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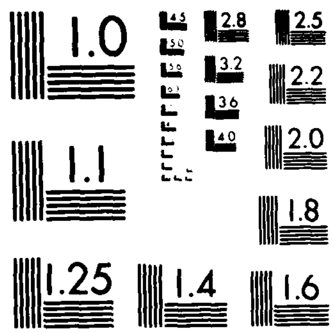

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This research seeks to *unify* certain problems of distributed parameter control: model reduction, control, sensor/actuator selections/failure, and decentralized control. These topics are all related and are to be unified through the quadratic performance metric with use of cost decomposition methods. The final research topic on model error estimation is required to make vernier adjustments after "best" models and controllers are developed, to absorb remaining modeling errors.

Summary of Progress

The following is a reference list of conference and journal papers supported in part by the Grant AFOSR 82-0209.

Article

Skelton, R.E., "Computer Aided Design of Suboptimal LQG Controller," IFAC Symposium on Computer Aided Design of Multivariable Technological Systems, Purdue Univ. Sept. 15-17, 1982.

Skelton, R.E., "Algorithm Development for the Control Design of Flexible Structures," Air Force/NASA Workshop on Modeling, Analysis and Optimization Issues for Large Space Structures, May 13-14, 1982, Williamsburg, Va.

Skelton, R.E., "Analysis of Structural Perturbations in Systems via Cost Decomposition Methods," IFAC Workshop on Singular Perturbations and Robustness of Control Systems, Ohrid, Yugoslavia, July 13-16, 1982.

Skelton, R.E., Frazho, A.E., and Wagie, D.A., "Generalization of Cost-Equivalent Realizations," IEEE ISCAS, May 2-4, 1983, Newport Beach, Ca.

Yousuff, A. and Skelton, R.E., "Cost-Equivalent Realizations of Stochastic Processes," 1982 ACC, June 14-16, 1982, Arlington, Va.

Skelton, R.E., "On the Selection of Optimal Bandwidths for LSS Controllers," 1982 ACC, June 14-16, 1982, Arlington, Va.

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Skelton, R.E., "On the Selection of Controller Order for the Control of Linear Dynamic Systems," 6th IFAC Symposium on Identification and System Parameter Estimation, June 7-11, 1982, Washington, D.C.

Yousuff, A. and Skelton, R.E., "A Solar Optical Telescope Controller Design by Component Cost Analysis," 3rd IFAC Symposium on Control of Distributed Parameter Systems, June 29-July 2, 1982 Toulouse, France.

Skelton, R.E. and Wagie, D.A., "Minimal Root Sensitivity in Linear Systems," IFAC Workshop on Applications of Nonlinear Programming and Optimal Control, June 20-22, 1983, San Francisco.

Skelton, R.E. and DeLorenzo, M.L., "On Selection of Weighting Matrices in the LQG Problem," 20th Annual Allerton Conf. on Communication Control and Computing, Oct. 6-8, 1982, Monticello, Ill.

Skelton, R.E. and Davis, J.A., "Model Error Estimation," IFAC Workshop for Adaptive Systems in Control and Signal Processing, June 20-22, 1983, San Francisco, Ca.

Yousuff, A. and Skelton, R.E., "Balanced Controller Reduction," 17th Annual Allerton Conf. on Communication, Control, and Computing, Oct. 1982, Monticello, Ill.

Skelton, R.E. and Yousuff, A., "Component Cost Analysis of Large Scale Systems," Int. J. Control, scheduled to appear 1983.

Skelton, R.E. and DeLorenzo, M.L., "Selection of Noisy Actuators and Sensors in Linear Stochastic Systems," to appear, J. Large Scale Systems Theory and Applications, 1983.

Yousuff, A. and Skelton, R.E., "Controller Reduction by Component Cost Analysis," to appear, IEEE Trans. Auto. Control, 1983.

Skelton, R.E. and Davis, J.S., "Comments on Realizations and Reduction of Markovian Models from Nonstationary Data," to appear IEEE Trans. Auto. Control, April 1983, AC-28, No. 4.

The contributions are summarized in the attached Appendices.

APPENDIX A

Minimal Root Sensitivity in Linear Systems
(IFAC Workshop for Applications of Nonlinear Programming and Optimal Control, June 20-22, 1983, San Francisco.)

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Abstract

A lower bound is derived for root sensitivity and an explicit criteria for achieving this minimum is given. Secondly, an optimal output feedback control problem is discussed which penalizes an index related to root sensitivity.

1.0 Introduction

The modal data for physical systems is rarely well known. This can make stability predictions unreliable in *feedback control* problems and can make the behavior far from the analytical predictions. This paper documents the smallest possible sensitivity of eigenvalues λ_i with respect to the independent plant parameters in linear systems of the form

$$\dot{x} = Ax \ , \quad x \varepsilon R^n \tag{1}$$

That is, the norm of $\frac{\partial \lambda_i}{\partial A}$ and the lower bound of its norm are of interest. Secondly, a metric related to root sensitivity is added to the optimal output feedback problem to achieve a compromise between performance and root sensitivity.

The norm of a matrix shall be denoted by

$$\|[\cdot]\|_Q^2 = \mathrm{tr}\ [\cdot]^* Q[\cdot] \ , \quad \mathrm{tr}\ [\cdot] \triangleq \mathrm{trace}\ [\cdot] \tag{2}$$

and the norm of a vector shall be denoted by

$$\|(\cdot)\|_Q^2 = (\cdot)^* Q(\cdot) \tag{3}$$

where * denotes complex conjugate transpose. Results herein are limited to the case of distinct eigenvalues for A.

2.0 The Construction of a Root Sensitivity Metric

The sensitivity of the i^{th} eigenvalue $\partial\lambda_i/\partial A$ is a nxn matrix denoted by $S_i \triangleq \partial\lambda_i/\partial A$. The norm of S_i from (3) is

$$\|S_i\|_Q^2 \triangleq \mathrm{tr}\ S_i^* Q_i S_i = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} [(\frac{\partial \lambda_i}{\partial A_{\alpha row}})(\frac{\partial \lambda_i}{\partial A_{\beta row}})^*] Q_{i_{\alpha\beta}} \tag{4}$$

where $A_{\alpha row} \stackrel{\Delta}{=} (A_{\alpha_1}, \ldots, A_{\alpha_n})$, and $Q_i = Q_i^T \geq 0$. The weighting matrix Q_i is to be chosen larger than Q_j if root shifts in λ_i are of greater concern than roots shifts in λ_j. *Within* the matrix Q_i, the weight $Q_{i_{\alpha\alpha}}$ is to be chosen larger than $Q_{i_{\beta\beta}}$ if the parameters of the α^{th} row of A (these are associated with $\dot{x}_\alpha$ in (1)) are more uncertain than those of the β^{th} row of A. (Hence, a logical choice for $Q_{i_{\alpha\alpha}}$ is a value proportional to the variance of parameters $A_{\alpha row}$). Finally, the complete root sensitivity metric of interest is

$$s \stackrel{\Delta}{=} \sum_{i=1}^{n} \|S_i\|^2_{Q_i} \stackrel{\Delta}{=} \sum_{i=1}^{n} \left\| \frac{\partial \lambda_i}{\partial A} \right\|^2_{Q_i} \tag{5}$$

Thus, from the point of view of robustness, a system design with a large value of s might be considered less desirable than a system design with a small value of s.

3.0 Computation of the Root Sensitivity Metric

It is assumed that A has a linearly independent set of eigenvectors e_i

$$Ae_i = e_i \lambda_i, \quad i = 1, \ldots, n \tag{6}$$

The reciprocal basis vectors ℓ_i are defined by

$$E \stackrel{\Delta}{=} [e_1, \ldots, e_n], \quad \begin{bmatrix} \ell_1^* \\ \vdots \\ \ell_n^* \end{bmatrix} \stackrel{\Delta}{=} E^{-1}, \text{ hence } \ell_i^* e_j = \delta_{ij} \tag{7}$$

Multiplying (6) from the left by ℓ_i^*, using (7), yields the eigenvalues in terms of A, its eigenvectors, and its reciprocal basis vectors,

$$\lambda_i = \ell_i^* A e_i \quad . \tag{8}$$

Differentiation of the scalar (8) with respect to A provides the required sensitivity $\partial\lambda_i/\partial A$. To derive this result two identities from linear algebra are required

$$\mathrm{tr}\, AB = \mathrm{tr}\, BA \tag{9}$$

$$\frac{\partial}{\partial A}(\mathrm{tr}\, AB) = \frac{\partial}{\partial A}(\mathrm{tr}\, BA) = B^T \tag{10}$$

where (9), (10) hold for real or complex matrices B and A, and (10) holds if and only if the elements of A are independent. Hence from (8), using (9), (10),

$$\frac{\partial\lambda_i}{\partial A} = \frac{\partial}{\partial A}[\mathrm{tr}\, A(e_i\ \ell_i^{\,*})] = (e_i\ \ell_i^{\,*})^T = \bar{\ell}_i e_i^{\,T} \tag{11}$$

where $\bar{}$ denotes complex conjugate. The weighted norm (4) may now be written

$$\|S_i\|^2_{Q_i} \overset{\Delta}{=} \mathrm{tr}\, S_i^{\,*} Q_i S_i = \mathrm{tr}(\bar{\ell}_i e_i^{\,T})^* \, Q_i(\bar{\ell}_i\ e_i^{\,T}) = \|\ell_i\|^2_{Q_i}\|e_i\|^2 \tag{12}$$

where the last equality requires use of identity (9) again, and where

$$\|\ell_i\|^2_Q = \ell_i^{\,*} Q \ell_i\ , \quad \|e_i\|^2 = e_i^{\,*} e_i \tag{13}$$

One may note the similarity between (11) and the weak differential of λ_i given by Eq. (6.2.3) of [1]. Also note the similarity between (12) and the *upper* bound of the weak derivative of λ_i provided on the top of p. 235 in [1]. This paper seeks *lower* bounds rather than the upper bounds of [1]. Otherwise the nature of the results are similar (see theorem 6.2.4 of [1]).

Note also that without loss of generality eigenvectors e_i may always be normalized to unit length. In which case (12) may be written

$$\|S_i\|^2_{Q_i} = \|\ell_i\|^2_{Q_i}\ , \quad \|e_i\|^2 = 1 \quad . \tag{14}$$

Define by use of the Cholesky [1] decomposition of $Q_i = \mathcal{Q}_i^T \mathcal{Q}_i$,

$$r_1 \stackrel{\Delta}{=} \mathcal{Q}_i \ell_i \ , \quad Q_i = \mathcal{Q}_i^T \mathcal{Q}_i \tag{15}$$

Then (12) may be written

$$\|s_i\|_{Q_i}^2 = \|\ell_i\|_Q^2 \ \|e_i\|^2 = \|r_i\|^2 \ \ \|e_i\|^2 \tag{16}$$

The Schwartz inequality [1] holds for *any* two vectors ℓ_i, e_i

$$|\ell_i^* \ e_i| \leq \|\ell_i\| \ \ \|e_i\| \tag{17}$$

Since the *particular* vectors ℓ_i, e_i are related by (8),

$$\ell_i^* \ e_i = 1 \tag{18}$$

(17) and (18) lead immediately to

$$\|\ell_i\| \ \ \|e_i\| \geq 1 \tag{19}$$

From (15) note also that

$$\|\mathcal{Q}_i \ell_i\| = \|r_i\| \quad . \tag{20a}$$

Hence,

$$\|\mathcal{Q}_i\| \ \|\ell_i\| \geq \|r_i\| \ . \tag{20b}$$

Multiply (19) by $\|\mathcal{Q}_i\|$ and use (20) to obtain

$$\|\mathcal{Q}_i\| \ \|\ell_i\| \ \|e_i\| \geq \|r_i\| \ \ \|e_i\| \geq \|\mathcal{Q}_i\| \tag{21}$$

Squaring both sides of (21), using the fact $\|\mathcal{Q}_i\|^2 = \mathrm{tr}\,\mathcal{Q}_i^T\mathcal{Q}_i = \mathrm{tr}\ Q_i = \|Q_i\|$ leads to

$$\|r_i\|^2 \ \|e_i\|^2 \geq \|Q_i\| \tag{22}$$

The equality in (19), and hence in (22), holds if and only if ℓ_i and e_i are colinear ($\ell_i = e_i$ for normalized e_i), [1]. From linear algebra [2], $\ell_i = e_i$ if and only if A is normal ($AA^* = A^*A$). Thus, these results are summarized as follows.

Theorem 1

Let (λ_i, e_i, ℓ_i) *be the* i^{th} *eigenvalue, eigenvector and its reciprocal basis vector associated with the real matrix* A. *If* A *has a linearly independent set of eigenvectors* e_i, $i=1, \ldots, n$, *then*

$$\left\|\frac{\partial\lambda_i}{\partial A}\right\|^2_{Q_i} \geq \|Q_i\| \tag{23}$$

where the lower bound

$$\left\|\frac{\partial\lambda_i}{\partial A}\right\|^2_{Q_i} = \|Q_i\|$$

is achieved (for $Q_i \neq 0$*) if and only if* $AA^* = A^*A$.

Corollary to Theorem 1:

The sensitivity metric (5) *is bounded from below by*

$$s \geq \sum_{i=1}^{n} ||Q_i|| \tag{24}$$

and, for the case $Q_i \neq 0$ $(i=1,2,\ldots,n)$, *the minimum sensitivity* $s = \sum_{i=1}^{n} ||Q_i||$ *is achieved if and only if* A *is normal* ($AA^* = A^*A$).

The corollary provides necessary and sufficient conditions for minimum root sensitivity. The next section suggests a means to incorporate this information into the output feedback control design problem.

4.0 Output Feedback Design

It has been shown [3]-[5] that the necessary conditions for the output feedback for the system

$$\begin{aligned}
\dot{x} &= Ax + Bu + Dw & & E\binom{w}{v} = 0, \\
y &= Cx & & E\binom{w(t)}{v(t)}\left(w^T(\tau), v^T(\tau)\right) = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta\,(t-\tau) \\
z &= Mx + v & & \\
u &= Gz & & E\binom{w(t)}{v(t)}\, x^T(o) = 0,\ t > 0
\end{aligned} \tag{25}$$

to minimize

$$V = \lim_{t\to\infty} E\left(\|y\|_Q^2 + \|u\|_R^2\right) \tag{26 a}$$

$$= \operatorname{tr} P(C^TQC + M^TG^TRGM) + \operatorname{tr} VG^TRG \tag{26 b}$$

are

$$0 = K(A+BGM) + (A+BGM)^TK + M^TG^TRGM + C^TQC \tag{27}$$

$$0 = P(A+BGM)^T + (A+BGM)P + BGVG^TB^T + DWD^T \tag{28}$$

$$0 = RGMPM^T + RGV + B^TKPM^T + B^TKBGV + \Psi \tag{29}$$

where $\psi = 0$ for the standard measurement feedback problem in [3]-[5]. Various suboptimal strategies for approximating the solution of (27)-(29) may be found in the literature [4]-[5].

Parameter sensitivity has long been a concern in optimal control. Some authors [6]-[7] have suggested modifying the performance index (26a) by the addition of trajectory sensitivity terms $\sum_i \left\|\frac{\partial y}{\partial p_i}\right\|^2$ (where p_i, $i = 1, \ldots, r$ represent the uncertain parameters). The resulting computational burdens are very great indeed, since the dimension of the constraint (state) equations becomes n(1+r). There is also the fact that minimizing output sensitivity does not necessarily keep root sensitivity small. Hence, the concerns of stability in the presence of parameter uncertainty are not addressed by trajectory sensitivity methods. In Section 3.0 we showed that root sensitivity is minimized when A is normal. Motivated by Section 3.0, we note that some of the shortcomings of the trajectory sensitivity methods are therefore avoided by minimizing an "abnormality" index related to root sensitivity. It has also been shown [8] that the robustness bound for a certain class of parameter errors is maximized when the plant matrix, A, is normal.

Motivated, therefore, by Theorem 1 and [8] we pose a new performance index for optimization that includes an "abnormality" penalty

$$V' = \lim_{t\to\infty} E(\|y\|_Q^2 + \|u\|_R^2) + \beta\,\|(A+BGM)(A+BGM)^T-(A+BGM)^T(A+GBM)\|^2_{C^TQC} \quad (30)$$

When β is much smaller than the norms of Q and R the solution tends toward the standard optimal control result (27)-(29) with $\psi = 0$. On the other extreme, when β is chosen much larger than the norms of Q and R, the closed-loop system approaches the smallest possible root sensitivity. (From Theorem 1 and its corollary note that root sensitivity is minimized if and only if the latter term in (30) is zero. Other choices of weights on the matrix norm may be chosen besides C^TQC. This choice is suggested only to make the sensitivity weight C^TQC the same as the state weight in $y^TQy = x^T[C^TQC]x$.

Using the same matrix norm as in previous sections, and defining $A \overset{\Delta}{=} A+BGM$, V' becomes

$$V' = \mathrm{tr}\, P(C^TQC + M^TG^TRGM) + \mathrm{tr}\, VG^TRG +$$
$$\beta\, \mathrm{tr}[(AA^T - A^TA)]C^TQC[(AA^T - A^TA)\ ] \qquad (31)$$

The necessary conditions for the optimum G are obtained by augmenting the constraint equation (28) [which defines the P in (26) and (29)] to (31) via Lagrange multiplier matrix K and differentiating the augmented V' with respect to P, K, and G. The equations (27) - (29) result, with the following definition of ψ:

$$\psi \triangleq \beta B^T\{[(AA^T - A^TA)C^TQC + C^TQC(AA^T - A^TA)]A$$
$$- A[(AA^T - A^TA)C^TQC + C^TQC(AA^T - A^TA)]\}M^T \qquad (32)$$

These results are summarized as follows:

Theorem 2

The necessary conditions for minimizing (30) *subject to the constraints* (25) *are given by* (27) - (29) *and* (32).

The following conclusion should also be clear, since for an arbitrary A the matrix A+BGM can be made normal (by choice of G) only if rank B = rank M = n.

Theorem 3

Suppose $Q_i = I$, $i = 1, \ldots, n$ *in* (5). *The minimum sensitivity* $s = n$ *can be guaranteed by output feedback control for arbitrary* A *if and only if rank* B = *rank* M = n. *Furthermore, the control gain in this case is not unique. Two gains that provide minimum sensitivity are*

$$G = -B^{-1}AM^{-1} \tag{33}$$

$$G = B^{-1}A^{T}M^{-1} \tag{34}$$

Proof:

Substitute (33), (34) into the normality condition for minimum sensitivity

$$(A + BGM)(A + BGM)^T - (A + BGM)^T(A + BGM) = 0 \tag{35}$$

to see that the condition (35) holds.

5.0 Application of Closed-Loop Root Sensitivity Design

Example 1:

The pitch motion of a rigid aircraft is governed by, [9],

$$\begin{pmatrix} \dot{\alpha} \\ \dot{q} \end{pmatrix} = \begin{bmatrix} -1/\tau & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{pmatrix} \alpha \\ q \end{pmatrix} + \begin{bmatrix} 0 \\ \omega_0^2 Q \end{bmatrix} (u + w) \tag{36}$$

where α is the angle of attack, q is the pitch rate, u is the elevator angle, τ is the lifting time constant, ω_0 is the undamped pitch natural frequency, and Q is the elevator effectiveness. In the open-loop case, $(u+w) = 0$, we are interested in the root sensitivity properties of (36).

Note from the corollary to Theorem 1 that minimum sensitivity $s = \sum_{i=1}^{2} \| \frac{\partial \lambda_i}{\partial A} \|^2 = 2$ is achieved if and only if A is normal. Computing the "abnormality" matrix $(AA^* - A^*A)$ yields

$$AA^* - A^*A = \left[\begin{array}{c|c} 1-\omega_0^4 & \frac{1}{\tau}(1+\omega_0^2) \\ \hline \frac{1}{\tau}(1+\omega_0^2) & \omega_0^4 - 1 \end{array}\right] \tag{37}$$

Thus, root sensitivity takes on its absolute minimum when $\omega_0 = 1$, $\frac{1}{\tau} = 0$. This, of course, is not a practical possibility for the aircraft. Now consider the output feedback design of Section 4.

Example 2: Let the angle of attack measurement be made $z = \alpha + v$ where $Ev = 0$, $E[v(t)v(\tau)] = \delta(t-\tau)$ describes the white measurement noise v and $Ew = 0$ $E[w(t)w(\tau)] = \delta(t-\tau)$ describes the white actuator noise w. For the system (36) design a measurement feedback control law for regulating u such that

$$V = \lim_{t\to\infty} \{E(q^2+\rho u^2) + \beta \|(A+BGM)(A+BGM)^T - (A+BGM)^T(A+BGM)\|^2_{C^TC}\} \quad (38)$$

is minimized. Assume $\mathcal{Q} = 1$, $Q = I$, and $C = [0, 1]$.

The solution is provided by (29), where P is obtained from (28),

$$P = \begin{bmatrix} \tau & 1 \\ 1 & \frac{1}{\tau} + \omega_0^2\tau(1-G) \end{bmatrix} \frac{\omega_0^2 (G^2+1)}{2(1-G)} \quad (39)$$

and K is obtained from (27) (For this example $R = \rho$)

$$K = \begin{bmatrix} \frac{1}{2}[G^2\rho\tau - \omega_0^2\tau(G-1)] & -1/2 \\ -1/2 & \dfrac{\rho G^2\tau^2+1 - \omega_0^2\tau^2(G-1)}{2\tau\omega_0^2(1-G)} \end{bmatrix} \quad (40)$$

For the aircraft example, (29) yields a fifth order equation in G as a function of β, ρ, ω_0^2 and τ. For given values of ρ, β, ω^2 and τ this equation will yield five candidate values of G. The optimal G is that root that minimizes the cost function. In the present case $G < 1$ is required for stability. If none of the real G are < 1, then analysis of this problem shows that a decrease in β will drive G toward more stable values. After changing β, the equation can then be solved once again for G. This iteration is repeated until a stable (G < 1) solution is achieved. (One may also increase ρ to seek stability, although the direction of change of ρ required depends on β).

Assume $\omega_0^2=1$, $\tau=.7$ (this corresponds to damping ratio = .71). Setting $\beta=0$, $\rho=1$ yields the standard optimal measurement feedback control

$$G = -0.118 \tag{41}$$

and setting $\beta = \infty$, $\rho < \infty$ yields the optimally sensitive design

$$G = 2.0 \quad . \tag{42}$$

This choice of G in (63) forces the closed-loop system matrix to be symmetric

$$A + BGM = \begin{bmatrix} -1/\tau & 1 \\ -\omega_0^2+\omega_0^2 G & 0 \end{bmatrix} = \begin{bmatrix} -1/\tau & 1 \\ 1 & 0 \end{bmatrix} \tag{43}$$

and by Theorem 1 and its corollary, the sensitivity is at its minimum in this case. Note, however, that stability is lost by this minimum sensitivity design, (G < 1 is required for stability). Thus, minimally sensitive designs might not be stable.

Figures 1 and 2 show the tradeoff between output performance $V_y \overset{\Delta}{=} \lim_{t\to\infty} E\|y\|_Q^2$ versus the control effort $V_u \overset{\Delta}{=} \lim_{t\to\infty} E\|u\|_R^2$ (Fig.1) and abnormality of $A = (A+BGM)$, $V_A = \| AA^* - A^* A \|_{C^TQC}$, versus the control effort (Fig.2). For both figures $\omega_0^2 = 1$, $\tau=.7$, β and ρ vary. In the standard output feedback design ($\beta=0$), the output performance is improved with an increase in control effort (Fig. 1) whereas the abnormality index greatly increases with control effort (Fig. 2 with $\beta = 0$). Note also that $\beta > 2$ is not desired, since larger values of β do not yield substantially larger abnormality reductions (Fig. 2) but do accelerate the degradation of the nominal output performance (Fig. 1).

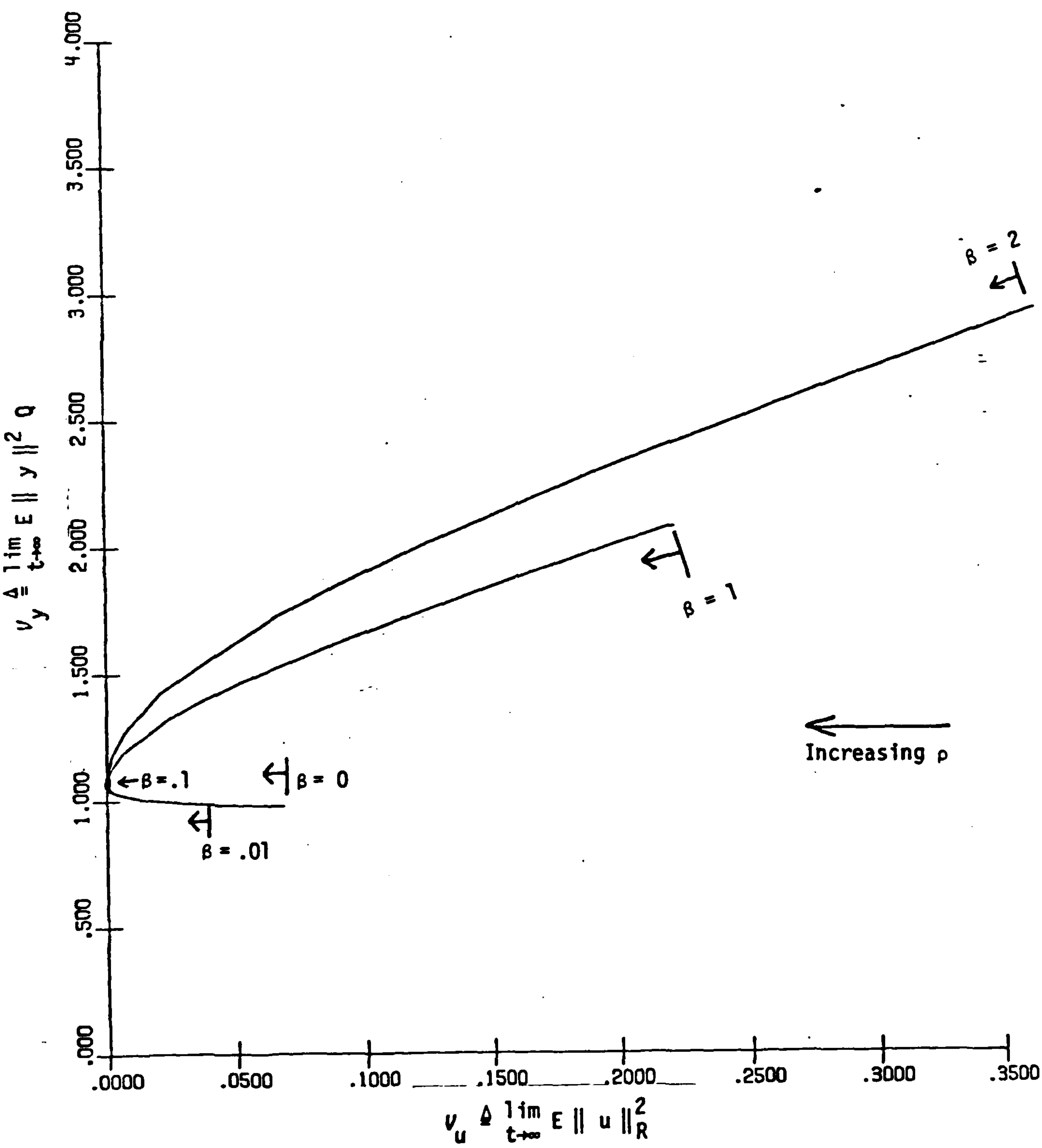

Figure 1. Output Cost (V_y) vs Input Cost (V_u) [τ=.7, ω_0= 1.0]

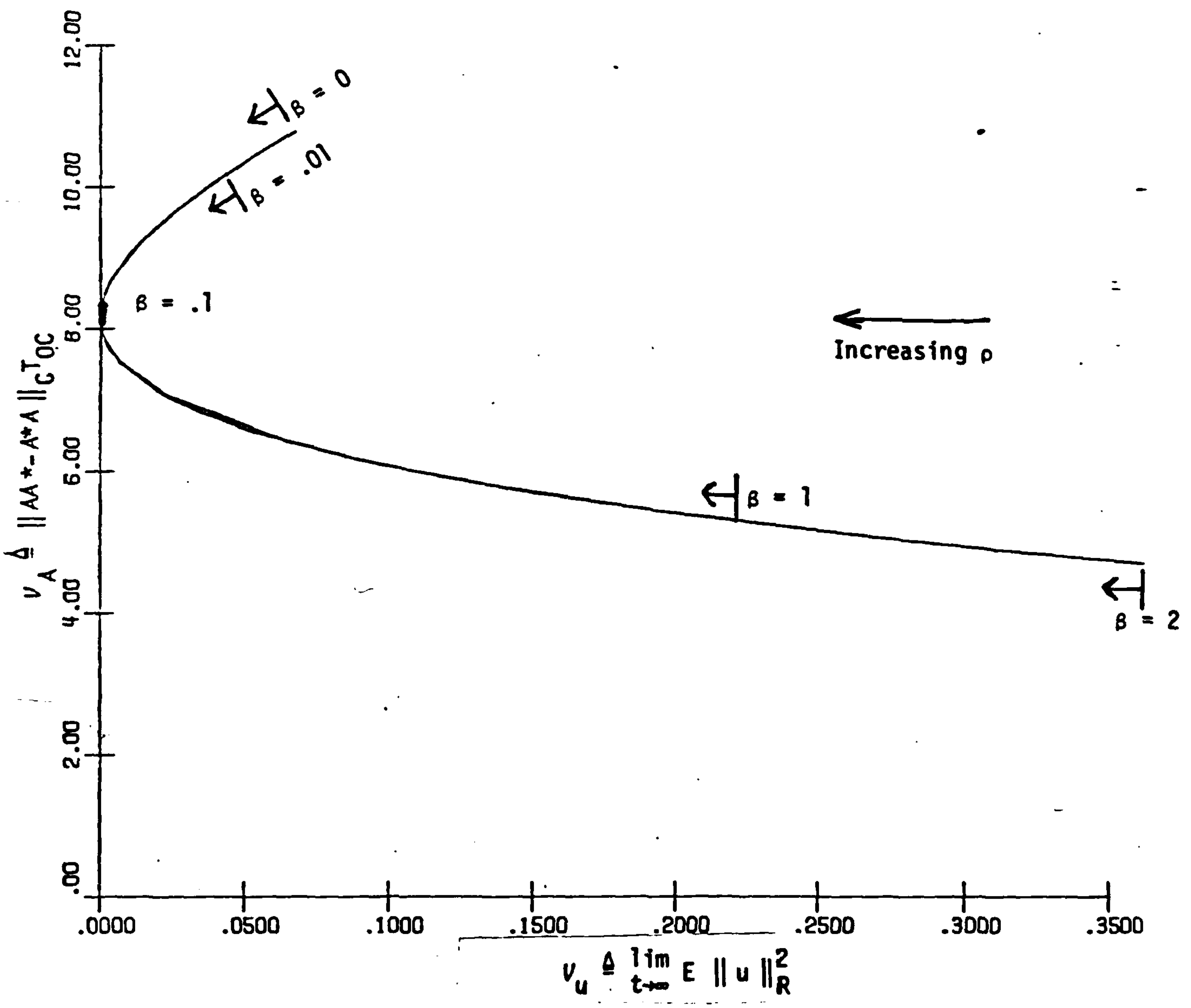

Figure 2. Abnormality Cost (V_A) vs Input Cost (V_u) [τ=.7, ω_0=1.0]

5.0 Conclusions

Explicit expressions for a scalar metric of root sensitivity is given in terms of the left and right eigenvectors of the system, so that sensitivity of each eigenvalue with respect to the plant matrix may be readily computed. A necessary and sufficient condition for minimum root sensitivity is that the plant matrix of the state equations be normal.

However, root sensitivity alone is not a sufficient design goal. Thus, an "abnormality" term is added to the traditional quadratic performance metric of optimal control. The necessary conditions are given for the solution of this problem and an example gives some practical insights.

