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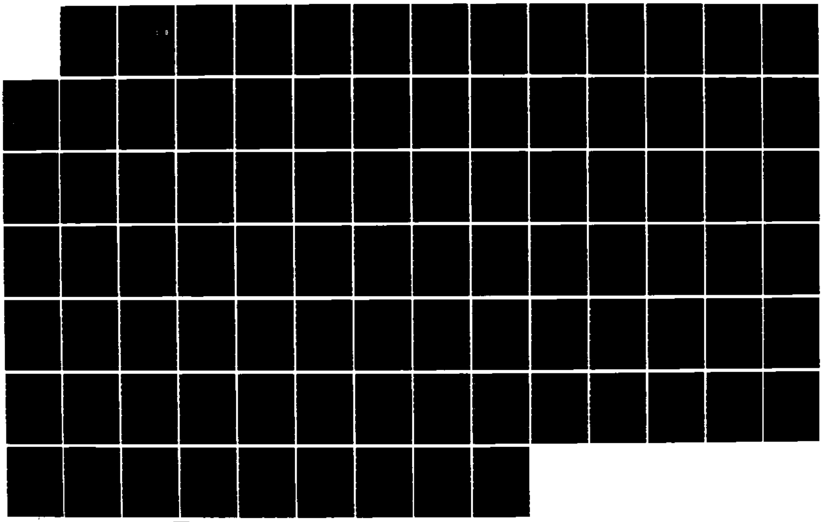

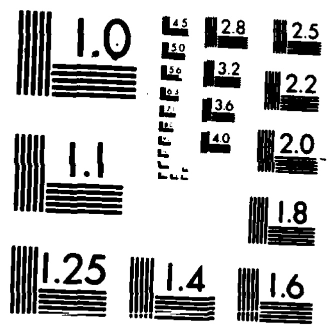

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Control Theory for Autonomously Guided Missile Platforms

Final Report
ARO Contract: DAAG29-83-C-0028

J.S. Baras G.L. Blankenship H.G. Kwatny

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Submitted by:

SYSTEMS ENGINEERING, INC.
7833 Walker Drive, Suite 308
Greenbelt, MD 20770

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P.O. Box 12211
Research Triangle Park, NC 27709-2211
Attn: Richard O. Ulsh

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This Final Report summarizes an effort to develop analytical and computational techniques for performance evaluation of Autonomously Guided Platforms with Multiple Sensors (AGPMS). A classical analysis of seeker pointing and tracking servomechanism performance for missile guidance problem is developed as a baseline for comparison with theoretical studies of nonlinear stochastic control laws.

(SEE REVERSE SIDE FOR CONTINUATION OF ABSTRACT.)

Continuation of Block 20 (Abstract):

Stochastic control theory is applied to the AGPMS problem. Analysis of time-scale separation of multiple control loops for AGPMS problem leads to effective simplifying control structures. Results of these studies include a substantially new theory for singular perturbation of stochastic control. This theory suggests significantly different control structures for AGPMS problem than suggested by the deterministic formulation of optimal control theory.

The problem of multiple target detection and tracking is considered as a problem of simultaneous detection and estimation of signals. We develop a set of sufficient statistics which can be computed recursively by linear equations. Examples in radar tracking and target discrimination are considered.

TABLE OF CONTENTS

Chapter 1

Introduction and Summary

1.1 Summary

The purpose of this study was to develop analytical and computational techniques for performance evaluation of Autonomously Guided Platforms with Multiple Sensors (AGPMS).

The fundamental principles used are: modeling of the many feedback sensors, modeling of the sensor data, advanced estimation and detection techniques, sensor scheduling problems, regulator theory and design, stochastic control techniques, careful analysis of multiple time scales.

When multiple sensors are present, such as radar, various types of IR sensors and others, one has to consider carefully the "fusion" of the data from the various sensors in a dynamically changing environment. These problems are essential in the success of the overall design and have not been investigated systematically before with dynamic signal models.

Design of tracking control loops for each sensor class is a stochastic control problem (not just a nonlinear filtering problem). When all loops are treated simultaneously, simplifications in the analysis and the resulting implementation occur when one exploits the different time scales present in the various feedback loops.

In addition, AGPMS must have an adaptive control-decision: sensors employed have diverse performance characteristics. This fact necessitates a careful analysis of sensor models and target representations in those sensor models.

The techniques and models used in our analysis are fairly sophisticated, vis-a-vis the classical treatment of these problems. In the classical treatment, one ignores the combined performance index for missile guidance and tracking loops which is the *miss distance at interception*, and instead one considers separately several subproblems:

(a) selection of guidance loop configuration,

(b) setting loop gains for steady state accuracy requirements,

(c) stabilization for acceptable "gain and phase margins", and

(d) study effects of noise and parametric uncertainties.

One iterates through this sequence of subproblems in the order described until a satisfactory design is achieved. This approach has many deficiencies. In this research, we exploit stochastic control and estimation to study several interrelated problems.

In Chapter 2, we consider the design of pointing and tracking servomechanisms for a seeker using an imaging FLIR with a gimbaled platform from a more or less conventional perspective. We specifically consider the application of classical, single-input single-output servo theory and the extended Kalman filter techniques. Our intent is to establish a basis for meaningful comparison of the performance improvement achieved with the nonlinear stochastic control theory which is the main subject of this research project. Performance objectives for these systems are stated primarily in classical terms, and it is essential to fully appreciate their intent and their implications in order to formulate well posed stochastic control problems which are meaningful in the context of this application.

In Chapter 3, we summarize our research in stochastic control theory relevant to tracking and missile guidance problems. Two classes of problems are addressed: (i) optimal stochastic control of nonlinear systems with "fast" and "slow" states; and (ii) stochastic scheduling and stability of systems (linear and nonlinear) with Poisson noise disturbances (in the coefficients).

The work on (i) has been led to a rather complete theory for singularly perturbed optimal stochastic control problems. The theory encompasses several classes of models, including systems with states taking values in bounded sets (e.g., angular variables) and systems with unbounded states. Stability criteria for the "fast" states play a key role in the second class of systems. Our main focus is on the existence and nature of "composite" control laws for the fast and slow subsystems like those defined by Chow and Kokotovic for singularly perturbed deterministic control problems. One of the most important findings of this research is that composite control laws for singularly perturbed stochastic control problems generally do not exist in the simple form suggested by the deterministic case. In fact, the limiting optimal control law for the slow subsystem retains a dependence on the states of the fast subsystem.

Stochastic control problems with fast and slow states are common in the design and evaluation of tracking loops and missile guidance systems. They occur whenever it is necessary to retain the interdependence of subsystems operating on different time scales (e.g., sampling rates) such as the interaction of sensor tracking loops and guidance control loops in autonomously guided missiles.

The second class of problems treated in this chapter concerns stochastic dynamical systems with Poisson noise disturbances. These systems arise as models of physical processes with intermittent noise disturbances. We have obtained results on the control, scheduling, and stability of such systems. The control results are not discussed here. The results on scheduling are primarily concerned with the derivation of optimality conditions and the verification that these conditions are well posed.

We also consider the asymptotic stability of linear systems with Poisson noise coefficients. Criteria for stability of the moments of such systems have been available for some time. As is the case with diffusion processes, criteria for almost sure stability of the sample paths are much more delicate. In the present case, a key result is a deep theorem of Furstenburg on the (ergodic) limit properties of products of random matrices. This result allows us to develop an exact expression for the asymptotic, exponential growth (decay) rate of the paths in terms of an ergodic measure. We give several examples to illustrate the nature of the computations and criteria. We also give tight estimates on the probability of a large deviation in a stable process; and we give a condition for stabilization of linear systems with state and control dependent Poisson noises.

In Chapter 4, we consider the problem of simultaneous detection and estimation when the signals corresponding to the M different hypotheses can be modelled as outputs of M distinct stochastic dynamical systems of the Ito type. Under very mild assumptions on the models and on the cost structure, we show that there exists a set of sufficient statistics for the simultaneous detection-estimation problem that can be computed recursively by linear equations. Furthermore, we show that te structure of the detector nd estimator is completely determined by the cost structure. The methodology used employs recent advances in nonlinear filtering and stochastic control of partially observed stochastic systems of the Ito type. Specific examples and applications in radar tracking and discrimination problems are discussed.

1.2 Participating Scientific Personnel

SEI personnel contributing to this project include Drs. B. Avromovic, J. Baras, N. Barkakati, W. Bennett, G. Blankenship and H. Kwatny. Consulting services were provided by Professors Y. BarShalom, H. Kushner and W. Fleming.

1.3 Papers and Reports Published under Support of This Project

1. G. L. Blankenship, "Multi-time scale methods in Stochastic control," in Proc. Seventh Meeting Coordinating Group on Modern Control Theory, H. Cohen, ed., Redstone Arsenal, AL, October 1985, pp.111-138.

2. A. Bensoussan and G.L. Blankenship, "Singular perturbations in stochastic control," in Asymptotic Analysis and Singular Perturbations in Control, P. V. Kokotovic, A. Bensoussan, and G.L. Blankenship, eds., Springer-Verlag, Lecture Notes in Control and Information Systems, New York, accepted for publication.

3. G.L. Blankenship, "Stochastic optimal control of systems with fast and slow states," Proc. American Control Conference, Seattle, July 1986.

4. C.W. Li and G.L. Blankenship, "Almost sure stability of linear stochastic systems with Poisson process coefficients," SIAM J. Appl. Math, Vol. 46 (1986), pp. 875-911.

5. C.W. Li and G.L. Blankenship, "Optimal stochastic scheduling of systems with Poisson noise disturbances," Appl. Math. Optim., accepted for publication.

6. C.L. Li and G.L. Blankenship, "Optimal stochastic control of linear systems with Poisson noise disturbances," submitted for publication.

7. J.S. Baras and Alain Bensoussan, "Optimal Sensor Scheduling in Nonlinear Filtering of Diffusion Processes," in preparpation.

8. W.H. Bennett, N. Barkakati, J.S. Baras and H.G. Kwatny, "Generic Guidance Model for a Homing Missile," SEPI Technical Report TR-84-14, August 22, 1984.

9. J.S. Baras, W.H. Bennett, N. Barkakati and H.G. Kwatny, "Stochastic Control Formulation for Tracking and Guidance of Autonomous Missiles," SEPI Technical Report TR-84-16, December 10, 1984.

10. J.S. Baras, "Simultaneous Detection and Estimation for Diffusion Process Signals," 1984 INRIA Conference on System Optimization.

Chapter 2

Seeker Pointing and Tracking: Some Classical Considerations

In this chapter, we consider the design of pointing and tracking servomechanisms for a seeker using an imaging FLIR with a gimbaled platform from a more or less conventional perspective. We will specifically consider the application of classical, single input single output servo theory and the extended Kalman filter. Our intent is to establish a basis for meaningful comparison of the performance improvement achieved with the nonlinear stochastic control theory which is the main subject of this reseach project. Performance objectives for these systems are stated primarily in classical terms, and it is essential to fully appreciate their intent and their implications in order to formulate well posed stochastic control problems which are meaningful in the context of this application.

In the following paragraphs, we first discuss classical design methods and then control design based on the extended Kalman filter.

2.1 Classical Servomechanism Design

In the classical SISO approach, the seeker boresight angles – elevation, θ_s, and azimuth, ψ_s – are treated as independent control loops. We consider only the elevation angle θ_s loop. Figure 1 illustrates the general configuration of a servo-tracker in which it is desired that the boresight elevation angle track the target line of sight elevation angle, θ_t. The tracking error is defined as

$$e = \theta_t - \theta_s \tag{2.1}$$

The general control system objectives are twofold: (a) loop stability, and (b) error regulation. Loop stability requires, of course, that the closed loop system eigenvalues lie in an acceptable region of the open left half plane, and it is also typically required that specified stability margins (usually gain and phase margins) obtain. Error regulation usually refers to one or a combination of the following types of error specifications:

1. Provide acceptable ultimate state error coefficients for prescribed deterministic target trajectories. A common example would be the requirement that $e(t) \rightarrow 0$ as $t \rightarrow \infty$ when $\theta_t(t)$ is a step or a ramp function. It is also common to add other time response shape requirements, e.g., rise time and overshoot specifications.

2. With θ_t specified as a zero mean random signal with prescribed power density spectrum, provide an acceptable error power density spectrum – which is frequently specified as an upper bound over a given frequency band.

