the southern hemisphere or west of Greenwich, you must apply a minus sign. 75° 34' 20" would be entered as -75 execute, -34 execute, -20 execute. Once the latitude and longitude are entered the program will ask for the "MERIDIANO CENTRAL". This is entered as two digits. For example 69°W will be entered as -69 execute. After this the transformation calculations will be performed and the answer displayed.

The display sequence is as follows; "N=" and the Northing will appear. Press the continue key and "E=" and the Easting will appear.
After the Easting is displayed the program will begin again by asking for the latitude, longitude and the central meridian of the next point.

The inverse transformation program, like the forward program starts by asking for "A=" and "E CUADRADO=". The program then asks for the Northing as "NORTE CUTM=" and the Easting as "ESTE CUTM". After this the program will ask for the "MERIDIANO CENTRAL". This is entered as two digits, like in the forward program. Finally, the program will ask for the hemisphere in which the point lies. The statement "HEMISFERIO N O S" will appear. Enter N if you are in the northern hemisphere or S for the south. After the transformations are complete the results will be displayed in the following manner. "LATITUD=" will appear. Press the continue key to display the results. Press continue again and "LONGITUD=" will appear. Press continue to see the results. If you wish to transform more points press continue and the program will ask you for the Northing, Easting, central meridian and hemisphere of the next point.
The UTM Forward Program

LIST
10 \AC : MODE 5: INF
   "M": 3: INF "E"
   COMPRADO= "E"
20 INF "LAT DEL PUNTO"
   TCR\"X, 3: INF
   "LAT DEL PUNTO= "X:=X
30 GET(GR, R)/(1
50 \Ox\x01X)/\Ox\x01X
20 INF "MERIDIANO"
   \Ox\x01X = \Ox\x01X "E"
40 \Ox\x01X = \Ox\x01X \\Ox\x01X\Ox\x01X
50 \Ox\x01X = \Ox\x01X \Ox\x01X
   \Ox\x01X = \Ox\x01X \Ox\x01X
"COG"
60 \Ox\x01X = \Ox\x01X \Ox\x01X
70 \Ox\x01X = \Ox\x01X \Ox\x01X
80 \Ox\x01X = \Ox\x01X \Ox\x01X
90 \Ox\x01X = \Ox\x01X \Ox\x01X
100 \Ox\x01X = \Ox\x01X \Ox\x01X
110 \Ox\x01X = \Ox\x01X \Ox\x01X
120 \Ox\x01X = \Ox\x01X \Ox\x01X
130 \Ox\x01X = \Ox\x01X \Ox\x01X
140 \Ox\x01X = \Ox\x01X \Ox\x01X
150 \Ox\x01X = \Ox\x01X \Ox\x01X
160 \Ox\x01X = \Ox\x01X \Ox\x01X

The UTM Inverse Program

LIST
10 \AC : INF "M": E
   INF "E" COMPRADO .
   C='E'
20 INF "NORTE C.U.
   T.R. = 0: INF "E"
   T.R. = 0: INF "E"
   T.R. = 0: INF "E"
   0 = 0
30 INF "MERIDIANO"
   CENTRAL "E": INF
   CENTRAL "E": INF
40 \Ox\x01X = \Ox\x01X /\Ox\x01X
50 \Ox\x01X = \Ox\x01X \Ox\x01X
60 \Ox\x01X = \Ox\x01X \Ox\x01X
70 \Ox\x01X = \Ox\x01X \Ox\x01X
80 \Ox\x01X = \Ox\x01X \Ox\x01X
90 \Ox\x01X = \Ox\x01X \Ox\x01X
100 \Ox\x01X = \Ox\x01X \Ox\x01X
110 \Ox\x01X = \Ox\x01X \Ox\x01X
120 \Ox\x01X = \Ox\x01X \Ox\x01X
130 \Ox\x01X = \Ox\x01X \Ox\x01X
140 \Ox\x01X = \Ox\x01X \Ox\x01X
150 \Ox\x01X = \Ox\x01X \Ox\x01X
160 \Ox\x01X = \Ox\x01X \Ox\x01X

Programs written by Liam P. O'Brien
Numerical Examples

UTM Forward Equations

Given: Clarke 1866 ellipsoid: 
\[ a = 6378206.4 \text{m}, \quad e^2 = 0.00676866 \]

Central meridian: 
UTM Zone 18: 
\[ \lambda_0 = 75^\circ \text{ West longitude} \]

Central meridian Scale factor: 
\[ k_0 = 0.9996 \]

Point: 
\[ \varphi = 40^\circ 30' \text{ North latitude} \]
\[ \lambda = 73^\circ 30' \text{ West longitude} \]