The weaknesses in these results include the fact that the necessary and sufficient condition (normality) for minimal root sensitivity are only local results and also that they apply only for non-defective plant matrices (A has linearly independent eigenvectors).

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ON SELECTION OF WEIGHTING MATRICIES IN THE LQG PROBLEM

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ABSTRACT

The weighting matrices in the standard Linear Quadratic Gaussian (LQG) theory are most often used to achieve pole assignment. Instead, this paper proposes to select these weighting matrices to achieve RMS bounds on inputs and outputs.

1.0 INTRODUCTION

One of the most frequent complaints about the application of the standard Linear Quadratic Gaussian (LQG) theory is the "arbitrariness" in the choice of weights Q and R in the quadratic cost function

$$V = \mathcal{E} \lim_{t\to\infty} (\|y(t)\|_Q^2 + \|u(t)\|_R^2) \tag{1.1}$$

which is minimized subject to the constraints

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw\,, && x\epsilon R^n\,,\quad u\epsilon R^m\,,\quad w\epsilon R^p \\ y &= Cx && y\epsilon R^k \text{ (system outputs)} \\ z &= Mx + v && z\epsilon R^\ell \text{ (system measurements)} \end{aligned} \tag{1.2}$$

$$\mathcal{E}w = 0\,,\quad \mathcal{E}v = 0\,,$$

$$\mathcal{E}\begin{bmatrix} x^T(t_o) \\ w(t) \\ v(t) \end{bmatrix} \begin{bmatrix} x^T(t_o) & w^T(\tau), & v^T(\tau) \end{bmatrix} = \begin{bmatrix} X_o & 0 & 0 \\ 0 & W\delta(t-\tau) & 0 \\ 0 & 0 & V\delta(t-\tau) \end{bmatrix}$$

$$W > 0\,,\quad V > 0$$

where w(t) and v(t) are zero-mean white noise processes with intensities W and V respectively, and $\mathcal{E}$ is the expectation operator and T denotes matrix transposition. In the literature, there have been two basic suggestions for the choice of Q, R.

The first suggestion, made by Bryson [1], relates Q and R to the input-output specifications σ_i^2, μ_i^2; where the *desired* values of the response of the closed-loop system are

$$y_i^2(t) \le \sigma_i^2 \quad \text{or} \quad \left(\frac{y_i}{\sigma_i}\right)^2 \le 1$$
$$u_i^2(t) \le \mu_i^2 \quad \text{or} \quad \left(\frac{u_i}{\mu_i}\right)^2 \le 1 \tag{1.3}$$

leading to Bryson's suggestion for the choices of Q and R

$$Q = \begin{bmatrix} 1/\sigma_1^2 & & \\ & \ddots & \\ & & 1/\sigma_k^2 \end{bmatrix}, \quad R = \begin{bmatrix} 1/\mu_1^2 & & \\ & \ddots & \\ & & 1/\mu_m^2 \end{bmatrix} \tag{1.4}$$

Hence with this choice of Q and R (1.1) becomes

$$V = \lim_{t\to\infty} \{ \sum_{i=1}^{k} Q_{ii}(Ey_i^2) + \sum_{j=1}^{m} R_{jj}(Eu_j^2) \} \tag{1.5}$$

It usually happens, however, that even though this choice of Q and R has some physical motivation the *actual* $y_i^2(t)$, $u_i^2(t)$ do not satisfy the *desired* bounds (1.3). Furthermore, there is no theory available which will show how to choose Q and R such that (1.3) is satisfied. Neither is there a method available which will guarantee satisfaction of (1.3) in the mean squared sense

$$Ey_i^2(t) \le \sigma_i^2 \ , \quad i = 1, \ldots, k$$
$$Eu_j^2(t) \le \mu_j^2 \ . \quad j = 1, \ldots, m \tag{1.6}$$

The second basic suggestion about the choice of Q, R in the literature has been to choose Q and R to achieve a desired pole assignment in the closed-loop system [2],[3]. In [4] a gradient technique adjusts the elements of Q, R to maximize the singular values of selected return difference matrices which are related to stability margins and disturbance rejection. There has been much more written about the pole assignment role for Q and R than about the time response assignment (1.3) or the variance assignment (1.6). This is indeed curious since LQG is a theory steeped in the *time-domain* and is *directly* concerned with the time responses y(t) and u(t). One might even argue that it is a bit obtuse to use LQG theory to do pole assignment since there are more direct methods to assign poles which do not use the artifice of LQG theory, [5].

This paper deals with the more natural use of LQG theory in the time-domain question relating to objective (1.6). The ideal goal is to find a linear dynamical feedback controller which satisfies the variance constraints (1.6). We shall alter this problem statement to make it more tractable by using the steady approximation of the constraints (1.6)

$$\lim_{t\to\infty} Ey_i^2(t) \le \sigma_i^2$$
$$\lim_{t\to\infty} Eu_i^2(t) \le \mu_i^2 \tag{1.7}$$

Then, a penalty function approach (i.e. LQG theory) is employed to accommodate the constraints (1.7), in the manner (1.5) by proper choice of Q_{ii}, and R_{ii}. This formulation automatically satisfies the constraint for a linear dynamical feedback controller, since the optimal LQG controller for (1.5) employs the standard Kalman filter and state feedback control gains. This will be called the

<u>Constrained Variance LQG Problem (LQG_{cv})</u>:

Find Q_{ii} *and* R_{ii} *in* (1.5) *so that* (1.7) *is satisfied, subject to* (1.2).

The LQG_{cv} problem does not have a solution for every set of requirements σ_i^2, (i=1, ..., k), μ_j^2, (j=1, ..., m). To see that this is true, consider a single input-single output system. Hence, $y\epsilon R^k(k=1)$, $u\epsilon R^m(m=1)$. The LQG theory promises an inverse relationship between output regulation and control effort as the weighting R on control effort varies from 0 to infinity, as depicted in Fig. 1.

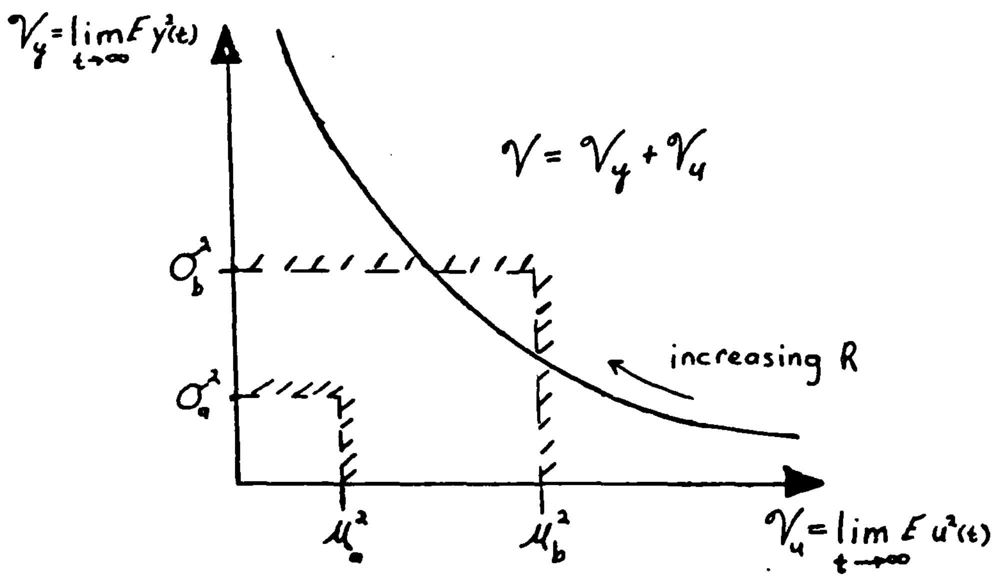

Fig. 1 Results of LQG theory for SISO Systems

Clearly no solution to the LQG_{cv} problem exists for the problem imposed by the values (σ_a, μ_a) but an infinite number of solutions exist for the values (σ_b, μ_b).

The key to progress beyond the Bryson rule (1.4) is to provide a relationship between the *actual* variances Ey_i^2 , Eu_i^2 and the choices of Q_{ii} and R_{ii} in (1.5). This step is provided in Section 2 and is based upon the cost decomposition results of [6]. An iterative algorithm based upon these results is given in Section 3. Section 4 applies the algorithm to a Large Space Structure.

2.0 INPUT/OUTPUT COST ANALYSIS

Under the assumptions (2.1)

$$\left.\begin{array}{ll} (A, B)\ ,\ (A, D) & \text{stabilizable pairs} \\ (A, C)\ ,\ (A, M) & \text{detectable pairs} \end{array}\right\} \tag{2.1}$$

the LQG problem (1.1) (1.2) has the solution [ ]

$$u = G\hat{x} \quad , \quad G = -R^{-1}B^TK\hat{x} \tag{2.2a}$$

$$KA + A^TK - KBR^{-1}B^TK + C^TQC = 0 \tag{2.2b}$$

$$\dot{\hat{x}} = (A+BG)\hat{x} + F(z-M\hat{x}) \quad , \quad F = PM^TV^{-1} \tag{2.2c}$$

$$PA^T+AP - PM^TV^{-1}MP + DWD^T = 0 \tag{2.2d}$$

yielding the closed-loop regulator

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{bmatrix} A & BG \\ FM & A+BG-FM \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{bmatrix} D & 0 \\ 0 & F \end{bmatrix} \begin{pmatrix} w \\ v \end{pmatrix} \\ \mathcal{y} \triangleq \begin{pmatrix} y \\ u \end{pmatrix} &= \begin{bmatrix} C & 0 \\ 0 & G \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \end{aligned} \tag{2.3}$$

System (2.3) is a linear system driven by white noise and with output $\mathcal{y}$. It has been established in [6] that such system outputs $\mathcal{y}_i$ satisfy a "cost-decomposition" property

$$\mathcal{V} = \sum_{i=1}^{k+m} \mathcal{V}_i^{\mathcal{y}} = \lim_{t\to\infty} E\|\mathcal{y}\|_{\mathcal{Q}}^2 \tag{2.4}$$

where

$$V_i^y \triangleq \lim_{t\to\infty} \frac{1}{2} E(\frac{\partial ||y||_Q^2}{\partial y_i} y_i) \tag{2.5}$$

The calculation and study of the "output costs" V_i^y is called "Output Cost Analysis" in [6]. We momentarily postpone the calculation of V_i^y.

Now the choice $Q = I$ in (2.5) leads to

$$V_i^y \triangleq \lim_{t\to\infty} \frac{1}{2} E(\frac{\partial ||y||^2}{\partial y_i} y_i) = \lim_{t\to\infty} E(y_i^2) \quad i=1, \ldots, k \tag{2.6}$$

$$V_i^u \triangleq \lim_{t\to\infty} \frac{1}{2} E(\frac{\partial ||y||^2}{\partial y_j} y_j) = \lim_{t\to\infty} E(u_i^2) \ , \quad j=k+i \quad i=1, \ldots, m \tag{2.7}$$

and for $Q = I$, note that (2.4) becomes the *unweighted* cost $V°$

$$V° = \lim_{t\to\infty} E(||y||^2 + ||u||^2) = \sum_{i=1}^{k} V_i^y + \sum_{j=1}^{m} V_j^u \ . \tag{2.8}$$

The system (2.3) evaluated according to (2.8) yields the following output cost analysis, using the output cost formula given by Eq. (4.14) of [6].

$$V_i^y = \lim_{t\to\infty} E(y_i^2) = ||c_i||_X^2 = c_i^T X c_i \tag{2.9a}$$

$$V_i^u = \lim_{t\to\infty} E(u_i^2) = R_{ii}^{-2} \ ||b_i||_K^2 = R_{ii}^{-2} \ b_i^T \ K b_i \tag{2.9b}$$

where X and K satisfy

$$X = \hat{X}+P \quad , \quad PA^T + AP - PM^T V^{-1} MP + DWD^T = 0 \tag{2.10}$$

$$K = K\hat{X}K \quad , \quad KA + A^T K - KBR^{-1} B^T K + C^T QC = 0 \tag{2.11}$$

$$G = -R^{-1} B^T K, \quad \hat{X}(A+BG)^T + (A+BG)^T \hat{X} + FVF^T = 0 \tag{2.12}$$

$$F = PM^T V^{-1}$$

These "output costs" V_i^y and "input costs" V_i^u provided by (2.9) show the *actual* contributions of y_i and u_i as opposed to the *desired* contributions σ_i and μ_i.

These results now allow a precise statement of the Q, R selection problem. Note that the weights Q and R appear in the cost (1.1) which is to be minimized to obtain the linear controller (2.2), but the weights do not appear in the cost (2.6)-(2.8) used for *evaluation* of the controller. Thus, using (2.9)-(2.12) the LQG_{cv} problem reduces to

The LQG_{cv} Problem:

Find Q, R *such that*

$$\|c_i\|_X^2 \le \sigma_i^2 \tag{2.13a}$$

$$R_{ii}^{-2}\ \|b_i\|_K^2 \le \mu_i^2 \tag{2.13b}$$

where X *and* K *are given by* (2.10)-(2.12).

3.0 AN ITERATIVE ALGORITHM FOR THE LQC_{cv} PROBLEM

The parameter optimization problem posed by (2.13) requires the simultaneous solution of $(n^2+n+2k+2m)$ algebraic equations. Gradient schemes may be developed for such purposes. However, we wish to avoid gradient schemes and the attendant numerical problems. Instead, we introduce an iterative scheme for the selection of Q_{ii}, R_{ii} such that the weighted output, input costs are constant. That is, if $V_i^y(k)$ denotes the value of V_i^y on the k^{th} iteration of $Q_{ii}(k)$ and $R_{ii}(k)$, then

$$V_i^y(k+1)\ Q_{ii}(k+1) = V_i^y(k)\ Q(k) \qquad \text{for all } k \tag{3.1a}$$

and similarly

$$V_i^u(k+1)\ R_{ii}(k+1) = V_i^u(k)\ R_{ii}(k) \qquad \text{for all } k \tag{3.1b}$$

Now *if* these iterations converge to the maximum allowable values ($V_i^y(k+1)\to \sigma_i^2$ and $V_i^u(k+1)\to \mu_i^2$) on the $(k+1)^{st}$ iteration then (3.1) reduces to

$$Q_{ii}\ (k+1) = \frac{V_i^y(k)}{\sigma_i^2}\ Q_{ii}(k) = \phi_i(k)\ Q_{ii}(k) \tag{3.2a}$$

$$R_{ii}(k+1) = \frac{V_i^u(k)}{\mu_i^2}\ R_{ii}(k) = \psi_i(k)\ R_{ii}(k) \tag{3.2b}$$

where $\Phi(k)$ and $\psi(k)$ are non-negative numbers [6]. Note that the constraint (1.7) is satisfied if $\phi(k) \le 1$ and $\psi(k) \le 1$. The proposed algorithm for Q, R selection is now summarized.

<u>The LQG_{cv} Algorithm for Q, R Selection:</u>

STEP I: Set k=0, guess an initial value for Q(0), R(0) and solve

$$PA^T+AP - PM^TV^{-1}MP + DWD^T = 0 \ , \quad F = PM^TV^{-1} \tag{3.3}$$

STEP II: Compute $\mathcal{V}_i^y(k)$, $\mathcal{V}_i^u(k)$ and check for solution

$$\mathcal{V}_i^y(k) = \|c_i\|^2_{X(k)} \quad , \quad \mathcal{V}_i^u(k) = R_{ii}^{-2}(k) \ \|b_i\|^2_{\mathcal{K}} \tag{3.4}$$

where X(k), $\mathcal{K}(k)$ satisfy $X(k) \triangleq \hat{X}(k) + P$, $\mathcal{K}(k) \triangleq K(k)\hat{X}(k)K(k)$ and K(k), $\hat{X}(k)$ satisfy

$$0 = K(k)A+A^TK(k) - K(k)BR^{-1}(k)B^TK(k) + C^TQ(k)C \tag{3.5}$$

$$0 = \hat{X}(k)[A+BG(k)]^T + [A+BG(k)]\,\hat{X}(k) + FVF^T \tag{3.6}$$

$$G(k) = R^{-1}(k)B^TK(k) \quad , \tag{3.7}$$

STEP III: If $\Phi_i(k) \triangleq \dfrac{\mathcal{V}_i^y(k)}{\sigma_i^2} < 1$ do not change Q_{ii}

If $\psi_i(k) \triangleq \dfrac{\mathcal{V}_i^u(k)}{\mu_i^2} > 1$ do not change R_{ii}

If $\Phi_i(k) > 1$ change Q_{ii} according to

$$Q_{ii}(k+1) = \Phi_i(k) \ Q_{ii}(k) \tag{3.8}$$

If $\psi_i(k) < 1$ change R_{ii} according to

$$R_{ii}(k+1) = \psi_i(k) \ R_{ii}(k) \tag{3.9}$$

Set k=k+1 in (3.4)-(3.7) and return to STEP II until $\Phi_i(k) \le 1$ for all i=1, ... k and $\psi_i(k) > 1$ for some i=1, ... m.

STEP IV: Change Q_{ii} according to (3.8). If $\psi_i(k) < 1$ change R_{ii} according to (3.9). Go to STEP II until $|\psi_i(k+1) - \psi_i(k)| \le \varepsilon$ for $\psi_i(k+1) > 1$ (i.e. no more change in out of spec actuators).

There are many more choices for a Q and R algorithm other than the one posed above. To date, this algorithm has produced the best results. However, there is currently no proof that failure of this algorithm to converge means that a choice for Q and R does *not* exist. Research is continuing in this area. The algorithm does, however, attempt to use the "general" (Fig. 1) nature of LQG theory to its benefit. Specifically, choosing to adjust the r_{ii}'s for those actuators with $\psi_i(k) < 1$, using (3.2b), reduces r_{ii} and should serve to reduce $E_\infty\{y_i^2\}$ and $E_\infty\{u_i^2\}$ for actuators with $\psi_i(k) > 1$. Using (3.2b) to adjust the r_{ii}'s on actuators with $\psi_i(k) > 1$ does *not* have this *double* beneficial effect. In addition, once $\psi_i(k) \leq 1$ for all $i=1, \ldots k$, using (3.2a) to adjust q_{ii} will reduce q_{ii} which should serve to reduce $E_\infty\{u_i^2\}$ for all $i=1, \ldots m$.

4.0 HOOP-COLUMN ANTENNA EXAMPLE

The LQG_{cv} algorithm has been applied to a model for a hoop-column antenna which has 26 states, 39 measurements, 12 actuators, and 24 outputs. A detailed description of the model is presented in [6]. The initial guesses for Q and R were appropriately sized identity matricies. Figures 2, 3, and 4 are plots of $\phi_i(k)$ and $\psi_i(k)$ and reflect 14 iterations of the algorithm. The numbers appearing along the horizontal axis represent the output or actuator number. The 14 points associated with each horizontal number are the values of $\phi_i(k)$ or $\psi_i(\kappa)$ for each iteration from 1 to 14. As was expected, Figs. 2 and 3 show that the algorithm forced all outputs to be at or below their maximum values (i.e. 1 or below on the plot). The fact that the algorithm tried but could not drive all outputs to their maximum allowable σ_i^2 is mathematically predictable but can not be discussed at this time. It is, however directly related to the sensitivity of $\hat{X}$ to changes in q_{ii}.

Figure 4 indicates that $E_\infty\{u_i^2\}$ for all actuators converged to levels above μ_i^2, and the algorithm does *not* provide a solution. However, a solution is obtained if μ_i^2 is changed to the converged value of $E_\infty\{u_i^2\}$. As mentioned earlier, the failure of the LQG_{cv} algorithm to converge has yet to be proven as sufficient for no choice of Q and R to exist.

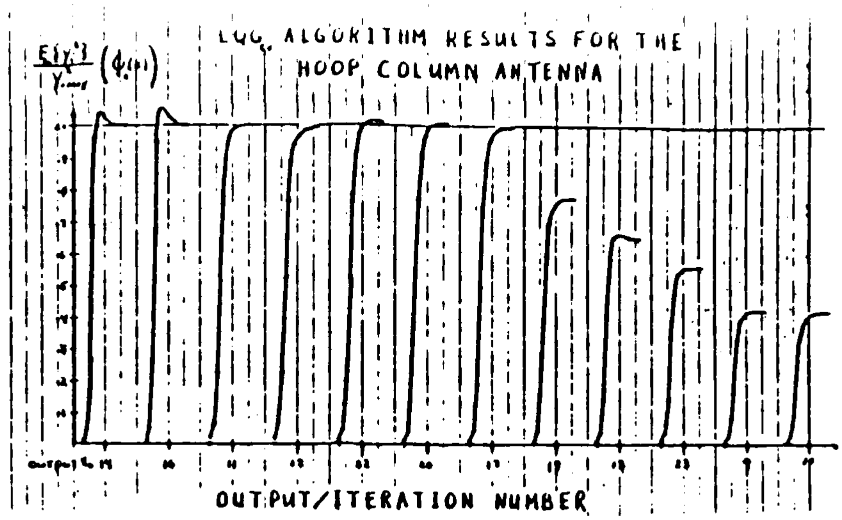

Fig. 2 Output results for the LQG_{cv} algorithm

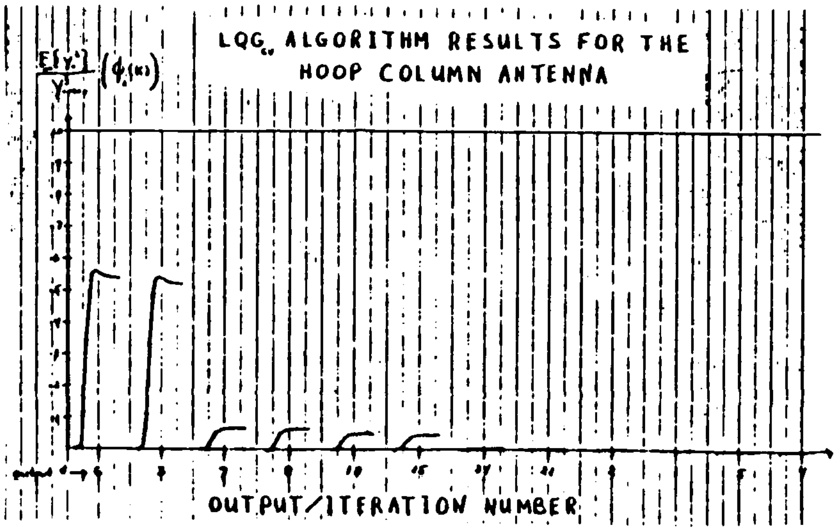

Fig. 3 Output results for the LQG_{cv} algorithm

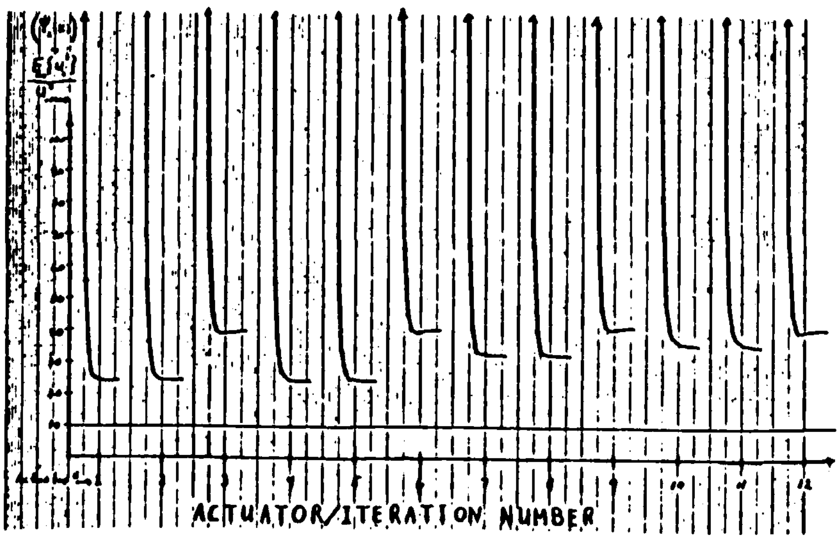

Fig. 4 Actuator results for the LQG_{cv} algorithm

5.0 CONCLUSION

This paper has presented an algorithm for selecting the Q and R weighting matrices so that the steady-state LQG regulator operates within specified RMS bounds on the input and output variables. The algorithm is iterative and requires the calculation of the steady state control Ricatti equation and a steady state Lyapunov equation of dimension n (i.e. # of states) at each iteration. This is a considerable computational savings when compared to the requirements of a standard parameter optimization gradient technique. Research is continuing to determine if failure of the algorithm to converge to an appropriate Q and R is necessary and sufficient condition for no choice of Q and R to exist. As pointed out by the hoop-column example of Section 4, even when the algorithm does not determine the required Q and R it provides the RMS regulator specifications for which Q and R weights have been found.

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GENERALIZATIONS OF COST-EQUIVALENT REALIZATIONS

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ABSTRACT

This paper bridges a gap between two known results in model reduction theory. Cost Equivalent Realizations [1] are known to match the first two output covariances of the full-order system, while we will define a Stochastically Equivalent Realization as one that matches all the output covariances [2]. The purpose of this paper is twofold. First, we describe methods for obtaining the minimal realization that matches the first two output covariances; this realization is shown to be a Cost Equivalent Realization. Second, we describe a method for obtaining a realization (not necessarily minimal) which matches any specified number of output covariances; in the limit, we can therefore obtain a Stochastically Equivalent Realization. In addition, certain generalizations of Cost Equivalent Realizations are described which apply to infinite dimensional systems.

I. INTRODUCTION

Assume $y(n)$ is a stable stationary Gaussian k-vector random process. The time marker n is an integer in $(-\infty, \infty)$. We may consider $y(n)$ as a vector of independent outputs (rank C = dim $y(n)$) generated by a state space realization of the form

$$x(n+1) = Ax(n) + Bu(n) \qquad x \in C^n,\quad u \in R^m \tag{1}$$
$$y(n) = Cx(n) \qquad y \in R^k$$

where A, B, and C are matrices of the appropriate dimension. Assume $u(n)$ is a stationary vector-valued Gaussian white noise process such that

$$E[u(n)\,u^*(j)] = [I_m]\,\delta_{nj} \tag{2}$$

where I_m is the $m \times m$ identity matrix, δ is the Kronecker Delta, and E is the expectation.

Define R_n as the Toeplitz covariance matrix

$$\mathcal{R}_n \triangleq \begin{bmatrix} R_0 & R_1^* & \cdots & R_n^* \\ R_1 & R_0 & \cdots & R_{n-1}^* \\ \vdots & & & \vdots \\ R_n & \cdots & \cdots & R_0 \end{bmatrix} \geq 0 \tag{3}$$

where

$$R_n = E[y(n+j)\,y^*(j)] \tag{4}$$

It is always assumed that $R_0 > 0$.

Recall (for $n \geq 0$)

$$R_n = CA^n XC^* \quad \text{where } X \triangleq E[x(n)\,x^*(n)] \tag{5}$$

It is well known that X solves the Lyapunov equation:

$$X = AXA^* + BB^* \tag{6}$$

Finally, system (1), with state covariance X will be denoted by (A, B, C, X).

We now define a "q-COVER" (q - COVariance Equivalent Realization).

Definition: A state space realization of the form

$$x_r(n+1) = A_r x_r(n) + B_r u(n) \qquad x_r \in C^p$$
$$y_r(n) = C_r x_r(n) \qquad u \in R^m \tag{7}$$
$$y_r \in R^k$$

where $u(n)$ *is a white noise process with identity covariance, is a* q-COVER *of* (1) *if*

$$E[y_r(n+j)\,y_r^*(j)] = R_n \quad \text{for } n \leq q \tag{8}$$

A minimal q-COVER is a q-COVER whose state space dimension p is minimal over the class of all q-COVERs satisfying (8). A minimal 1-COVER is relatively easy to obtain and has been done via Component Cost Analysis (CCA) [1]. A minimal ∞-COVER is the minimal Stochastically Equivalent Realization (SER). In general, however, the task of finding a minimal q-COVER for $1 < q < \infty$ is still an unsolved partial realization problem [3]. The purpose of this paper is to describe methods for obtaining the minimal 1-COVER, and to show that it is the Cost

Equivalent Realization (CER) in [1]. For completeness we also describe a method for finding a general q-COVER (not necessarily minimal) of (1) for $1 \le q \le \infty$. This is essentially a generalization of the CER.

We feel these methods are valuable for a number of reasons. First, they produce a CER much faster than the method in [1]. Second, the approach is much more general than previous methods and can be applied to infinite-dimensional systems. Third, a q-COVER ($q > 1$) will provide a more faithful model than a CER. Finally, this problem is really a special case of the more general Partial Realization Problem.

II. MINIMAL 1-COVER: DIRECT METHOD

In this section we obtain the minimal 1-COVER for any system (A, B, C, X). Assume R_0 and R_1 are the first two output covariances of (A, B, C, X) which we wish to match with a reduced system (A_r, B_r, C_r, X_r).

To begin, define C_r as the positive definite square root of R_0. Using the fact that $R_1^* \ge 0$, we have

$$\begin{bmatrix} C_r^{-1} & 0 \\ 0 & C_r^{-1} \end{bmatrix} \begin{bmatrix} R_0 & R_1 \\ R_1^* & R_0 \end{bmatrix} \begin{bmatrix} C_r^{*-1} & 0 \\ 0 & C_r^{*-1} \end{bmatrix} = \begin{bmatrix} I & A_r \\ A_r^* & I \end{bmatrix} \ge 0 \quad (9)$$

where $A_r \triangleq C_r^{-1} R_1 C_r^{*-1}$. It can easily be shown that Equation (9) implies that $\|A_r\| \le 1$ or, equivalently, that $(I - A_r A_r^*) \ge 0$. We can therefore define B_r as the positive square root of $(I - A_r A_r^*)$.

We claim that (A_r, B_r, C_r, X_r) is a minimal 1-COVER of (A, B, C, X). To show this, note that $X_r = I$, and therefore

$$R_{0_r} = C_r C_r^* = R_0 \quad (10)$$

$$R_{1_r} = C_r A_r C_r^* = C_r (C_r^{-1} R_1 C_r^{*-1}) C_r^* = R_1$$

The system (A_r, B_r, C_r, I) is therefore a 1-COVER. It is minimal because the state and output vectors are of the same dimension. We can now restate the above as

Theorem 1: The system (A_r, B_r, C_r, I) with parameters defined by

$$C_r = +(R_0)^{1/2} \quad A_r = C_r^{-1} R_1 C_r^{*-1} \quad B_r = +(I - A_r A_r^*)^{1/2} \quad (11)$$

is a minimal 1-COVER of (A, B, C, X).

Now consider a different minimal 1-COVER of (A, B, C, X) denoted by ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{X}$). Since both (A_r, B_r, C_r, I) and ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{X}$) are minimal 1-COVERs of (A, B, C, X) they must be of the same order. We say that any two systems of the same order are "equivalent" if there exists a nonsingular transformation P such that

$$C_r = \bar{C}P \qquad A_r = P^{-1}\bar{A}P \quad (12)$$

and

$$E[\bar{y}(n+j)\bar{y}^*(j)] = E[y_r(n+j)y_r^*(j)] \text{ for all } n,j \quad (13)$$

If P is a unitary matrix, we say that (A_r, B_r, C_r, I) and ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{X}$) are "unitarily equivalent".

Theorem 2: All minimal 1-COVERs are equivalent.

Proof: (The proof is similar to well known arguments in [4] concerning equivalent systems. For completeness it is provided here.)

Assume that (A_r, B_r, C_r, I) and ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{X}$) are both minimal 1-COVERs of (A, B, C, X). We will show that a nonsingular P exists to satisfy (12) and (13).