For example, a typical FLIR performance specification defines normal dynamic inputs to be those with line of sight rates less than $0.5 rad/sec$ and angular accelerations less than $0.5 rad/sec^2$ (see Interface Control Document 5801647A, 30 September 1983). It further requires that the line of sight angular deviations remain within the bounds indicated in Figure 2. We will consider the design of a servomechanism to meet this deterministic performance objective and then examine the implications of restating the design objectives in terms of a stochastic control problem.

Figure 3 illustrates a choice of inner loop and series compensation which allows the stated objectives to be achieved. Various choices of the parameters satisfy the tracking requirement, and the final selection would be made by analysis of the tradeoff between tracking performance and stability margins. Note that the performance specification as stated requires that the control loop be at least a type 1 servomechanism. This guarantees zero ultimate state error following step input signals and bounded ultimate state error following ramp input signals. The ramp input error bound is controlled by the lead/lag ratio $\frac{\alpha_1}{\alpha_2}$. Increasing the type number of the loop or increasing the lead/lag ratio will improve the ultimate state error response but substantially reduce stability margins.

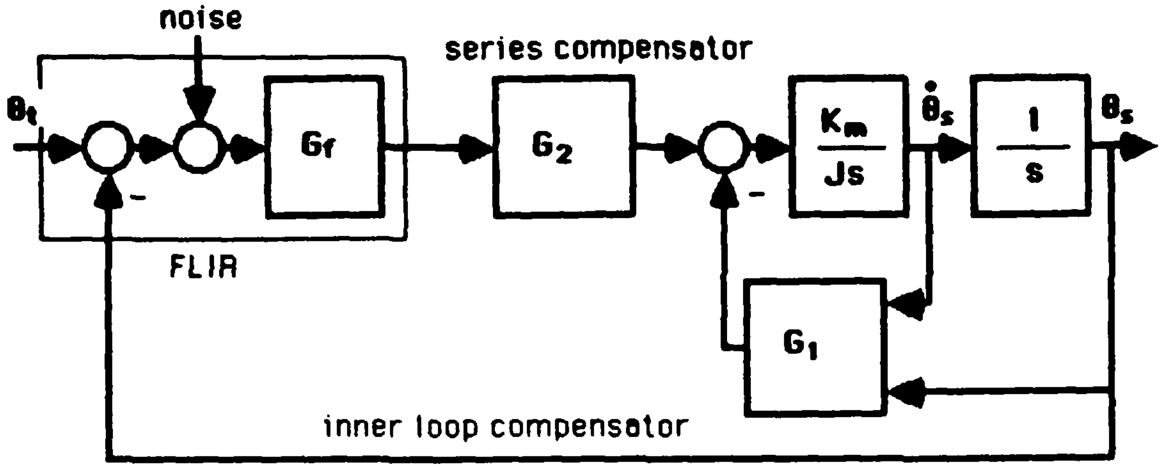

Figure 1

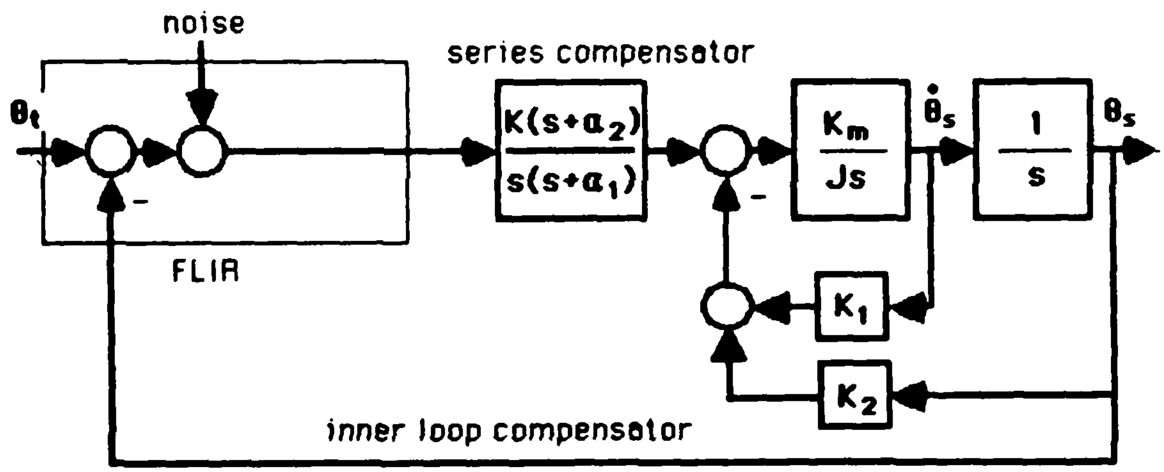

Figure 2

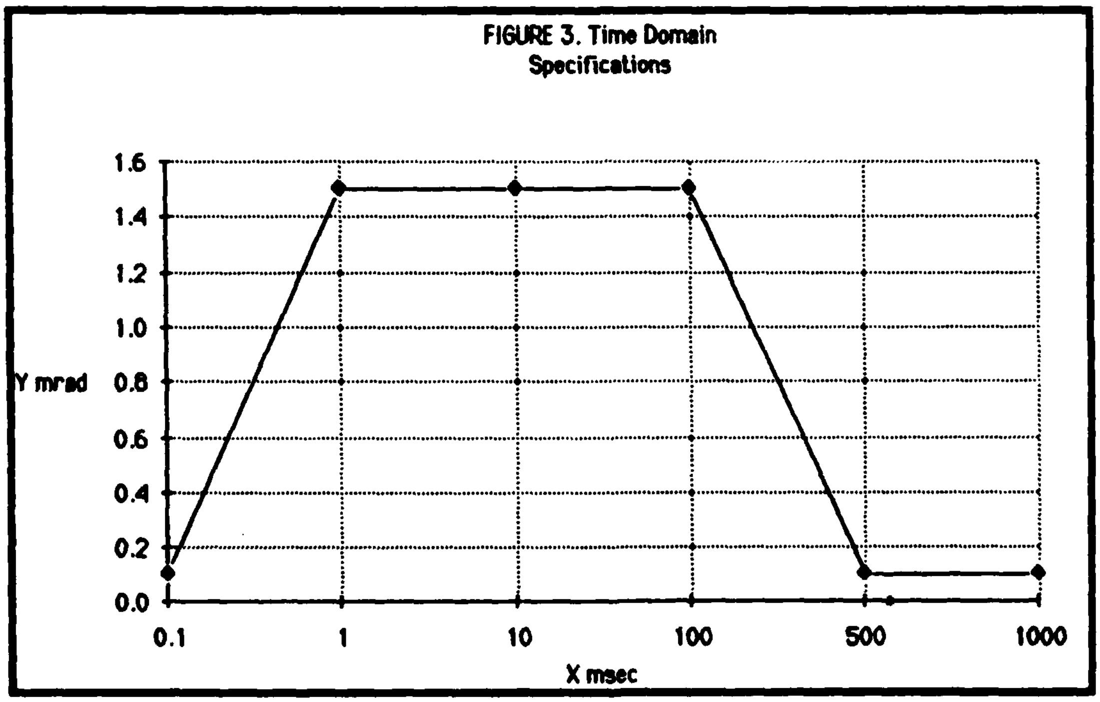

FIGURE 3. Time Domain
Specifications
1.6
1.4
1.2
1.0
Y mrad
0.8
0.6
0.4
0.2
0.0
0.1
1
10
100
500
1000
X msec

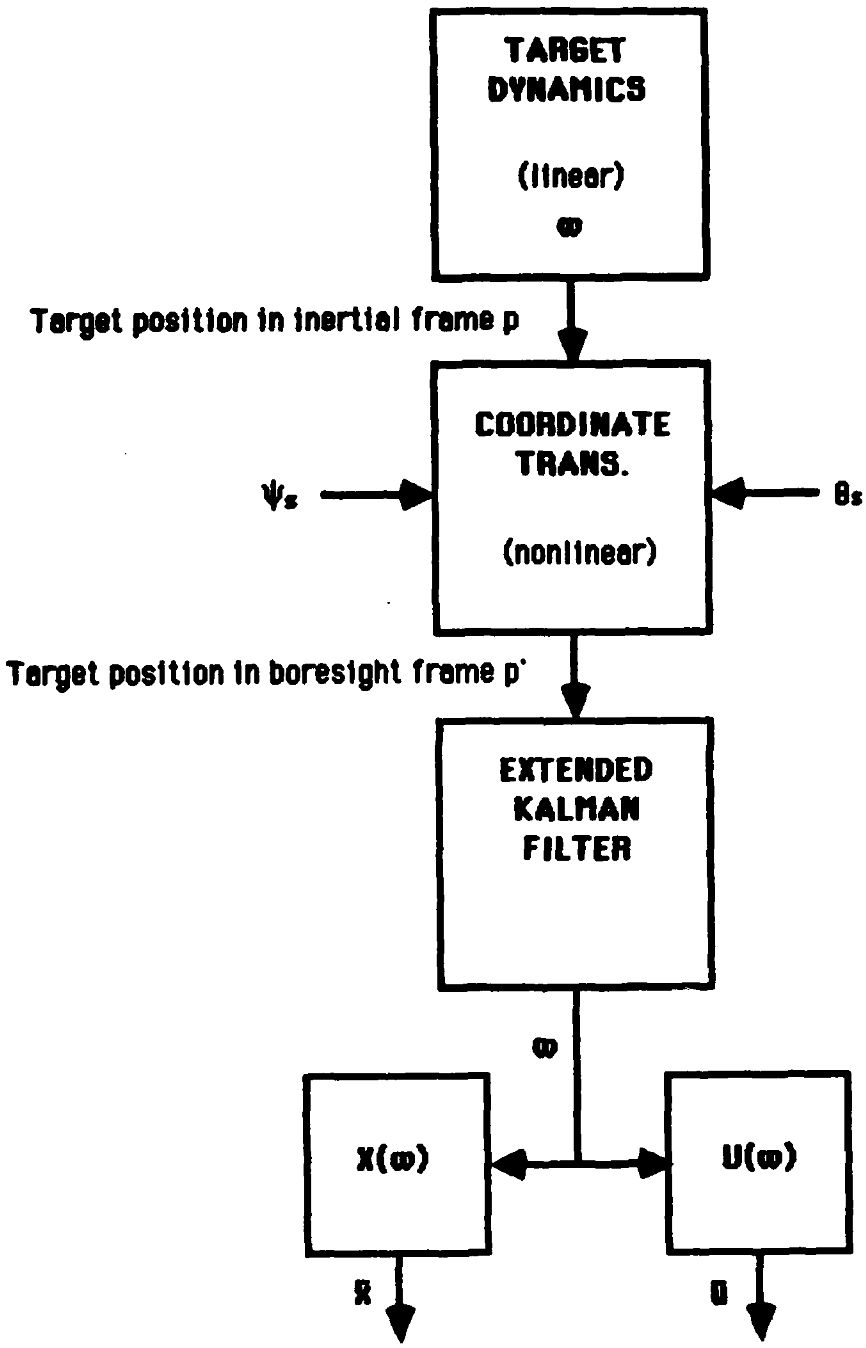
TARGET
DYNAMICS
(linear)
ω
Target position in inertial frame p
COORDINATE
TRANS.
(nonlinear)
ψs
θs
Target position in boresight frame p'
EXTENDED
KALMAN
FILTER
ω
X(ω)
U(ω)
X̂
Û

Figure 4

Suppose now that we consider the following stochastic version of the above design problem. The target line of sight elevation angle is modeled by the stochastic differential equation

$$\frac{d^2}{dt^2}\theta_t = \nu \tag{2.2}$$

where ν is a zero mean Gaussian white noise. The motivation for such a model is provided in the report [2]. It is easy to show that

$$E\{e^2\} = \int G_{ee}(\omega)d\omega < \infty \tag{2.3}$$

only if the control loop is at least a type 2 servomechanism. This is an obvious consequence of the fact that the target model is not asymptotically stable. It has important implications, however, with respect to the formulation of well-posed stochastic control problems for this class of models.