Find: \( x, y, k \)

\[ e^2 = 0.00676866/(1 - 0.00676866) = 0.0068148 \]

\[ N = 6378206.4/(1 - 0.00676866 \sin^2 40.5^\circ) = 387330.5 \text{m} \]

\[ T = \sin^2 40.5^\circ = 0.724523 \]

\[ C = 0.0068148 \cos^2 40.5^\circ = 0.0039404 \]

\[ A = (\cos 40^\circ 30') \times (73.5^\circ - (-75^\circ)) \pi/180^\circ = 0.0199074 \]

\[ M = 6378206.4 \left(1 - 0.00676866/64 - 5x0.00676866^4/256 - (3x0.00676866^6/1024) + 0.0039404 \right) \times (4x40^\circ 30') \]

\[ = 0.9996 \times 6378206.4 \times [0.0199074 + (1 - 0.724523 + 0.0039404) \times 0.0068148] \times 0.0199074 \]

\[ = 127,106.5 \text{m} \]

\[ y = 0.9996 \times [484,837.7 - 6378206.4 \times 0.854067 \times 0.0199074^2/2 - (5 - 0.729416 - 4\times0.0039404 \times 0.0068148 \times 0.0199074^3) + 0.0039404 \times 0.0199074^2/24 + (61 - 55x0.729416 - 4\times0.0039404 \times 0.0068148 \times 0.0199074^3) - 600 \times 0.0039404 \times 0.0199074 \times 0.0068148 \times 0.0199074^6/270]] \]

\[ = 4,484,124.4 \text{m} \]

\[ k = 0.9996 \times [1 + (1 - 0.0068148 \cos^2 40.5^\circ) \times 127,106.5^2/(2 \times 0.9996^2 \times 637330.5^2)] \]

\[ = 0.9997999 \]

Northing = 4,484,124.4 m
Easting = 500,000 + 127,106.5 = 627,106.5 m
UTM Inverse Equations

Given: Clarke 1866 ellipsoid: 
\( a = 6378206.4 \text{ m} \)
\( e^2 = 0.00676866 \)

Central meridian:
UTM Zone 18: 
\( \lambda_e = 75^\circ \) West longitude

Central meridian
Scale factor
Point:
Northing = 4484124.4 m
Easting = 627106.5 m

Find: \( \varphi, \lambda \)
\( x = 627106.5 - 500000 = 127106.5 \text{ m} \)
\( y = 4484124.4 \text{ m} \)
\( e^2 = 0.00676866/(1-0.00676866) = 0.0068148 \)
\( M = 0.7045135 \text{ radian} \)
\( C_1 = [1-(1-0.00676866)^2]/[1+(1-0.00676866)^2] = 0.0016979 \)
\( \mu = 448918.8/([6378206.4x(1-0.00676866/4-3x0.00676866^2/64-5x0.00676866^3/256)] = 0.0016979 \text{ radian} \)
\( \gamma = 0.0068148 \cos^2(40.5097362^\circ) = 0.0039393 \)
\( T = \tan^{-1}(40.5097362^\circ/0.7229560 \) 
\( \gamma = 6378206.4/(1-0.00676866\sin^2(40.5097362^\circ)) = 6378206.4 \text{ m} \)
\( K_1 = 6378206.4x(1-0.00676866/4-0.5097362^\circ)^3/2 = 6362,271.4 \text{ m} \)
\( D = 127106.5/(637334.2x0.99996) = 0.0199077 \)
\( \gamma = 40.5097362^\circ-(6387334.2x0.8543746/6362271.4)x[0.0199077^2/2-(5+3x0.7299560+10x0.0393935-4x0.0393935^2-9x0.0068148)x0.0199077^2/24+0.7299560+298x0.0393935-45x0.7299560^2-252x0.0068148-3x0.0393935^2)x0.0199077^2/720x180/\pi = 40.5097362^\circ = 40^\circ30' \) North latitude
\( \lambda = -75^\circ - \{[0.0199077-(1-2x0.7299560+0.0039393)x0.0199077^3/6+(5-2x0.0039393-25x0.7299560^2-3x0.0039393^2+6x0.0068148+4x0.7299560^2)x0.0199077^5/120]/\cos(40.5097362^\circ) \} \times 180^\circ/\pi = -75^\circ+1.5000000^\circ = 73.5^\circ30' \) West longitude
<table>
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<tr>
<th>Reference Ellipsoid</th>
<th>Semimajor Axis $a$</th>
<th>Excentricity Squared $e^2$</th>
</tr>
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<td>Clarke 1866</td>
<td>6378206.4</td>
<td>0.0067586580</td>
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<tr>
<td>Internacional</td>
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<td>0.0067226700</td>
</tr>
<tr>
<td>South American 1969</td>
<td>6378160</td>
<td>0.0066945419</td>
</tr>
<tr>
<td>WGS 72</td>
<td>6378135</td>
<td>0.0066943178</td>
</tr>
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References

The cylindrical developable surface as seen in its regular and transverse forms.
**Figure 2**

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<th>ZONE</th>
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</tbody>
</table>

The UTM numbering system and corresponding central meridians.
Figure 3

Graph of scale factor deterioration as a function of distance when the central meridian is a standard line in the Transverse Mercator projection.

Graph of scale factor deterioration as a function of distance when two standard lines are spaced 180 km East and West of the central meridian in the UTM system.

Graphs are based on Figures 7.9 and 7.10, pages 139 and 140 of reference three.
CM = The Central Meridian.
AB, DE = The Standard Lines formed by the intersections of the cylinder and the spheroid.
END

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