Now, since (A_r, B_r, C_r, I) and ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{X}$) both have the same R_0 and R_1 by hypothesis, using (8) and the definition $\bar{X} = E[x(n)\,x^*(n)]$ yields

$$R_0 = C_r C_r^* = \bar{C}\bar{X}\bar{C}^* \qquad R_1 = C_r A_r C_r^* = \bar{C}\bar{A}\bar{X}\bar{C}^* \quad (14)$$

Let $\bar{P}$ be the positive square root of $\bar{X} > 0$. Then the first equation in (14) yields

$$\| C_r^* f \|^2 = \| \bar{P}^*\bar{C}^* f \|^2 \qquad f \in \mathbb{C}^k \quad (15)$$

This implies that there exists a unitary operator U on $\mathbb{C}^k$ such that $C_r^* = U^*\bar{P}^*\bar{C}^*$. Hence $C_r = \bar{C}\bar{P}U$. Letting $P = \bar{P}U$ gives $C_r = \bar{C}P$, as required in the first equation in (12). Since, by hypothesis, $R_{1_r} = R_1 = \bar{R}_1$,

$$C_r A_r C_r^* = \bar{C}\bar{A}\bar{X}\bar{C}^* = \bar{C}\bar{A}\bar{P}\bar{P}^*\bar{C}^* = (C_r P^{-1})\bar{A}\bar{P}UU^*\bar{P}^*\bar{C}^*$$

$$= C_r P^{-1}\bar{A}PP^*\bar{C}^* = C_r P^{-1}\bar{A}PC_r^* \quad (16)$$

Since C_r is invertible, this leads directly to $A_r = P^{-1}\bar{A}P$ as required in (12). Proof of (13) follows from

$$E[y_r(n)y_r^*(0)] = C_r A_r^n C_r^* = \bar{C}PA_r^n P^*\bar{C}^*$$

$$= \bar{C}PA_r^n P^{-1}PP^*\bar{C}^*$$

$$= \bar{C}\bar{A}^n PP^*\bar{C}^* = \bar{C}\bar{A}^n\bar{P}UU^*\bar{P}^*\bar{C}^* = \bar{C}\bar{A}^n\bar{P}\bar{P}^*\bar{C}^*$$

$$= \bar{C}\bar{A}^n\bar{X}\bar{C}^* = E[\bar{y}(n)\bar{y}^*(0)] \quad (17)$$

where $n > 0$. A similar argument holds for $n < 0$. This completes the proof.

Remark 1. By Theorem 1, if $\bar{X} = I$ then the operator P in (12) is unitary. This follows from the proof where

$$\bar{P}\bar{P}^* = \bar{X} = I \quad (18)$$

Hence, (A_r, B_r, C_r, I) and ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{X}$) are unitarily equivalent.

Remark 2. Assume, as before, that $C_R > 0$. Following [5] a random process y(n) is Markov if y(n) admits a representation of the form (5) such that the dimension of x(n) equals the dimension of y(n).

Theorem 2 shows that $y(n) = y_r(n)$ when $y(n)$ is Markov. Notice the operator P in (12) does not intertwine the operators B_r and $\bar{B}$ in (A_r, B_r, C_r, I) and $(\bar{A}, \bar{B}, \bar{C}, \bar{X})$. This is not unusual for stochastic systems [5].

Remark 3. Theorem 2 cannot be generalized for q-COVERs where $q > 1$.

III. MINIMAL 1-COVER: PROJECTION METHOD

Given an initial state space realization (A, B, C, X), we wish to obtain a minimal 1-COVER directly from it. The approach in this section is equivalent to the method in [1] of obtaining a Cost Equivalent Realization of (A, B, C, X).

Consider a system (A, B, C, X). Without loss of generality, assume $X = I_n$. (This can always be done by defining a new state $\bar{x} = Px$ through the appropriate transformation P.) Now define H as the range space of C^* by

$$H = N^{\perp}(C) \tag{19}$$

where $N^{\perp}$ denotes the nullspace perpendicular. Define the following operators, A on H and C_1: $H \to R^k$ by

$$A_1 \triangleq P_H\, A|H \qquad C_1 \triangleq C|H \tag{20}$$

where P_H denotes the orthogonal projection onto the subspace H, and the symbol "|" means restricted to.

Now define B_1 as the positive square root of $(I - A_1A_1^*)$. Note that B_1 is well defined since the Lyapunov equation (6) with $X_1 = I$ shows that $(I-A_1A_1^*) \geq 0$. Obviously, with $y_1 \triangleq C_1X_1$, we have

$$R_0 = CC^* = C_1C_1^* = E[y_1(n)y_1^*(n)] \tag{21}$$

$$R_1 = CAC^* = C_1A_1C_1^* = E[y_1(n+1)y_1^*(n)] \tag{22}$$

Therefore (A_1, B_1, C_1, I) is a minimal 1-COVER of (A, B, C, I). By Theorem 2, it is unitarily equivalent to (A_r, B_r, C_r, I) found in Section II.

We will now demonstrate this approach with the following generic example:

Example:
Since every state space realization can be put into a Generalized Hessenberg form [6], then transformed to coordinates where X=I, we will assume, with no loss of generality, a system of the form (A, B, C, I) where

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ 0 & A_{32} & A_{33} & \cdots & \\ 0 & 0 & A_{43} & \cdots & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & A_{nn} \end{bmatrix} \quad \begin{matrix} x \in \mathcal{E}^n \\ u \in R^m \\ y \in R^k \end{matrix} \tag{23}$$

$$C = [C_k \quad 0 \ . \ . \ . \ . \ . \ 0]$$

and B is the positive square root of $I-AA^*$ to satisfy (6), with $X = I$.

Now, using the projection approach in this example yields

$$H = N^{\perp}C = \{[R^k \quad 0 \ . \ . \ . \ . \ . \ . \ 0]\}^* \tag{24}$$

$$A_1 = P_H(A|H) = A_{11} \quad C_1 = C|H = C_k \quad B_1 = +(I-A_1A_1^*)^{1/2} \tag{25}$$

Obviously

$$E[y_1(n)y_1^*(n)] = R_0 \quad E[y_1(n+1)y_1^*(n)] = R_1 \tag{26}$$

The system (A_1, B_1, C_1, I) of dimension k is therefore a minimal 1-COVER of (A, B, C, I).

NOTE: The method used in Section II only required R_0 and R_1; no initial state space realization was required to obtain a minimal 1-COVER. However, the projection method above (or equivalently in [1]), requires an initial state space realization (A, B, C, X) to obtain the minimal 1-COVER. The results are identical.

Now, compare the methods in Sections II and III assuming both start with an initial state space representation (A, B, C, X). The method in Section II for obtaining a minimal 1-COVER is much faster than the projection method or, equivalently, that in [1]. Both the projection method and [1] require the original n^{th} order system (A, B, C, X) to be transformed to (A', B', C', I). This requires finding an n-dimensional transformation $X^{1/2}$ and its <u>n-dimensional inverse</u>. The method in Section II allows <u>one to go immediately</u> to the smaller k (output)-dimensional space, in which one must then only compute C_r and its <u>k-dimensional inverse</u>. Since always $k < n$, and in many cases $k << n$, it is computationally much faster to obtain a 1-COVER by the method of Section II than by the projection method and [1].

IV. q-COVER BY PROJECTION

Given a system (A, B, C, I) we wish to construct a realization (A_q, B_q, C_q, I) of lower order that matches up through the qth covariance of the original system. This can be done in two different ways. First, by using the results in [7] and elsewhere with $R_0 \ldots R_q$, one can obtain the well known maximal entropy realization of R_q, which is a q-COVER. In fact, the 1-COVER is precisely the maximal entropy realization of R_0, R_1 (i.e., the rest of the Schur parameters are zero). The other approach, which we wish to pursue here, is a generalization of the projection approach used in the previous section. We expect the more general q-COVER obtained by projection to provide a better reduced order realization of (A, B, C, I) than either the 1-COVER or the maximum entropy realization.

Consider a system (A, B, C, I). Define the subspace H_{qc} by

$$H_{qc} = \bigvee_{i=0}^{q-1} A^i H \tag{27}$$

where H, the range space of C^*, is defined in (19). The subscript "qc" denotes that we are obtaining a realization that matches up through the qth covariance by defining H_{qc} as part of the "q-controllability" subspace of (A, C^*).

Now define the following operators, A_{qc} on H_{qc} and C_{qc}: $H_{qc} \to R^k$ by

$$A_{qc} \triangleq P_{H_{qc}}(A|H_{qc}) \qquad C_{qc} \triangleq C|H_{qc} \tag{28}$$

Noting that $\| A_{qc} \| \leq \|A\| \leq 1$, we can therefore define

$$B_{qc} \triangleq +(I - A_{qc}A_{qc}^*)^{1/2} \tag{29}$$

We claim that $(A_{qc}, B_{qc}, C_{qc}, I)$ is a realization (not necessarily minimal) which matches up through the qth covariance of (A, B, C, I). To see this, consider the case q = 3. Using (6) with X=I yields

$$R_3 = CA^3C^* \tag{30}$$

Now, using the fact that $H_{1c} \subset H_{2c} \subset H_{3c}$, and, from (27) that

$$A^iH \subset H_{(i+k)c} \qquad k = 1,2,\ldots\infty \tag{31}$$

we can see that

$$R_3 = CP_{H_{3c}}(A^3C^*) = CP_{H_{3c}}AP_{H_{3c}}(A^2C^*)$$
$$= CP_{H_{3c}}AP_{H_{3c}}AP_{H_{3c}}AC^* = C_{3c}A_{3c}^3C^*_{3c}=R_{3c} \tag{32}$$

Similarly, we can show that $R_2 = R_{2c}$, $R_1=R_{1c}$, $R_0=R_{0c}$. We therefore see that $(A_{3c}, B_{3c}, C_{3c}, I)$ is a 3-COVER for (A, B, C, I). For arbitrary q, the proof is analogous.

We could have obtained a q-COVER by defining

$$H_{qo} = \bigvee_{i=0}^{q-1} A^{*i}H \tag{33}$$

where H_{qo} is part of the "q-observability" subspace of (A,C). We then would have used H_{qo} in place of H_{qc} in (28). The resulting q-COVER $(A_{qo}, B_{qo}, C_{qo}, I)$ could be of either larger or smaller dimension than $(A_{qc}, B_{qc}, C_{qc}, I)$.

To assure that one obtains the smallest dimension q-COVER possible by this method (though not necessarily minimal), one should take the q-COVER $(A_{qc}, B_{qc}, C_{qc}, I)$ obtained initially and use this realization to project onto its "q-observability" subspace. In other words, after obtaining $(A_{qc}, B_{qc}, C_{qc}, I)$ we form

$$H_{qco} = \bigvee_{i=0}^{q-1} A_{qc}^{*i} H \tag{34}$$

We can then obtain

$$A_{qco}= P_{H_{qco}}A_{qc}|H_{qco} \qquad C_{qco}=C_{qc}|H_{qco} \tag{35}$$
$$B_{qco} = +(I - A_{qco}A_{qco}^*)^{1/2}$$

Therefore $(A_{qco}, B_{qco}, C_{qco}, I)$ is a q-COVER for (A, B, C, I) of dimension less than or equal to the dimension of both $(A_{qc}, B_{qc}, C_{qc}, I)$ and $(A_{qo}, B_{qo}, C_{qo}, I)$.

The example in Section III can be easily extended to demonstrate this method. Due to lack of space, however, we are not able to include that demonstration in this paper.

NOTE 1: If, at any time, we find

$$H_{(q+1)co} = H_{qco} \tag{37}$$

then, obviously $H_{qco} = H_{\infty co}$. In particular, the Cayley-Hamilton Theorem implies that once $q \geq n$,(37) will automatically hold. When (37) does apply, then the q-COVER is an ∞-COVER, and is therefore an SER. For the smallest q satisfying (37), the applicable $(A_{qco}, B_{qco}, C_{qco}, I)$ is a Minimal SER of R_n.

NOTE 2: The maximal entropy approach will always yield a q-COVER of order q, even when $q > n$, while our approach often gives a lower order q-COVER.

V. CONCLUSION

A q-COVER (q COVariance Equivalent Realization) has been defined as a realization that matches up through the R_q covariance of a system or process. Two methods have been developed for obtaining a minimal 1-COVER. One only requires R_0 and R_1, while the other requires an initial state space representation and is essentially equivalent to the Cost Equivalent Realization method in [1]. Finally, a projection method was described which can be used to obtain a q-COVER when q = 2,3,...∞. When q = ∞, the ∞-COVER is a Stochastically Equivalent Realization.

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APPENDIX D

CONTROLLER REDUCTION BY COMPONENT COST ANALYSIS

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ABSTRACT

Component Cost Analysis [1] is used to develop a method for controller reduction. The reduction of the controller is based upon the participation of the controller states in the value of a quadratic performance metric. The controller states which have the smallest contribution to the performance metric are truncated to produce the reduced controllers. An error index is defined to evaluate the reduced controller compared with optimal LQG controller, and bounds on this index are derived. A numerical example is included to illustrate the procedure.

I. INTRODUCTION

The straightforward application of Linear-Quadratic-Gaussian (LQG) theory in practical applications is hindered by these limitations:

(i) The plant model which is accurate enough to serve in the evaluation of candidate controllers is often too complex for direct LQG computations (beyond "Riccati-solvable" dimension).

(ii) A traditional approach to synthesize a controller of a specified order is to first apply (one's favorite) model reduction theory to obtain a low order model (compatible with on-line controller hardware/software limitations), and then to apply optimal control and estimation theory to obtain a LQG controller which is optimal for the reduced model. However, this optimization is based upon a reduced order model which was guaranteed "close" to the plant only in the *open*-loop. Since the control inputs can drastically affect the behavior of the system (and the quality of the reduced model), reliable model reduction cannot be performed without some knowledge of the inputs. But since the control inputs are yet to be determined, this is tantamount to admitting that *the modeling problem and the control problem are not independent problems*, as was pointed out in [1b].

Component Cost Analysis (CCA) [1-3] was introduced as an attempt to unify the model reduction and control design problems. In the open-loop model reduction versions of the CCA theory, the state dependent term in the quadratic performance measure (which is intended for use in later control design) is used in the model reduction decisions. In this way, the modeling and control problems were "integrated" in [1-3]. In these references, the following LQG control design strategy is suggested to obtain a controller of order n_c, beginning with a model of order $n>>n_c$, where n is too large for solutions of Riccati equations.

Suboptimal LQG Design Strategy

(a) Apply open-loop CCA model reduction methods [1-3] to the high-order evaluation model of order n. Reduce the order from n to n_r, where n_r is the "Riccati-solvable" dimension of the local offline computer.

(b) Solve for the optimal LQG controller for this model of order n_r. This yields a controller of order n_r, where $n_c < n_r < n$.

(c) Apply *controller* reduction CCA methods to reduce the controller from order n_r to order $n_c<n_r$.

It is emphasized that model reduction [step (a)] and controller-reduction [step (c)] are *different* mathematical procedures.

The intended advantage of this strategy over the traditional approach (ii) [which sets $n_r = n_c$ and skips step (c)] is that more information about the higher order system and its would-be optimal controller is made available for the design of the reduced controller.

The purpose of this paper is to present a reliable *controller* reduction scheme to accomplish step (c) for infinite time, stationary LQG problems. This paper assumes that a reliable reduced model of order n_r is available. We will denote this model, by $S(n_r)$. Let this *reduced* model of order n_r be

$$S(n_r): \begin{cases} \dot{x} = Ax+Bu+Dw \\ y = Cx \\ z = Mx+v \end{cases} \tag{1.1}$$

where $x \epsilon R^{n_r}$, $y \epsilon R^k$, $u \epsilon R^m$ and $z \epsilon R^\ell$. The disturbances $w \epsilon R^d$ and $v \epsilon R^\ell$ are assumed to be uncorrelated zero-mean white noise processes with intensities $W > 0$ and $V > 0$ respectively, and under the standard assumptions the matrix pairs $\{C,A\}$ and $\{M,A\}$ are observable and the matrix pairs $\{A,B\}$ and $\{A,D\}$ are controllable. The vector z is composed of the measurements corrupted by the noise v. The vector v contains only the variables which are used to measure the performance of the system via a cost function V defined as follows

$$V \triangleq \lim_{t \to \infty} E\{V(t)\} \tag{1.2a}$$

where

$$V(t) \triangleq \|y(t)\|_Q^2 + \|u(t)\|_R^2 \tag{1.2b}$$

The notation $\|y\|_Q^2$ denotes $y^T Q y$, and $Q > 0$ and $R > 0$ are weighting matrices. E denotes the expectation operator.

Step (b) of the Suboptimal LQG Design Strategy is to obtain the optimal controller [which minimizes the cost function V, in (1.2)] for (1.1). This controller, denoted by $S_c(n_r)$, is given by [4].

$$S_c(n_r): \begin{cases} \dot{\hat{x}} = A_c \hat{x} + Fz \\ u = G\hat{x} \end{cases}, \quad \hat{x} \epsilon R^{n_r} \tag{1.3a}$$

where

$$A_c \stackrel{\Delta}{=} A+BG-FM \;, \quad G \stackrel{\Delta}{=} -R^{-1}B^TK; \quad F \stackrel{\Delta}{=} PM^TV^{-1} \qquad (1.3b)$$

$$KA+A^TK - KBR^{-1}B^TK + C^TQC = 0 \qquad (1.3c)$$

$$PA^T + AP - PM^TV^{-1}MP + DWD^T = 0 \qquad (1.3d)$$

Since the controller (1.3) is of order $n_r > n_c$, a reduction of this *controller* (1.3) to order n_c is now required. [Step (c) of the strategy].

Past Approaches

Since the original version of this paper was submitted, Verriest [5,6] has proposed to select a set of coordinates, named "LQG-balanced" coordinates, in which the Riccati solutions K and P of (1.3c) and (1.3d) are 'balanced' in the sense, $K = P = \Pi = \text{diag}\,\{\pi_1, \pi_2, \ldots, \pi_{n_r}\}$, and to delete those n_r-n_c controller states that have the smallest π_i-s, yielding a reduced controller of order n_c. These deleted states are interpreted as those states that are estimated with the least uncertainty (measured by π_i of P) and have the least contribution to a 'fictitious' cost function (measured by π_i of K). This 'fictitious' cost function is evaluated by *assuming that all the plant states are available for feedback*. But, when the plant states are not available, one could feedback only the estimates [which is precisely why an estimator (1.3a) is constructed], and the contribution of the states to the *actual* cost function V may be quite different from π_i, as can be seen from Eq. 11 of [6].

The work of Kosut [7] and Wenk and Knapp [8] should also be mentioned since they also deal with *closed*-loop methods for controller simplification. However, since they treat parameter optimization approaches, their work is along totally different lines than the work herein.

We consider the contribution of these controller states to the cost function (1.2), and delete those controller states having the smallest contribution to the cost function (1.2). Hence the proposed controller reduction algorithm follows the following three steps.

Controller-reduction-algorithm

1. Select a suitable basis for the controller. The contribution of each of the controller states to the cost function (1.2) is precisely measured by a metric called the 'component cost', $V_i(\hat{x})$; i=1, 2, ..., n_r. Compute the component costs.
2. Rank the controller states so that

$$V_1(\hat{x}) \geq V_2(\hat{x}) \geq \cdots \geq V_{n_r}(\hat{x}) .$$

3. Delete the last n_r-n_c states to obtain the reduced controller.

Of course, this algorithm is similar to the "CCA Model Reduction Algorithm" given in [2], where *plant*-states x_i are considered for truncation, unlike the above controller-reduction algorithm where the *controller* states $\hat{x}_i$ are the only candidates for truncation. Both these algorithms use the basic concepts of Component Cost Analysis [1].

The organization of the paper is as follows. Section II presents some preliminary results pertaining to controller-reduction schemes to aid in the evaluation of reduced controllers, and develops a 'controller-error-index' for this purpose. The main contribution of this paper is in Section III where the concepts of CCA are used to systematically develop a set of coordinates to be used for controller reduction studies. Reduced controllers obtained in this representation are evaluated with respect to the performance of the system (1.1). A Solar Optical Telescope is considered as a numerical example in Section IV to illustrate the proposed method. A comparison of these controllers with those obtained by the LQG-balanced method [5,6] is also made. The numerical scheme for the computation of the above mentioned coordinates is presented in Appendix A.

II. PRELIMINARIES TO CONTROLLER REDUCTION

Let the model $S(n_r)$ in (1.1) be partitioned as follows:

$$\begin{bmatrix} \dot{x}_R \\ \dot{x}_T \end{bmatrix} = \begin{bmatrix} A_R & A_{RT} \\ A_{TR} & A_T \end{bmatrix} \begin{bmatrix} x_R \\ x_T \end{bmatrix} + \begin{bmatrix} B_R \\ B_T \end{bmatrix} u + \begin{bmatrix} D_R \\ D_T \end{bmatrix} w$$

$$y = [C_R \quad C_T] \begin{bmatrix} x_R \\ x_T \end{bmatrix} \tag{2.1}$$

$$z = [M_R \quad M_T] \begin{bmatrix} x_R \\ x_T \end{bmatrix} + v$$

where $x_R \varepsilon R^{n_c}$ and $x_T \varepsilon R^t$, $t+n_c=n_r$, with all the matrices appropriately dimensioned. The corresponding partitioned form of the controller $S_c(n_r)$ in (1.3a) is

$$\begin{bmatrix} \dot{\hat{x}}_R \\ \dot{\hat{x}}_T \end{bmatrix} = \begin{bmatrix} A_{C_R} & A_{C_{RT}} \\ A_{C_{TR}} & A_{C_T} \end{bmatrix} \begin{bmatrix} \hat{x}_R \\ \hat{x}_T \end{bmatrix} + \begin{bmatrix} F_R \\ F_T \end{bmatrix} z$$

$$u = [G_R \quad G_T] \begin{bmatrix} \hat{x}_R \\ \hat{x}_T \end{bmatrix} \tag{2.2}$$

where $x_R \epsilon \mathcal{R}^{n_c}$ (note that, $A_{C_R} = A_R + B_R G_R - F_R M_R$). Now augmenting (2.1) and (2.2) together, the closed loop system can be written as

$$\begin{bmatrix} \dot{x}_R \\ \dot{\hat{x}}_T \end{bmatrix} = \begin{bmatrix} \mathcal{A}_R & \mathcal{A}_{RT} \\ \mathcal{A}_{TR} & \mathcal{A}_T \end{bmatrix} \begin{bmatrix} x_R \\ \hat{x}_T \end{bmatrix} + \begin{bmatrix} \mathcal{D}_R \\ \mathcal{D}_T \end{bmatrix} \xi$$

$$\mathcal{y} = [\mathcal{C}_R \quad \mathcal{C}_T] \begin{bmatrix} x_R \\ \hat{x}_T \end{bmatrix} \tag{2.3}$$

where

$$x_R^T \triangleq [x_R^T \quad x_T^T \quad \hat{x}_R^T] \; , \quad x_R \epsilon \mathcal{R}^r \; ; \quad r \triangleq n_r + n_c \tag{2.4a}$$

$$\xi^T \triangleq [w^T, \; v^T] \; , \quad \mathcal{y}^T \triangleq [y^T, \; u^T]$$

$$\begin{bmatrix} \mathcal{A}_R & \mathcal{A}_{RT} \\ \mathcal{A}_{TR} & \mathcal{A}_T \end{bmatrix} \triangleq \left[\begin{array}{ccc|c} A_R & A_{RT} & B_R G_R & B_R G_T \\ A_{TR} & A_T & B_T G_R & B_T G_T \\ F_R M_R & F_R M_T & A_{C_R} & A_{C_{RT}} \\ \hline F_T M_R & F_T M_T & A_{C_{TR}} & A_{C_T} \end{array}\right] \tag{2.4b}$$

$$\begin{bmatrix} \mathcal{D}_R \\ \mathcal{D}_T \end{bmatrix} \triangleq \left[\begin{array}{cc} D_R & 0 \\ D_T & 0 \\ 0 & F_R \\ \hline 0 & F_T \end{array}\right] \tag{2.4c}$$

and

$$[C_R \quad C_T] \triangleq \left[\begin{array}{ccc|c} C_R & C_T & 0 & 0 \\ 0 & 0 & G_R & G_T \end{array}\right] \tag{2.4d}$$

Now assume that a reduced controller of order n_c denoted by $S_c(n_c)$ is obtained from (2.2) by deleting $\hat{x}_T$, to yield,

$$S_c(n_c): \quad \begin{cases} \dot{\rho} = A_{C_R}\rho + F_R z \ , & \rho \epsilon R^{n_c} \\ u_R = G_R\rho \end{cases} \tag{2.5}$$

The rest of this section devotes attention to the evaluation of this controller by considering the value of the cost function (1.2) when $S(n_r)$ of (2.1) is driven by (2.5). For the convenience of evaluation, $S(n_r)$ and $S_c(n_c)$ are augmented to obtain

$$\begin{aligned} \dot{\hat{x}}_R &= A_R\hat{x}_R + \mathcal{D}_R\,\xi \ , \quad \hat{x}_R \epsilon R^r \\ y_R &= C_R\hat{x}_R \end{aligned} \tag{2.6}$$

which results from the truncation of $\hat{x}_T$ from (2.3), where

$$\hat{x}_R^T \triangleq [x_R^T x_T^T \rho^T]$$

and

$$y_R^T = [y_R^T, u_R^T] \ .$$

It is easy to verify that $\{A_R, \mathcal{D}_R, C_R\}$ satisfy the definitions in (2.4b, c, d), and that the cost function for the evaluation of (2.6) is

$$V_R = \lim_{t\to\infty} E\{V_R(t)\} \tag{2.7a}$$

where

$$V_R(t) \triangleq \|y_R(t)\|_Q^2 + \|u_R(t)\|_R^2 \quad . \tag{2.7b}$$

The cost functions V and V_R are computed by [4]

$$V = \mathrm{Tr}[C^T \mathcal{Q} C X] \tag{2.8a}$$

$$V_R = \mathrm{Tr}[C_R{}^T \mathcal{Q} C_R X_R] \tag{2.8b}$$

with X and X_R satisfying

$$XA^T + AX + \mathcal{D}\mathcal{W}\mathcal{D}^T = 0 \tag{2.9a}$$

$$X_R A_R^T + A_R X_R + \mathcal{D}_R \mathcal{W} \mathcal{D}_R{}^T = 0 \tag{2.9b}$$

where

$$A \triangleq \begin{bmatrix} A_R & A_{RT} \\ A_{TR} & A_T \end{bmatrix}, \quad \mathcal{D} \triangleq \begin{bmatrix} \mathcal{D}_R \\ \mathcal{D}_T \end{bmatrix}, \quad C \triangleq [C_R \quad C_T] \tag{2.9c}$$

$$\mathcal{W} \triangleq \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}, \quad \mathcal{Q} \triangleq \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \quad .$$

The following definitions and lemmas will prove useful in the subsequent development of the controller reduction algorithm proposed herein. (The associated proofs are given in Appendix B).

Definitions:

1. *The Predicted Controller Index* $\hat{I}(n_r, n_c)$ *is defined by*

$$\hat{I}(n_r, n_c) \triangleq \frac{1}{V} (\hat{V}_R - V) \tag{2.10a}$$

where

$$\hat{V}_R \triangleq \mathrm{Tr}[C_R{}^T Q C_R \hat{X}_R] + \mathrm{Tr}[C_R{}^T Q C_T \hat{X}_{RT}{}^T] \tag{2.10b}$$

and where $\hat{X}_R$ *and* $\hat{X}_{RT}$ *are obtained from partitioning* X *as*

$$X = \begin{bmatrix} \hat{X}_R & \hat{X}_{RT} \\ \hat{X}_{RT}^T & \hat{X}_T \end{bmatrix} .$$

2. *The* <u>*Controller Error Index*</u> $I(n_r, n_c)$ *is defined as*

$$I(n_r, n_c) \triangleq \frac{1}{V} (V_R - V) . \tag{2.11}$$

In the definitions above, the arguments n_r *and* n_c *indicate the order of the plant and the controller respectively.*

<u>*Lemma 1:*</u>

1. *The error indices* $\hat{I}(n_r, n_c)$ *and* $I(n_r, n_c)$ *satisfy the following*

(i) $$\hat{I}(n_r, n_c) = -\frac{1}{V} (\mathrm{Tr}[C_T{}^T Q C_T \hat{X}_T] + \mathrm{Tr}[C_R{}^T Q C_T \hat{X}_{RT}^T]) \tag{2.12a}$$

(ii) $$I(n_r, n_c) = \frac{1}{V}(\mathrm{Tr}[C_R{}^T Q C_R (X_R - \hat{X}_R)] - \mathrm{Tr}[C_T{}^T Q C_T \hat{X}_T] - 2\mathrm{Tr}[C_R^T Q C_T \hat{X}_{RT}{}^T]) \tag{2.12b}$$

where $(X_R - \hat{X}_R)$ *satisfies*

$$(X_R - \hat{X}_R) A_R{}^T + A_R (X_R - \hat{X}_R) - U = 0 \tag{2.12c}$$

and where

$$U \triangleq [A_{RT} \hat{X}_{RT}{}^T + \hat{X}_{RT} A_{RT}{}^T] . \tag{2.12d}$$

(iii) *max* $\{0, \mu(n_c)\lambda_m(X_R)-1\} \leq I(n_r,n_c) \leq$ *max* $\{0,\mu(n_c)\lambda_M(X_R)-1\}$ (2.13a)

where $\lambda_m(\cdot)$ $[\lambda_M(\cdot)]$ *denotes the minimum [maximum] eigenvalue of the matrix* $(\cdot)$, *and*

$$\mu(n_c) \triangleq \frac{1}{V} \mathrm{Tr}[C_R^T Q C_R] \qquad (2.13b)$$

where the argument n_c *is used to denote the dependence of* μ *on* n_c.

2. *The predicted controller error index is exact* $\hat{I}(n_r,n_c) = I(n_r,n_c)$, *under any of the following conditions:*

   (i) *necessary and sufficient condition* .

$$\mathrm{Tr}[C_R^T Q C_R(X_R-\hat{X}_R)]-\mathrm{Tr}[C_R^T Q C_T \hat{X}_{RT}^T] = 0 \qquad (2.14a)$$

   (ii) *sufficient conditions*

   a) $\hat{x}_T$ *is unobservable in* u *in* $S_c(n_r)$ (2.14b)

   b) $\hat{x}_T$ *is uncontrollable from* z *in* $S_c(n_r)$ (2.14c)

   c) $\hat{X}_{RT} = 0$ *with* A_R *stable* (2.14d)

3. *The controller error index* $I(n_r,n_c)$ *is zero under any of the following conditions*

   i) *necessary and sufficient condition*

$$\mathrm{Tr}[C_R^T Q C_R(X_R-\hat{X}_R)] - \mathrm{Tr}[C_T^T Q C_T \hat{X}_T] - 2\mathrm{Tr}[C_R^T Q C_T \hat{X}_{RT}^T] = 0 \qquad (2.15a)$$

   ii) *sufficient conditions*

   a) $\hat{x}_T$ *is unobservable in* u *in* $S_c(n_r)$ (2.15b)

   b) $\hat{x}_T$ *is undisturbable from* z *in* $S_c(n_r)$ (2.15c)

III. CONTROLLER REDUCTION BY COMPONENT COST ANALYSIS

The idea of Component Cost Analysis (CCA), [1-3] is to determine the significance of each 'component' (in this paper, these 'components' are the individual controller states $\hat{x}_i$) by assigning a metric, called 'component-cost', to each component. Then, a reduced order controller is obtained by deleting those controller states that have the smallest component costs. These ideas are extended to controller-reductions as follows.