2.2 Control Design Based on the Extended Kalman Filter

In this section, we consider the application of the extended Kalman filter (EKF) to seeker servomechanism design. The general configuration of the control system is illustrated in Figure 4. The configuration shown is based on an extension of linear disturbance accommodating/tracking servomechanism theory (see Kwatny and Kalnitsky [3] and the references therein). The EKF provides continuous. on-line estimates of a linear target/platform model in relative coordinates, given observations involving nonlinear transformations in the presence of additive measurement noise. These estimates are then used by a robust disturbance accommodating servomechanism, where the controller is optimal for the case of full state observations. In the following paragraphs, we define the model, describe the design of the EKF, and describe the computation of the feedforward matrix functions $U(w)$ and $X(w)$.

2.2.1 The Model

The model details depend, of course, on the specific configuration of the seeker. We consider a simple, reasonably generic situation. The FLIR is mounted via two sets of gimbals on an inertial base and is therefore free to rotate about a fixed point 0 in inertial space about two axes. We define the following three coordinate systems all with origin at 0:

1. the *inertial frame* with coordinates X, Y, Z

2. the *target LOS frame* with coordinates x, y, z

3. the *boresight LOS frame* with coordinates x', y', z'

The relative position of any two reference frames can be defined in terms of the conventional elevation angle θ and azimuth angle ψ. We will use the following notation:

1. θ_s, ψ_s – boresight LOS angles, relative angles between the boresight LOS frame and the inertial frame.

2. θ_t, ψ_t – target LOS angles, relative angles between the target LOS frame and the inertial frame.

3. $\Delta\theta_t, \Delta\psi_t$ – boresight/target deviation, relative angles between the target LOS frame and the boresight LOS frame.

For a system without a rotor and assuming that the inertia about the x' axis and z' axis are the same, the equations of motion for the boresight angles take the form

$$\begin{aligned} J_\psi \frac{d^2}{dt^2}\psi_s &= \tau_\psi \\ J_\theta \frac{d^2}{dt^2}\theta_s &= \tau_\theta \end{aligned} \tag{2.4}$$

We assume that the torque τ is related to the corresponding control input u by the linear relation

$$\tau_\alpha = K'_\alpha u_\alpha, \ \alpha = \psi, \theta \tag{2.5}$$

The Target Kinematic Model

The target kinematics in inertial space are defined by

$$\begin{aligned} d\overline{a}_T(t) &= -\Lambda \overline{a}_T(t)dt + \Sigma d\overline{w}(t) \\ \dot{\overline{V}}_T(t) &= \overline{a}_T(t) \\ \dot{\overline{P}}_T(t) &= \overline{V}_T(t) \end{aligned} \tag{2.6}$$

where the three vectors $\overline{a}_T$ = target acceleration, $\overline{V}_T$ = target velocity, $\overline{P}_T$ = target position, and

$$\Lambda = \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & 0 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma & & 0 \\ & \sigma & \\ 0 & & 0 \end{bmatrix}$$

$\overline{w}(t)$ is a three-vector valued Gaussian process with independent components with mean zero and

$$E\{\overline{w}(t_1)\overline{w}^T(t_2)\} = \begin{cases} I_3, & \text{if } t_1 = t_2 \\ O_3, & \text{else} \end{cases}$$

The Platform Kinematic Model

A platform vibration model is included for resonant vibration characteristics of the airframe

$$\begin{aligned} d\overline{s}(t) &= A_m\overline{s}(t)dt + B_m d\overline{v}(t) \\ \overline{a}_m(t) &= T_m\overline{s}(t) \\ \dot{\overline{V}}_m &= \overline{a}_m(t) \\ \dot{\overline{P}}_m(t) &= \overline{V}_m(t) \end{aligned} \tag{2.7}$$

where $\overline{a}_m$ = platform acceleration, $\overline{V}_m$ = platform velocity, $\overline{P}_m$ = platform position, $\overline{s}$ = a fictitious six-vector of states s.t.

$$\overline{s} = \begin{bmatrix} a_{mx} \\ s_2 \\ a_{my} \\ s_4 \\ a_{mz} \\ s_6 \end{bmatrix} \quad \overline{a}_m = \begin{bmatrix} a_{mx} \\ a_{my} \\ a_{mz} \end{bmatrix}$$

The model parameters are

$$A_m = \begin{bmatrix} A_1 & & 0 \\ & A_1 & \\ 0 & & A_1 \end{bmatrix} \quad A_1 = \begin{bmatrix} -a & -b \\ b & -a \end{bmatrix} \tag{2.8}$$

$$B_m = \begin{bmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix} \quad T_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

where $a = 2\varsigma b(1 - 2\varsigma^2)^{-\frac{1}{2}}$; $b = 2\pi f$; $c = a_{res}b2\varsigma\sqrt{1-\varsigma^2}$; f = resonant frequency in H_z; $0 < \varsigma < 1$ = damping ratio; a_{res} = peak-to-peak vibration acceleration in $[m/sec^2]$.

The Relative Target-Platform Kinematic Model

Since the dimensions of the state models for target and platform are not the same, we augment the former with some trivial states as follows.

The platform acceleration state equation can be written from Equation (2.7) as

$$\begin{pmatrix} \dot{\overline{a}}_m \\ \underset{\sim}{\overline{\dot{s}}} \end{pmatrix} = PA_mP^{-1}\begin{pmatrix} \overline{a}_m \\ \underset{\sim}{\overline{s}} \end{pmatrix} + PB_m\dot{\overline{V}}(t) \qquad (2.9)$$

where P is a 6x6 permutation matrix

$$P : \begin{pmatrix} a_{mx} \\ s_2 \\ a_{my} \\ s_4 \\ a_{mz} \\ s_6 \end{pmatrix} \longrightarrow \begin{pmatrix} a_{mx} \\ a_{my} \\ a_{mz} \\ s_2 \\ s_4 \\ s_6 \end{pmatrix} = \begin{pmatrix} \overline{a}_m \\ \underset{\sim}{\overline{s}} \end{pmatrix}$$

We next write an equivalent state space model for the target acceleration as

$$\begin{pmatrix} \dot{\overline{a}}_T \\ \dot{\overline{\delta}} \end{pmatrix} = \begin{bmatrix} -\Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \overline{a}_T \\ \overline{\delta} \end{pmatrix} + \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \dot{\overline{w}}(t) \qquad (2.10)$$

Let

$$\overline{\varsigma}(t) = \begin{pmatrix} \overline{a}_T \\ \overline{\delta} \end{pmatrix} - \begin{pmatrix} \overline{a}_m \\ \underset{\sim}{\overline{s}} \end{pmatrix}$$

$$\dot{\overline{\varsigma}}(t) = \left\{ \begin{bmatrix} -\Lambda & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P & A_m & P^{-1} \end{bmatrix} \right\} \left[\begin{array}{c|c} \Sigma & -PB_m \\ 0 & \end{array} \right] \begin{pmatrix} \dot{\overline{w}} \\ \dot{\overline{v}} \end{pmatrix} \qquad (2.11)$$

Thus the relative kinematics can be written in the form

$$\begin{aligned} \dot{\overline{\varsigma}}(t) &= A_{rel}\overline{\varsigma}(t) + B_{rel}\dot{\overline{\nu}} \\ \overline{a}_{rel}(t) &= T_{rel}\overline{\varsigma}(t) = [I_3|0_3]\overline{\varsigma}(t) \\ \dot{\overline{V}}_{rel}(t) &= \overline{a}_{rel}(t) \\ \dot{\overline{P}}_{rel}(t) &= V_{rel}(t) \end{aligned} \qquad (2.12)$$

where

$$A_{rel} = \begin{bmatrix} -(\lambda+a) & 0 & 0 & -b & 0 & 0 \\ 0 & -(\lambda+a) & 0 & 0 & -b & 0 \\ 0 & 0 & -a & 0 & 0 & -b \\ b & 0 & 0 & -a & 0 & 0 \\ 0 & b & 0 & 0 & -a & 0 \\ 0 & 0 & b & 0 & 0 & -a \end{bmatrix}$$

$$B_{rel} = \begin{bmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & -c \end{bmatrix}$$

Finally, Equation (2.12) is written compactly as

$$\dot{w}(t) = Zw(t) + W\dot{\nu}(t) \tag{2.13}$$

where w is 12-vector, $\dot{\nu}(t)$ is 6-vector. $w = (\overline{P}_{rel}, \overline{V}_{rel}, \overline{\varsigma})^T$ where

$$Z_1 = \left[\begin{array}{cc|cc} O_3 & I_3 & O_3 & O_3 \\ O_3 & O_3 & I_3 & O_3 \\ \hline \multicolumn{2}{c|}{O_6} & \multicolumn{2}{c}{A_{rel}} \end{array}\right] \quad ,W = \begin{bmatrix} O_6 \\ --- \\ B_{rel} \end{bmatrix}$$

The observations: For simplicity of notation, take $w = (x, y, x, w)^t$ where x,y,z are the relative position coordinates of target weight platform in inertial frame fixed to the platform with z-axis pointing down.

The target location in the seeker boresight frame is given in terms of the angles Θs, Ψs

$$P_r' = T(\Psi_s, \Theta_s) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.14}$$

where the rotation matrix T is given as

$$T(\Psi_s, \Theta s) = \begin{bmatrix} cos\Theta_s cost\Psi_s & cos\Theta_s sin\Psi_s & sin\Theta_s \\ -sin\Psi_s & cos\Psi_s & 0 \\ -sin\Theta_s cos\Psi_s & -sin\Theta_s sin\Psi_s & cos\Theta_s \end{bmatrix} \tag{2.15}$$

coordinates of the target trackpoint in the FLIR image plane are given by

$$P_i = \frac{f_o}{r(t)} P_2 T(\Psi_s, \Theta_s) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.16}$$

where

$P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a projection onto the y-z plane in the boresight frame

$R(t) = \sqrt{x^2 + y^2 + z^2}$ is range which is available by separate laser range finder measurement

f_o is focal length

Thus the observation equation is in discrete time

$$
\begin{aligned}
y(t_R) &= \begin{bmatrix} r(X,Y,Z) \\ \hline \frac{f_o}{r} P_2 T(\Psi_s, \Theta_s) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{bmatrix} \qquad (2.17) \\
&\quad + \dot{\xi}(t_k) \qquad k = 1, 2, \ldots \\
&= h(X,Y,Z) + \dot{\xi}
\end{aligned}
$$

where

y is a 3-vector and the measurement noise

$\dot{\xi}$ is 3-vector, Gaussian, zero mean, white noise process with $R_k = E\{\xi_k \xi_k^T\}$

2.2.2 The Extended Kalman Filter

To implement the EKF, we will need the 12x12 Jacobian matrix

$$H(w) = \frac{\partial h(w)}{\partial \overline{w}}$$

Let $\overline{w} = (x, y, z)^t$ and note that h depends only on $\overline{w}$. Define the 3x3 matrix

$$\tilde{H}(\overline{w}) = \frac{\partial h(w)}{\partial \overline{w}}$$

The

$$H(w) = [\tilde{H}(\overline{w}), O_{3x9}]. \qquad (2.18)$$

We implement the EKF in *continuous time* but with the observations $y(t_k)$ available at discrete times only. (cf. *Applied Optimal Estimation*, A. Gelb, pg. 188)

$$\begin{aligned} \dot{\hat{w}}(t) &= Z\hat{w}(t) \\ \dot{P}(t) &= ZP(t) + P(t)Z^T + Q \end{aligned} \tag{2.19}$$

Now integrate over $t_k < t < t_{k+1}$, with initial conditions given at t_k by the update equations:

$$\hat{w}(t_k+) = \hat{w}(t_k-) + K_k[y(t_k) - h_k(\hat{w}(t_k-))] \tag{2.20}$$

Define $H_k = H(w)|w = \hat{w}(t_k+)$ then

$$P(t_k+) = [I - K_kH_k]P(t_k-) \tag{2.21}$$

$$K_k = P(t_k-)H_k^T[H_kP(t_k-)H_k^T + R_k]^{-1} \tag{2.22}$$

and the matrix Q is defined by

$$Q = E\{\eta(t)\eta^T(t)\}$$

where $\eta(t) = W\dot{\nu}(t)$.

Remarks:

1. Equation (2.19a) is a 12-dimensional linear differential equation with the same parameters, Z, as in Equation (2.13). It is the "on-line" model.

2. Equation (2.19b) is a matrix-valued differential Riccati equation with symmetric solution $P(t)$, which must be propagated from t_k to t_{k+1}.