From the structure of $\mathcal{C}$ in (2.4d) and $\mathcal{Q}$ in (2.9c), and by partitioning $\mathcal{X}$ as

$$\mathcal{X} = \begin{bmatrix} X & X_{12} \\ X_{12}^T & \hat{X} \end{bmatrix} ; \quad \begin{aligned} \hat{X} &\triangleq \lim_{t\to\infty} E\{\hat{x}(t)\hat{x}^T(t)\} \\ X &\triangleq \lim_{t\to\infty} E\{x(t)x^T(t)\} \end{aligned} \tag{3.1}$$

the expression for the cost functions $\mathcal{V}$ in (2.8a) can be rewritten as

$$\mathcal{V} = \mathrm{Tr}[C^TQCX] + \mathrm{Tr}[G^TRG\hat{X}] \ . \tag{3.2}$$

We note that the 'Control Cost' ($\mathrm{Tr}[G^TRG\hat{X}]$) may be decomposed into contributions from each controller state

$$\mathcal{V}_i(\hat{x}) \triangleq \frac{1}{2} \lim_{t\to\infty} E\left\{ \frac{\partial \mathcal{V}(t)}{\partial \hat{x}_i} \hat{x}_i \right\} \tag{3.3a}$$

such that

$$\mathcal{V}(\hat{x}) \triangleq \lim_{t\to\infty} E\|u(t)\|_R^2 = \mathrm{Tr}[G^TRG\hat{X}] = \sum_{i=1}^{n_r} \mathcal{V}_i(\hat{x}) \ . \tag{3.3b}$$

The 'Regulation Cost' $\mathcal{V}(x)$ is the remaining term in (3.2)

$$V(x) \stackrel{\Delta}{=} \lim_{t\to\infty} E\|y(t)\|_Q^2 = \mathrm{Tr}[C^T QCX] \qquad (3.3c)$$

so that

$$V = V(x) + V(\hat{x}) \ . \qquad (3.3d)$$

As a passing remark we mention that when $S_c(n_r)$ is optimal LQG-controller, we have $X_{12} = \hat{X}$ and $X = P+\hat{X}$, with P satisfying (1.3d) and $\hat{X}$ satisfying [4]

$$\hat{X}(A+BG)^T + (A+BG)\hat{X} + FVF^T = 0 \quad . \qquad (3.3\ )$$

It follows from the definition (3.3a) and the derivations in [2] that

$$V_i(\hat{x}) = [G^T RG\hat{X}]_{ii} \ , \quad i = 1, 2, \ldots, n_r. \qquad (3.4)$$

According to the CCA theory the controller states $\hat{x}_i$ associated with the smallest values of $V_i(\hat{x})$ are deleted from the controller.

Now, the optimal controller $S_c(n_r)$ may not be minimal (in the sense of Kalman), even if the plant $S(n_r)$ is minimal [9]. This means that the sufficient conditions of lemma 1 may be satisfied and hence, there may exist reduced order controllers which are still optimal, i.e., $I(n_r,n_c) = 0$ for $n_c < n_r$. However, for brevity in the presentation, it will be assumed henceforth that the controller (1.2) is completely controllable. (Refer to [10] for extension of this work to the general case). This assumption implies that the condition (2.15c) of lemma 1 will not be satisfied, but (2.15b) may still be satisfied.

In the context of *model* reduction, a set of coordinates called Cost Decoupled Coordinates was presented in [2]. These coordinates cause the largest number of component costs $V_i(x)$ to be zero. Application of the CCA theory in these coordinates yields a reduced *model* of order n_r = rank [C] with the property that $V_R = V$. Hence such reduced order models are called "cost-equivalent realizations" [2]. Analogously, we define Controller Cost Decoupled Coordinates as follows.

Definition 3. The Controller COst-DEcoupled (CODE) coordinates are defined by the following properties:

$$\text{(P1)} \qquad \hat{X} = I_{n_r} \tag{3.5a}$$

$$\text{(P2)} \quad G^T R G = \text{diag}\,\{\gamma_1^2, \cdots, \gamma_{r_1}^2, 0, \cdots, 0\} \tag{3.5b}$$

with

$$\gamma_1^2 \geq \gamma_2^2 \geq \cdots \geq \gamma_{r_1}^2 > 0 \quad , \tag{3.5c}$$

where $r_1 \overset{\Delta}{=} rank[G] \leq m$.

These CODE-coordinates are non unique as established by the following proposition.

Proposition 1: Under the assumption of distinct γ_i^2, i=1, 2, .., *in* (3.5b), *the CODE-coordinates are unique within a similarity transformation* $\hat{x} = T\,\hat{x}_N$, *of the form*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

where

$$T_1 = \text{diag}\ \{\pm 1, \pm 1, \ldots, \pm 1\}\ \varepsilon R^{r_1 x r_1}$$

and

$$T_2 \varepsilon R^{(n_r - r_1) x (n_r - r_1)}$$

is any orthonormal matrix.

Proof: Since the covariance of the transformed coordinates is given by

$$\lim_{t\to\infty} E\{\hat{x}_N(t)\hat{x}_N^T(t)\} = \lim_{t\to\infty} T^{-1}\ E\{\hat{x}(t)\hat{x}^T(t)\}T^{-T} ,$$

the matrix T must be orthonormal to satisfy (3.5a). The state weighting matrix G^TRG in the transformed coordinates is T^TG^TRGT, which should satisfy (3.5b,c), i.e.

$$T^TG^TRGT = \text{diag } \{\gamma_1^2, \gamma_2^2, \ldots, \gamma_{r_1}^2, 0, \ldots, 0\} . \tag{3.6a}$$

Now (3.5b,c) imply that

$$G = [G_1, 0]; \quad G_1 \ \varepsilon R^{mxr_1} \tag{3.6b}$$

and

$$G_1^T RG_1 = \Lambda_1 , \tag{3.6c}$$

where

$$\Lambda_1 \triangleq \text{diag } \{\gamma_1^2, \gamma_2^2, \ldots \gamma_{r_1}^2\}.$$

Hence partitioning T (which must be orthonormal) as

$$T = \begin{bmatrix} T_1 & T_{12} \\ T_{21}^T & T_2 \end{bmatrix}$$

rewrite (3.6a) as

$$\begin{bmatrix} T_1^T & T_{21}^T \\ T_{12}^T & T_2^T \end{bmatrix} \begin{bmatrix} G_1^T R G_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_{12} \\ T_{21}^T & T_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (3.6d)$$

Two of the equations resulting from (3.6c) are

$$T_1^T G_1^T R G_1 T_1 = T_1^T \Lambda_1 T_1 = \Lambda_1 \qquad (3.6e)$$

and

$$T_{12}^T G_1^T R G_1 T_{12} = T_{12}^T \Lambda_1 T_{12} = 0 \ , \qquad (3.6f)$$

which can be satisfied if and only if $T_{12} = 0$, $T_{21} = 0$, and T_{21} is as defined in Proposition 1. #

Hence, there exists a considerable flexibility in the cost decoupled coordinates: from the structure of the transformation in proposition 1 and from (3.6b), notice that the CODE-coordinates are not uniquely defined within the null space of G. This non uniqueness will later be used to obtain a generalized Hessenberg representation while remaining within the class of CODE-coordinates.

Now, in any CODE-coordinates, the controller has the following property (in *addition* to property (3.5)).

Proposition 2: Any controller $S_c(n_r)$ *in CODE-coordinates satisfies the following property in addition to* (P1) *and* (P2)

$$V_i(\hat{x}) = \begin{cases} \gamma_i^2 & i \le r_1 \\ 0 & i > r_1 \end{cases} .$$

The proof follows immediately from the substitution of (3.5a) and (3.5b) into (3.4). #

Proposition 3: The component costs associated with the cost decoupled controller coordinates are minimally sensitive to perturbations in the weighting matrix G^TRG.

Proof: The proof relies on a result derived by Skelton and Wagie [12] which is restated here.

Lemma 2 [11] *Let* λ_i *be an eigenvalue of a matrix* A. *Then the sensitivity of* λ_i *to perturbations in* A, *measured by*

$$s_i \triangleq \left\|\frac{\partial \lambda_i}{\partial A}\right\|^2 \quad ,$$

where $\|(\cdot)\|^2 \triangleq \mathrm{Tr}[(\cdot)^T(\cdot)]$, *is bounded from below by* $s_i \geq 1$ *and* s_i *takes on its minimum value* $s_i = 1$ *if and only if the matrix* A *is normal* (i.e. $AA^T = A^TA$).

Now note from proposition 1, that the component costs $V_i(\hat{x})$ are the eigenvalues of G^TRG, i.e.,

$$V_i(\hat{x}) = \lambda_i(G^TRG) \ , \ i = 1, 2, \ldots, n_r \ . \qquad (3.7)$$

Since G^TRG is symmetric (hence, it is normal), the proof follows from lemma 2 and (3.7). #

The implications of proposition 3 is the following. Consider the case when the *controller state* weighting matrix G^TRG in the cost function is subject to perturbation - this may happen, for example, when the *control* weighting matrix R is changed. In this situation, it is shown that the choice of the reduced controller is least sensitive to these perturbations.

Note from Proposition 2, that all the controller components $\hat{x}_i$, $i > r_1$ are candidates for truncation, since their corresponding component costs are zero. Hence, a reduced controller obtained by truncating some of these zero cost states seems attractive. We will now evaluate such a reduced controller, $S_c(n_c)$.

Theorem 1: The error indices $\hat{I}(n_r,n_c)$ *and* $I(n_r,n_c)$ *associated with the reduced controllers* $S_c(n_c \geq r_1)$ *obtained by truncating* $t = n_r - n_c$ *CODE-controller states from* $S_c(n_r)$ *satisfy the following*

(i) $\hat{I}(n_r, n_c) = 0 \qquad n_c \geq r_1$ (3.8)

(ii) *If* (2.6) *is stable, then*

$$I(n_r, n_c) = \frac{1}{V}\, \mathrm{tr}[C_R^T\, Q C_R (X_R - \hat{X}_R)] \;, \quad n_c \geq r_1 \tag{3.9}$$

where $(X_R - \hat{X}_R)$ *satisfies*

$$(X_R - \hat{X}_R)A_R^T + A_R(X_R - \hat{X}_R) - U(n_c) = 0 \tag{3.10a}$$

and where A_R, C_R *and* Q *are as defined in* (2.4), (2.9c) *and*

$$U(n_c) \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (A_{RT} - F_R M_T)^T \\ 0 & (A_{RT} - F_R M_T) & 0 \end{bmatrix} \tag{3.10b}$$

(iii) *In* (2.13b), $\mu(n_c) = \frac{1}{V}\, \mathrm{Tr}[C_R^T Q C_R]$ *is a constant for* $n_c \geq r_1$

(3.11)

Proof: Since $n_c \geq r_1$ in view of (3.6b) and (2.4d), (2.12a) yields (3.8). Now since $C_T = 0$, (2.12b) reduces to (3.9) with $(X_R - \hat{X}_R)$ satisfying (2.12c). Comparing (3.10a,b) with (2.12c), we need to show that $U = U(n_c)$ as defined in (3.10b). Recall from (2.4b) that

$$A_{RT} = \begin{bmatrix} B_R G_T \\ B_T G_T \\ A_{C_{RT}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ A_{RT} - F_R M_T \end{bmatrix} \tag{3.12a}$$

since $G_T = 0$ and $A_{C_{RT}} = A_{RT} + B_R G_T - F_R M_T$. Also recall from (2.4a) that

$$\hat{X}_{RT} = \lim_{t\to\infty} E\{x_R(t)\ \hat{x}_T^T(t)\}$$

$$= \lim_{t\to\infty} E\{ \begin{bmatrix} x_R(t)\ \hat{x}_T^T(t) \\ x_T(t)\ \hat{x}_T^T(t) \\ \hat{x}_R(t)\ \hat{x}_T^T(t) \end{bmatrix} \}.$$

Now, using the property that

$$X_{12} \overset{\Delta}{=} \lim_{t\to\infty} E\{ \begin{bmatrix} x_R(t) \\ x_T(t) \end{bmatrix} [\hat{x}_R^{\ T}(t)\ \hat{x}_T^{\ T}(t)]\} = \hat{X} = \lim_{t\to\infty} E\{ \begin{bmatrix} \hat{x}_R(t) \\ \hat{x}_T(t) \end{bmatrix} [\hat{x}_R^T(t)\hat{x}_T^T(t)]\}$$

and (3.5a), $\hat{X}_{RT}$ is shown to be

$$\hat{X}_{RT}^T = [0\quad I_t\quad 0]\ . \tag{3.12b}$$

Substituting (3.12) in (2.12d) to show that $U = U(n_c)$ proves (ii). Since $C_T = 0$ for $n_c \geq r_1$ the $\mu(n_c)$ defined in (2.13b) is the same constant for all $n_c \geq r_1$ and hence (iii) holds. #

Now, since $C_T = 0$, $\hat{x}_T$ lies in the nullspace of the output matrix of (2.3), its observability is measured by A_{RT}, i.e., the "smaller" (in a sense to be made precise later) is A_{RT}, the less observable is $\hat{x}_T$. In fact, if $A_{RT} \equiv 0$, then $\hat{x}_T$ is unobservable. Furthermore, $A_{RT} \equiv 0$ implies $A_{C_{RT}} = A_{RT} - F_R M_T = 0$ and $\hat{x}_T$ is unobservable in $S_c(n_r)$. Hence, by lemma 1,

$I(n_r,n_c) = 0$. Therefore, in CODE-coordinates notice that $I(n_r,n_c)$, is influenced by the observability of the 'truncated' controller states. However, in general, A_{RT} will not be zero. But there may exist some $\hat{x}_i \in \hat{x}_T$ which are either unobservable [in which case a truncation of them would yield $I(n_r,n_c)=0$] or "nearly" unobservable, [in which case $I(n_r,n_c)$ would be small].

In order to identify these "nearly" unobservable states, we make use of the transformation in proposition 1 to obtain a representation of $S_c(n_r)$ in a convenient set of CODE-coordinates having the following structure. [The computational details are given in Appendix A].

$$\begin{bmatrix} \dot{\hat{\underline{x}}}_1 \\ \dot{\hat{\underline{x}}}_2 \\ \dot{\hat{\underline{x}}}_3 \\ \vdots \\ \dot{\hat{\underline{x}}}_{p-1} \\ \dot{\hat{\underline{x}}}_p \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & & 0 \\ A_{21} & A_{22} & A_{23} & & 0 \\ A_{31} & A_{32} & A_{33} & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ A_{p-1,1} & A_{p-1,2} & A_{p-1,3} & \cdots & A_{p-1,p} \\ A_{p1} & A_{p2} & A_{p3} & & A_{pp} \end{bmatrix} \begin{bmatrix} \hat{\underline{x}}_1 \\ \hat{\underline{x}}_2 \\ \hat{\underline{x}}_3 \\ \vdots \\ \hat{\underline{x}}_{p-1} \\ \hat{\underline{x}}_p \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_{p-1} \\ F_p \end{bmatrix} z$$

$$u = [G_1 \quad 0 \quad 0 \quad \cdots \quad 0]\, \hat{x} \qquad (3.13)$$

where $\hat{\underline{x}}_i \in R^{r_i}$, $i = 1,2,\ldots,p$,

$$\sum_{i=1}^{p} r_i = n_r, \quad r_i \leq r_{i-1}, \quad i = 2, 3, \ldots, p, \; [13]$$

and where $r_i \triangleq \text{rank}\,(A_{i-1,i})$, $i = 2,3, \ldots, p$, and the matrices $A_{i-1,i}$ are compatibly dimensioned.

Representations of systems in the form of (3.13) have been used in literature in different contexts. For example, Tse et al. [13] and Perkin et al. [14] have used Generalized Hessenberg Representations (GHR) in the context of *model* simplifications. In [13] and [14] GHR is used "... for detecting weakly observable subsystems". Also, if $A_{i,i+1}$ is "small" in the sense of some reasonable criteria, a reduced model is obtained by neglecting (truncating) $\hat{\underline{x}}_j$, $j>i$. Clearly, from the presence of the trailing zeros in the matrix G in (3.13), it is seen that if for some i, $A_{i,i+1} = 0$ (or "small") then $\hat{\underline{x}}_j$, $j>i$ are unobservable (or nearly unobservable). The representation (3.13) differs from the GHR in the sense that $G_1 \neq I_m$ which is required of a GHR. However, the statements made above are also applicable for the controller in (3.13). Another context where such a representation is used is in extracting a minimal realization from a non-minimal realization of a given transfer function [12], where again it is used to detect the unobservable subspace. The algorithm presented in Appendix A is almost identical to that used in [12]. Due to these similarities some of the results from [13] can be specialized to the *controller* reduction context. In particular we have the following.

<u>Conjecture</u>: If for some i, $\|A_{i,i+1}\|^{\dagger}$ is much smaller than $\|A_{C_R}\|$, $\|A_{C_T}\|$, $\|A_{C_{TR}}\|$, $\|F_R\|$ and $\|F_T\|$, then the reduced controller $S_c(n_c)$

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† $\|X\| \triangleq \lambda_M^{1/2}(X^T X)$ = maximum singular value of X.

$$S_c(n_c): \quad \dot{\rho} = A_{C_R}\rho + F_R z \ , \quad \rho \varepsilon R^{n_c} \ , \quad n_c = \sum_{j=1}^{i} r_j$$

$$u_R = G_R \, \rho$$

where $G_R = [G_1, 0, \ldots, 0]$, obtained by deleting the controller states $\hat{\underline{x}}_j$, $j = i+1, \ldots, p$ yields a small controller error index $I(n_r, n_c)$.

To show this, note from the structure of the controller in (3.13) and from the definition of $A_{C_{RT}}$ that

$$A_{C_{RT}} = A_{RT} - F_R M_T = \left[\begin{array}{c|c} 0_{(n_c - r_i) x r_{i+1}} & \\ \hline A_{i,i+1} & 0_{n_c x (t - r_{i+1})} \end{array}\right] .$$

Hence, from the definition of $U(n_c)$ in Theorem 1, small $\|A_{i,i+1}\|$ implies small $\|U(n_c)\|$. Furthermore, if A_R is stable, the solution to (3.10a) is

$$(\hat{X}_R - X_R) = \int_0^\infty e^{A_R t} \, U(n_c) \, e^{A_R^T t} \, dt \quad ,$$

from which we get

$$\| X_R - \hat{X}_R \| \le \int_0^\infty \| e^{A_R t} \| \cdot \| U(n_c) \| \cdot \| e^{A_R^T t} \| \cdot dt \quad .$$

Hence, from (3.10b) small $\|A_{i,i+1}\|$ leads to small $\|U(n_c)\|$ and hence from (3.8) to small $I(n_r, n_c)$.

Recall that all the controller states $\hat{x}_i$, $i > r_1$ have zero controller costs $V_i(\hat{x})$ and that the controller states $\hat{x}_i$, $i = 1, 2, \ldots, r_1$ are ordered according to (3.5b). This means, since $\gamma_i{}^2 = V_i(\hat{x})$, that $\hat{x}_i$ is more significant than $\hat{x}_{i+1}$; $i = 1, 2, \ldots, r_1$. Hence, if the order of the controller is predetermined to be $n_c \leq r_1$, then the controller states $\hat{x}_i$, $i=n_c+1, \ldots, n_r$ are to be truncated. Therefore, for $n_c \leq r_1$, there is *no* ambiguity of the controller states to be truncated, and the additional complexities of the GHR type structure (3.13) need *not* be computed. It is only when $n_c > r_1$ that one requires an additional criteria, such as observability of the components $\hat{x}_i$, $i > r_1$, to determine which of these $\hat{x}_i$ are to be truncated. It is this requirement that led to the development of (3.13). Clearly, more investigation is required in the study of the properties of the controller (3.13).

Stability Properties

Of primary concern to any controller reduction scheme is the stability properties of the closed loop system. The following theorem and corollaries present the known *instability* properties of $S_c(n_c)$. It is assumed that the controller is represented as in (3.13).

Theorem 2: Let $S_c(n_c)$ *be a reduced controller obtained by truncation from* $S_c(n_r)$. *Then the closed loop system* (2.6) *is asymptotically stable only if* $S_c(n_c)$ *is controllable, i.e., only if the matrix pair* $\{(A_R+B_RG_R), F_R\}$ *is controllable.*

Proof: This is proved by showing that if $S_c(n_c)$ is not controllable i.e., if the matrix pair $\{(A_R+B_RG_R-F_RM_R), F_R\}$ is not controllable, then A_R in (2.6) is not asymptotically stable. (Note that uncontrollability of $\{(A_R+B_RG_R), F_R\}$ is equivalent to uncontrollability of $\{(A_R+B_RG_R-F_RM_R), F_R\}$ [15]).

Now, let $S_c(n_c)$ be uncontrollable and for simplicity assume that the matrix $(A_R+B_RG_R)$ has n_c distinct eigenvalues. Then there exists a left eigenvector $\alpha_i{}^*$ of $(A_R+B_RG_R)$ such that

$$\alpha_i{}^*F_R = 0 \; ; \qquad \alpha_i \varepsilon\, C^{n_c} \tag{3.14a}$$

and

$$\alpha_i{}^* \, (A_R+B_RG_R) = \lambda_i \, \alpha_i^* \tag{3.14b}$$

for some $i\varepsilon\{1,2,\ldots, n_c\}$, where $\lambda_i \, \varepsilon \Lambda\,(A_R+B_RG_R)$ and * denotes the conjugate transposition.

Now, since the controller states $\hat{x}$ are covariance-normalized, i.e.,

$$\hat{X} \triangleq \lim_{t\to\infty} E\{\hat{x}(t)\;\hat{x}^T(t)\} = I_{n_r} \;,$$

(3.3d) reduces to

$$(A+BG)^T + (A+BG) + FVF^T = 0 \tag{3.15}$$

which can be written in its partitioned form as

$$\begin{bmatrix} (A_R+B_RG_R)^T & (A_{TR}+B_TG_R)^T \\ (A_{RT}+B_RG_T)^T & (A_T+B_TG_T)^T \end{bmatrix}^T + \begin{bmatrix} (A_R+B_RG_R) & (A_{RT}+B_RG_T) \\ (A_{TR}+B_TG_R) & (A_T+B_TG_T) \end{bmatrix} + \begin{bmatrix} F_RVF_R{}^T & F_RVF_T{}^T \\ F_TVF_R{}^T & F_TVF_T^T \end{bmatrix} = 0$$

The upper left corner of this equation yields

$$(A_R+B_RG_R)^T + (A_R+B_RG_R) + F_RVF_R{}^T = 0 \quad . \tag{3.16}$$

Now, pre- and post-multiply (3.16) by $\alpha_i{}^*$ and α_i respectively to get

$$\alpha_i{}^*(A_R+B_RG_R)^T \, \alpha_i + \alpha_i{}^*(A_R+B_RG_R)\, \alpha_i + \alpha_i{}^* \, F_RVF_R{}^T\alpha_i = 0 \; . \tag{3.17a}$$

In view of (3.14) and the conjugate-transpose of (3.14b) this reduces to

$$\bar{\lambda}_i \ \alpha_i^{*}\alpha_i + \lambda_i \ \alpha_i^{*}\alpha_i = 2 \ \mathrm{Re}(\lambda_i)\cdot\|\alpha_i\|^2 = 0 \ , \tag{3.17b}$$

where Re(·) denotes the real part of (·). Clearly from (3.17b)

$$\mathrm{Re}(\lambda_i) = 0 \ . \tag{3.18}$$

Now consider $\eta_i^{*} A_R$ where

$$\eta_i^{*} \overset{\Delta}{=} [0, \ 0, \ \alpha_i^{*}] \ ; \quad \eta_i \in C^r \quad .$$

Then

$$\eta_i^{*} A_R = [0, \ 0, \ \alpha_i^{*}] \quad \begin{bmatrix} A_R & A_{RT} & B_R G_R \\ A_{TR} & A_T & B_T G_R \\ F_R M_R & F_R M_T & A_{C_R} \end{bmatrix}$$

$$= [\alpha_i^{*} \ F_R M_R, \ \ \alpha_i^{*} F_R M_T \ , \ \ \alpha_i^{*} A_{C_R}]$$

$$= [0, \ 0, \ \alpha_i^{*} A_{C_R}]$$

where (2.4b) and (3.14a) have been used. Now since $A_{C_R} = A_R + B_R G_R - F_R M_R$, we get

$$\begin{aligned} \eta_i^{*} A_R &= [0, \ 0, \ \alpha_i^{*}(A_R + B_R G_R - F_R M_R)] \\ &= [0, \ 0, \ \alpha_i^{*}(A_R + B_R G_R)] \\ &= [0, \ 0, \ \lambda_i \alpha_i^{*}] = \lambda_i \eta_i^{*} \end{aligned} \tag{3.19a}$$

where (3.14) has been used. Hence from (3.19a) we see that

$$\lambda_i \ \varepsilon \ \Lambda(A_R) \ . \tag{3.19b}$$

Therefore from (3.18) and (3.19b) A_R is not asymptotically stable. #

Corollary 1: If the zeroth order Markov Parameter[1] *of* $S_c(n_r)$ *is zero, then the resulting closed loop system* (2.8) *for any* $S_c(n_c \leq r_1)$, *is not asymptotically stable.*

Proof: Since the Markov Parameters are invariant under similarity transformations, the zeroth order Markov Parameter of $S_c(n_r)$ is zero if and only if

$$GF = [G_1, 0] \begin{bmatrix} F_1 \\ \\ F_{\bar{1}} \end{bmatrix} = G_1F_1 = 0 \, , \tag{3.20}$$

where the matrix F of $S_c(n_r)$ is partitioned as

$$F = \begin{bmatrix} F_1 \\ \\ F_{\bar{1}} \end{bmatrix}; \quad F_1 \in R^{r_1 \times \ell}, \quad F_{\bar{1}} \in R^{(n_r - r_1)\times \ell}$$

Now, since rank $(G_1) = r_1 \leq m$, (3.20) is satisfied if and only if $F_1 = 0$. Therefore from (2.2) and (2.5), $F_R = 0$ for all $n_c \leq r_1$ and hence $S_c(n_c \leq r_1)$ is not controllable. The proof now follows from Theorem 2. #

The final controller reduction procedure is summarized in algorithmic form in Appendix A.

We now demonstrate this controller design scheme with the aid of an example in the next section.

---

[1]The i-th order Markov Parameter J_i of $S_c(n_r)$ is defined as

$$J_i \overset{\Delta}{=} G(A+BG-FM)^i F \; ; \quad i = 0, 1, 2, \ldots .$$

IV. EXAMPLE

The example considered is a Solar Telescope (SOT), schematically represented by Fig. 1.

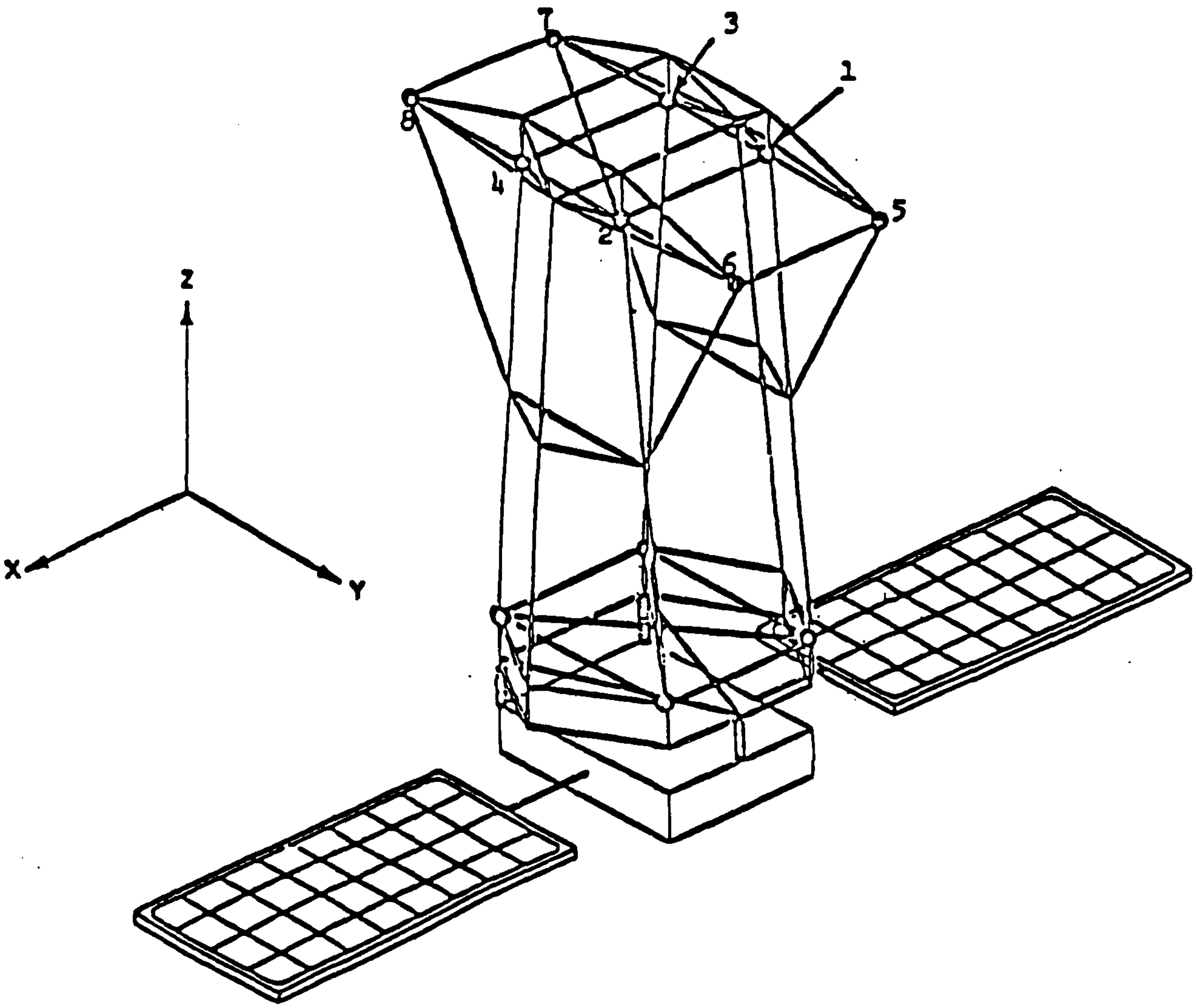


Fig. 1 Solar Optical Telescope Structure

The truss structure of Fig. 1. has been modeled by finite element methods in [11]. This SOT was originally described by 44 modes, but the model was reduced to 10 modes by a Modal Cost Analysis [16]. (This detail is not given here). The reduced model of 10 modes (8 elastic- and 2 rigid-modes) is described by

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = b_i^T(u + w), \; i=1,\ldots,10 \tag{4.1}$$

There are m (=8) force actuators whose control forces $u_1,\ldots,u_8$ act in the z-direction, located as shown in Fig. 1. The actuator noise (white) is denoted by w and has intensity $W = 10^{-4}I_8$. ω_i is the frequency of the i-th mode and ζ_i is the damping ratio of the i-th mode. The frequencies (ordered by modal cost) are given in Table 1, and the damping ratios are taken as $\zeta_i = 0.001$, i = 1, 2, ..., 8 and $\zeta_i = 0$, i = 9, 10 corresponding to rigid-modes. The control objective is

$$V = \lim_{t\to\infty} E\{\|y\|_Q^2 + \|u\|_R^2\} , \tag{4.2}$$

Table 1

Modal Frequencies

Mode #	1	2	3	4	5	6	7	8	9*	10*
ω_i(rad/sec)	14.853	0.914	10.817	3.652	153.43	53.861	3.630	149.37	0	0

(*: #9 and 10 are the rigid modes)

where the output $y\epsilon R^k$ (k=3) is

$$y^T = \{LOS_x, LOS_y, Defocus\} .$$

and where LOS_x is the angular displacement of the optical Line of Sight (LOS) error about the x axis. LOS_y is the optical Line of Sight error about the y axis. Defocus is caused by changes in the length of the optical axis (deflections in the z direction). The output y is related to the modes η_i by

$$y = \sum_{i=1}^{10} p_i \eta_i \tag{4.3}$$

The output weighting matrix is chosen as $Q = \text{diag}\,\{1,10,10^{-3}\}$ to indicate that line-of-sight errors about the y axis are most critical to the experiments. The available measurements for the control law implementation are

$$z = y+v \ , \quad z \varepsilon R^{\ell} \ , \ \ell = 3 \tag{4.4}$$

where the noise v is assumed to be a zero mean white noise with intensity $V = 10^{-15} I_3$ to reflect the uncertainties in the measurements.