3. Equation (2.20) is the update equation of the on-line model. It contains the "true" nonlinearity $h(\cdot)$ as it appears in Equation (2.17) except that the most current estimate of the range $\hat{R}(t_k)$ is used (instead of e.g. $\sqrt{x^2 + y^2 + z^2}$).

4. Equation (2.21) updates the Riccati matrix.

5. Equation (2.22) updates the optimal gain K_k for the current update evaluation.

2.2.3 Computation of the Feedforward Matrices

Let x represent the boresight state, i.e.,

$$x = (\Theta_s, \frac{d}{dt}\Theta_s, \Psi_s, \frac{d}{dt}\Psi_s)^t$$

and w the target state. We seek a control input $\overline{u}(t)$ and corresponding state trajectory $\overline{x}(t)$ so that perfect tracking occurs. That is

$$\Delta\Theta_t(t) \equiv 0 \tag{2.23}$$
$$\Delta\Psi_t(t) \equiv 0 \tag{2.24}$$

Moreover, we seek $\overline{u}, \overline{x}$ in the form

$$\overline{u} = U(w), \overline{x} = X(w) \tag{2.25}$$

$\underline{X}(\underline{w})$: Exact tracking requires that

$$\overline{x} \equiv (\Theta_t, \frac{d}{dt}\Theta_t, \Psi_t, \frac{d}{dt}\Psi_t)^t \tag{2.26}$$

Recall the transformation from rectangular to polar coordinates $(X, Y, Z) \leftrightarrow (R, \Theta_t, \Psi_t)$:

$$\begin{aligned} X &= Rcos\Theta_t cos\Psi_t \qquad (2.27)\\ Y &= Rcos\Theta_t sin\Psi_t \\ Z &= Rsin\Theta_t \end{aligned}$$

$$\begin{aligned} R &= (x^2 + y^2 + z^2)^{\frac{1}{2}} \qquad (2.28)\\ \Theta_t &= sin^{-1}(Z/R) \\ \Psi_t &= tan^{-1}(Y/X) \end{aligned}$$

Note that Equations (2.28b) and (2.28c) immediately provide Θ_t and Ψ_t as functions of w. We still need $\frac{d}{dt}\Theta_t(w)$ and $\frac{d}{dt}\Psi_t(w)$. To obtain these, let

V_r = target inertial velocity in target LOS coordinates
V_R = target inertial velocity in inertial frame coordinates
ω_t = target LOS frame angular velocity in target LOS coordinates

Then we have

$$\begin{aligned}
\omega_t &= (\frac{d}{dt}\Psi_t sin\Theta_t, \frac{d}{dt}\Theta_t, \frac{d}{dt}\Psi_t cos\Theta_t)^t && (2.29)\\
V_r &= T(-\Theta_t, \Psi_t)V_R && (2.30)\\
P_2\omega_t &= P_2 V_r && (2.31)
\end{aligned}$$

These lead to

$$\begin{bmatrix} \frac{d}{dt}\Theta_t \\ \frac{d}{dt}\Psi_t \end{bmatrix} = G(\Theta_t, \Psi_t)V_R = \begin{bmatrix} -sin\Psi_t & cos\Psi_t & 0 \\ -tan\Theta_t cos\Psi_t & -tan\Theta_t sin\Psi_t & 1 \end{bmatrix} V_R \quad (2.32)$$

which provide the required relations.

$\underline{U(\underline{w})}$: Exact tracking requires $(\frac{d}{dt}\Theta_t, \frac{d}{dt}\Psi_t) = (\frac{d}{dt}\Theta_s, \frac{d}{dt}\Psi_s)$. Using the equations of motion, we can write

$$\frac{d}{dt}(\frac{d}{dt}\Theta_t, \frac{d}{dt}\Psi_t)^t = K'\overline{U} \quad (2.33)$$

But from Equation (2.32), we have

$$\frac{d}{dt}(\frac{d}{dt}\Theta_t, \frac{d}{dt}\Psi_t)^t = [(\partial G/\partial\Theta_t)V_R|(\partial G/\partial\Psi_t)V_R]GV_R + Ga_R \quad (2.34)$$

where a_R is the target acceleration. Thus, we obtain from (2.33) and (2.34)

$$\begin{aligned}
\overline{u} &= U(w) = diag\,(1/K'_\Theta, 1/K'_\Psi)\,diag\,(J_\Theta, J_\Psi) && (2.35)\\
&\quad \{[(\partial G/\partial\Theta_t)\,V_R|\,(\partial G/\partial\Psi_t)\,V_R]\,GV_R + Ga_R\}
\end{aligned}$$

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[3] H.G. Kwatny and K.C. Kalnitsky, On Alternative Methodologies for the Design of Robust Linear Regulators, *IEEE Trans. Auto. Contr.*, Vol. AC-23, No. 5, pp. 930-933, October 1978.

Chapter 3

Stochastic Control of Dynamical Systems

Summary:

In this Chapter, we summarize our research in stochastic control theory relevant to tracking and missile guidance problems. Two classes of problems are addressed: (i) optimal stochastic control of nonlinear systems with "fast" and "slow" states; and (ii) stochastic scheduling and stability of systems (linear and nonlinear) with Poisson noise disturbances (in the coefficients).

The work on (i) has led to a rather complete theory for singularly perturbed optimal stochastic control problems. The theory encompasses several classes of models, including systems with states taking values in bounded sets (e.g., angular variables) and systems with unbounded states. Stability criteria for the "fast" states play a key role in the second class of systems. The theory includes both absorbing (Dirichlet) and reflecting (Neumann) boundary conditions for systems with bounded state spaces. Its main focus is on the existence and nature of "composite" control laws for the fast and slow subsystems like those defined by Chow and Kokotovic for singularly perturbed deterministic control problems. One of the most important findings of this research is that composite control laws for singularly perturbed stochastic control problems generally do not exist in the simple form suggested by the deterministic case.

In general, one cannot design an effective feedback control for the overall system (fast and slow states) based on optimization of the natural limiting

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This chapter was written by G.L. Blankenship. It is based on joint work with A. Bensoussan and C. W. Li.

system obtained by a standard asymptotic analysis of the model. That is, one cannot generally "separate" the processes of asymptotic analysis and optimization. In fact, the limiting optimal control law for the slow subsystem retains a dependence on the states of the fast subsystem.

Stochastic control problems with fast and slow states are common in the design and evaluation of tracking loops and missile guidance systems. They occur whenever it is necessary to retain the interdependence of subsystems operating on different time scales (e.g., sampling rates) such as the interaction of sensor tracking loops and guidance control loops in autonomously guided missiles.

The second class of problems treated in this chapter concerns stochastic dynamical systems with Poisson noise disturbances. These systems arise as models of physical processes with intermittent noise disturbances. We have obtained results on the control, scheduling, and stability of such systems. The control results are not discussed here. The results on scheduling are primarily concerned with the derivation of optimality conditions and the verification that these conditions are well-posed. We use a constructive limiting argument developed earlier for diffusion process models to obtain the optimal scheduling policy and cost as the limit of a sequence of optimal scheduling problems in which a finite number of switchings are permitted. The optimality conditions for these problems are *quasi-variational inequalities* (QVI's) introduced for scheduling and inventory control by Bensoussan and Lions. The properties of the Poisson noise disturbances cause the QVI's to be "first order" and "fully nonlinear" (in contrast to the classical case of diffusion processes). As a result, their analysis requires methods intermediate between those used for diffusion systems (elliptic models) and deterministic systems (first order). In particular, we use the method of *viscosity solutions* introduced by Crandall and Lions to establish uniqueness of the optimal cost when some of the switching costs are zero.

We also consider the asymptotic stability of linear systems with Poisson noise coefficients. Criteria for stability of the moments of such systems have been available for some time (S. Marcus). As is the case with diffusion processes, criteria for almost sure stability of the sample paths are much more delicate. In the present case, a key result is a deep theorem of Furstenburg on the (ergodic) limit properties of products of random matrices. This result allows us to to develop an exact expression for the asymptotic, exponential growth (decay) rate of the paths in terms of an ergodic measure. We give several examples to illustrate the nature of the computations and criteria. We also give tight estimates on the probability of a large deviation in a stable process; and we give a condition for stabilization of linear systems with state and control dependent Poisson noises.

In the first section we consider the problem of optimal stochastic control of diffusion processes containing "fast" and "slow" dynamics. The systems are considered on an unbounded state space. The analysis highlights the key role played by ergodicity of the fast state variables. We use a stochastic stability theorem of Khas'minskii to determine the conditions under which ergodicity holds and the optimal control problem is well posed. The limiting control problem obtained as the small parameter goes to zero retains an interesting interdependence between fast and slow variables. The work reported in the first section of this chapter is a summary of a portion of [3]. That paper should be consulted for details of the proofs and for other related problems and results.

In the second section of this chapter we present a summary of some work on the optimal stochastic scheduling of systems with jump process parameters. The work described in that section is abstracted from the paper [24]. The main results are a characterization of the optimality conditions in terms of *viscosity solutions* to a class of Bellman equations.

In the third section of this chapter we present a summary of our research on the stability properties of linear stochastic dynamical systems with Poisson noise disturbances as parameters. The main results in that section are expressions for the *exponential asymptotic growth (decay) rates* of the solutions.

3.1 Stochastic Optimal Control of Systems with Fast and Slow States

3.1.1 Introduction

In this section we address the following class of control problems. We have a system governed by

$$\begin{aligned} dx &= f(x,y,v)dt + \sqrt{2}dw \\ \epsilon dy &= g(x,y,v)dt + \sqrt{2\epsilon}db \qquad (3.1) \\ x(0) &= x, y(0) = y. \end{aligned}$$

where w and b are independent Wiener processes. The state $x(t)$ represents the *slow* system, while the state $y(t)$ represents the *fast* system. The scaling is such that the variations of the fast system per unit of time, in average as well as in variance, are of order $1/\epsilon$. The dynamics are controlled via the parameter $v(t)$. There is full information and the objective is to minimize the payoff

$$J^{\epsilon}_{x,y}(v(\cdot)) = E\int_0^{\tau} e^{-\beta t} l(x^{\epsilon}(t), y^{\epsilon}(t), v(t))dt \tag{3.2}$$

where τ denotes the first exit time of the process x from the boundary Γ of a domain O usually taken to be smooth and bounded. (We will, in fact, treat systems on unbounded domains.) Call

$$u_{\epsilon}(x,y) = \inf_{v(\cdot)}\{J^{\epsilon}_{x,y}(v(\cdot))\},$$

then u_{ϵ} is the solution of the Bellman equation

$$-\Delta_x u^{\epsilon} - \frac{1}{\epsilon}\Delta_y u^{\epsilon} + \beta u^{\epsilon} = H(x, D_x u^{\epsilon}, y, \frac{1}{\epsilon} D_y u^{\epsilon}) \tag{3.3}$$

$$u^{\epsilon} = 0 \quad \forall x \in \Gamma$$

with

$$H(x,p,y,q) = \inf_{v\in U_{ad}} [l(x,y,v) + p\cdot f(x,y,v) \tag{3.4}$$

$$+q\cdot g(x,v,v)] = \inf_{v\in U_{ad}} L(x,p,y,q)$$

We assume sufficient smoothness so that there exists a Borel map $\hat{V}(x,p,y,q)$ with values in U_{ad} such that

$$H(x,p,y,q) = L(x,p,y,q,\hat{V}) \tag{3.5}$$

We can then define an optimal feedback control for the problem by setting

$$\hat{v}_{\epsilon}(x,y) = \hat{V}(x, D_x u_{\epsilon}, y, D_y u_{\epsilon}) \tag{3.6}$$

and the process

$$\hat{v}_{\epsilon} = \hat{v}_{\epsilon}(x_{\epsilon}, y_{\epsilon}) \tag{3.7}$$

is an optimal control for (3.2).

Such systems arise in the design and analysis of tracking loop systems where the fast subsystem corresponds to the dynamics of the sensor control loop and the slow subsystem corresponds to the dynamics of the platform. Many other applications have models which exhibit similar features.

Our objective is to study the behavior of the equation (3.3) for ϵ small, and to interpret the results as a limit control problem approximating (3.1), (3.2). Let us explain the type of results which one can expect.