This second order representation (4.1-4) of the SOT is equivalently written in a first order state form as follows:

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw \ , \ x \varepsilon R^{n_r} \\ y &= Cx \\ z &= Mx + v \end{aligned} \tag{4.5}$$

where

$$x \triangleq [\eta_1, \eta_2, \ldots, \eta_{10}, \dot{\eta}_1, \dot{\eta}_2, \ldots, \dot{\eta}_{10}]^T , \; n_r = 20$$

$$A = \begin{bmatrix} 0 & I_{10} \\ -\omega^2 & -2\zeta\omega \end{bmatrix} , \; B = D = \begin{bmatrix} 0 \\ \mathcal{B} \end{bmatrix}$$

$$C = M = [\mathcal{P} \quad 0]$$

and where

$$\omega \triangleq \text{diag}\{\omega_1, \omega_2, \ldots, \omega_{10}\} , \; \zeta = 0.001$$

$$\mathcal{B} \triangleq \begin{bmatrix} b_1^T \\ \vdots \\ b_{10}^T \end{bmatrix} \quad \text{and} \quad \mathcal{P} \triangleq [p_1, p_2 \cdot \cdot \cdot p_{10}] .$$

The matrices $\mathcal{B}$ and $\mathcal{P}$ are given in Table 2. Having represented the SOT in the form (4.5) required by the Cost Decoupled Controller Design Algorithm (given in Appendix A), this algorithm was used to design reduced order controllers.

Reduced Order Controller Design:

The control weighting R in (4.2) was taken as $R = \rho I_8$ and ρ was varied to study controllers of different bandwidth. For each ρ, the Cost Decoupled Controller Design Algorithm was applied. This constitutes the following steps.

Table 2

Input and Output Matrices

Matrix B (10x8)

-1.169E-02	-1.365E-02	1.171E-02	1.361E-02	-1.074E-02	1.073E-02	-1.535E-02	1.537E-02
-1.517E-03	-1.510E-03	1.517E-03	1.510E-03	-1.522E-03	1.522E-03	-1.504E-03	1.503E-03
5.596E-03	8.348E-04	-5.614E-03	-8.592E-04	1.120E-02	-1.118E-02	-3.982E-03	4.023E-03
-3.473E-03	-3.540E-03	3.481E-03	3.519E-03	-3.405E-03	3.446E-03	-3.611E-03	3.558E-03
-3.733E-02	3.279E-03	1.389E-02	-3.592E-03	3.591E-03	-1.385E-03	-1.839E-03	1.019E-03
2.416E-03	-3.075E-03	2.568E-03	-3.059E-03	2.023E-02	2.074E-02	-1.440E-02	-1.512E-02
1.212E-03	-3.095E-03	1.190E-03	-3.124E-03	6.250E-03	6.236E-03	-8.125E-03	-8.162E-03
1.608E-02	-7.604E-04	3.905E-02	-6.677E-03	-1.909E-03	-4.554E-03	2.554E-04	1.491E-03
4.931E-06	3.908E-06	-4.849E-06	0.	4.931E-06	3.908E-06	-4.849E-06	0.
-4.576E-07	3.273E-06	3.273E-06	-5.119E-06	4.576E-07	-3.273E-06	-3.273E-06	1.148E-05

Matrix P (3x10)

2.212E-05	-2.532E-07	2.737E-05	-2.296E-06	-9.024E-03	1.866E-02	-7.338E-04	-2.265E-02
1.376E-02	3.253E-04	-2.775E-03	-8.727E-04	-3.714E-02	-1.143E-05	6.735E-06	1.665E-02
9.086E-06	-5.025E-08	-4.963E-05	4.061E-07	2.286E-02	1.098E-03	-1.016E-05	5.217E-02

1. 000E+00	0.
0.	1. 000E+00
0.	0.

i) Construct an optimal controller for (4.5).

ii) Transform the controller to the form in (3.13) with properties (3.5).

iii) Obtain reduced controllers of order n_c by truncating the last (n_r-n_c) controller states and evaluate these controllers. (Note, $n_c = n_r = 20$ is the full order optimal controller for this example).

The controller design algorithm was repeated for a range of $\rho = .01 \rightarrow 100$, and reduced controllers of different orders were obtained. The performance of the controllers was evaluated and is presented in Fig. 2. The different labels presented in the following figures are defined by

$$\text{CONTROL COST} \triangleq \lim_{t\to\infty} E\|u(t)\|^2$$

$$\text{REGULATION COST} \triangleq \lim_{t\to\infty} E\|y(t)\|_Q^2$$

$$\text{CONTROL EFFORT} \triangleq [\text{control cost}]^{1/2}$$

$$\text{LOS}(y) \triangleq \lim_{t\to\infty} E\|\text{LOS}_y(t)\| \ .$$

The dotted lines indicate lines of constant ρ. The solid lines are continued until instability occurs. Fig. 2 illustrates that the Regulation Cost asymptotically reaches a constant value ($\sim$1·0E-07) corresponding to a control cost of greater than 1·6E-06N^2 (control effort > 1.3E-03 Newtons). Fig. 2 and Table 3 show that for a *fixed* level of control effort (8·5E-4 Newtons for Table 3), controllers of smaller order perform worse than the controllers of larger order.

FIG 2 - PERFORMANCE PLOT

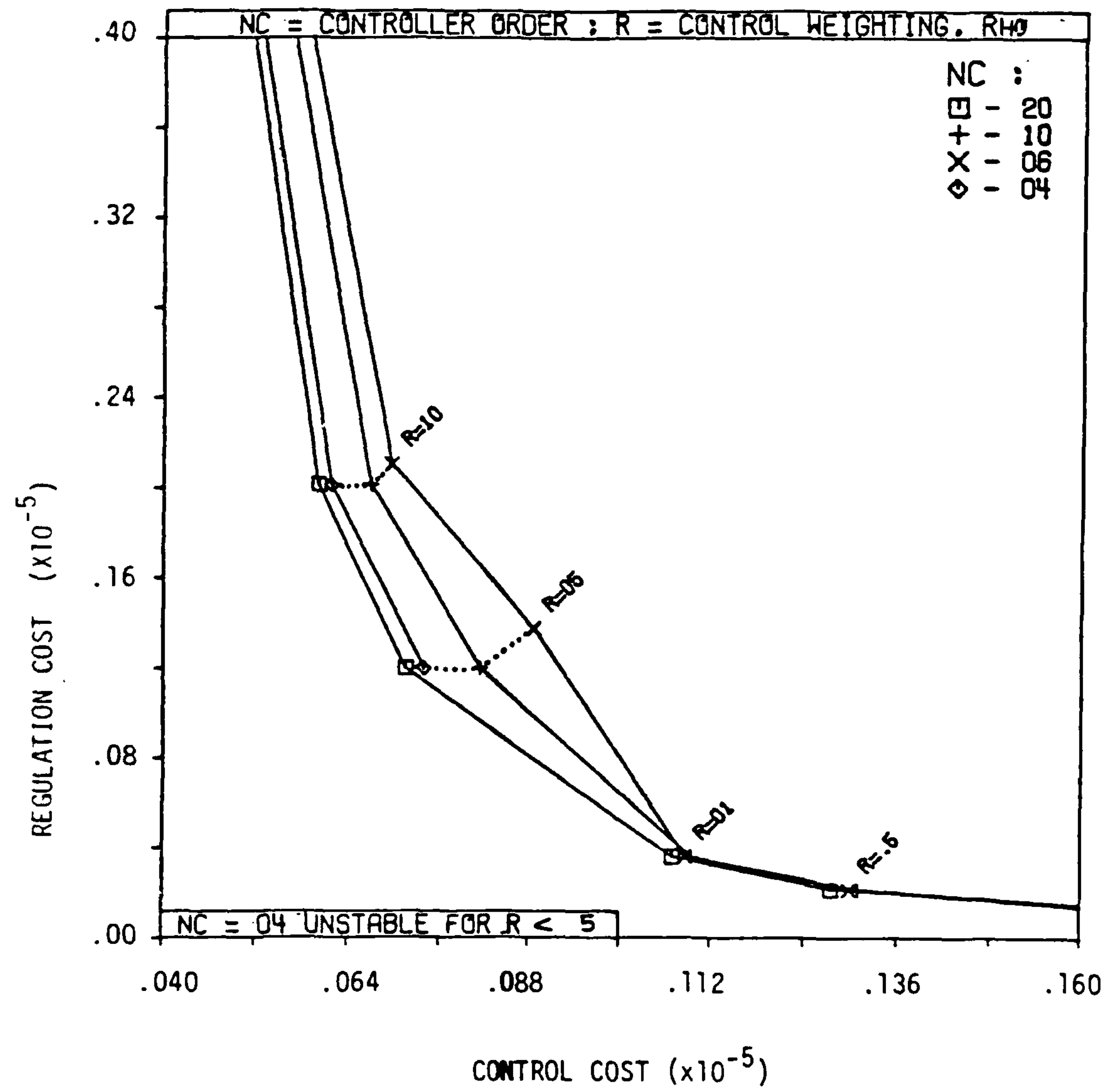


Table 3 shows that the 4-th order controller and the full order (n_c=20) controllers yield essentially the same RMS performance. Our explanation for this is that the relative observability conditions in the conjecture on p. 21 regarding small $I(n_r,n_c)$ happens to hold more accurately for the 4-th order controller than for other reduced controllers. However, the 4-th order controller has a lower margin of stability which is indicated by this controller becoming unstable even for a small increase in the control effort beyond CONTROL COST > $.07 \times 10^{-6} N^2$. (See Fig. 2).

Table 3

LOS(y) vs. n_c (Control Effort = 8.5E-4N)

n_c	20	18	12	10	6	4
LOS(y). Rad($\times 10^{-3}$)	0·28	0·28	0·29	0·35	0·37	0·29
$I(n_r,n_c)$	0	.0150	.0196	.0848	.1301	.0245

Controllers of order n_c =18 and 12 have not been included in Fig. 2. From these figures one could pick a design to meet the mission objective. For example, if the following is the mission objective:

(i) LOS(x) ≤ 5·0E-4 Rad.

(ii) LOS(y) ≤ 3·0E-4 Rad.

and

(iii) Control Effort ≤ 1·0E-3 Newtons.

one would pick a 6-th order controller instead of $n_c > 6$ so that the performance specifications are met with the least amount of on-line controller hardware/software.

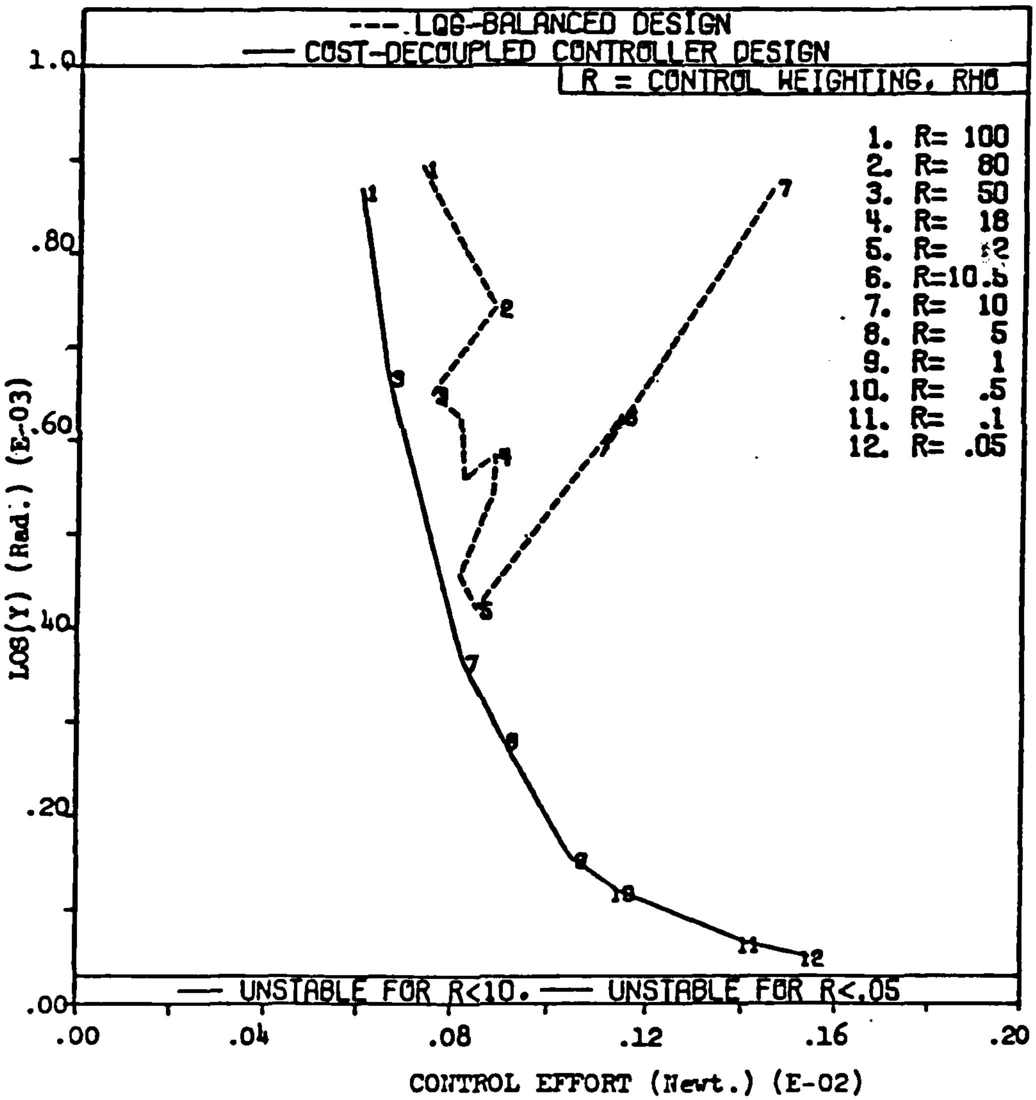
Obviously, none of the reduced controllers ($n_c < 20$) can perform better than the optimal controller ($n_c = 20$). One could only aim to get the corresponding performance curves close to the optimal curve. Note that the reduced order Cost-Decoupled-controller designs in Fig. 2 are close to the optimal curve until instability occurs. (The 4^{th} order controller (NC=4) eventually went unstable in Fig. 2).

<u>Comparison with Balanced [5,6] Controller Design</u>:

Since [5,6] also consider LQG-based design problems we will now compare the reduced controllers obtained by Cost-Decoupled (CODE) controller design algorithm presented herein with the LQG-balanced controller design method proposed by Verriest [5,6].

The results are presented in Fig. 3. Let CODE denote the controller design presented herein and BAL denote that of [5,6]. The points labeled by similar numbers indicate that the controllers were obtained from the *same* optimal controller ($n_c = 20$) but by different methods. In this example the CODE-controllers performed better than BAL-controllers and this difference increases with smaller order controllers. For the values of ρ tried the LQG-balanced method did not yield any stable controllers of orders less than 10 even for low control effort. The Cost Decoupled Controller Design Algorithm resulted in stable reduced controllers of order less than 10 as shown by Fig. 2. For a controller order $n_c = 10$, the smallest line-of-sight (LOS_y) error achieved by the LQG-BALANCED design was 0.4×10^{-3} rad., and the smallest achieved by the Cost-Decoupled design was 0.03×10^{-3} rad.

FIG3 CONTROLLER DIMENSION : 10



VI. CONCLUSIONS

The concepts of Component Cost Analysis (CCA) utilizes the contribution of states to a quadratic cost function as a metric to measure the significance of the states. Using these concepts, a controller-reduction algorithm is proposed. Controller error indices are defined to measure the 'quality' of reduced-controllers and expressions are derived for their computations. Upper bounds are also obtained for these indices. The dependence of these indices on the observability of the controller states that are truncated is also shown.

Employing CCA to reduce these error indices, a set of Cost Decoupled Controller Coordinates are developed. The representation of the optimal controller in these coordinates closely resembles the Generalized Hessenberg Representation. The *truncated* controller states in this representation have the following properties:

(i) smallest controller component costs,

(ii) the component cost is least sensitive to the control weighting matrix R,

(iii) least observability in the controller,

(iv) uncorrelation from the retained controller states, and

(v) they have the least dynamic interaction with the retained controller states.

Necessary conditions have been derived for the stability of the closed loop system when driven by the reduced order controllers. The conditions are shown to be related to the Markov Parameters of the full order optimal controller. A Solar Optical Telescope is used to illustrate the design procedure. The resulting controllers are compared with those obtained by the LQG-balanced method proposed by Verriest [5,6].

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APPENDIX A. The Cost-Decoupled (CODE) Controller Design Algorithm

Given a model,

$$\begin{aligned} \dot{x} &= A\,x + B\,u + D\,w; \quad w \sim n(o,W) \\ y &= C\,x \quad , \quad x\epsilon R^{n_r}, u\epsilon R^m, w\epsilon R^d, y\epsilon R^k, z\epsilon R^\ell \} \\ z &= M\,x + v\;;\; v \sim n(o,V) \end{aligned} \tag{A.1}$$

the objective of this algorithm is to construct a controller in the form (3.13) with the properties (3.5), to reduce

$$V = \lim_{t\to\infty} E\{\| y(t) \|_Q^2 + \| u(t) \|_R^2\} \quad . \tag{A.2}$$

ALGORITHM

I. Ia. Read {A, B, D, C, M, W, V, Q, R}

Ib. Construct the optimal controller,

$$\begin{aligned} \dot{\hat{x}} &= A_c\,\hat{x} + F\,z \\ u &= G\,\hat{x} \end{aligned} \tag{A.3a}$$

where

$$A_c \triangleq A + B\,G - F\,M \tag{A.3b}$$

$$F = P\,M^T\,V^{-1} \tag{A.3c}$$

$$G = -R^{-1}B^T K \tag{A.3d}$$

with K and P satisfying.

$$K\,A + A^T K - K\,B\,R^{-1}B^T K + C^T QC = 0 \tag{A.3e}$$

$$P A^T + A P - P M^T V^{-1} M P + D W D^T = 0 \qquad \text{(A.3f)}$$

Ie. Compute $\hat{X}$ by solving

$$\hat{X}(A + B G)^T + (A + B G)\hat{X} + F V F^T = 0 \qquad \text{(A.4)}$$

Id. Compute Θ_x, the square root of $\hat{X}^1$,

$$\hat{X} = \Theta_x \Theta_x^T \qquad \text{(A.5)}$$

Ie. Compute Θ_u, the orthonormal modal matrix[2] of

$\Theta_x^T G^T R G \Theta_x$ such that

$$\Theta_u^T \Theta_x^T G^T R G \Theta_x \Theta_u = \text{diag}\{\nu_1(\hat{x}), \nu_2(\hat{x}), \ldots, \nu_{r_1}(\hat{x}), 0 \ldots 0\} \qquad \text{(A.6)}$$

where $\nu_1(\hat{x}) \geq \nu_2(\hat{x}) \geq \cdot \cdot \cdot \geq \nu_{r_1}(\hat{x}) > 0$.

II. IIa. Define $T_1 \stackrel{\Delta}{=} \Theta_x \Theta_u$ (4.7)

IIb. Define

$$\begin{bmatrix} A_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix} = T_1^{-1} A_c T_1 \; ; \quad \begin{matrix} A_{11} \varepsilon R^{r_1 x r_1} \\ A'_{22} \varepsilon R^{(n_r - r_1) x (n_r - r_1)} \end{matrix} \qquad \text{(A.8a)}$$

$$\begin{bmatrix} F_1 \\ F'_2 \end{bmatrix} = T_1^{-1} F \quad , \; F_1 \varepsilon R^{r_1 x \ell} \quad . \qquad \text{(A.8b)}$$

---

1. In case of uncontrollable controllers, singular value decomposition may be used in (A.5), to help factor out the uncontrollable subspace.

2. For this task use singular value decomposition [17] or use an eigenvalue/eigenvector program specialized for symmetric matrices.

IIc. Set $i = 2$, $p = 2$, $r = r_1$

III. IIIa. Obtain singular value decomposition of $A'_{i-1,i}$ as

$$A'_{i-1,i} = U^i \begin{bmatrix} \sigma^i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1{}^{i^T} \\ V_2{}^{i^T} \end{bmatrix} \tag{A.9}$$

where $\sigma^i = \text{diag}\{\sigma_1{}^i, \sigma_2{}^i, \ldots, \sigma_{r_i}{}^i;\ \sigma_j{}^i \geq \sigma_{j+1}{}^i > 0\}$

and $V_1{}^i \varepsilon R^{n_r - r \times r_i}$

IIIb. Define $T_i \triangleq \begin{bmatrix} I_r & 0 \\ 0 & V^i \end{bmatrix}$; $V^i \triangleq [V_1^i \ , \ V_2^i]$ (A.10)

IIIc. Set $r = r + r_i$; if $r = n_r$ go to IV.

IIId. Define $A'_{i,i+1} \triangleq V_1{}^{i^T} A'_{i,i} V_2{}^i$ (A.11a)

$$A'_{i+1,i+1} \triangleq V_2^{i^T} A'_{i,i} V_2{}^i \tag{A.11b}$$

IIIe. Set $i = i+1$ and $p = p+1$ and repeat III.

IV. Define $T \triangleq \prod_{i=1}^{p} T_i$ (A.12)

V. The reduced controller of order r is

$$\dot{\rho} = A_{C_R}\rho + F_R z , \quad \rho \in R^r$$

$$u_R = G_R \rho \qquad \text{(A.13a)}$$

where

$$A_{C_R} \triangleq L_R A_C T_R , \qquad \text{(A.13b)}$$

$$F_R \triangleq L_R F , \qquad \text{(A.13c)}$$

$$G_R \triangleq G T_R , \qquad \text{(A.13d)}$$

and where T_R and L_R are obtained from

$$[T_R \;\; T_T] = T , \quad T_R \in R^{n_r \times r} \qquad \text{(A.13e)}$$

and

$$\begin{bmatrix} L_R \\ L_T \end{bmatrix} = L \triangleq T^{-1} , \quad L_R \in R^{r \times n_r} \qquad \text{(A.13f)}$$

#

APPENDIX B. Proof of Lemma 1

Rewrite (2.8a) as

$$V = \mathrm{Tr}[C_R^T Q C_R \hat{X}_R] + \mathrm{Tr}[C_T^T Q C_T \hat{X}_T] + 2\mathrm{Tr}[C_R^T Q C_T \hat{X}_{RT}^T] \quad \text{(B.1)}$$

where $\hat{X}_R$, $\hat{X}_{RT}$ and $\hat{X}_T$ satisfy

$$\hat{X}_R A_R^T + A_R \hat{X}_R + \hat{X}_{RT} A_{RT}^T + A_{RT} \hat{X}_{RT}^T + D_R W D_R^T = 0 \quad \text{(B.2a)}$$

$$\hat{X}_R A_{TR}^T + \hat{X}_{RT} A_T^T + A_R \hat{X}_{RT} + A_{RT} \hat{X}_T + D_R W D_T^T = 0 \quad \text{(B.2b)}$$

$$\hat{X}_T A_T^T + A_T \hat{X}_T + \hat{X}_{RT}^T A_{TR}^T + A_{TR} \hat{X}_{RT} + D_T W D_T^T = 0 \quad . \quad \text{(B.2c)}$$

The equations (B.2) are obtained by partitioning equation (2.9a).

Equation (2.12a) follows from the substitution of (B.1) and (2.10b) in (2.10a), and (2.12b) follows from (B.1), (2.11) and (2.8b). Equation (2.12c) is obtained by subtracting (B.2a) from (2.8b).

To prove (2.13a), use (2.8b) and (2.9b) to rewrite (2.11) as

$$I(n_r, n_c) = \frac{1}{V} \mathrm{Tr}[C_R^T Q C_R X_R] - 1 \ . \quad \text{(B.3)}$$

Now use the following identity [18]

$$\lambda_m(X)\mathrm{Tr}[Y] \leq \mathrm{Tr}[YX] \leq \lambda_M(X)\mathrm{Tr}[Y] \ , \text{ for } X \geq 0,\ Y \geq 0$$

in (B.3) to get

$$\frac{1}{V} \lambda_m(X_R)\mathrm{Tr}[C_R^T Q C_R] - 1 \leq I(n_r, n_c) \leq \frac{1}{V} \lambda_M(X_R)\mathrm{Tr}[C_R^T Q C_R] - 1 \ .$$

Clearly, from the definition of C_R in (2.4d), μ as defined in lemma 1 depends on n_c. (2.13a) is proved by observing that $I(n_r, n_c) \geq 0$.

(2.14a) is proved by equating (2.12a) and (2.12b). To prove the sufficient condition (2.14b) assume without loss of generality that (2.2) is in observable canonical form, so that $\hat{x}_T$ is unobservable (i.e., $A_{C_{RT}} = 0$, $G_T = 0$). Hence, from the definition of A_{RT} in (2.4b), $A_{RT} = 0$. Now, since (2.2) is the optimal controller, (2.3) is asymptotically stable. Therefore, with $A_{RT} = 0$,

$$\Lambda(A) = \Lambda(A_R) \cup \Lambda(A_T) \subset C^- \tag{B.4a}$$

where $\Lambda(\cdot)$ denotes the eigenvalues of $(\cdot)$ and C^- represents the open left half complex plane. Hence

$$\Lambda(A_R) \subset C^- . \tag{B.4b}$$

Stability of A_R guarantees that the solution to (2.12c) is $X_R - \hat{X}_R = 0$, since $A_{RT} = 0$. Hence recognizing that $C_T = 0$ if $G_T = 0$, (2.14a) is satisfied. To prove (2.14c) let (2.2) be in controllable canonical form so that $\hat{x}_T$ is uncontrollable (i.e., $A_{C_{TR}} = 0$, $F_T = 0$). From (2.4b) $A_{TR} = 0$, hence (B.4) holds. Also note that from (2.4c) $\mathcal{D}_T = 0$, yielding $\hat{X}_T = 0$ and $\hat{X}_{RT} = 0$ as the solutions to (B.2b) and (B.2c) since $\Lambda(A_T) \subset C^-$. Furthermore, the solution to (2.12c) is $X_R - \hat{X}_R = 0$, since $\hat{X}_{RT} = 0$. Since $\hat{X}_{RT} = 0$ and $X_R - \hat{X}_R = 0$, (2.14a) is satisfied. This proves (2.14c) and (2.14d).

The proof of the sufficient conditions (2.15b) and (2.15c) follows identical steps, and equation (2.15a) is obvious from (2.12b). #

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APPENDIX E

Selection of Noisy Actuators and Sensors in Linear Stochastic Systems

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Abstract

Algorithms are given to determine the critical inputs and outputs in a linear system evaluated by quadratic performance criteria. This type of analysis is referred to as "input cost analysis" (ICA) and "output cost analysis" (OCA). The fundamental concept is to decompose the quadratic performance metric into contributions from each input/output. This type of "cost-decomposition" has application in the selection of best sensors and actuators in engineering control systems, and that application is the focus of this paper.

1.0 INTRODUCTION

The problem under consideration is the time-invariant linear dynamic system described by

(1.1a) $\dot{x} = Ax + B(u + w)$

(1.1b) $y = Cx$

(1.1c) $z = Mx + v$

and evaluated by the performance metric

(1.2) $$V = \lim_{t\to\infty} E(\|y(t)\|_Q^2 + \|u(t)\|_R^2) \; , \quad \|y\|_Q^2 = y^TQy \; , \quad \begin{matrix} Q > 0 \\ R > 0 \end{matrix} \; .$$

Where the zero-mean white noise disturbances are described by $Ew(t) = 0$, $Ev(t) = 0$, $Ew(t)w^T(\tau) = W\delta(t-\tau)$, $Ex(o)\, w^T(t) = 0$, $Ex(o)v^T(t) = 0$, $Ev(t)v^T(\tau) = V\delta(t-\tau)$, $W > 0$, $V > 0$. The notation "$Q > 0$" means that "Q is a positive definite matrix". The superscript T denotes matrix transposition, and the notation $y\epsilon R^k$, $u\epsilon R^m$, $z\epsilon R^\ell$, $x\epsilon R^n$, $w\epsilon R^m$ indicates the dimensions k, m, ℓ, n, m of the real vectors y, u, z, x, w respectively. It is assumed that rank B = m, rank M = ℓ, and rank C = k. This eliminates the possibility of redundant inputs/outputs. Now suppose that $m > 1$. (In fact in some large scale systems m is quite large [7].) One might wish to know which of the inputs have a greater effect on the responses that contribute to the performance metric V in both the open loop (u=0) and closed loop (u≠0) cases. We shall refer to these inputs as the more "critical" inputs. Generally speaking, there are three reasons for using input cost analysis:

(i) For the open loop case, the "critical" inputs can be retained in the model and the others deleted if a simplified representation of the system is desired, using fewer inputs.

(ii) If the inputs evolve from actuator devices of varying degrees of reliability (i.e. closed-loop case), the "critical actuators suggest which actuator devices should be made more reliable.

(iii) If the inputs evolve from actuator devices with m admissible locations throughout the engineering system, only the $\bar{m} < m$ most "critical" actuators might be retained if the system is to be designed using only $\bar{m} < m$ actuators. In this way the optimal location of $\bar{m}$ actuators is sought from an admissible set of $m > \bar{m}$ actuators. In fact, when noisy actuators are considered, the set of $\bar{m}$ actuators may yield *better* performance than the total set of m actuators.

A similar set of circumstances and questions exist concerning *outputs* of the system.

The idea of decomposing the performance metric V in terms of contributions from each input was presented in [1]-[3] and this analysis is referred to as "Input Cost Analysis" (ICA). Similarly, "Output Cost Analysis" (OCA) was also presented in [1]-[3]. This paper develops ICA and OCA for the more difficult case of *closed-loop* systems, whereas [2] and [3] are limited to *open-loop* systems. In [1] and [2], the definition of the i^{th} "input cost" V_i^b is

(1.3a) $$V_i^b \triangleq \frac{1}{2} \frac{\partial V}{\partial b_i} b_i , \quad B = [b_1, \ldots, b_m]$$

and the definition of the i^{th} "output cost" V_i^c is

(1.3b) $$V_i^c \triangleq \frac{1}{2} \frac{\partial V}{\partial c_i} c_i \qquad C^T = [c_1, \ldots, c_k]$$

The definition for input costs employed herein and in [3] is

(1.4a) $$V_i^w = \lim_{t\to\infty} \frac{1}{2}\left\{E \frac{\partial V(t)}{\partial w_i} w_i\right\}, \quad V(t) \triangleq ||y||_Q^2$$

and the definition for output costs described herein and in [3] is

$$(1.4\text{b}) \qquad V_i^y = \lim_{t\to\infty} \frac{1}{2} E\left\{\frac{\partial V(t)}{\partial y_i} y_i\right\}, \quad V(t) \triangleq y^T Q y$$

To see clearly the distinctions between (1.3a) and (1.4a) note that the inputs and outputs of (1.1a) can be expanded in terms of the columns of B, and the rows of C i.e.

$$(1.5) \qquad Bw = \sum_{i=1}^{m} b_i w_i \quad \text{and}$$

$$(1.6) \qquad y_i = (Cx)_i = c_i^T x(t) .$$

Note that definitions (1.3) are given in terms of *parameters* b_i and c_i, whereas the definitions (1.4) are given in terms of the *variables* w_i and y_i. The early work on measurement optimization [8], [1] used the *parameters* c_i in the optimization process. However, the parameters b_i and c_i are only the *coefficients* of the variables for which we wish to determine a performance value. That is, the physical entity which might be deleted generates $w_i(t)$ or $y_i(t)$. Hence, it is reasonable to expect definitions (1.4) to be more accurate in predicting the effect of the deletion of an input or output. However, there is an important case when definitions (1.3) and (1.4) yield the *same* result.

Theorem 1

Consider the linear system (1.1), (1.2). *If* $u(t) \equiv 0$, *then* $V_i^b = V_i^w$ *and* $V_i^c = V_i^y$.

Proof:

To show that $V_i^c = V_i^y$, compute first

(1.7a) $$V_i^y \triangleq \frac{1}{2} \lim_{t\to\infty} E\{(\frac{\partial y^T Qy}{\partial y_i})\, y_i\}$$

(1.7b) $$= \lim_{t\to\infty} E\{\sum_{j=1}^{k} y_j Q_{ij} y_i\}$$

(1.7c) $$= \lim_{t\to\infty} E[yy^T Q]_{ii}$$

Then note from (1.3b) that

(1.8a) $$V_i^c = \frac{1}{2}\frac{\partial V}{\partial c_i} c_i = \lim_{t\to\infty} \frac{1}{2} (\frac{\partial Ey^T Qy}{\partial c_i}) c_i = \lim_{t\to o} \frac{1}{2}\frac{\partial}{\partial c_i} (\sum_{i,j=1}^{k} Ey_i Q_{ij} y_j) c_i$$

$$= \lim_{t\to\infty} \frac{1}{2}\frac{\partial}{\partial c_i} (\sum_{i,j=1}^{k} E(c_i^T x) Q_{ij} (c_j^T x))\ c_i$$

$$= \lim_{t\to\infty} E[\sum_{j=1}^{k} x^T c_j Q_{ij} x^T] c_i = \lim_{t\to\infty} E\{\sum_{j=1}^{k} y_j Q_{ij} y_i\}$$

(1.8b) $$= \lim_{t\to\infty} E[yy^T Q]_{ii} = \lim_{t\to\infty} E[cxx^T c^T Q]_{ii} = [CXC^T Q]_{ii}$$

where X is the steady state covariance satisfying

(1.9) $$0 = XA^T + AX + BWB^T$$

Hence, for open loop systems ($u(t) \equiv 0$), (1.7c) and (1.8b) prove that $V_i^c = V_i^y$. The proof that $V_i^w = V_i^b$ follows similar arguments, using the fact that

(1.10) $$V = \mathrm{tr}\ [SBWB^T]$$

where S satisfies

(1.11) $$0 = SA + A^T S + C^T QC\ ,$$

and leads to the calculation

$$(1.12) \qquad \mathcal{V}_i^{w} = \mathcal{V}_i^{b} = [B^T SBW]_{ii} \; . \qquad \#$$

Certainly the new definitions (1.4a) and (1.4b) offer no advantage in open-loop situations, as evidenced by Theorem 1. However, in closed-loop situations the optimal closed-loop plant matrix $\mathcal{A}$, which has the form,

$$(1.13a) \qquad \mathcal{A} \triangleq \begin{bmatrix} A & BG \\ FM & A+BG-FM \end{bmatrix}, \quad \begin{array}{ll} F = PM^T V^{-1}, & 0 = PA^T + AP - PM^T V^{-1} MP + BWB^T \\ G = -R^{-1}B^T K, & 0 = KA + A^T K - KBR^{-1}B^T K + C^T Q \end{array}$$

$$(1.13b) \qquad \begin{array}{lll} \dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\omega, & \mathcal{X}^T \triangleq (x^T, \hat{x}^T), & \omega^T = (w^T, v^T) \\ \mathcal{Y} = \mathcal{C}\mathcal{X}, & \mathcal{Y}^T = (y^T, u^T) & \end{array}$$

$$(1.13c) \qquad E\omega(t)\omega^T(\tau) = \mathcal{W}\delta(t-\tau) = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t-\tau), \; E\mathcal{X}(o)\omega^T(t) = 0.$$

$$\mathcal{B} = \begin{bmatrix} B & 0 \\ 0 & F \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} C & 0 \\ 0 & G \end{bmatrix}$$

is a function of the input/output parameters b_i, c_i. This situation does not occur in the open-loop system since A in (1.1a) is not a function of the parameters of B or C. Thus the definitions (1.3) and (1.4) will yield *different* results for closed-loop applications. The closed-loop results which follow from (1.4) are developed in this paper.

The properties of input costs (1.4a) and output costs (1.4b) for open-loop systems are discussed in sections 2 and 3 respectively. The essential results of these two sections are combined in section 4 to produce the closed-loop ICA/OCA. Closed-loop ICA/OCA is then used in the selection of noisy actuators (section 5) and the selection of noisy sensors (section 6), while the problem of simultaneous selection of noisy actuators and sensors is addressed in section 7.

In Section 8, the methods suggested in this paper are applied to a substantial example of a large space structure. Concluding comments appear in Section 9.

2.0 Properties of Input Costs (1.4a) for Open-Loop Systems ($u(t) \equiv 0$)

Before applying results (1.4) to the closed loop system it will prove useful to know the properties of the input cost V_i^W for the open-loop system ($u(t) \equiv 0$). This Section is devoted exclusively to the case $u(t) \equiv 0$ in (1.1), (1.2) with A asymptotically stable. Proofs of all remaining theorems appear in Appendix A.

Theorem 2

The open-loop input costs $V_i^W, i = 1, \ldots, m$, *defined by* (1.4a) *and calculated by* (1.12), *satisfy the cost-decomposition property*

$$(2.1) \qquad V = \sum_{i=1}^{m} V_i^W \quad , \qquad (u(t) \equiv 0) \; .$$

Where the total value of the system performance metric is V, *and the in situ contribution from* w_i *is* V_i^W.

The sign of V_i^W is nonnegative under these conditions.

Theorem 3

For the stable open-loop system (1.1), 1.2), $u(t) \equiv 0$, $W > 0$, *if* $W_{ij} = 0$ *for all* $j \neq i$ *then* $V_i^W \geq 0$. *If, in addition* (A, C) *is an observable pair, then* $V_i^W > 0$.

If the number of active inputs is not perturbed it is sufficient to know the *in situ* input cost V_i^W. However, additional information is required in order to determine the amount by which the performance metric is perturbed *after* an input is deleted. This amount is defined by

$$(2.2) \qquad \Delta V_i^W \triangleq V - V_{Ri}$$

where V_{Ri} is the value of V *after* the i^{th} input is removed. Clearly, V_i^W is the information available to the analyst *prior* to deletion of the i^{th}

input and ΔV_i^W is the information available *after* such deletion. If V_i^W is to be used *a priori* as a prediction of ΔV_i^W, it is of interest then to know how accurately V_i^W predicts the actual value of ΔV_i^W. Furthermore, it is of interest to know when this prediction is *exact* ($V_i^W = \Delta V_i^W$).

Theorem 4

If $u(t) \equiv 0$ *in* (1.1), (1.2) *then*

$$\Delta V_i^W = 2V_i^W - b_i^T S b_i W_{ii} \quad . \tag{2.3}$$

If W *is a diagonal matrix then* V_i^W *has the following properties:*

$$V_i^W = \Delta V_i^W \ , \quad i = 1, \ldots, m \tag{2.4}$$

$$V(\bar{m}) = \sum_{i \in R} V_i^W \ , \quad R \stackrel{\Delta}{=} \textit{set of } \bar{m} \textit{ retained inputs} \tag{2.5}$$

where $V(\bar{m})$ *is the value of* V *with only the reduced set of* $\bar{m}$ *inputs acting.*

The circumstances (2.4), (2.5) are very valuable, since they allow the performance of the reduced input system $V(\bar{m})$ to be evaluated on the basis of information computed *prior* to deletion of the inputs (V_i^W). Hence, when (2.4) holds, then (2.5) follows. Eq. (2.5) is referred to as the *cost-superposition* property of inputs costs, whereas (2.1) is referred to as the *cost-decomposition* property. The cost-decomposition property (2.1) *always* holds, whereas (2.5) holds only under certain restricted conditions, such as the diagonal W condition of Theorem 4.

Finally, it should be noted that the input costs V_i^W ar(invariant under coordinate transformation.

Theorem 5

If $u(t) \equiv 0$ *in* (1.1), (1.2), *the input costs* V_i^W *defined by* (1.4a) *are invariant under state transformation* $x = Ts$, $|T| \neq 0$.

The Open-Loop Input Selection Problem

Since the input costs V_i^W provide one with information concerning the individual contributions of each input to the performance metric, the next logical use of this information is to describe an algorithm which provides an answer to problem (i) in the Introduction. This shall be referred to as the Open-Loop Input Selection Problem.

There are two different statements of the Open-Loop Input Selection Problem. We shall call them problem (ICA-1) and (ICA-2):

(ICA-1) *Let* m *and* $\bar{m} < m$ *be specified integers. From an admissible set of* m *inputs find a reduced set of* $\bar{m}$ *inputs such that the input error index* I,

(2.6) $$I \triangleq \left|\frac{V-V(\bar{m})}{V}\right|$$

is minimized.

(ICA-2) *Let* m *and* $\bar{I}$ *be specified numbers. From an admissible set of inputs find the smallest number,* $\bar{m}$, *of inputs such that*

$$I \leq \bar{I}$$

The following algorithm provides a systematic means to solve problems (ICA-1) or (ICA-2).

The ICA Algorithm

STEP 1: Specify system data (A, B, W) and output objectives (C, Q) for (1.1), (1.2) with $u \equiv 0$. Specify either $\bar{m}$ or $\bar{I}$ (to solve either problem (ICA-1) or (ICA-2).

STEP 2: Compute the input cost perturbation, from (2.3), for i=1, ..., m.

(2.7) $\quad \Delta V_i^W = 2[B^TSBW]_{ii} - [B^TSBW_D]_{ii}\ , \quad W_D \overset{\Delta}{=} \text{diag}\ \{W_{11}, \ldots, W_{mm}\}$

(2.8) $\quad 0 = SA + A^TS + C^TQC$

Of course, if W is diagonal, then $W = W_D$ and according to Theorem 4 the calculation (2.7) can be *replaced* by

(2.9) $\quad \Delta V_i^W = V_i^W = [B^TSBW]_{ii}$.

Rearrange the inputs w_i such that $\Delta V_1^W \geq \Delta V_2^W \geq \ldots, \geq \Delta V_m^W$.

STEP 3: If $\bar{I}$ is specified, go to STEP 4. If $\bar{m}$ is Specified and W is diagonal, then retain those $\bar{m}$ inputs having the $\bar{m}$ largest values of ΔV_i^W and stop. If W is *not* diagonal, delete the input having the smallest value of ΔV_i^W. For this reduced $B = [b_1, \ldots, b_{m-1}]$ return to STEP 2. (Note that 2.8 does not have to be recomputed.) Repeat this cycle $m-\bar{m}$ times. END.

STEP 4: If $\bar{I}$ is specified, delete the input w_m with the smallest ΔV_i^W. With this reduced input matrix $B = [b_1, \ldots, b_{m-1}]$, return to STEP 2 unless the following condition is satisfied: (Note that (2.8) does not have to be recomputed.)

(2.10) $\quad I \overset{\Delta}{=} \left|\frac{V-V(\bar{m})}{V}\right| > \bar{I}$;

where V equals the value of performance metric with all inputs present and $\bar{m}$ refers to the number of inputs of the *current* iteration. If 2.10 is satisfied, the required set of reduced inputs is the set of inputs from the *previous* iteration (i.e. a set of $\bar{m}$ + 1 inputs). END. (Note: if W is diagonal, the STEP 4 iteration cycle is unnecessary, and $V-V(\bar{m}) = \sum_{i \in J} \Delta V_i^W$; J = set of lowest ΔV_i^W's calculated from the 1st iteration.

3.0 Properties of Output Costs (1.4b) for Open-Loop Systems ($u(t) \equiv 0$).

All of the questions of Section 2.0 can be applied to *outputs* instead of inputs. To make the presentation of these similar results concise, the following duality can be used. It is straightforward to show that substitution of the parameters of TABLE 1 converts all of the ICA results of Section 2.0 to results which hold for Output Cost Analysis (OCA).

With the help of this duality the following results are listed as corollaries to the theorems corresponding to their ICA application of Section 2. The proof of these corollaries is the exact dual (Table 1) of the proofs for Theorems 1-5.

Corollary to Theorem 2:

The open-loop output costs V_i^y, $i=1, \ldots, k$ *defined by* (1.4b) *and calculated by*

$$(3.1) \qquad V_i^y = [CXC^TQ]_{ii}$$

where X *satisfies*

$$(3.2) \qquad 0 = XA^T + AX + BWB^T \ ,$$

satisfy the cost-decomposition property

$$(3.3) \qquad V = \sum_{i=1}^{k} V_i^y \quad .$$

Corollary to Theorem 3:

For the stable open-loop system (1.1), (1.2), ($u(t) \equiv 0$), $Q > 0$, *if* $Q_{ij} = 0$ *for all* $j \neq i$ *then* $V_i^y \geq 0$. *If, in addition,* (A, B) *is a controllable pair, then* $V_i^y > 0$.

Corollary to Theorem 4:

If $u(t) \equiv 0$ *in* (1.1), (1.2) *then*

TABLE 1 ICA/OCA DUALITY

ICA MATRICES	DUAL	OCA MATRICES
A	$\longleftrightarrow$	A^T
B	$\longleftrightarrow$	C^T
S	$\longleftrightarrow$	X
W	$\longleftrightarrow$	Q

(3.4) $\quad \Delta V_i^y = 2V_i^y - c_i^T X c_i Q_{ii}$; where c_i is the i^{th} column of C^T

If Q *is a diagonal matrix then* V_i^y *has the following properties:*

(3.5) $\quad V_i^y = \Delta V_i^y$

(3.6) $\quad V(\bar{k}) = \sum_{i \varepsilon R} V_i^y \quad , \qquad R \overset{\Delta}{=}$ set of $\bar{k}$ retained outputs

where $V(\bar{k})$ *is the value of* V *with only the reduced set of outputs.*

Corollary to Theorem 5:

If $u(t) \equiv 0$ *in* (1.1), (1.2), *the output costs* V_i^y *defined by* (1.4b) *are invariant under state transformation* $x = Ts$, $|T| \neq 0$.

The Open-Loop Output Selection Problem

Since the output costs V_i^y provide one with information concerning the individual contributions of each output to the performance metric, the next logical use of this information is to describe an algorithm which provides an answer to questions (OCA-1) and (OCA-2) below.

(OCA-1) *Let* k *and* $\bar{k}$ < k *be specified integers. From an admissible set of* k *outputs find a reduced set of* $\bar{k}$ *outputs such that*

(3.7) $\quad \theta \overset{\Delta}{=} \left|\frac{V-V(\bar{k})}{V}\right|$

is minimized.

(OCA-2) *Let* k *and* $\bar{\theta}$ *be specified numbers. From an admissible set of* k *outputs find the smallest number,* $\bar{k}$, *of outputs such that*

$$\theta \leq \bar{\theta}$$

The following algorithm provides a systematic means to solve problems (OCA-1) and (OCA-2).

The OCA Algorithm

STEP 1: Specify system data (A,B,W) and output objectives (C,Q) for (1.1), (1.2) with $u \equiv 0$. Specify $\bar{k}$ (to solve problem (OCA-1)) or specify $\bar{\Theta}$ (to solve problem (OCA-2)).

STEP 2: Compute the output cost perturbation, from (3.4), for i=1, ..., k

$$\Delta V_i^y = 2[CXC^TQ]_{ii} - [CXC^TQ_D]_{ii} \ , \quad Q_D \overset{\Delta}{=} \text{diag} \ \{Q_{11}, \dots, Q_{kk}\} \tag{3.8}$$

$$0 = XA^T + AX + BWB^T \tag{3.8a}$$

Of course, if Q is diagonal, then $Q = Q_D$ and according to the corollary to theorem 4 the calculation (3.8) can be replaced by

$$\Delta V_i^y = V_i^y = [CXC^TQ]_{ii} \tag{3.9}$$

Rearrange the outputs y_i such that

$$\Delta V_1^y \geq \Delta V_2^y \geq \cdots \geq \Delta V_k^y$$

STEP 3: If $\bar{\Theta}$ is specified go to STEP 4. If $\bar{k}$ is specified, and Q is diagonal, then retain those $\bar{k}$ outputs having the $\bar{k}$ largest values of ΔV_i^y and END. If Q is *not* diagonal, delete the output having the smallest value of ΔV_i^y. For this reduced $C^T = [c_1, \dots, c_{n-1}]$ return to STEP 2. (note that 3.8a does not have to be recomputed). Repeat this cycle $k-\bar{k}$ times. END.

STEP 4: If $\bar{\Theta}$ is specified, delete the output y_k with the smallest ΔV_i^y. With this reduced output matrix $C^T = [c_1, \ldots, c_{m-1}]$ return to STEP 2 unless the following condition is satisfied. (note that (3.8a) does not have to be recomputed).

$$\Theta \triangleq \left|\frac{V-V(\bar{k})}{V}\right| > \bar{\Theta} ; \qquad (3.10)$$

where $\bar{k}$ refers to the number of outputs in the *current* iteration. If 3.10 is satisfied, the required reduced set of outputs is the set of outputs from the previous iteration (i.e. a set of $\bar{k}+1$ outputs) END. (Note: if Q is diagonal, the STEP 4 iteration cycle is unnecessary, and $V-V(\bar{k}) = \sum_{i \epsilon J} \Delta V_i^y$; J = set of lowest ΔV_i^{w}'s calculated from the 1st iteration.

4.0 Closed-Loop ICA/OCA

Sections 2 and 3 assumed $u(t) \equiv 0$ in (1.1), (1.2). However, it was the *closed-loop* situation (1.13) which motivated the definitions (1.4) of input and output costs. To treat the closed-loop system (1.13) the following will be assumed:

$$\begin{array}{l} (A,\ C) \text{ observable} \\ (A,\ M) \text{ observable} \\ (A,\ B) \text{ controlable} \end{array} \qquad (4.0)$$

Under these conditions the matrix A in (1.13) is guaranteed stable [4]. Now the concepts of Sections 2 and 3 will be applied to (1.13). The practical value of such analysis is the determination of sensor and actuator devices which are most critical to the performance metric (1.2). Thus, reasons (ii) and (iii) in the Introduction are the paramount goals of Section 4.0.

The performance metric V associated with the closed-loop system (1.13) is given by (1.2), and can be rewritten in the form

(4.1) $$V = \lim_{t\to\infty} E\|y\|_{\mathcal{Q}}^2 , \qquad \mathcal{Q} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$$

where y is defined in (1.13b). Thus, input cost analysis (ICA) applied to (1.13) yields

(4.2) $$V_i^{\omega} = [B^T S B W]_{ii} = \begin{Bmatrix} V_i^{w} & i=1, & \ldots, & m & \\ V_j^{v} & j=1, & \ldots, & \ell, & i=m+j \end{Bmatrix}$$

(4.3) $$0 = SA + A^T S + C^T \mathcal{Q} C$$

(4.4) $$V = \sum_{i=1}^{m+\ell} V_i^{\omega} = \sum_{i=1}^{m} V_i^{w} + \sum_{i=1}^{\ell} V_i^{v}$$

and output cost analysis (OCA) applied to (1.13) yields

(4.5) $$V_i^{y} = [C X C^T \mathcal{Q}]_{ii} = \begin{Bmatrix} V_i^{y}, & i=1, & \ldots, k & \\ V_j^{u}, & j=1, & \ldots, m, & i=k+j \end{Bmatrix}$$

(4.6) $$0 = XA^T + AX + BWB^T$$

(4.7) $$V = \sum_{i=1}^{k+m} V_i^{y} = \sum_{i=1}^{k} V_i^{y} + \sum_{i=1}^{m} V_i^{u}$$

where V_i^{w} may be interpreted as the effect of the i^{th} plant noise on the closed-loop performance, V_i^{v} is the effect of the i^{th} measurement noise on the closed-loop performance, V_i^{y} is the effect of the i^{th} output y_i on the closed-loop performance, and V_i^{u} is the effect of the i^{th} control input u_i on the closed-loop performance. The computation of V_i^{y}, V_i^{u}, V_i^{w}, V_i^{v} follow from (4.2) and (4.5), but the 2nx2n matrix equations (4.3) and (4.6) actually represent an excessive burden. The following result shows that the special structure of A in (1.13a) allows

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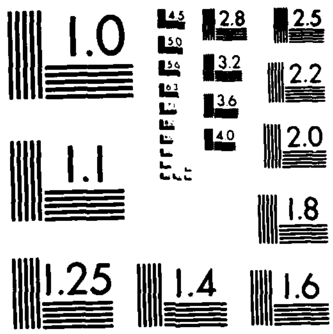

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the solutions of (4.3) and (4.6) to be expressed in terms of the Riccati matrices K and P already obtained in (1.13a).

Theorem 6

The solutions S *and* X *of* (4.3) *and* (4.6) *respectively may be written*

$$(4.8) \qquad S = \begin{bmatrix} K+L & -L \\ -L & L \end{bmatrix}$$

where K *satisfies* $0 = KA + A^TK - KBR^{-1}B^TK + C^TQC$, *and* L *satisfies*

$$(4.9) \qquad 0 = L(A-FM) + (A-FM)^TL + G^TRG$$

and

$$(4.10) \qquad X = \begin{bmatrix} P+N & N \\ N & N \end{bmatrix}$$

where P *satisfies* $0 = PA^T + AP - PM^TV^{-1}MP + BWB^T$, *and where* N *satisfies*

$$(4.11) \qquad 0 = N(A+BG)^T + (A+BG)N + FVF^T$$

Theorem 6 and the special structure of the matrices C, Q, B and W in (1.13), (4.1), allow these expressions as simplifications of (4.2) and (4.5):

$$(4.12) \qquad V_i^{w} = [B^T SBW]_{ii} = [B^T(K+L)BW]_{ii}, \quad i = 1, \ldots, m$$

$$(4.13) \qquad V_i^{v} = [B^T KBW]_{jj} = [F^TLFV]_{ii}, \quad j = m+i, \; i = 1, \ldots, \ell$$

$$(4.14) \qquad V_i^{y} = [CXC^TQ]_{ii} = [C(P+N)C^TQ]_{ii}, \quad i = 1, \ldots, k$$

$$(4.15) \qquad V_i^{u} = [CXC^TQ]_{jj} = [GNG^TR]_{ii}, \quad j = k+i, \; i = 1, \ldots, m$$

Theorems 7-9 follow in the same manner as in Section 2. Their proofs are contained in appendix A.

Theorem 7

For the closed loop system (1.13) *under the conditions* (4.0), *if* $W_{ij} = 0$ *for all* $j \neq i$ *then* $V_i^w > 0$. *If* $V_{ij} = 0$ *for all* $j \neq i$ *then* $V_i^v \geq 0$. *In addition, if* (A-FM,G) *is an observable pair then* $V_i^v > 0$.

Theorem 8

For the closed-loop system (1.13) *under the conditions* (4.0), *if* $Q_{ij} = 0$ *for all* $j \neq i$ *then* $V_i^y > 0$. *If* $R_{ij} = 0$ *for all* $j \neq i$ *then* $V_i^u \geq 0$. *In addition, if* (A+BG,F) *is a controllable pair, then* $V_i^u > 0$.

Theorem 9

The closed-loop input and output costs, V_i^y, V_i^u, V_i^w, V_i^v *are invariant under state transformation* $x = Ts$, $|T| \neq 0$.

Equations (4.2)-(4.7) and (4.12)-(4.15) can be used to set up the duality of TABLE 2 applicable to closed-loop systems.

5.0 Selection of Noisy Actuators in Closed-Loop LQG Problems

A simplifying assumption made in (1.1) is that the only source of disturbances w(t) is the actuator noises. Thus, the disturbance distribution matrix and the control distribution matrix are the same (B=D). Questions of the type (ICA-1) and (ICA-2) from Section 2 are now posed for the *closed-loop* system. The questions are complicated by the fact that for the closed-loop system ($u(t) \neq 0$) there are *two* costs, V_i^w and V_i^u associated with inputs to (1.1a). Since V_i^w is the "bad" effect (from noisy disturbances) and V_i^u is the "good" effect (from optimal control action), it is reasonable to choose the difference between the good and bad effects

TABLE 2 DUALITY OF CLOSED-LOOP ICA/OCA

CLOSED-LOOP ICA		CLOSED-LOOP OCA
$\mathcal{S}$	$\longleftrightarrow$	X
L	$\longleftrightarrow$	N
$A-FM$	$\longleftrightarrow$	$(A+BG)^T$
G	$\longleftrightarrow$	F^T
R	$\longleftrightarrow$	V
$\mathcal{B}$	$\longleftrightarrow$	$\mathcal{C}^T$
B	$\longleftrightarrow$	C^T
K	$\longleftrightarrow$	P
$\mathcal{W}$	$\longleftrightarrow$	$\mathcal{Q}$
W	$\longleftrightarrow$	Q
A	$\longleftrightarrow$	A^T
A	$\longleftrightarrow$	A^T

$$(5.1) \qquad V_i^{act} \triangleq V_i^{u} - V_i^{w}$$

as the "value" of the i^{th} actuator. A negative value of V_i^{act} indicates that the i^{th} actuator is contributing more noise than control action and should either be deleted or its signal to noise ratio should be improved. If one defines $\eta_i \triangleq V_i^{u}/V_i^{w}$ as the "effective signal to noise ratio" of actuator i, then (5.1) becomes

$$(5.2) \qquad V_i^{act} = (\eta_i - 1)\, V_i^{w}$$

If (A,C) is an observable pair, and (A+BG,F) is a controllable pair, and the matrices (W,R) have no negative elements then theorems 7 and 8 guarantee that $\eta_i > 0$ (since V_i^{u} and V_i^{w} are both > 0). Under these conditions *improvement* in performance is expected by *deleting* the actuators with effective signal to noise ratios less than 1($\eta_i < 1$) as long as controllability of (A,B) is maintained. The two closed-loop versions of actuator selection problems (ASP-1 and ASP-2) are now defined.

Problem ASP-1: *Given the optimal system* (1.13) *using* m *admissible but noisy actuators, find the set of* $\bar{\bar{m}}$ *actuators which lead to the smallest value of the closed loop system performance metric V as defined by* (1.2) *without losing controllability. Label this minimum value as* $V(\bar{\bar{m}})$. The

The computations suggested to approximate the solution of problem (ASP-1) are as follows:

Algorithm ASP-1: *Using* (5.1), (4.12), (4.15), *solve for* V_i^{act}, i = 1, ...m, *and rearrange so that* $V_i^{act} \geq \ldots V_m^{act}$. *Delete the actuator with the smallest value of* V_i^{act} *if that value is negative or zero and if deletion of that actuator does not reduce the rank* [B, AB, ... A^{n-1}B]. *Solve the optimal* LQG *problem for the reduced number of actuators* (i.e.

$B = [b_1, \ldots b_{m-1}]$, $W = [w_1, \ldots w_{m-1}]$). *Repeat the above cycle until no actuator satisfies the deletion criteria. The number of actuators left is the set of* $\bar{\bar{m}}$ *actuators to be used.*

Note: At present, no guarantee exists that algorithm ASP-1 provides an optimal solution for Problem ASP-1; however, the results of Section 8 do lend some empirical support, and in addition they suggest a possibility for greatly reducing the computational burden of the algorithm.

Problem ASP-2: *Specify the integer* m *and the number* $\bar{I}_c \geq 1.0$. *Given the optimal system* (1.13) *using* m *admissible but noisy actuators, find the smallest* $\bar{m}$ *set of actuators such that*

$$\frac{V(\bar{m})}{V(\bar{\bar{m}})} \leq \bar{I}_c \quad \text{(and (A,B) a controllable pair)} \tag{5.3}$$

where $V(\bar{\bar{m}})$ *is the smallest value of* V *obtained in* (ASP-1) *and* $V(m)$ *is the value of* V *obtained with* $m \leq \bar{\bar{m}}$ *actuators.*

Although no proof of optimality is currently available, the following algorithm is suggested for the solution of ASP-2.

Algorithm ASP-2: *Solve algorithm* ASP-1 *and use this set of* $\bar{\bar{m}}$ *actuators in the solution of the* LQG *system* (1.13). *Compute* V_i^{act}, $i = 1, \ldots \bar{\bar{m}}$, *and rearrange so that* $V_i^{act} \geq \ldots \geq V_{\bar{\bar{m}}}^{act}$. *Delete the actuator with the smallest value of* V_i^{act} *if deletion of that actuator does not effect the rank of* $[B, \ldots\ldots A^{n-1}B]$. *Solve the optimal* LQG *problem for the reduced number of actuators. Repeat the above cycle until either* $\frac{V(\bar{m})}{V(\bar{\bar{m}})} \leq \bar{I}_c$ *or until all remaining actuators are necessary for controllability. The previous set of* $\bar{m}$ *actuators is then the number desired.*

6.0 Selection of Noisy Sensors in Closed-Loop LQG Problems

Before discussing noisy sensor selection, it is useful to consider the following theorem:

Theorem 10

Adding noisy sensors to the LQG *system described by* (1.13) *cannot increase the performance metric* V (i.e. 4.1), *of the system.*

Heuristically speaking, Theorem 10 is plausable since the sensor measurements, z, are being passed through a Kalman filter and the purpose of the Kalman filter is to "de-emphasize" or "throw-out" the measurements which have more noise than estimation information, [4]. (Note: A result such as Theorem 10 does not exist for actuators. In fact it will be shown later that adding actuators can degrade performance (Likewise deleting actuators can help).) As a result of this property of the Kalman filter, it is not surprising that any sensor noise source (v_i) that is making a large contribution to the closed loop performance metric (i.e. large V_i^v) eminates from a sensor which is making an even *larger* contribution in estimation information! Based upon these points, the following definition for the i^{th} sensor value is offered.

$$V_i^{sen} \triangleq V_i^v \tag{6.1}$$

Large values of V_i^v indicate sensors which are providing a "large" amount of estimation information and are therefore critical to the performance of the closed loop system. Sensors with smaller values of V_i^v are providing less information and are therefore candidates for deletion.

In light of this discussion the following closed loop sensor selection problems and algorithms are defined.

Problem SSP-1: *Let ℓ and $\bar{\ell} < \ell$ be specified integers. From an admissible set of ℓ sensors find a reduced set of $\bar{\ell}$ sensors such that*

(6.2) $$\phi \triangleq \frac{V-V(\bar{\ell})}{V}$$

is minimized. Where $V(\bar{\ell})$ *is the system performance metric with only* $\bar{\ell}$ *sensors operating.*

The computations suggested to approximate the solution of problem SSP-1 are as follows:

Algorithm SSP-1: *Solve for* V_i^{sen}, $i=1, \ldots \ell$ *from* (6.1), (4.13) *and rearrange so that* $V_i^{sen} \geq \ldots \geq V_\ell^{sen}$. *Delete the sensor with the smallest value of* V_i^{sen} *if deletion of that sensor does not reduce the rank of* $[M^T, \ldots A^{T^{n-1}}M^T]$. *Solve the optimal* LQG *problem for the reduced number of sensors* (i.e. $M^T = [M_1, \ldots M_{\ell-1}]$, $V = \text{diag}\,[V_1, \ldots V_{\ell-1}]$). *Repeat the above cycle* $\ell-\bar{\ell}$ *times or until all remaining measurements are necessary for observability of* (A, M).

At present, no guarantee exists that algorithm SSP-1 is optimal. However, it can be shown (See Appendix B) that V_i^V (i.e. V_i^{sen}) is closely related to the switching functions of the extended-Chen-Seinfeld method for optimal selection of sensors, [1]. The switching functions are used to indicate the sensors which satisfy the necessary conditions for optimality [1]. (i.e. minimization of (6.2)). The above fact, together with the limited results presented in section 8 indicate that algorithm SSP-1 may in fact be optimal. Research is continuing in this area.

A second version of the closed-loop sensor selection problem is presented below.

Problem SSP-2: *Let* ℓ *and* $\bar{\phi}$ *be specified numbers. From an admissible set of* ℓ *sensors, find the smallest number,* $\bar{\ell}$, *of sensors such that*

$$\phi \leq \bar{\phi}$$

and observability of (A, M) *is maintained.*

The computations suggested to approximate the solutions to SSP-2 are as follows:

Algorithm SSP-2: *Solve for* V_i^{sen}, $i = 1, \ldots \ell$, *and rearrange so that* $V_i^{sen} \ldots \geq \ldots \geq V_\ell^{sen}$. *Delete the sensor with the smallest value of* V_i^{sen} *if deletion of that sensor does not effect observability of* (A, M). *Solve the optimal* LQG *problem for the reduced number of sensors* (i.e. $M^T = [M_1, \ldots M_{\ell-1}]$, $V = \text{diag}\,[V_1, \ldots V_{\ell-1}]$). *Repeat the above cycle until* ϕ *is no longer less than or equal to* $\bar{\phi}$ *or* (A, M) *is no longer observable. At this point the last set of sensors is the desired set of* $\bar{\ell}$ *sensors.*

Further research is required to establish the relative degree of optimality of Algorithm SSP-2.