Proceed formally with an asymptotic expansion

$$u^\epsilon(x,y) = u(x) + \epsilon\phi(x,y).$$

$$-\Delta u - \Delta_y \phi + \beta u = H(x, D_x u, y, D_y \phi) \tag{3.8}$$

which we try to match for any x, y by a convenient choice of u and ϕ. Consider x in (3.8) as a parameter, as well as $p = D_x u$; set

$$\begin{aligned} L(y,v) &= l(x,y,v) + p \cdot f(x,y,v) \\ G(y,v) &= g(x,y,v) \\ \mathbf{H}(y,q) &= \inf_{v \in U_{ad}} [L(y,v) + q \cdot G(y,v)] \end{aligned} \tag{3.9}$$

which also depend parametrically on x and p.

One can then consider the Bellman equation of ergodic control relative to (3.9). It is defined are follows: pick a constant χ (constant with respect to y) and a function ϕ such that

$$-\Delta_y \phi + \chi = \mathbf{H}(y, D_y \phi). \tag{3.10}$$

Suppose one can find such a pair χ, ϕ depending parametrically on x, p; hence,

$$\chi = \chi(x,p).$$

$$-\Delta u + \beta u = \chi(x, D_x u), \tag{3.11}$$

then the pair u, ϕ will satisfy (3.8). One can thus expect a solution of (3.11), vanishing on the boundary Γ of $\mathbf{O}_2$ to be the limit of u^ϵ.

This procedure depends on the possibility of being able to solve ergodic control problems of the type (3.10). This control problem itself is as follows: Consider

$$dy = G(y,v)d\tau + \sqrt{2}db, y(0) = 0 \tag{3.12}$$

$$k_y(v(\cdot)) = \lim_{T\to\infty} \frac{1}{T} E \int_0^T L(y,v)d\tau$$

then in general

$$\chi = \inf_{v(\cdot)} \{k_y(v(\cdot))\}$$

independent of y. The interpretation of ϕ is more delicate. Pick a feedback $v(y)$ and consider the controlled state

$$dy = G(y, v(y))d\tau + \sqrt{2}db, y(0) = y. \tag{3.13}$$

It seems inevitable to require ergodicity of the process y to define a well-posed control problem.

This means that as $\tau \to \infty$, $y(\tau)$ behaves like a random variable following a probability $m_x^{v(\cdot)}(y)$, depending on the choice of $v(\cdot)$ and of the parameter entering into the definition of G. Suppose, moreover, that m is a probability density with respect to Lebesgue measure; it is possible to give another interpretation of χ as follows:

$$\chi = \inf_{v(\cdot)}\{\int_Y L(y, v(y))m_x^{v(\cdot)}(y)dy\}. \tag{3.14}$$

In fact, taking account of

$$EL(y(\tau), v(\tau)) \to \int_Y L(y, v(y))m_x^{v(\cdot)}(y)dy \text{ as } \tau \to \infty$$

one understands the relations between both interpretations of χ. Formula (3.14) permits a better interpretation of (3.11), which turns out to be a Bellman equation for the slow system.

Indeed

$$\chi(x,p) = \inf_{v(\cdot)}\{\int_Y (l(x, y, v(y)) + p \cdot f(x, y, v(y))m_x^{v(\cdot)}(y)dy\}$$

Setting

$$\tilde{l}(x, v(\cdot)) = \int_Y l(x, y, v(y))m_x^{v(\cdot)}(y)dy$$

$$\tilde{f}(x, v(\cdot)) = \int_Y f(x, y, v(y))m_x^{v(\cdot)}(y)dy$$

then the limit problem is described by

$$\inf_{v(\cdot)} J(v) = E_x\{\int_0^T e^{-\beta t}\tilde{l}(x, v(\cdot))dt\}$$

$$dx = \tilde{f}(x, v(\cdot))dt + \sqrt{2}dw \tag{3.15}$$

$$x(0) = 0$$

It is interesting to note that the set of controls in (3.15) is changed from the original definition. One must consider feedback laws $v \equiv v(y)$. A control defined by a feedback with respect to the slow system is thus a function $v(x, y)$. To justify these considerations, it is thus important to make assumptions in order that the ergodicity of the process (3.13) is guaranteed. There must be one way or another a Markov chain defined on a compact set for which Doeblin's theorem holds (see J.L. Doob [1]). This is achieved when one assumes that G is periodic in y together with the feedback or when one considers instead of (3.13) a reflected diffusion. The first case was treated in the paper [2]. In this section we shall consider the case of diffusions on the whole space. Reflected diffusions are treated in [3]. This section contains a treatment of most cases where a natural ergodic fast system governs the evolution of the state. There are other situations where different techniques of singular perturbations are used. Examples of such situations may be found in the paper of R. Jensen and P.L. Lions [4]. For other approaches to ergodic control, see [5].

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3.1.2 Ergodic control for diffusions in the whole space

Assumptions - Notation

We consider

$$\begin{gathered} g(y,v) : \mathcal{R}^d \times U \to \mathcal{R}^d \\ l(y,v) : \mathcal{R}^d \times U \to \mathcal{R}^d \\ \text{continuous and bounded} \end{gathered} \tag{3.16}$$

$$U_{ad} \quad \text{(compact)} \quad \subset U \tag{3.17}$$

U a metric space. For a given feedback $v(y)$, which is a Borel function with values in U_{ad}, we shall solve in a weak sense the stochastic differential equation

$$dy = (Fy + g(y, v(y))dt + \sqrt{2}db_r(t), y(0) = y. \tag{3.18}$$

The linear term Fy will be useful to ensure an ergodicity property later on (F a stable matrix). The Brownian motion b_r is defined through a Girsanov transformation. We can find a system (Ω, A, F^t, P_v^y) such that (3.18) holds.[1]

[1] We limit ourselves to feedback controls, since only those will appear in the singular perturbation problem that we shall eventually solve. Of course, this is not at all necessary for the ergodic control itself.

We then consider the function

$$k_y^\alpha(v(\cdot)) = E_v^y \int_0^\infty e^{-\alpha t} l(y(t), v(t))dt \tag{3.19}$$

where

$$v(t) = v(y(t)),$$

and we set

$$\phi_\alpha(y) = \inf_{v(\cdot)} k_y^\alpha(v(\cdot)). \tag{3.20}$$

Setting $A = -\Delta - Fy \cdot D$, we can assert that ϕ_α is the solution of

$$A\phi_\alpha + \alpha\phi_\alpha = H(y, D\phi_\alpha) \tag{3.21}$$

$$\phi_\alpha \text{ bounded }, \phi_\alpha \in W^{2,p,\mu}(\mathcal{R}^n), 2 \le p < \infty$$

where $W^{2,p,\mu}(\mathcal{R}^n)$ denotes a Sobolev space with weight

$$\beta_\mu(y) = e^{-\mu(1+|y|^2)^{1/2}} \tag{3.22}$$

and

$$L^{p,\mu} = \{z(y) | z\beta_\mu \in L^p(\mathcal{R}^d)\}$$

$$W^{2,p,\mu} = \{z \in L^{p,\mu} | \frac{\partial z}{\partial y_i}, \frac{\partial^2 z}{\partial y_i \partial y_j} \in L^{p,\mu}\}$$

Invariant measures

Since the diffusion $y(\cdot)$ does not lie in a compact set, some assumptions on the drift g are necessary to ensure ergodicity. We shall mainly use the results of Khas'minskii [6]. We make the following assumption:

(A) There exists a bounded smooth domain $\mathbf{D}$ *and a function* ψ *which is continuous and locally bounded on* $\mathcal{R}^d - \mathbf{D}, \ge 0, \psi \in W_{loc}^{2,p}(\mathcal{R}^d - \mathbf{D})$, *and*

$$A\psi - g(y,v) \cdot D\psi \ge 1, \forall v, y \in \mathcal{R}^d - \mathbf{D} \tag{3.23}$$

$$\psi > 0 \quad \psi \to \infty \text{ as } |y| \to \infty \text{ and } \frac{|D\psi|^2}{\psi} \text{ bounded}$$

In general, one can try to find ψ of the form

$$\psi(y) = \log Q(y) + k \tag{3.24}$$

where

$$Q(y) = \frac{1}{2}My \cdot y + m \cdot y + \rho \tag{3.25}$$

M symmetric and positive definite and $Q > 0$; and $\mathbf{D}$ is a region containing the zeros of Q.

The following condition must hold to have (3.23):

$$\frac{|My+m|^2}{\frac{1}{2}My^2 + m \cdot y + \rho} - \operatorname{tr} M \tag{3.26}$$

$$-(Fy + g(y,v)) \cdot (My + m)$$

$$\geq \frac{1}{2}My \cdot y + m \cdot y + \rho, \forall y \in \mathcal{R}^d - \mathbf{D};$$

for a convenient choice of $M, m,$ and ρ. For instance, if $d = 2$, we can take $M = I, m = 0, \rho = 0$ and (3.24) is satisfied provided that, for instance

$$F \leq (-\frac{1}{2} - \Delta)I \tag{3.27}$$

and $\mathbf{D}$ is a sufficiently large neighborhood of 0.

Consider a domain $\mathbf{D}_1$ such that $\overline{\mathbf{D}} \subset \mathbf{D}_1, \mathbf{D}_1$ smooth and bounded. Let Γ and Γ_1 be the boundaries of $\mathbf{D}, \mathbf{D}_1$, respectively. We shall construct a Markov chain on Γ_1. Let $x \in \mathcal{R}^d$, we define

$$\theta'(x;\Omega) = \inf\{t | y_x(t) \in \mathbf{D}\} \tag{3.28}$$

$$\theta(x;\Omega) = \inf\{t \geq \theta'(x;\Omega) | y_x(t) \notin \mathbf{D}_1\} \tag{3.29}$$

In (3.28), (3.29) $y_x(t)$ is the diffusion (3.18) with initial condition x. Using $\psi(x)$, we can write

$$E_v^x \theta'(x) \leq \psi(x). \tag{3.30}$$

This implies also that the exterior Dirichlet problem

$$A\eta - g(y, v(y)) \cdot D\eta = 0, y \in \mathcal{R}^d - \mathbf{D} \tag{3.31}$$

$$\eta|_\Gamma = h, h \in L^\infty(\Gamma)$$

has a bounded solution given explicitly by

$$\eta(x) = E_v^x h(y_x(\theta'(x))). \tag{3.32}$$

The Markov chain on Γ_1 is then constructed as follows. We define two sequences of stopping times (relative to F^t),

$$\tau_0, \tau_1, \tau_2, \dots$$
$$\tau'_1, \tau'_2, \dots$$

such that

$$\tau_0 = 0$$
$$\tau_n = \inf\{t > \tau'_n | y(t) \notin \mathbf{D}_1\}, n \geq 1$$
$$\tau'_{n+1} = \inf\{t \geq \tau_n | y(t) \in \mathbf{D}\}, n \geq 0$$

The process $y(t)$ in the brackets is the process defined by (3.18), i.e., with initial condition y. Let us set $Y_n = y(\tau_n), n \geq 1$. Then $Y_n \in \Gamma_1$ and is a Markov chain with transition probability defined by

$$E_v^y \left[\phi(Y_{n+1}) | F^{\tau_n}\right] = E_v^x \phi(y_x(\theta(x)))|_{x=Y_n}. \tag{3.33}$$

We define the following operator on Borel bounded functions on Γ_1

$$\mathbf{P}\phi(x) = E_v^x \phi(y_x(\theta(x))) \tag{3.34}$$

We can give an analytic formula as follows. Consider the problem

$$A\varsigma - g(y, v(y)) \cdot D\varsigma = 0 \text{ in } \mathbf{D}_1, \varsigma|_{\Gamma_1} = \phi. \tag{3.35}$$

We first note that

$$E_v^x \phi(y_x(\theta(x))) = E_v^x \varsigma(y_x(\theta'(x))$$

therefore taking account of (3.32) , we have

$$\mathbf{P}\phi(x) = \eta(x) \tag{3.36}$$

where η denotes the solution of (3.31) corresponding to the boundary condition $h = \varsigma$. Of course, in (3.36) $x \in \Gamma_1$ are the only relevant points. We then have

Lemma 1.1. *The operator* $\mathbf{P}$ *is ergodic.*

Proof. See [3] for the proof of this and all the remaining lemmas in this section.

From ergodic theory, it follows

$$|\mathbf{P}^n\phi(y) - \int_{\Gamma_1} \phi(\eta)\pi(d\sigma)| \leq K||\phi||e^{-\rho n}, x \in \Gamma_1 \tag{3.37}$$

where K, ρ are uniform with respect to the feedback control $v(\cdot)$, and $\pi = \pi^v$ denotes the invariant probability on Γ_1.

It follows that, since

$$\mathbf{P}^n\phi(y) = E_v^y\phi(y(\tau_n))$$

we can write

$$|E_v^y\phi(y(\tau_n)) - \int_{\Gamma_1}^{\phi}(\eta)\pi(d\sigma)| \leq K||\phi||e^{-\rho n}. \tag{3.38}$$

We can then define a probability on $\mathcal{R}^d$, by the formula

$$\int_{\mathcal{R}^d} \Lambda(y)d\mu(y) = \frac{\int_{\Gamma_1}[E_v^\eta \int_0^{\theta(\eta)} \Lambda(y_\eta(t))dt]\pi(d\sigma)}{\int_{\Gamma_1} E_v^\eta\theta(\eta)\pi(d\sigma)} \tag{3.39}$$

$$\forall \Lambda \quad \text{Borel bounded in} \quad \mathcal{R}^d.$$

Following Khas'minskii, one can then prove that the invariant probability is unique, has a density with respect to Lebesgue measure, denoted by $m = m^v$ which is the solution of

$$A^* m + \operatorname{div}(mg^v) = 0, m > 0, \tag{3.40}$$

$$\int_{\mathcal{R}^d} m(y)dy = 1.$$

where

$$A^* = -\Delta + \operatorname{div}(Fy\cdot).$$

Consider now the Cauchy problem

$$\frac{\partial z}{\partial t} + Az - g^v Dz = 0 \tag{3.41}$$

$$z(y,0) = \phi(y)$$

Lemma 1.2. *We have*

$$z(y,1) \leq c|\phi|_{L_1} \tag{3.42}$$

We deduce from Lemma 1.2 an estimate on the invariant probability solution of (3.40). Using

$$\int_{\mathcal{R}^d} m^v(y)z(y,1)dy = \int_{\mathcal{R}^d} m^v(y)\phi(y)dy$$

we deduce easily that

$$m^v(y) \le \Delta, \forall y, \forall v(\cdot) \tag{3.43}$$

It follows that m^v is uniformly bounded in $L^p(\mathcal{R}^d), \forall p,\ \ 1 \le p < \infty$. Let θ be an element of $C_0^\infty(\mathcal{R}^d)$, we have

$$-\Delta(m\theta) + \operatorname{div}(m\theta g) + Fy \cdot D(\theta m) \tag{3.44}$$

$$= m(D\theta \cdot Fy - g\operatorname{div}\theta - \theta \operatorname{tr} F) = f$$

and $f \in L^p(\mathcal{R}^d), \forall p, 1 \le p \le \infty$.

From results on the Dirichlet problem, it follows that $m\theta$ belongs to $W^{1,p}(\mathcal{R}^d), \forall p, 1 < p < \infty$. In particular, $m\theta$ is continuous. Therefore, we deduce that

$$m^v(y) \ge \Delta_k > 0, \forall y \in K, \text{ compact} \tag{3.45}$$

where the constant Δ_k does not depend on $v(\cdot)$.

Remark 1.2. The assumption (3.23) requires D nonempty. Otherwise (3.23) and (3.40) yield $\int m dy = 0$, which is impossible.

We also shall consider the following approximation to m. Let B_R be the ball of radius R, centered at 0. Let us consider m_R defined by

$$A^* m_R + \operatorname{div}(m_R g^v) + \Lambda m_R = \Lambda \tau_R m \tag{3.46}$$

$$m_R|_{\partial B_R} = 0$$

$$m_R \in W_0^{1,p}(B_R)$$

in which Λ is sufficiently large so that

$$|\xi|^2 - \xi \cdot g\theta + (\Lambda + \frac{1}{2}\operatorname{tr} F)\theta^2 \ge C(|\xi|^2 + \theta^2)$$

$$\forall \xi \in \mathcal{R}^d, \theta \in \mathcal{R}$$

Moreover, $\tau_R(y) = \tau(y/R)$ where $\tau(y)$ is smooth $\tau(y) = 0$ for $|y| \ge 1$,

$$\tau(y) = 1, \text{ for } |y| \le \frac{1}{2} \text{ and } 0 \le \tau \le 1.$$

We have

Lemma 1.3. *$\tilde{m}_R$ the extension of m_R by 0 outside B_R, converges to m in $H^1(\mathcal{R}^d)$ strongly and $\tilde{m}_q = \tilde{m}_{2q}$ converges monotonically increasing to m.*

Hamilton-Jacobi-Bellman equation of ergodic control

We consider the following problem: Find a pair χ, ϕ such that

$$\chi \in \mathcal{R} \quad \phi \in W^{2,p}_{loc}(\mathcal{R}^d), \tag{3.47}$$

with ϕ/ψ bounded at ∞

$$A\phi + \chi = H(y, D\phi) \tag{3.48}$$

Our objective is to prove the following

Theorem 1.1. *We assume (3.16), (3.17), (3.23). Then there is one and only one ϕ (up to an additive constant) and a scalar χ such that (3.47), (3.48) hold.*

We begin with some preliminary steps. Let us consider a feedback $v_\alpha(\cdot)$ such that (c.f., (3.21)) we may write

$$A\phi_\alpha + \alpha\phi_\alpha = l(y, v_\alpha) + D\phi_\alpha \cdot g(y, v_\alpha). \tag{3.49}$$

Then let m_α be the invariant probability corresponding to the feedback v_α in equation (3.40). We then have

Lemma 1.4. *The following relation holds*

$$\int (\alpha\phi_\alpha - l(y, v_\alpha)) m_\alpha dy = 0 \tag{3.50}$$

Lemma 1.5. *We have*

$$|\phi_\alpha(y) - \int_{\Gamma_1} \phi_\alpha(\eta)\pi_\alpha(d\sigma)| \leq \begin{cases} C \quad \psi(y) \text{ in } \mathcal{R}^d - \mathbf{D} \\ C \quad \psi(y) \text{ in } \mathbf{D} \end{cases} \tag{3.51}$$

where the constant does not depend on α, nor y.

Proof of Theorem 1.1.

Existence

Let us set $\tilde{\phi}_\alpha = \phi_\alpha - \int_{\Lambda_1} \phi_\alpha(\eta)\pi_\alpha(d\sigma)$. Then $||\tilde{\phi}_\alpha/\psi||_{L^\infty} \leq C$. Moreover, from (3.21) we also have

$$A\tilde{\phi}_\alpha + \alpha\tilde{\phi}_\alpha + \chi_\alpha = H(y, D\tilde{\phi}_\alpha), \tag{3.52}$$

in which

$$\chi_\alpha = \alpha \int_{\Lambda_1} \phi_\alpha(\eta)\pi_\alpha(d\sigma).$$

It readily follows from (3.52) that

$$\frac{\tilde{\phi}_\alpha}{\psi} \quad \text{bounded in } W^{2,p,\mu}(\mathcal{R}^d), 2 \le p \le \infty, \mu > 0.$$

We can extract a subsequence such that

$$\chi_\alpha \to \chi$$
$$\tilde{\phi}_\alpha \to \phi \text{ in } W^{2,p,\mu}(\mathcal{R}^d) \text{ weakly.}$$

We can assert that

$$\tilde{\phi}_\alpha, D\tilde{\phi}_\alpha \to \phi, D\phi \text{ pointwise,}$$

hence,

$$H(y, D\tilde{\phi}_\alpha) \to H(y, D\phi) \quad \text{pointwise,}$$

Noting that $H(y, D\tilde{\phi}_\alpha)$ is bounded in $L^{p,\mu}$, we can pass to the limit in (3.52), and the pair ϕ, χ satisfies (3.47), (3.48).

See [3] for the proof of uniqueness.

3.1.3 Singular perturbations with diffusions in the whole space

Setting of the problem

We consider

$$f(x,y,v) : \mathcal{R}^n \times \mathcal{R}^d \times U \to \mathcal{R}^n \tag{3.53}$$
$$g(x,y,v) : \mathcal{R}^n \times \mathcal{R}^d \times U \to \mathcal{R}^d$$
$$l(x,y,v) : \mathcal{R}^n \times \mathcal{R}^d \times U \to \mathcal{R}$$
$$\text{continuous and bounded}$$

$$U_{ad} \text{ compact } \quad \subset U \quad \text{(a metric space).} \tag{3.54}$$

On a convenient set $(\Omega, A, F^t, P^\epsilon)$ we define a dynamic system, composed of a slow and a fast system described by the equations (3.1), with g replaced by $Fy + g(x,y,v)$. The cost function is defined by (3.2), and we are interested in the behavior of the value function $u_\epsilon(x,y)$. It is given as the solution of the Hamilton Jacobi Bellman equation (noting $A_y = -\Delta_y - Fy \cdot D$)

$$-\Delta_x u_\epsilon - \frac{1}{\epsilon} A_y u_\epsilon + \beta u_\epsilon \tag{3.55}$$

$$= H(x, D_x u_\epsilon, y, \frac{1}{\epsilon} D_y u_\epsilon)$$

$$u_\epsilon = 0, \quad \forall x \in \Gamma, \forall y$$

$$u_\epsilon \in W^{2,p,\mu}(\mathbf{O} \times \mathcal{R}^d), 2 \le p < \infty$$

By $W^{2,p,\mu}(\mathbf{O} \times \mathcal{R}^d)$ we mean in fact, (since $\mathbf{O}$ is bounded) the set of functions z such that $z\beta_\mu(y)$ belongs to $W^{2,p}(\mathbf{O} \times \mathcal{R}^d)$.

We shall denote by $v_\epsilon(x, y)$ the optimal feedback. The assumption (3.23) is replaced by

$$A\psi - g(x, y, v) \cdot D\psi \ge 1, \forall x, v, y \in \mathcal{R}^d - \mathbf{D} \tag{3.56}$$

and the requirement that $\mathbf{D}, \psi$ have the same properties as in (3.23).

Approximation to the invariant measure

We shall consider the following invariant measures. For a feedback $v(y)$, consider $m^v(x, y)$ which is the solution of

$$A_y^* m + \mathrm{div}_y(mg^v) = 0 \tag{3.57}$$

$$m > 0, \int_{\mathcal{R}^d} m(x, y)dy = 1, m \in H^1(\mathcal{R}^d), \forall x.$$

For a feedback $v(x, y)$ we shall consider $m_\epsilon^v(x, y)$ which is the solution of

$$-\epsilon\Delta_x m_\epsilon + A_y^* m_\epsilon + \mathrm{div}_y(mg^v) = 0 \tag{3.58}$$

$$\frac{\partial m_\epsilon}{\partial \nu}|_\Gamma = 0, m_\epsilon \in H^1(\mathbf{O} \times \mathcal{R}^d)$$

$$m_\epsilon > 0, \int_{\mathcal{R}^d} m_\epsilon(x, y)dy = 1, \quad \forall x.$$

In particular, we shall call m_ϵ the solution of (3.58) corresponding to the feedback $v_\epsilon(x, y)$ as defined in the preceding paragraph. The construction of the invariant probability m_ϵ is done in a way similar to that of m. Let us consider $\mathbf{D}, \mathbf{D}_1$ as in (3.23). To avoid confusion in the notation, let us call Γ, Γ_1 the respective boundaries of $\mathbf{D}, \mathbf{D}_1$ (instead of Γ, Γ_1, since now Γ denotes the boundary of $\mathbf{O}$). We consider the stochastic processes

$$dx = \sqrt{\epsilon}dw - \chi_\Gamma(x, t)\nu d\xi, x(0) = x$$

$$dy = Fy + g(x, y, v(x, y))dt + \sqrt{2}db_v(t), y(0) = y$$

which are defined on a system $(\Omega, A, F^t, P_v^{x,y})$ and w, b are independent standard Wiener processes.