7.0 The Sensor/Actuator Selection (SAS) Algorithm

The actuator selection problems and algorithms presented in section 5 assumed that no change takes place in the number, type, and location of the sensors. The sensor selection problems and algorithms presented in section 6 assumed that no change takes place in the number, type, and location of the actuators. In practical design problems one needs an algorithm for *combined* actuator and sensor selection. The fact that actuator and sensor selection are coupled is well evidenced by equations (4.9), (4.11) and (4.12-4.15). As a result of this coupling, it is necessary to have an algorithm which simultaneously solves problems of type (ASP-1, ASP-2) and (SSP-1, SSP-2). As our first attempt for generating such an algorithm it is suggested that the appropriate actuator algorithm and sensor algorithm of Sections 5 and 6 be solved simultaneously. More specifically, if it is desired to solve ASP-1 and SSP-1, algorithms ASP-1 and SSP-1 should be implimented simultaneously. The optimality of such an algorithm or the possibility of simplified variations of such an algorithm is currently under study.

8.0 Hoop-Antenna Example

The ASP-1 Algorithm of Section 6 was applied to a 26-state model of a Hoop-Antenna Satellite which has 12 actuators, 39 sensors, and 27 outputs. Data was also collected to provide some verification of the validity of algorithm SSP-1. For the sake of brevity, only the general details of the model are presented in (8.1). A complete model description is provided in Appendix C along with a schematic describing sensor and actuator locations.

$$\dot{x} = Ax + B(u + w); \quad x \epsilon R^{26}; \quad u \epsilon R^{12}; \quad w \epsilon R^{12}$$

$$y = Cx; \; y \epsilon R^{27}$$

$$V = \lim_{t \to \infty} E\{ \|y\|_Q^2 + \|u\|_R^2 \}; \; Q > 0, \; R > 0 \; (Q, \text{ and } R \text{ diagonal})$$

More specifically,

$$V = \lim_{t \to \infty} \left[\sum_{i=1}^{27} E\{y_i^2\} \frac{q_i}{y_{im}^2} + \sum_{i=1}^{12} E\{u_i^2\} \frac{r_i}{u_{im}^2} \right] \tag{8.1}$$

where,

$$Q_{ii} = \frac{q_i}{y_{im}^2}, \quad \begin{cases} y_{im} = \text{Max allowable value of } i^{th} \text{ output} \\ q_i = \text{dimensionless output weight} \end{cases}$$

$$R_{ii} = \frac{r_i}{u_{im}^2}, \quad \begin{cases} u_{im} = \text{Max allowable control effort for } i^{th} \text{ actuator} \\ r_i = \text{dimensionless control weight} \end{cases}$$

$$z = Mx + v; \; z \epsilon R^{39}$$

$$E\{w(t)\} = E\{v(t)\} = 0$$

$$E\{w(t)w^T(\tau) = W\delta(t-\tau); \; W > 0$$

$$E\{v(t)v^T(\tau)\} = V\delta(t-\tau); \; V > 0$$

$$E\{v(t)w^T(\tau)\} = 0, E\{x(t)w^T(\tau)\} = 0, \; E\{x(t)v^T(\tau)\} = 0, \; \tau \geq t$$

A closed loop analysis of (8.1) was performed and Q and R were chosen such that system specifications were met (i.e. $y_{im}^2 \geq E\{y_i^2\}$, i=1, ...27, $u_{im}^2 \geq E\{u_i^2\}$, i=1, ...12). Algorithm ASP-1 was then applied. Specifically, the actuators with negative values of V_i^{act} (and not necessary for controllability) were deleted one at a time. Figure 1a shows a graph of the total system cost versus the number of closed loop actuators as the algorithm progressed. The numbers above the bar graphs indicate deleted actuators. Figure 1b shows the total output cost V^y versus the number of closed loop actuators.

Note that the algorithm recommends a 6-actuator closed loop system and that this system is almost 25% better in terms of total cost and performance cost than the original 12-actuator system. It is important to note also that the control effort of each actuator in the reduced actuator system was still within its specifications.

As a partial check on the optimality of the ASP-1 algorithm, the system was iteratively operated with each *one* of the 12 actuators deleted while the 11 other actuators remained. The results are shown in Figure 2.

From Figure 2 it is apparent that deleting actuator 10 would be the optimal decision if only 1 actuator were to be deleted. This result agrees with the decision of Algorithm ASP-1 for one actuator (See Fig. 1). To check the complete optimality of Algorithm ASP-1 empirically, the analysis of Fig. 2 would have to be repeated for any 2 actuators deleted, any 3 actuators deleted, etc. until the entire set of possible combinations were exercised. Clearly, this brute force approach is, computationally, an undesirable alternative to Algorithm ASP-1.

The *in situ* contribution or "value" of the i^{th} actuator is given by V_i^{act}. However, the actual perturbation of the cost *after* the deletion of the i^{th} actuator will be labeled ΔV_i^{act}.

Mathematically speaking,

$$\Delta V_i^{act} \triangleq V(m-1) - V(m) \tag{8.2}$$

where $V(m-1)$ is the cost of the system operating without the i^{th} actuator and $V(m)$ is the cost of the system operating with all actuators.

Since we intend to use V_i^{act} as an approximation of ΔV_i^{act} in the actuator selection process, we wish to know how good the approximation is.

Figure 3 indicates that V_i^{act} tracks the actual ΔV_i^{act} with some bias. However, a more important feature of actuator costs would be

$$(8.4) \qquad (V_i^{act} \geq V_j^{act}) \Rightarrow (\Delta V_i^{act} \geq \Delta V_j^{act})$$

Verification of (8.4) would be sufficient for verifying the optimality of Algorithm ASP-1. By comparing the data of Figure 2 to the ordering of V_i^{act} with all actuators present, condition (8.4) was verified for the example problem.

Another interesting fact noticed in the data for this example was that if the deletion criteria of Algorithm ASP-1 was applied only to the original ranking of V_i^{act}, (i.e. a new ranking was not calculated after each deletion) the algorithm still retained the same six actuators as before. If this result is shown to be a property of V_i^{act}, Algorithm ASP-1 would become non-iterative and this would greatly reduce its computational burden. In summary, the results presented above indicate the usefulness and potential optimality of Algorithm ASP-1 for solving the noisy actuator selection problem.

In addition to applying ASP-1 to the hoop-antenna model, some verification data was collected for Algorithm SSP-1. The data consisted of 7 simulations with only 38 of the 39 sensors acting (See Appendix C) and a different sensor deleted for each simulation. Table 3 shows the original V_i^V ranking of the seven sensors when all 39 sensors were acting.

Sensor #	Ranking
10	1
37	13
38	14
34	15
29	17
14	28
1	34

Table 3: V_i^V ranking

Figure 4 is a plot of $\Delta V = V - V(\ell-1)$ and V_i^V versus the deleted sensor where $V(\ell-1)$ is the value of the performance metric with only 38 sensors acting.

Figure 4 indicates that V_i^v is a good prediction of the increase in the performance metric that results when the i^{th} sensor is deleted. Comparing Figure 4 to Table 3 the following observation can be made

$$(8.5) \qquad V_i^v \geq V_j^v \Rightarrow \Delta V_i \geq \Delta V_j$$

If (8.5) can be shown to be a generic property of the SSP-1 algorithm, it would be sufficient to prove the optimality of the Algorithm.

9.0 Conclusion

This paper has defined Input Cost Analysis (ICA) and Output Cost Analysis (OCA) and shown the basic properties of each. Algorithms using ICA and OCA in an open loop setting were then presented and they provided an optimal solution to the open-loop input and output selection problems posed in Sections 2 and 3. In Section 4, closed loop versions of ICA and OCA were developed and their properties discussed. Selection of noisy actuators was discussed in Section 5.0 and algorithms using closed-loop ICA and OCA were suggested for the solution of ASP-1 and ASP-2. In Section 6 it was shown that adding sensors *cannot* degrade LQG performance. Noisy sensor selection problems were defined and algorithms using closed-loop OCA and ICA were posed for the solution of SSP-1 and SSP-2. In Section 7, a combined sensor and actuator selection algorithm was suggested. The noisy actuator selection algorithm, ASP-1, of Section 5, was applied to a model of a hoop antenna satellite in Section 8. The results indicated that the system performed better with a *fewer* number of noisy actuators. Research is continuing in these areas. In addition, data was also presented in Section 8 which supported the optimality of algorithms SSP-1 and SSP-2. Both empirical and analytical research on the optimality of the algorithms posed in this paper are continuing with particular focus on the combined algorithms suggested in Section 7.

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Appendix A: Proofs

Proof of Theorem 2:

For $u \equiv 0$, it can be shown [4] that the V defined in (1.1) is equal to value given by (1.10), (1.11). Using the fact that trAB = trBA, (1.10) becomes

(A1) $$V = tr[SBWB^T] = tr[B^TSBW]$$

But, from (1.12)

(A2) $$\sum_{i=1}^{m} V_i^W = \sum_{i=1}^{m} [B^TSBW]_{ii} = tr[B^TSBW]$$

Therefore, from (A1) and (A2)

$$V = \sum_{i=1}^{m} V_i^W \qquad \#$$

Proof of Theorem 3:

The expression (1.12) can be written as follows, if W is diagonal,

(A3) $$V_i^W = b_i^TSb_iW_{ii} \quad (b_i = i^{th} \text{ col. of } B)$$

The stability of A guarantees that S is the unique and at least positive semi-definite solution of the Lyapunov equation (1.11). Therefore, since B is of maximal rank (i.e. $\|b_i\| > 0$ for all i = 1, ... m) and since $W > 0$ implies $W_{ii} > 0$, then (A3) cannot be negative and the first part of the theorem is proved.

The observability of (A, C) guarantees that S is positive definite. Hence (A3) is strictly positive and the theorem is proved. #

Proof of Theorem 4:

Without loss of generality but with considerably more ease in representation, assume the last input, w_m, is to be deleted. The following partitions are defined:

(A4) $$B = [B_R\ b_m];\ B_R \in R^{nx(m-1)};\ b_m \in R^{nx1}$$

(A5) $$W = \begin{bmatrix} W_R & w_{Rm} \\ w_{Rm}^T & w_{mm} \end{bmatrix} ;\ W_R \in R^{(m-1)x(m-1)};\ w_{Rm} \in R^{m-1x1};\ w_{mm} \in R^1$$

using (A4) and (A5), (1.12) can be written as

(A6) $$V = tr[SBWB^T] = tr[SB_RW_RB_R^T + Sb_mw_{Rm}^TB_R^T + SB_Rw_{Rm}b_m^T + Sb_mw_{mm}b_m^T]$$

after deletion of the m^{th} input,

(A7) $$V(m-1) = tr[SB_RW_RB_R^T]$$

and

(A8) $$V_m^w = [B^TSBW]_{mm} = b_m^TSB_Rw_{Rm} + b_m^TSb_mw_{mm}$$

note that (A8) can be written as follows:

(A9) $$V_m^w = tr[SB_Rw_{Rm}b_m^T + Sb_mw_{mm}b_m^T]$$

using (A6) and (A7),

(A10) $$\Delta V_m^w \triangleq V-V(m-1)=tr[Sb_mw_{Rm}^TB_R^T]+tr[SB_Rw_{Rm}b_m^T]+tr[Sb_mw_{mm}b_m^T]$$

substituting (A9) into (A10) yields:

(A11) $$\Delta V_m^w = tr[Sb_mw_{Rm}^TB_R^T] + V_m^w$$

Now, using $tr(AB) = tr(A^T B^T)$ and the fact that S is symmetric, (A11) becomes

$$\text{(A12)} \qquad \Delta V_m^W = tr[SB_R w_{Rm} b_m^T] + V_m^W$$

Adding and subtracting $tr[Sb_m w_{mm} b_m^T]$ to (A12) results in:

$$\text{(A13)} \qquad \Delta V_m^W = tr[SB_R w_{Rm} b_m^T + Sb_m w_{mm} b_m^T] + V_m^W - tr[Sb_m w_{mm} b_m^T]$$

using (A9) and rewriting $tr[Sb_m w_{mm} b_m^T] = b_m^T Sb_m w_{mm}$ gives

$$\text{(A14)} \qquad \Delta V_m^W = 2V_m^W - b_m^T Sb_m w_{mm}$$

letting m = i in (A14) completes the proof of (2.3)

For Proof of (2.4) look again at (A11)

$$\text{(A15)} \qquad \Delta V_m^W = tr[Sb_m w_{Rm}^T B_R^T] + V_m^W$$

Where $w_{Rm} = 0$ if W is diagonal. Hence (A15) leads immediately to (2.4).

for Proof of (2.5),

If W is diagonal, the expression for V_i^W becomes:

$$\text{(A16)} \qquad V_i^W = \sum_{j=1}^{m} b_i^T Sb_j W_{ji} = b_i^T Sb_i W_{ii}$$

since S <u>does</u> <u>not</u> depend on B, (A16) implies that the input cost for the i^{th} input *is dependent strictly upon the* i^{th} *column of the* B *matrix and the* i^{th} *diagonal entry of* W. Therefore, no matter how many inputs are deleted, the cost of the i^{th} input in the reduced system will *not* change. Therefore, invoking Theorem 2 for the reduced input system

$$V(\bar{m}) = \sum_{i \varepsilon R} V_i^{w}$$ where R is the set of $\bar{m}$ inputs now acting on the system.

#

Proof of Theorem 5:

For the transformed system,

$$\dot{x} = T^{-1}ATx + T^{-1}Bw$$

(A17) $$y = CTx$$

$$V = \lim_{t \to \infty} E\{\| y \|_Q\}, \quad Q > 0$$

the input cost is

(A18) $$V_i^{w} = [B^T T^{-T} K T^{-1} B W]_{ii} \quad \text{(from 1.12)}$$

where from (1.11)

(A19) $$KT^{-1}AT + T^T A^T T^{-T} K + T^T C^T QCT = 0$$

post multiply by T^{-1} and premultiply by T^{-T} in (A19)

(A20) $$T^{-T} K T^{-1} A + A^T T^{-T} K T^{-1} + C^T QC = 0$$

Because A is stable, (A20) and (1.11) both have a unique solution which in (1.11) was defined to be S. Therefore,

(A21) $$S = T^{-T} K T^{-1}$$

or

(A22) $$K = T^T S T$$

substituting (A22) into (A18) gives:

(A23) $$V_i^W = [B^T T^{-T} T^T S T T^{-1} B W]_{ii} = [B^T S B W]_{ii}$$

which is identical to V_i^W in (1.12), and the proposition is proved. #

Outline of Proof of Theorem 6:

The most straight forward proof of the validity of (4.8 and 4.10) is to substitute them into (4.3 and 4.6) respectively and then multiply out the partitioned forms of the matrices which are given in (1.13). The details are omitted.

Proof of Theorem 7:

For diagonal W and V, equations (4.12) and (4.13) can be rewritten as follows:

(A24) $$V_i^W = b_i^T(K+L)b_i W_{ii}, \quad i=1, \ldots m$$

(A25) $$V_i^V = f_i^T L f_i V_{ii} \quad (f_i = Pm_i V_{ii}^{-1}), \; m_i = i^{th} \text{ col. of } M^T$$

and P is defined by (1.13).

The conditions (4.0) when applied to the closed loop system (1.13) guarantee that K is positive definite and L defined by (4.9) is at least positive semi-definite. [4] Therefore, since $W > 0$ and $V > 0$ (A24) can never be zero or negative and (A25) can never be negative. The matrix L defined in (4.9) will be positive definite if the pair (A-FM, G) is observable. Hence, with this condition, (A25) can never be zero or negative and the theorems proof is complete. It should be noted that the full rank of B and M implies $\|b_i\| \neq 0$ and $\|f_i\| \neq 0$.

Proof of Theorem 8:

The proof is the exact dual (Table 2) of the proof of Theorem 7.

Proof of Theorem 9:

Since the closed loop system of 1.13b is in the same form as 1.1 the proof of Theorem 5 and its corollary applies directly to Theorem 9.

Proof of Theorem 10:

From [4] it is known that the closed loop performance metric V (4.1) of (1.13) can be expressed as follows:

(A26) $$V = tr[KBWB^T + PG^TRG]$$

Now let V_+ equal the system performance metric for the system operating with one additional sensor. Therefore,

(A27) $$V_+ = tr[KBKB^T + P_+G^TRG]$$

where

(A28) $$P_+A^T + AP_+ - P_+M_+{}^TV_+{}^{-1}M_+P_+ + BWB^T = 0$$

(A29) $$M_+ = \begin{bmatrix} M \\ m^T \end{bmatrix} \quad ; m\varepsilon R^n$$ (i.e. added column of M^T matrix)

(A30) $$V_+ = \begin{bmatrix} V & 0 \\ 0 & v_+ \end{bmatrix} \quad ; v_+\varepsilon R^{1x1}$$ (i.e. variance of new sensor noise)

Subtracting (A27) from (A26) gives the following:

(A31) $$\Delta V \triangleq V-V_+ = tr[(P-P_+)G^TRG]$$

Equation (A31) can be rewritten as follows:

(A32) $$\Delta V = \mathrm{tr}[(P-P_+)G^TRG] = \mathrm{tr}[\sqrt{RG}(P-P_+)\sqrt{RG}^T]$$

Therefore, if $(P-P_+)$ is at least positive semi-definite the theorem is proved.

Recall that the matrix P used here and in (1.13) is defined by the following:

(A33) $$PA^T + AP - PM^TV^{-1}MP + BWB^T = 0$$

Now, subtracting (A28) from (A33) gives:

(A34) $$(P-P_+)A^T + A(P-P_+) - PM^TV^{-1}MP + P_+M_+{}^TV_+{}^{-1}M_+P_+ = 0$$

adding $\pm\ PM_+{}^TV_+{}^{-1}M_+P_+$ to (A34) yields:

(A35) $$(P-P_+)(A^T-M_+{}^TV_+{}^{-1}M_+P_+) + A(P-P_+) - PM^TV^{-1}MP + PM_+{}^TV_+{}^{-1}M_+P_+ = 0$$

adding $\pm\ P_+M_+{}^TV_+{}^{-1}M_+P_+$, $\pm\ P_+M_+{}^TV_+{}^{-1}M_+P$ to (A35) results in:

(A36) $$(P-P_+)(A^T-M_+{}^TV_+{}^{-1}M_+P_+) + (A-P_+M_+{}^TV_+{}^{-1}M_+)(P-P_+) - PM^TV^{-1}MP + PM_+{}^TV_+{}^{-1}M_+P_+ + P_+M_+{}^TV_+{}^{-1}M_+(P-P_+) = 0$$

adding $\pm\ Pmv_+{}^{-1}m^TP$ to (A36), making use of (A29) and (A30) and the definition $F_+ = P_+M_+{}^TV_+{}^{-1}$ gives:

(A37) $$(P-P_+)(A-F_+M_+)^T + (A-F_+M_+)(P-P_+) - PM_+{}^TV_+{}^{-1}M_+(P-P_+) + P_+M_+{}^TV_+{}^{-1}M_+(P-P_+) + Pmv_+{}^{-1}m^TP = 0$$

collecting terms gives:

(A38) $$(P-P_+)(A-F_+M_+)^T + (A-F_+M_+)(P-P_+) - (P-P_+)M_+{}^TV_+{}^{-1}M_+(P-P_+) + Pmv_+{}^{-1}m^TP = 0$$

Equation (A38) is a standard matrix Ricatti eq. It is well known, [4], that the soln. to (A38) (i.e. $P-P_+$) is at least positive semi-definite if the matrix $(A-F_+M_+)$ is stable. The matrix $(A-F_+M_+)$ will be stable if the matrix pair (A, B) is controllable and the pairs (A, C) and (A, M^+) are observable. From the conditions (4.0) (A, B) is controllable and (A, C) and (A, M) are observable. Therefore (A, M_+) must be observable since adding a Row to M (i.e. generating M_+ cannot effect the observability of (A, M). #

Appendix B: $\nu_i^{\,v}$ and the Chen-Seinfeld Switching-Function

In [1] it was shown that the switching function for the extended Chen-Seinfeld method of optimal selection of sensors in systems of type (1.13) was

$$\text{(B1)} \qquad \sigma_{c_i}^{\,S} = \mathrm{tr}(\hat{P}m_i V^{-1} m_i^{\,T}\hat{P}\Lambda_2)$$

where m_i is a column of M^T and Λ_2 is defined by:

$$\text{(B2)} \qquad \Lambda_2(A-\hat{P}M^TV^{-1}M) + (A^T-M^TV^{-1}M\hat{P})\Lambda_2 + KBR^{-1}B^TK = 0$$

where $\hat{P}$ is defined by:

$$\text{(B3)} \qquad \hat{P}A^T + A\hat{P} - \sum_{i-1}^{\ell} q_i\hat{P}m_iV^{-1}m_i\hat{P} + BWB^T = 0$$

and

$$q_i = \begin{cases} 1 \text{ if } i^{th} \text{ sensor is to be used} \\ 0 \text{ if } i^{th} \text{ sensor is to be deleted} \end{cases}$$

The expression for $\nu_i^{\,v}$ is given by (4.3), using $F = PM^TV^{-1}$ gives,

$$\text{(B4)} \qquad \nu_i^{\,v} = [V^{-1}MPLPM^T]_{ii} = V_{ii}^{\,-1}m_i^{\,T}PLPm_i$$

where

$$\text{(B5)} \qquad L(A-PM^TV^{-1}M) + (A-PM^TV^{-1}M)^TL + KBR^{-1}B^TK = 0$$

$$\text{(B6)} \qquad PA^T + AP - PM^TV^{-1}MP + BWB^T = 0$$

using the tr AB = tr BA (B1) becomes:

$$\text{(B7)} \qquad \sigma_{c_i}^{\,S} = \mathrm{tR}[V^{-1}m_i^{\,T}\hat{P}\Lambda_2\hat{P}m_i] = m_i^{\,T}\hat{P}\Lambda_2\hat{P}m_i\,\mathrm{tr}V^{-1}$$

comparing eqs. (B2), (B3) and (B7) to eqs. (B4), (B5) and (B6) it is apparent that, with the exception of the trace operation on V^{-1} in (B7), V_i^V is equivalent to calculating $\sigma_{c_i}^S$ for the system with *all* admissible measurements present. (i.e. all $q_i = 1$ in (B3)).

Appendix C: Hoop Antenna Model

Figure C-1 is a schematic of the hoop antenna model (8.1).

Table C-1 describes the actuator types and locations.

Actuator (Torquer) #	Location	Direction of Torque
1	2	X
2	2	Y
3	2	Z
4	6	X
5	6	Y
6	6	Z
7	9	X
8	9	Y
9	9	Z
10	10	X
11	10	Y
12	10	Z

Table C-1: Actuator description

Table C-2 describes the sensor types and locations.

Sensor #	Type	Location	Direction
1	Inertial Angle	2	X
2	"	"	Y
3	"	"	Z
4	Relative Linear Disp. Between	6 and 2	X
5	"	"	Y
6	"	"	Z
7	"	9 and 2	X
8	"	"	Y
9	"	"	Z
10	"	10 and 2	X
11	"	"	Y
12	"	"	Z
13	Inertial Angle	10	X
14	"	"	Y
15	"	"	Z
16	Relative Linear Displ. between	101 and 10	X
17	"	"	Y
18	"	"	Z
19	"	107 and 10	X
20	"	"	Y
21	"	"	Z
22	"	113 and 10	X
23	"	"	Y
24	"	"	Z
25	"	119 and 10	X
26	"	"	Y
27	"	"	Z
28	Inertial Angular Rate	2	X
29	"	"	Y
30	"	"	Z
31	"	6	X
32	"	"	Y
33	"	"	Z
34	"	9	X
35	"	"	Y
36	"	"	Z
37	"	10	X
38	"	"	Y
39	"	"	Z

Table C-2: Sensor description

Table C-3 describes the output types and locations (the Y vector in (8.1).)

Output #	Type	Location	Direction
Y_1	Inertial Angle	2	X
Y_2	"	"	Y
Y_3	"	"	Z
Y_4	Relative Angle Between	10 and 2	X
Y_5	"	"	Y
Y_6	Inertial Angle	10	Z
Y_7	Relative Linear Displ. Between	6 and 2	X
Y_8	"	"	Y
Y_9	"	"	Z
Y_{10}	"	9 and 2	X
Y_{11}	"	"	Y
Y_{12}	"	"	Z
Y_{13}	"	10 and 2	X
Y_{14}	"	"	Y
Y_{15}	"	"	Z
Y_{16}	"	101 and 10	X
Y_{17}	"	"	Y
Y_{18}	"	"	Z
Y_{19}	"	107 and 10	X
Y_{20}	"	"	Y
Y_{21}	"	"	Z
Y_{22}	"	113 and 10	X
Y_{23}	"	"	Y
Y_{24}	"	"	Z
Y_{25}	"	119 and 10	X
Y_{26}	"	"	Y
Y_{27}	"	"	Z

Table C-3: Output description

The contents of the matrices (A, B, C, Q, M, R, W, V) specified in (8.1) are described below.

(C1a)
$$A = \left[\begin{array}{cccc} 0 & I_{10} & \multicolumn{2}{c}{0} \\ -\omega^2 & -2\zeta\omega & & \\ \multicolumn{2}{c}{\multirow{2}{*}{0}} & 0 & I_3 \\ & & \underbrace{0 \quad}_{} & \underbrace{\quad 0}_{} \end{array}\right] \begin{array}{l} \left.\begin{matrix} \\ \\ \end{matrix}\right\} 20 \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} 6 \end{array}, \qquad B = \begin{array}{cc} 12 & \\ \left[\begin{array}{c} 0 \\ BE \\ 0 \\ BR \end{array}\right] & \begin{array}{c} 10 \\ 10 \\ 3 \\ 3 \end{array} \end{array}$$

(the last two block columns of A together span 6)

ω^2 = diag [.40579, 7.2090, 7.2362, 13.277, 44.834, 132.14, 142.66, 445.01, 448.69, 775.86] (radians^2/sec^2)

(C1b) I_{10} = 10x10 Identity Matrix, I_3 = 3x3 Identity Matrix

$2\zeta\omega$ = diag [.0127, .053699, .0538, .07286, .26283, .45981, .47777 , .84381, .8473, 1.1142] (radians/sec)

(C1c)
$$M = \begin{array}{cccc} 10 & & 3 & \\ \multicolumn{4}{c}{\left[\begin{array}{cccc} ME & 0 & MR & 0 \\ 0 & MER & 0 & MRR \end{array}\right]} \\ & 10 & & 3 \end{array}, \qquad C = 27 \begin{array}{cccc} 10 & 10 & 3 & 3 \\ [CE & 0 & CR & 0] \end{array}$$

The contents of the above defined submatricies are listed on the next several pages.

THE BE MATRIX (10 BY 12)

	1	2	3	4	5	6
1	-1.5980E-12	2.0391E-11	-1.1045E-02	-1.5494E-12	2.4122E-12	-1.0918E-02
2	9.8045E-10	1.6688E-03	-4.9651E-12	9.8565E-10	1.6354E-03	-1.7175E-12
3	-1.6599E-03	7.6073E-10	-1.0812E-12	-1.6433E-03	1.1351E-09	1.4252E-12
4	-4.6650E-14	1.2761E-11	-1.3452E-02	-1.1506E-14	-8.4481E-12	4.0239E-03
5	3.9018E-14	6.8600E-12	-7.2776E-03	-1.0655E-14	-4.5237E-12	2.4678E-02
6	-1.9443E-10	7.6551E-03	9.1886E-13	9.8361E-12	-3.2844E-05	2.6411E-13
7	-7.3458E-03	-4.9316E-10	1.6507E-13	-3.5997E-04	2.0989E-10	-1.1448E-13
8	-2.0518E-11	2.8388E-03	3.1986E-12	2.2558E-12	-1.4116E-03	5.5322E-15
9	-2.1316E-03	5.1427E-11	8.4627E-15	8.0639E-04	-3.4740E-11	5.2626E-15
10	1.1447E-10	-1.4643E-02	-2.2370E-11	-8.9715E-14	1.0167E-02	2.5578E-15

	7	8	9	10	11	12
1	-2.4578E-12	1.7996E-11	-1.0738E-02	-3.6592E-12	2.3622E-11	-1.0775E-02
2	-7.7906E-10	-1.3052E-03	-4.6410E-13	-2.8930E-09	-4.8232E-03	-4.9025E-13
3	1.3046E-03	-8.1944E-10	-2.8724E-13	4.8327E-03	-2.8561E-09	-6.8225E-13
4	-2.9036E-14	2.2303E-12	2.1881E-02	-1.6287E-13	-9.8949E-13	2.4690E-02
5	-9.2454E-14	1.2231E-12	-8.3798E-03	-4.5523E-13	-6.5593E-13	-1.3609E-02
6	7.6056E-11	-3.3330E-03	5.4994E-15	-7.2756E-11	3.1574E-03	-3.9383E-14
7	3.6726E-03	1.9284E-11	-1.0802E-13	-3.2853E-03	-3.6427E-11	3.7379E-14
8	-3.9170E-13	5.5245E-04	-3.3571E-15	-1.0075E-13	9.1035E-04	9.1567E-16
9	-2.3104E-05	9.4759E-12	3.6510E-14	-1.2039E-03	-4.6522E-12	1.2247E-15
10	2.2502E-14	-9.3469E-03	7.4532E-15	1.3914E-14	4.5114E-03	-1.4097E-15

THE BR MATRIX (3 BY 12)

	1	2	3	4	5	6
1	-6.5117E-04	8.8738E-08	3.3814E-11	-6.5118E-04	8.9906E-08	3.3492E-11
2	8.9529E-08	6.5131E-04	2.3734E-10	8.9528E-08	6.5122E-04	2.3745E-10
3	7.7548E-13	-1.2353E-11	6.0894E-03	7.6575E-13	-3.0140E-12	6.1336E-03

	7	8	9	10	11	12
1	-7.1776E-04	9.8472E-08	3.1958E-11	-7.9365E-04	1.0907E-07	3.2475E-11
2	9.8675E-08	7.1774E-04	2.3657E-10	1.0910E-07	7.9356E-04	2.3681E-10
3	4.3753E-13	-5.2949E-12	6.2153E-03	1.6303E-14	-6.8254E-13	6.1993E-03

HE CE

	1	2	3	4	5
1	-1.5980E-12	9.8045E-10	-1.6539E-03	-4.6550E-14	3.9014E-14
2	2.0391E-11	1.6688E-03	7.5073E-10	1.2781E-11	6.4877E-12
3	-1.1045E-02	-4.9651E-12	-1.7812E-12	-1.3452E-12	-7.2776E-03
4	-2.0612E-12	-3.8735E-09	6.4926E-13	-1.1622E-13	-5.7425E-13
5	3.2310E-12	-6.4920E-03	-3.6168E-09	-1.3770E-11	-7.5359E-12
6	-1.0775E-02	-4.9025E-13	-6.8225E-13	2.4590E-02	-1.3639E-02
7	1.1011E-10	2.3764E-02	1.4877E-04	-3.4696E-11	-1.8537E-11
8	2.2648E-11	-1.4311E-08	2.3764E-02	3.7395E-13	-1.3531E-13
9	-1.8000E-14	6.8493E-10	-7.8900E-15	3.4700E-15	7.2520E-15
10	3.7640E-10	4.3917E-02	2.6324E-08	-2.5765E-12	-2.9303E-13
11	6.1152E-11	-2.6451E-08	4.4763E-02	5.3362E-13	-2.6354E-13
12	-3.7000E-14	-3.3290E-12	6.4320E-14	9.5700E-15	1.6079E-14
13	7.3325E-10	-1.5985E-02	-9.4022E-09	-1.2017E-12	-7.7871E-13
14	1.1462E-10	9.4569E-09	-1.5939E-02	2.4442E-12	5.3315E-12
15	-5.1000E-14	-3.3120E-12	5.5790E-14	1.3350E-14	2.4573E-14
16	6.2644E-03	3.7333E-02	4.4234E-05	-2.6232E-04	2.7193E-03
17	-1.0850E-02	4.8142E-06	3.7289E-02	4.5439E-04	-4.7082E-03
18	4.1555E-08	1.7946E-02	1.0356E-02	-1.4920E-08	-1.8021E-08
19	1.0850E-02	3.7330E-02	-4.8139E-06	-4.5439E-04	4.7082E-03
20	6.2644E-03	-4.4221E-05	3.7293E-02	-2.6232E-04	2.7193E-03
21	4.1399E-08	-1.0361E-02	1.7938E-02	-1.4919E-08	-1.8021E-08
22	-4.1128E-04	3.7247E-02	-6.0024E-05	-3.4072E-04	-5.1321E-03
23	7.1243E-04	-6.1945E-06	3.7214E-02	5.9011E-04	8.8890E-03
24	5.7982E-08	-7.7130E-03	-4.4493E-03	-2.0848E-08	-2.5147E-08
25	-7.1243E-04	3.7256E-02	5.7730E-06	-5.9011E-04	-8.8890E-03
26	-4.1128E-04	6.4295E-06	3.7206E-02	-3.4072E-04	-5.1321E-03
27	5.8020E-08	4.4525E-03	-7.7274E-03	-2.0881E-08	-2.5165E-08