We define

$$\theta'(x,y;\Omega) = \inf\{t | y(t) \in \mathbf{D}\}$$

$$\theta(x,y;\Omega) = \inf\{t \geq \theta' | y(t) \notin \mathbf{D}_1\}$$

and we have (c.f. (3.30))[2]

$$E\theta'(x,y) \leq \psi(y).$$

Define the sequence of stopping times $\tau_0 = 0, \tau_n, \tau'_{n+1}$ as in section 1.2, and the Markov chain $X_n = x(\tau_n), Y_n = y(\tau_n)$ which is a Markov chain on $\mathbf{O} \times \Gamma_1$. We then define the linear operator on Borel bounded functions on $\mathbf{O} \times \Gamma_1$ by the relation

$$P^\epsilon \phi(x,y) = E_v^{xy} \phi(x(\theta), y(\theta)). \tag{3.59}$$

We deduce the analytic formula (c.f. (3.36))

$$P^\epsilon \phi(x,y) = \eta_\epsilon(x,y) \tag{3.60}$$

where

$$-\epsilon\Delta_x \eta + A_y \eta - g^v \cdot D_y^\eta = 0, \tag{3.61}$$

$$\text{on} \quad \mathbf{O} \times (\mathcal{R}^d - \mathbf{D})$$

$$\eta|_\Gamma = \varsigma, \frac{\partial \eta}{\partial \nu}|_\Gamma = 0$$

$$-\epsilon\Delta_x \varsigma + A_y \varsigma - g^v \cdot D_y \varsigma = 0 \quad \text{on} \quad \mathbf{O} \times \mathbf{D}_1 \tag{3.62}$$

$$\varsigma|_{\Gamma_1} = \phi, \frac{\partial \varsigma}{\partial \nu}|_\Gamma = 0$$

The ergodicity of P^ϵ is proved like that of P (c.f. Lemma 1.1). Let $\pi^\epsilon(dx, d\sigma)$ be the corresponding invariant probability on $\mathbf{O} \times \Gamma_1$. We then define the probability $\mu^\epsilon(dx,dy)$ on $\mathbf{O} \times \mathcal{R}^d$ by the formula

$$\int_{\mathbf{O}} \int_{\mathcal{R}^d} \Lambda(x,y) d\mu^\epsilon(x,y) \tag{3.63}$$

$$= \frac{\int_{\mathbf{O}} \int_{\Gamma_1} [E_v^{\xi\eta} \int_0^{\theta(\xi,\eta)} \Lambda(x(t), y(t)) dt] \pi^\epsilon(d\xi, d\eta)}{\int_{\mathbf{O}} \int_{\Gamma_1} E_v^{\xi\eta} \theta(\xi,\eta) \pi^\epsilon(d\xi, d\eta)}$$

[2]Here $E = E_v^{x,y}$ for short.

for any Λ Borel bounded on $\mathbf{O} \times \mathcal{R}^d$. Let us note that we can also give an analytic formula for the quantity

$$\alpha^\epsilon(x,y) = E_v^{xy} \int_0^{\theta(x,y)} \Lambda(x(t), y(t))dt$$

namely

$$-\epsilon\Delta_x\alpha + A_y\alpha - g^v \cdot D_y\alpha = \Lambda \tag{3.64}$$

$$\text{in } \mathbf{O} \times (\mathcal{R}^d - \mathbf{D})$$

$$\alpha|_\Gamma = \beta, \frac{\partial\alpha}{\partial\nu}|_\Gamma = 0$$

$$-\epsilon\Delta_x\beta + A_y\beta - g^v \cdot D_y\beta = \Lambda \text{ in } \mathbf{D}_1$$

$$\beta|_{\Gamma_1} = 0, \frac{\partial\beta}{\partial\nu}|_\Gamma = 0.$$

We have

$$d\mu^\epsilon(x,y) = m^\epsilon(x,y)dxdy. \tag{3.65}$$

Moreover, considering the Cauchy problem

$$\frac{\partial z}{\partial t} - \epsilon\Delta_x z + A_y z - g^v \cdot D_y z = 0 \tag{3.66}$$

$$\frac{\partial z}{\partial\nu}|_\Gamma = 0, z(x,y,0) = \Lambda(x,y)$$

we have

$$\int_{\mathbf{O}}\int_{\mathcal{R}^d} \Lambda(x,y)m^\epsilon(x,y)dxdy = \int_{\mathbf{O}}\int_{\mathcal{R}^d} m^\epsilon(x,y)z(x,y,t)dxdy,$$

$$\forall t > 0$$

and we deduce from this

$$0 < \Delta_k < m_\epsilon(x,y) < \Delta_1, \tag{3.67}$$

$$\forall x \in \mathbf{O}, \forall y \in K, \quad \text{compact of } \mathcal{R}^d$$

with constants uniform with respect to $v(\cdot)$, the left constant (but not the right) depending on the compact K.

To proceed we shall slightly reinforce the assumption (3.66) as follows

$$A\psi - k_0|D\psi| \geq 1, \forall y \in \mathcal{R}^d - \mathbf{D} \tag{3.68}$$

and $\mathbf{D}, \psi$ have the same properties as in (3.23). In (3.68) k_0 is a constant such that

$$|g(x, y, v)| \leq k_0 \tag{3.69}$$

Note that (3.68) is satisfied in the example (3.27).

Lemma 2.1. *Let B_ρ be the ball of radius ρ in $\mathcal{R}^d$, and $\overline{B}_\rho = \mathcal{R}^d - B_\rho$. Then*

$$\int_{\mathbf{O}} \int_{\overline{B}_\rho} m_\epsilon(x, y) dx dy \leq \Delta(\rho) \tag{3.70}$$

where $\Delta(\rho) \to 0$ as $\rho \to \infty$.

Consider also as in (3.46) the solution $m_{\epsilon R}$ of

$$- \epsilon \Delta_x m_{\epsilon R} + A_y^* m_{\epsilon R} + \operatorname{div}_y(m_{\epsilon R} g^v) \tag{3.71}$$

$$+ \Lambda m_{\epsilon R} = \Lambda r_R m_{\epsilon R}$$

$$\frac{\partial m_{\epsilon R}}{\partial \nu}|_\Gamma, m_{\epsilon R}|_{\partial B_R} = 0$$

then we have

$$m_{\epsilon R} \to m_\epsilon \text{ in } L^1 \subset H^1 \text{ as } R \to \infty. \tag{3.72}$$

A priori estimate

We shall need the approximation of u_ϵ given by

$$- \Delta_x u_{\epsilon R} - \frac{1}{\epsilon} A_y u_{\epsilon R} + \beta u_{\epsilon R} \tag{3.73}$$

$$= H(x, D_x u_{\epsilon R}, y, \frac{1}{\epsilon} D_y u_{\epsilon R})$$

$$u_\epsilon = 0 \text{ on } \partial(\mathbf{O} \times B_R)$$

and

$$u_{\epsilon R} \to u_\epsilon \text{ in } W^{2,p}_{\text{loc}} \text{ weakly and in } L^\infty \text{ weak star} \tag{3.74}$$

where *loc* is meant only for the y variable. We shall need also a similar approximation in the case of explicit feedbacks; in particular v_ϵ

Lemma 2.2. *The following estimates hold*

$$|D_x u_\epsilon|^2_{L^2_{\text{loc}}} \leq C, |u^\epsilon|_{L^\infty} \leq C \tag{3.75}$$

$$|D_y u_\epsilon|^2_{L^2_{\text{loc}}} \leq C^\epsilon$$

Lemma 2.3. *Let $\phi(x) \in H_0^1(O) \subset H^2(O)$, then we have the inequality*

$$\int\int m_\epsilon |D_x(u_\epsilon - \phi)|^2 dxdy \tag{3.76}$$

$$+\frac{1}{\epsilon}\int\int m_\epsilon |D_y u_\epsilon|^2 dxdy$$

$$+\int\int \beta m_\epsilon (u_\epsilon - \phi)^2 dxdy \leq \int\int m_\epsilon u_\epsilon (\Delta\phi - \beta\phi) dxdy$$

$$+\int (|D_x\phi|^2 + \beta\phi^2) dx$$

Convergence

Lemma 2.4. *Let us consider a subsequence of u_ϵ such that*

$$u_\epsilon \to u \text{ in } H^2_{\text{loc}}(O \times \mathcal{R}^d) \text{ weakly.} \tag{3.77}$$

Then u is a function of x only, belongs to $H_0^1(O)$, and the convergence (3.77) is strong.

We now identify the limit. Let us recall the definition of m^v given in (3.57). Define $\chi(x,p)$ by the formula

$$\chi(x,p) = \inf_{v(\cdot)} \int_{\mathcal{R}^d} m^{v(x,y)} (l(x,y,v(y)) \tag{3.78}$$

$$+ p \cdot f(x,y,v(y))) dy$$

and consider the Dirichlet problem

$$-\Delta u + \beta u = \chi(x, Du), \tag{3.79}$$

$$u|_\Gamma = 0, u \in W^{2,p}(O)$$

We can then state the following

Theorem 2.1. *We assume (3.53), (3.54) and (3.68). Then we have*

$$u_\epsilon \to u \text{ in } H^2_{\text{loc}}(O \times \mathcal{R}^d) \text{ strongly} \tag{3.80}$$

See [3] for details of the proof.

Interpretation of the limit problem

The limit problem is written as

$$-\Delta u + \beta u = \inf_{v(\cdot)} \{\tilde{l}(x, v(\cdot)) \tag{3.81}$$

$$+Du \cdot \tilde{f}(x, v(\cdot))\} \quad u|_\Gamma = 0$$

where we have set

$$\tilde{l}(x, v(\cdot)) = \int_Y m^v(x,y) l(x, y, v(y)) dy \tag{3.82}$$

$$\tilde{f}(x, v(\cdot)) = \int_Y m^v(x,y) f(x, y, v(y)) dy.$$

It is clear that (3.81) is a Hamilton Jacobi Bellman equation for a slow system whose drift if $\tilde{f}$, and integral cost is $\tilde{l}$. For this problem the set of controls is the set of Borel functions $v(y)$ with values in U_{ad}. A feedback on the slow system is thus still a function $v(x, y)$. There exists an optimal feedback for the limit problem, namely $\hat{v}(x, y)$ obtained in (3.6). Indeed consider the function $\hat{V}$ defined in (3.5), then

$$\hat{v}(x, y) = \hat{V}(x, Du, y, D_y\phi)$$

is an optimal feedback for the limit problem. In fact, this is the feedback to be applied on the real system as a surrogate for $v_\epsilon(x, y)$ defined in (3.58). One can show by techniques similar to those used in previous paragraphs to obtain Theorem 2.1, that the corresponding cost function will converge as ϵ tends 0 to u in $H^1(O \times Y)$. Note that unlike the deterministic situation the optimal feedback for the limit problem is not a function of x only. In fact (3.83) corresponds to the composite feedback of Chow-Kokotovic [7] (c.f. also [8] in the deterministic case).

3.2 Optimal Stochastic Scheduling of Systems with Poisson Noises

In this section we consider the problem of optimal stochastic scheduling for nonlinear systems with Poisson noise disturbances and a performance index including both operating costs and costs for scheduling changes. In general, the value functions of the dynamic programming, quasi-variational inequalities which define the optimality conditions for such problems are not differentiable. However, we can treat them as "viscosity solutions" as introduced by Crandall and Lions. Existence and uniqueness questions are studied from this point of view.

3.2.1 Introduction

Optimal scheduling problems arise in many contexts, including inventory control systems and resource allocation problems in military systems planning. These problems typically involve stochastic dynamical systems, admitting discrete state transitions at random times as control actions, and incurring both switching costs and continuous running costs. Using the dynamic programming principle, one can show that the optimality conditions for these problems are expressed mathematically by *quasi-variational inequalities* (QVI). It is difficult to treat QVI's explicitly, and most of the work has focussed on proving existence, uniqueness, and regularity of solutions.

In our case, the state system is forced by Poisson noises. Since the infinitesimal generator of the state process is first order and has a translation in the argument, the associated QVI is first order and fully nonlinear; and so, the standard existence and uniqueness theory developed for diffusion - parabolic systems does not apply. To treat the problem, we use the method of *viscosity solutions* introduced by M. G. Crandall and P. L. Lions [9]. Various properties of viscosity solutions are developed in Crandall - Evans - Lions [11]. We use the approach in Capuzzo Dolcetta - Evans [12] developed for deterministic systems.[3]

We prove that the value function u associated with the optimization problem is a viscosity solution of the corresponding (QVI). Existence of solutions to the (QVI) is shown by using a discrete approximation to an associated penalized system and then using results for accretive operators as in [15]. On the other hand, we use dynamic programming to obtain a decreasing sequence of value functions u^ℓ optimal for controls with at most l switches, which converges uniformly. This approach was used to obtain a maximum solution of certain (QVI's) in Menaldi [10-11] without nondegeneracy assumptions. In Blankenship - Menaldi [20], related problems were treated involving the application of (QVI) to power generation systems with scheduling delays. See also [21][22] for a survey of viscosity methods for the

[3]Cases with white noise models are treated in [13] and [14], while control problems for diffusion processes with jumps are treated in Bensoussan [15]. See also [16] for an introduction to the subject.

control of diffusions.