	6	7	8	9	10
1	-1.9443E-10	-7.3458E-03	-2.0518E-11	-2.1316E-03	1.1447E-10
2	7.6551E-03	-4.9316E-10	2.8388E-03	5.1427E-11	-1.4543E-02
3	9.1886E-13	1.6507E-13	3.1986E-12	8.4627E-15	-2.2370E-11
4	1.2167E-10	4.0605E-03	2.0337E-11	9.2770E-04	-1.1446E-10
5	-4.4977E-03	4.5673E-10	-1.9285E-03	-5.6289E-11	1.9154E-02
6	-3.9383E-14	3.7079E-14	9.1557E-16	1.2247E-15	-1.4097E-15
7	6.6063E-02	-6.3608E-10	7.4294E-03	-1.5739E-10	1.6830E-04
8	1.5715E-09	6.8980E-02	2.9092E-10	9.8576E-03	-2.0204E-09
9	3.1619E-09	-8.4540E-14	6.0273E-10	-8.2300E-14	-8.4909E-10
10	-9.7582E-03	1.9237E-10	-1.4196E-02	-2.4179E-12	4.4242E-03
11	-1.0049E-10	-8.8818E-03	2.8761E-10	-1.4117E-02	-2.0119E-09
12	-3.2403E-11	-2.0090E-14	-1.1865E-11	-3.5000E-15	5.2654E-11
13	6.5518E-03	-1.0065E-10	-1.2157E-03	-3.7776E-12	2.6514E-03
14	2.7943E-10	6.9694E-03	2.9173E-10	-8.2168E-04	-2.0122E-09
15	-3.2405E-11	-5.0430E-14	-1.1377E-11	-1.7610E-13	5.2645E-11
16	5.1493E-02	-5.1044E-05	7.3549E-03	6.4942E-04	4.4598E-03
17	-7.5693E-05	5.4072E-02	1.0799E-03	9.2917E-03	-1.0650E-04
18	5.1511E-02	3.0689E-02	6.5278E-03	4.2834E-03	-1.1151E-02
19	5.1524E-02	9.0562E-05	6.9553E-03	-1.0816E-03	4.6821E-03
20	4.1696E-05	5.4033E-02	-6.4726E-04	9.6874E-03	-9.3106E-06
21	-2.9739E-02	5.3157E-02	-3.7301E-03	7.4901E-03	6.5574E-03
22	5.1311E-02	-1.7229E-04	7.3096E-03	3.3921E-04	3.2006E-03
23	-2.0998E-04	5.3937E-02	5.7177E-04	9.3256E-03	5.7443E-04
24	-2.2569E-02	-1.3332E-02	-2.6076E-03	-1.5122E-03	8.5358E-03
25	5.1518E-02	1.9710E-04	7.1797E-03	-6.1042E-04	2.6757E-03
26	1.8871E-04	5.3750E-02	-3.0786E-04	9.4765E-03	-5.4280E-04
27	1.3029E-02	-2.3094E-02	1.4526E-03	-2.7153E-03	-5.1253E-03

THE CR MATRIX (27 BY 3)

	1	2	3
1	-6.5117E-04	8.9529E-08	7.7548E-1[illegible]
2	8.8738E-08	6.5131E-04	-1.2353E-11
3	3.3814E-11	2.3734E-10	6.04[illegible]
4	-1.4249E-04	1.9571E-08	-7.5914E-1[illegible]
5	2.0332E-08	1.4225E-04	1.1570E-11
6	3.2475E-11	2.3681E-10	6.1993E-1[illegible]
7	1.2908E-06	9.3852E-03	-8.77[illegible]
8	9.3833E-03	-1.2901E-06	-1.1053E-1[illegible]
9	-1.0300E-13	-2.0928E-10	7.0000E-1[illegible]
10	3.2372E-06	2.4006E-02	-2.3233E-1[illegible]
11	2.4304E-02	-3.3092E-06	-2.5191E-11
12	5.0000E-14	2.7900E-12	2.1000E-1[illegible]
13	5.1293E-06	3.6613E-02	-2.6876E-1[illegible]
14	3.6612E-02	-5.0334E-06	-2.8753E-11
15	3.0300E-14	2.7900E-12	2.9[illegible]
16	-3.9225E-06	-2.8101E-02	5.8961E-0[illegible]
17	-2.4101E-02	3.0125E-06	-1.0212E-0[illegible]
18	4.0575E-03	7.0266E-03	-1.8058E-0[illegible]
19	-3.8071E-06	-2.8101E-02	1.0212E-0[illegible]
20	-2.8101E-02	3.9265E-06	5.9961E-0[illegible]
21	7.0257E-03	-4.0581E-03	-1.7954E-0[illegible]
22	-3.6891E-06	-2.8166E-02	-3.4592E-0[illegible]
23	-2.8167E-02	4.0548E-06	6.0388E-0[illegible]
24	-1.7160E-03	-2.9717E-03	-2.5153E-0[illegible]
25	-4.0539E-06	-2.8166E-02	-6.[illegible]
26	-2.8167E-02	3.6969E-06	-3.4592E-0[illegible]
27	-2.9713E-03	1.7162E-03	-2.5195E-0[illegible]

HE ME MATRIX (27 BY 10)

	1	2	3	4	5
1	-1.5980E-12	9.8045E-10	-1.6599E-03	-4.6550E-14	3.9018E-14
2	2.0391E-11	1.6698E-03	7.6073E-10	1.2781E-11	6.8900E-12
3	-1.1045E-02	-4.9651E-12	-1.0812E-12	-1.3452E-02	-7.2776E-03
4	1.1011E-10	2.3764E-02	1.4977E-04	-3.4696E-11	-1.8577E-11
5	2.2681E-11	-1.4311E-08	2.3764E-02	3.7395E-13	-1.3531E-13
6	-1.8000E-14	6.8493E-10	-7.8900E-15	3.4700E-15	7.2520E-15
7	3.7640E-10	4.3917E-02	2.6324E-08	-2.5766E-12	-2.9333E-13
8	6.1152E-11	-2.6451E-08	4.4553E-02	5.3362E-13	-2.6354E-13
9	-3.7000E-14	-3.3290E-12	6.4320E-14	9.5700E-15	1.6079E-14
10	7.3325E-13	-1.5985E-02	-9.4922E-09	-1.2117E-12	-7.7871E-13
11	1.1462E-10	9.4569E-09	-1.5939E-02	2.4442E-12	5.3315E-12
12	-5.1000E-14	-3.3120E-12	5.5790E-14	1.6350E-14	2.4573E-14
13	-3.6592E-12	-2.8930E-09	4.8327E-03	-1.6287E-13	-4.6523E-13
14	2.3622E-11	-4.8232E-03	-2.8361E-09	-9.8949E-13	-6.5593E-13
15	-1.0775E-02	-4.9925E-13	-6.8225E-13	2.4590E-02	-1.3619E-02
16	6.2644E-03	3.7333E-02	4.4234E-05	-2.6232E-04	2.7183E-03
17	-1.0850E-02	4.8142E-06	3.7289E-02	4.5439E-04	-4.7082E-03
18	4.1555E-08	1.7946E-02	1.0356E-02	-1.4920E-08	-1.9321E-08
19	1.0850E-02	3.7330E-02	-4.8139E-06	-4.5439E-04	4.7092E-03
20	6.2644E-03	-4.4221E-06	3.7293E-02	-2.6232E-04	2.7183E-03
21	4.1399E-08	-1.0361E-02	1.7938E-02	-1.4919E-08	-1.8021E-08
22	-4.1128E-04	3.7247E-02	-6.0624E-05	-3.4072E-04	-5.1321E-03
23	7.1243E-04	-6.1945E-06	3.7214E-02	5.9011E-04	8.4490E-03
24	5.7942E-08	-7.7130E-03	-4.4493E-03	-2.0848E-08	-2.5147E-08
25	-7.1243E-04	3.7256E-02	5.7700E-06	-5.9011E-04	-8.4490E-03
26	-4.1128E-04	6.4295E-06	3.7236E-02	-3.4072E-04	-5.1321E-03
27	5.8020E-08	4.4525E-03	-7.7774E-03	-2.0881E-08	-2.5165E-08

	6	7	8	9	10
1	-1.9443E-10	-7.3458E-03	-2.0518E-11	-2.1316E-03	1.1447E-10
2	7.6551E-03	-4.9316E-10	2.8398E-03	5.1427E-11	-1.4643E-02
3	9.1886E-13	1.6507E-13	3.1986E-12	8.4627E-15	-2.2372E-11
4	6.6063E-02	-6.3608E-10	7.4294E-03	-1.5739E-10	1.5830E-04
5	1.5715E-09	6.8980E-02	2.9192E-10	9.8576E-03	-2.2234E-09
6	3.1619E-09	-8.4540E-14	6.0273E-10	-8.2300E-14	-9.4809E-10
7	-9.7582E-03	1.9237E-10	-1.4196E-02	-2.4179E-12	4.4242E-03
8	-1.0048E-10	-8.8818E-03	2.8761E-10	-1.4117E-02	-2.0119E-09
9	-3.2403E-11	-2.0090E-14	-1.1865E-11	-3.5300E-15	5.2554E-11
10	6.5518E-03	-1.0065E-10	-1.2157E-03	-3.7776E-12	2.6514E-03
11	2.7943E-10	6.9694E-03	2.9173E-10	-8.2168E-04	-2.0122E-09
12	-3.2405E-11	-5.0430E-14	-1.1977E-11	-1.7610E-13	5.2546E-11
13	-7.2756E-11	-3.2953E-03	-1.8075E-13	-1.2039E-03	1.3914E-14
14	3.1574E-03	-3.6427E-11	9.1035E-04	-4.8622E-12	4.5114E-03
15	-3.9383E-14	3.7779E-14	9.1567E-15	1.2247E-15	-1.4037E-15
16	5.1493E-02	-5.1044E-05	7.3548E-03	6.4842E-04	4.4588E-03
17	-7.5693E-05	5.4072E-02	1.0799E-03	9.2917E-03	-1.0657E-04
18	5.1511E-02	3.0689E-02	6.5278E-03	4.2934E-03	-1.1151E-02
19	5.1524E-02	9.0562E-05	6.9553E-03	-1.0916E-03	4.5421E-03
20	4.1596E-05	5.4033E-02	-6.4726E-04	9.5874E-03	-9.3136E-06
21	-2.9739E-02	5.3157E-02	-3.7301E-03	7.4991E-03	5.5504E-03
22	5.1311E-02	-1.7229E-04	7.3096E-03	3.3721E-04	3.2036E-03
23	-2.0998E-04	5.3937E-02	5.7177E-04	9.3265E-03	5.0883E-04
24	-2.2569E-02	-1.3332E-02	-2.6076E-03	-1.5122E-03	8.5758E-03
25	5.1518E-02	1.9710E-04	7.1797E-03	-6.1042E-04	2.5757E-03
26	1.3471E-04	5.3750E-02	-3.0086E-04	9.4965E-03	-5.4283E-04
27	1.3329E-02	-2.3094E-02	1.4526E-03	-2.7153E-03	-5.1253E-03

THE MR MATRIX (27 BY 3)

	1	2	
1	-6.5117E-04	8.9529E-08	7.7548E-[illegible]
2	8.8738E-08	6.5131E-04	-1.2353[illegible]
3	3.3814E-11	2.3734E-10	6.0494[illegible]
4	1.2308E-06	9.3852E-03	-8.0779[illegible]
5	9.3833E-03	-1.2901E-06	-1.10[illegible]
6	-1.0000E-13	-2.0928E-10	7.[illegible]
7	3.2972E-06	2.4006E-02	-2.3233[illegible]
8	2.4004E-02	-3.3002E-06	-2.6171[illegible]
9	5.0000E-14	2.7900E-12	2.1099[illegible]
10	5.0290E-06	3.6613E-02	-2.5975[illegible]
11	3.6612E-02	-5.0334E-06	-2.8753[illegible]
12	3.0000E-14	2.7900E-12	2.[illegible]
13	-7.9365E-04	1.0910E-07	1.5393[illegible]
14	1.0907E-07	7.9356E-04	-6.8259[illegible]
15	3.2475E-11	2.3681E-10	6.1993[illegible]
16	-3.9225E-06	-2.8101E-02	5.3451[illegible]
17	-2.8101E-02	3.8125E-06	-1.0212[illegible]
18	4.0575E-03	7.0266E-03	-1.8768[illegible]
19	-3.8071E-06	-2.8101E-02	1.0212[illegible]
20	-2.8101E-02	3.9265E-06	5.9451[illegible]
21	7.0257E-03	-4.0581E-03	-1.7958[illegible]
22	-3.6891E-06	-2.8166E-02	-3.4592[illegible]
23	-2.8157E-02	4.0548E-06	6.[illegible]
24	-1.7160E-03	-2.9717E-03	-2.5153[illegible]
25	-4.0539E-06	-2.8166E-02	-6.7498[illegible]
26	-2.9167E-02	3.6969E-06	-3.4592[illegible]
27	-2.9713E-03	1.7162E-03	-2.5195[illegible]

THE MER MATRIX (12 BY 10)

	1	2	3	4	5
1	-1.5980E-12	9.8[illegible]45E-10	-1.6599E-03	-4.6650E-14	3.9[illegible]18E-14
2	2.0391E-11	1.6698E-03	7.6[illegible]73E-10	1.2781E-11	6.440[illegible]E-12
3	-1.1045E-02	-4.9651E-12	-1.0912E-12	-1.3452E-02	-7.2776E-03
4	-1.5494E-12	9.8565E-10	-1.6433E-03	-1.1506E-14	-1.0665E-14
5	2.4122E-12	1.6354E-03	1.1361E-09	-8.4481E-12	-4.5237E-12
6	-1.0918E-02	-1.7175E-12	1.4252E-12	4.0239E-03	2.4678E-02
7	-2.4578E-12	-7.7906E-10	1.3[illegible]46E-03	-2.9[illegible]36E-14	-9.2454E-14
8	1.7996E-11	-1.3052E-03	-8.1944E-10	2.2303E-12	1.2231E-12
9	-1.0738E-02	-4.6410E-13	-2.9729E-13	2.1981E-02	-4.3778E-03
10	-3.6592E-12	-2.8930E-09	4.8327E-03	-1.6287E-13	-4.6523E-13
11	2.3622E-11	-4.8232E-03	-2.8561E-09	-9.4949E-13	-6.5593E-13
12	-1.0775E-02	-4.9025E-13	-6.8225E-13	2.4690E-02	-1.3609E-02

	6	7	8	9	10
1	-1.9443E-10	-7.3458E-03	-2.0518E-11	-2.1316E-13	1.1447E-10
2	7.6551E-03	-4.9316E-10	2.8388E-03	5.1427E-11	-1.4643E-02
3	9.1886E-13	1.6507E-13	3.1986E-12	8.4627E-15	-2.237[illegible]E-11
4	9.8361E-12	-3.5997E-04	2.2558E-12	9.0539E-04	-8.9715E-14
5	-3.2844E-05	2.0989E-10	-1.4116E-03	-3.4740E-11	1.0167E-02
6	2.6411E-13	-1.1448E-13	6.5322E-15	5.2626E-15	2.5578E-15
7	7.6056E-11	3.6726E-03	-3.9170E-13	-2.3104E-05	2.2502E-14
8	-3.3330E-03	1.9284E-11	5.5245E-04	9.4759E-12	-9.3469E-03
9	5.4994E-15	-1.0802E-13	-3.3571E-15	3.6510E-14	7.4532E-15
10	-7.2756E-11	-3.2853E-03	-1.8075E-13	-1.2039E-03	1.3914E-14
11	3.1574E-03	-3.6427E-11	9.1035E-04	-4.8622E-12	4.5114E-03
12	-3.9383E-14	3.7079E-14	9.1567E-16	1.2247E-15	-1.4097E-15

THE MRR MATRIX (12 BY 3)

	1	2	3
1	-6.5117E-04	8.9529E-08	7.7548E-13
2	8.8738E-08	6.5131E-04	-1.2353E-11
3	3.3914E-11	2.3734E-10	6.0894E-03
4	-6.5118E-04	8.9528E-08	7.6575E-13
5	8.9306E-08	6.5122E-04	-3.0140E-12
6	3.0492E-11	2.3745E-10	6.1339E-03
7	-7.1776E-04	9.8675E-08	4.3753E-13
8	9.8472E-08	7.1774E-04	-5.2949E-12
9	3.1958E-11	2.3657E-10	6.2153E-03
10	-7.9365E-04	1.0910E-07	1.6303E-14
11	1.0907E-07	7.9356E-04	-6.8254E-13
12	3.2475E-11	2.3681E-10	6.1993E-03

System Performance Requirements:

$$Y_{im} = .02°,\ i = 1, 6$$

(C2) $$Y_{im} = .0005 \text{ meters},\ i = 7, 8, \ldots 27$$

$$u_{im} = 1 \text{ n-m}$$

Weights:

$$q_i = 1,\ i = 1, 2, 4, 5$$

$$q_i = .01,\ i = 3, 6$$

(C3) $$q_i = .1,\ i = 7, 8, \ldots 27$$

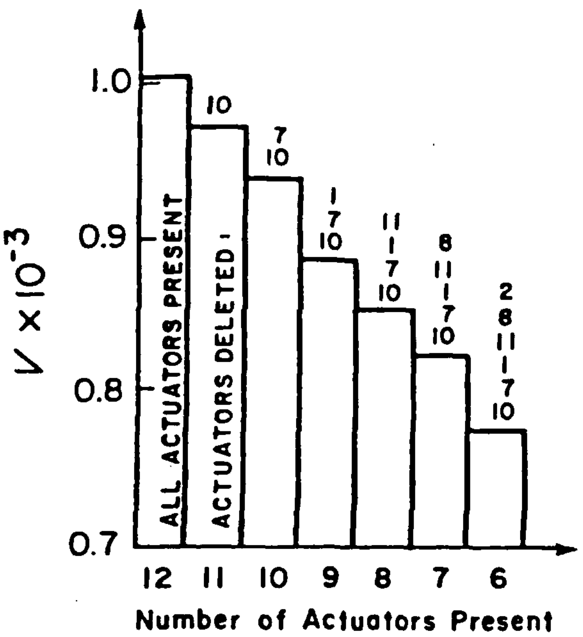
$$r_i = 10^{-5},\ i = 1, \ldots 12$$

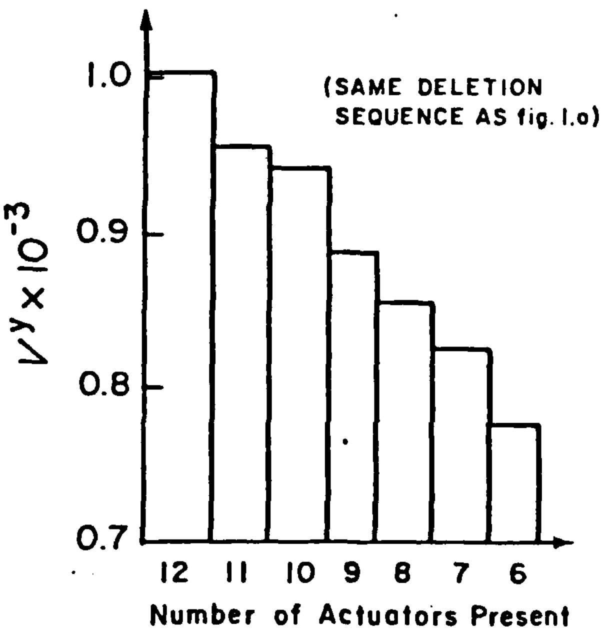
(C4) $$Q = \text{diag}\,[82.07, 82.07, .8207, 82.07, 82.07, .8207, \underbrace{400{,}000, \ldots\ldots 400{,}000]}_{21 \text{ elements}}\ 10^5$$

(C5) $$R = 10^{-5} I_{12} \quad , \quad W = 10^{-8} I_{12} \quad .$$

(C6) $$V = \begin{bmatrix} V_1 & & & & \\ & V_2 & & & \\ & & V_1 & & \\ & & & V_3 & \\ & & & & V_4 \end{bmatrix} \quad ; \quad \begin{aligned} V_1 &= 7.6154 \times 10^{-9} I_3 \\ V_2 &= 2.5 \times 10^{-7} I_9 \\ V_3 &= 2.5 \times 10^{-7} I_{12} \\ V_4 &= 4.7597 \times 10^{-15} I_{12} \end{aligned}$$

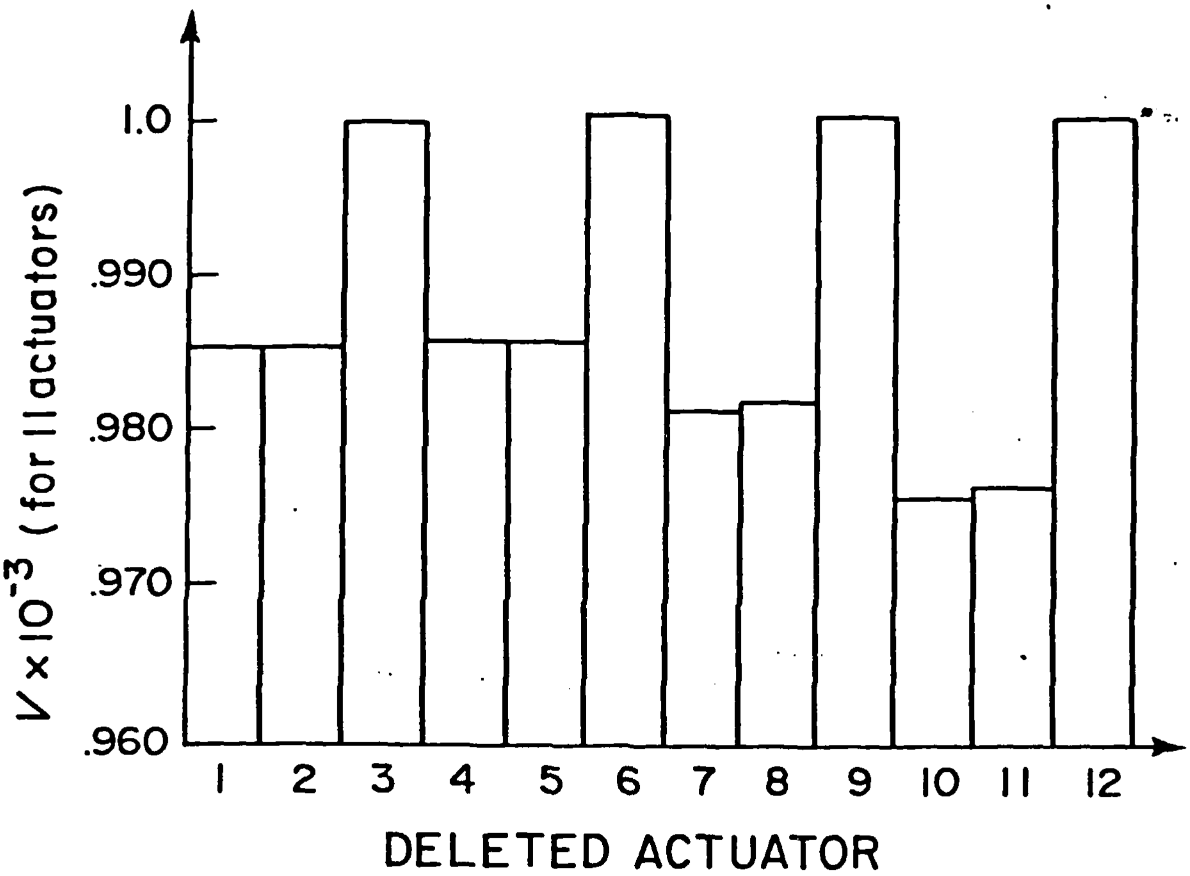
LIST OF FIGURES

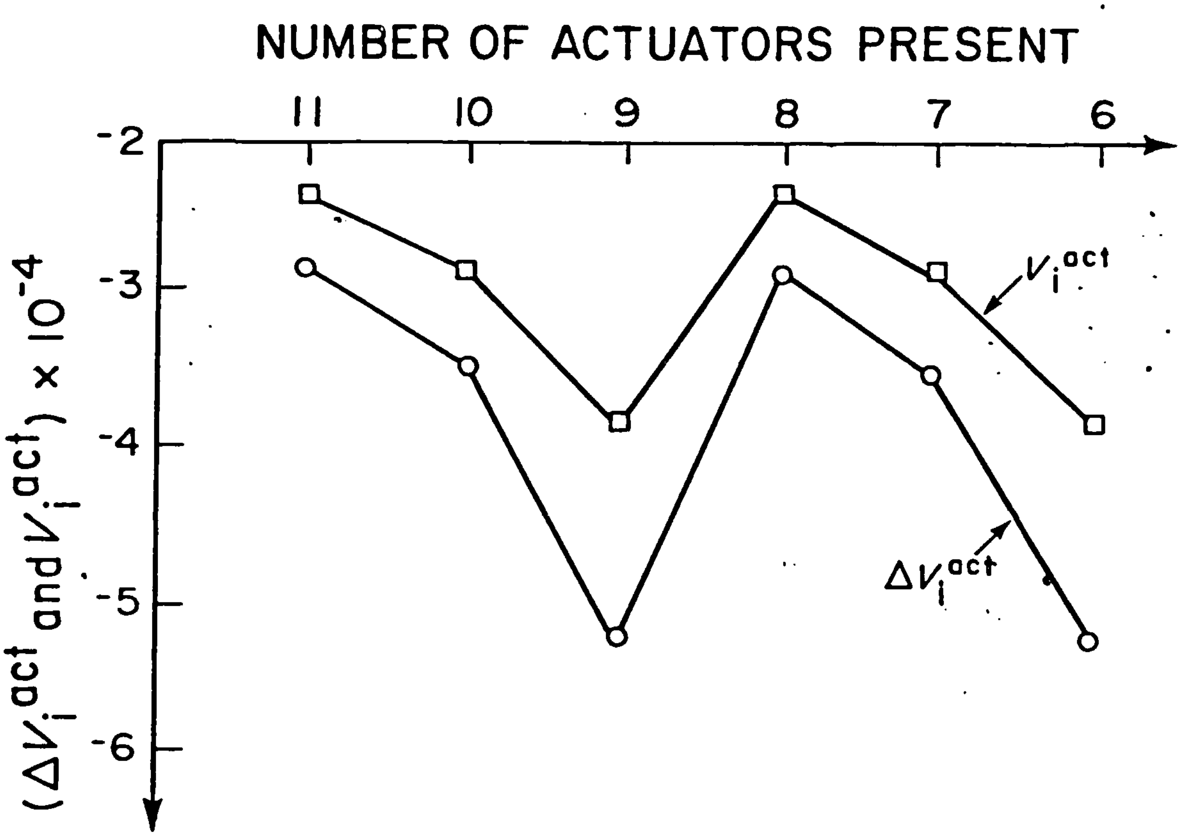

1.0
0.9
0.8
0.7
$V \times 10^{-3}$
ALL ACTUATORS PRESENT
ACTUATORS DELETED:
10
7 10
1 7 10
11 1 7 10
8 11 1 7 10
2 8 11 1 7 10
12 11 10 9 8 7 6
Number of Actuators Present


1.0
0.9
0.8
0.7
$V^y \times 10^{-3}$
(SAME DELETION SEQUENCE AS fig. 1.a)
12 11 10 9 8 7 6
Number of Actuators Present

·ig. 1A

Fig. 1B


1.0
.990
.980
.970
.960
$V \times 10^{-3}$ (for 11 actuators)
1
2
3
4
5
6
7
8
9
10
11
12
DELETED ACTUATOR


NUMBER OF ACTUATORS PRESENT
11
10
9
8
7
6
-2
-3
-4
-5
-6
$(\Delta V_i^{act}$ and $V_i^{act}) \times 10^{-4}$
V_i^{act}
ΔV_i^{act}

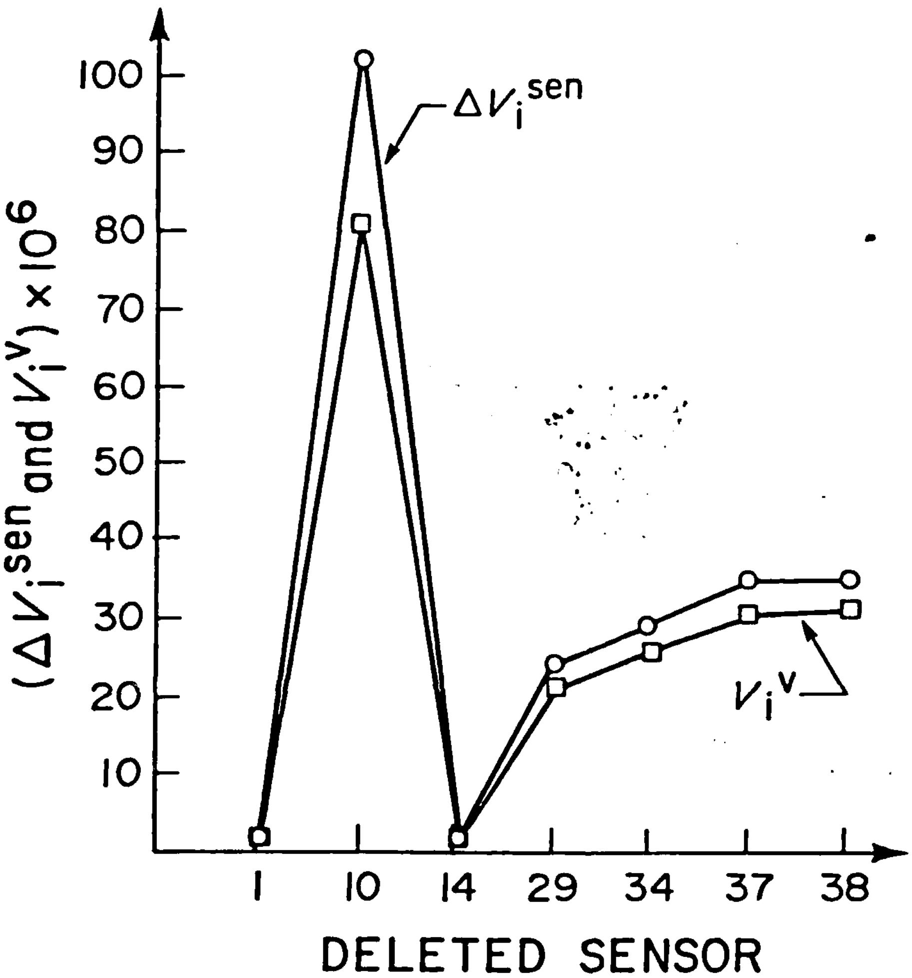

100
90
80
70
60
50
40
30
20
10
$(\Delta V_i^{sen} \text{ and } V_i^{v}) \times 10^6$
ΔV_i^{sen}
V_i^{v}
1
10
14
29
34
37
38
DELETED SENSOR

Fig 4

END

1-87

DTIC