The optimal stochastic control of linear regulator systems with Poisson noise disturbances is considered in [23]; stochastic stability properties of linear systems with multiplicative Poisson noises are derived in [25]. See also [26].

Problem Statement

Let (Ω, F, P) be a probability space and $F_t, t \geq 0$ a non-decreasing, right-continuous family of completed sub σ-fields of F such that $F_t \uparrow F_\infty := F, t \geq 0$. Consider the general nonlinear dynamical system

$$\begin{cases} dy_x(t) = & g(y_x(t), \alpha(t))dt + h(y_x(t), \alpha(t))dN_{\alpha(t)}(t) \\ y_x(0) = & x \end{cases} \tag{3.84}$$

where $N_i(t), i = 1, \ldots, m$, are independent Poisson processes with intensities λ_i, $i = 1, \ldots, m$. $\alpha(t)$ is a right continuous, piecewise constant random function with finite range $1, \ldots, m$, and is measurable with respect to $F_t, t \geq 0$. Actually, α is an admissible control consisting of random switching times θ_i and random switching decisions d_i such that θ_i are adapted to F_t and d_i are F_{θ_i} - measurable so that

$$0 \equiv \theta_0 \leq \theta_1 \leq \ldots \leq \theta_{i-1} \leq \theta_i \leq \theta_{i+1}, \theta_i \to +\infty \text{ a.s.}$$

$$d_i \in \{1, \ldots, m\}, d_i \neq d_{i-1} \text{ if } \theta_i < \infty \tag{3.85}$$

And so

$$\alpha(t) := d_i \text{ if } \theta_i \leq t < \theta_{i+1}, i \geq 0$$

is indeed F_t - measurable.

Let the set of all admissible controls with initial setting d be

$$A^d := \{\alpha | \alpha = \{\theta_i, d_i\} \text{ satisfies the "above" properties} \tag{3.86}$$

$$\text{with initial setting } d_0 = d\}.$$

We take the performance index to be

$$J_x^d(\alpha) := E_{x,d}\{\int_0^\infty f(y_x(t), \alpha(t))e^{-\beta t}dt + \sum_{i=1}^\infty k(d_{i-1}, d_i)e^{-\beta\theta_i}\}$$

$$= E_{x,d}\{\sum_{i=1}^\infty [\int_{\theta_{i-1}}^{\theta_i} f(y_x(t), d_{i-1})e^{-\beta t}dt + k(d_{i-1}, d_i)e^{-\beta\theta_i}]\} \tag{3.87}$$

where $\beta > 0$ is a discount factor and $k(d, \hat{d})$ is the cost of switching from d to $\hat{d}$ such that[4]

$$k(d, \hat{d}) > 0 \text{ if } d \neq \hat{d};\, k(d, d) = 0 \tag{3.88}$$

$$k(d, \hat{d}) < k(d, \tilde{d}) + k(\tilde{d}, \hat{d}) \text{ if } d \neq \tilde{d} \neq \hat{d}.$$

Without loss of generality, we can define $k_0 := \min k(d, \hat{d}),\ d \neq \hat{d}$. We assume $f \geq 0$, g and h are bounded and Lipschitz continuous

$$|q(x, d)| \leq ||q|| < \infty$$

$$|q(x, d) - q(\hat{x}, d)| \leq L|x - \hat{x}| \tag{3.89}$$

with $q = f$, g and h, for all x, $\hat{x} \in \mathcal{R}^n$, $d \in 1, \ldots, m$.

Under these assumptions, (1.1) has a unique solution. Defining the value function

$$u^d(x) := \inf_{\alpha \in A^d} J_x^d(\alpha), x \in \mathcal{R}^n, d \in \{1, \ldots, m\} \tag{3.90}$$

we want to design an optimal control α^* such that

$$u^d(x) = J_x^d(\alpha^*) = \inf_{\alpha \in A^d} J_x^d(\alpha). \tag{3.91}$$

Remark. $N_{\alpha(t)}(t)$ is an inhomogeneous Poisson process with intensity function $\lambda_{\alpha(t)}$.

Summary of Results.

In subsection 2.2 we show that the optimal value function $u^d(x)$ in (3.21) maybe defined as the limit of the value functions $u_\ell^d(x)$ of systems with a finite number ℓ of switches as $\ell \to \infty$ (Theorem 2.3). We show that the convergence is uniform (Theorem 2.5); and we derive two representations of $u^d(x)$ as the *optimal* value function (Theorems 2.6 and 2.7). We describe the associated optimal (control) switching policy (Theorem 2.8), and we use it to obtain an additional estimate on the convergence of u_ℓ^d to u^d.

In subsection 2.3 we derive the QVI which must be satisfied by the optimal value function (equation (3.113)). We show that the optimal value function is a viscosity solution of the QVI (Theorem 3.1). Then we show that the solution is unique.

In subsection 2.4 we prove that the QVI has a viscosity solution by constructing a sequence of solutions to a penalized system (equation (3.119))

[4]The case when the switching costs can be zero is treated in subsection 2.5.

and proving that these solutions are uniformly bounded and uniformly Holder continuous (Theorem 4.4). We show that the limit of the sequence of solutions to the penalized system is a viscosity solution of the QVI.

In subsection 2.5 we consider the case when the switching costs vanish ($k(d,\hat{d}) = 0$ for $d \neq \hat{d}$ in (3.19). In this case the optimal value function u is independent of the initial control configuration d (since we can switch without cost at any time), and it (formally) satisfies a Hamilton - Jacobi - Bellman equation which is fully nonlinear in ∇u. The method of viscosity solutions is required to treat this case. We show that the optimal value function corresponding to non-zero switching costs will converge to u as the switching costs tend to zero, and that u is the unique viscosity solution of the Hamilton - Jacobi - Bellman equation. The result is analogous to those in Capuzzo Dolcetta - Evans [12]. Thus, the method of viscosity solutions provides a *complete* framework for the treatment of the optimal control problem (3.16) - (3.23) over the full range of parameter values and operating regimes.

3.2.2 Dynamic Programming and Preliminary Results.

Before using dynamic programming to investigate the properties of the value function $u^d(x)$, we need some preliminary results.

Lemma 2.1. *For any stopping time τ which is adapted to F_t and any measurable bounded function q, we have*

$$E[q(y_x(t+\tau))|F_\tau] = E_{y_x(\tau)}q(y_{y_x(\tau)}(t)). \tag{3.92}$$

Proof See [24] for the proof of this and all other lemmas and theorems in this section.

Lemma 2.2. *(i) For each $d \in \{1,\ldots,m\}$ and $x \in \mathcal{R}^n$,*

$$u^d(x) \leq \min_{\hat{d}\neq d}\{u^{\hat{d}}(x) + k(d,\hat{d})\} \tag{3.93}$$

(ii) For any stopping time $\theta \geq 0$,

$$u^d(x) \leq E\{\int_0^\theta f(y_x(s),d)e^{-\beta s}ds + u^d(y_x(\theta))e^{-\beta\theta}\} \tag{3.94}$$

Notation. For $x \in \mathcal{R}^n$, $d \in 1,\ldots,m$,

$$M^d[u](x) := \min_{\hat{d}\neq d}\{u^{\hat{d}}(x) + k(d,\hat{d})\}. \tag{3.95}$$

Now, we want to use the dynamic programming principle to show there exists a convergent sequence u_ℓ^d of optimal solutions of the problem with respect to controls which have at most ℓ switches.

For each $x \in \mathcal{R}^n$, $d \in 1, \ldots, m$, let

$$u_0^d(x) := \int_0^\infty f(y_x(s), d)e^{-\beta s}ds. \tag{3.96}$$

Notation. If $u, v \in C(\mathcal{R}^n)^m$, then we say $u \geq v$ if $u^d \geq v^d$, $\forall d = 1, \ldots, m$.

Define an operator $\Gamma_d : C(\mathcal{R}^n)^m \to C(\mathcal{R}^n)$ by

$$\Gamma_d u(x) := \inf_{\theta \geq 0} E\left\{\int_0^\theta f(y_x(s), d)e^{-\beta s}ds + e^{-\beta\theta}M^d[u](y_x(\theta))\right\}. \tag{3.97}$$

Here we understand the *infimum* is taken for all stopping times $\theta \geq 0$ adapted to F_t. If $u \geq v$, then for each $\epsilon > 0$, there exists a stopping time $\theta_\epsilon \geq 0$ and d_ϵ F_{θ_ϵ} - measurable such that

$$\Gamma_d u(x) > E\left\{\int_0^{\theta_\epsilon} f(y_x(s), d)e^{-\beta s}ds + e^{-\beta\theta_\epsilon}[u_\epsilon^d(y_x(\theta_\epsilon)) + k(d, d_\epsilon)]\right\} - \epsilon$$

$$\geq E\left\{\int_0^{\theta_\epsilon} f(y_x(s), d)e^{-\beta s}ds + e^{-\beta\theta_\epsilon}[v_\epsilon^d(y_x(\theta_\epsilon)) + k(d, d_\epsilon)]\right\} - \epsilon$$

$$\geq \Gamma_d v(x) - \epsilon.$$

Let $\epsilon \downarrow 0$, we have $\Gamma_d u \geq \Gamma_d v$. Let $0 \leq \eta \leq 1$, then

$$\Gamma_d[(1-\eta)u + \eta v]$$

$$= \inf_{\theta \geq 0} E\left\{\int_0^\theta f(y_x(s), d)e^{-\beta s}ds + e^{-\beta\theta}M^d[(1-\eta)u + \eta v](y_x(\theta))\right\}$$

$$\geq \inf_{\theta \geq 0} E\left\{\int_0^\theta f(y_x(s), d)e^{-\beta s}ds + e^{-\beta\theta}\{(1-\eta)M^d[u](y_x(\theta))\right.$$

$$\left. + \eta M^d[v](y_x(\theta))\}\right\}$$

$$\geq (1-\eta)\Gamma_d u(x) + \eta\Gamma_d v(x).$$

Thus, Γ_d is a non-decreasing, concave function.

Suppose we are given $u_{\ell-1}$. We can define

$$u_\ell^d(x) := \Gamma_d u_{\ell-1}(x). \tag{3.98}$$

Since $u_1^d(x) = \Gamma_d u_0(x) \le u_0^d(x)$, then by the non-decreasing property of Γ_d, we have $u_2^d(x) = \Gamma_d^2 u_0(x) \le \Gamma_d u_0(x)$, and so

$$0 \le u_\ell^d(x) \le u_{\ell-1}^d(x) \le \ldots \le u_0^d(x) \le \frac{||f||}{\beta}. \tag{3.99}$$

Thus, $u_\ell^d(x)$ converges. We can define

$$u_\infty^d(x) := \lim_{\ell\to\infty} u_\ell^d(x). \tag{3.100}$$

Theorem 2.3.

$$u_\ell^d(x) = \inf\{J_x^d(\alpha_\ell) | \alpha_\ell \in A^d \text{ has at most } \ell \text{ switches }\} \tag{3.101}$$

and thus

$$u_\infty^d(x) = u^d(x) := \inf_{\alpha\in A^d} J_x^d(\alpha). \tag{3.102}$$

Lemma 2.4. *For each* $0 < \gamma < \min\left\{1, \frac{\beta}{L(1+\lambda_{\max})}\right\}$,

$$|u_\ell^d(x) - u_\ell^d(\hat{x})| \le C_\gamma |x - \hat{x}|^\gamma \tag{3.103}$$

for all $1 \le l \le \infty$ *and* $x, \hat{x} \in \mathcal{R}^n$ *with*

$$C_\gamma = \frac{||f||^{1-\gamma} L^\gamma}{\beta - \gamma L(1+\lambda_{\max})} \tag{3.104}$$

where

$$\lambda_{\max} = \max\{\lambda_1, \ldots, \lambda_m\}.$$

If $\beta > L(1+\lambda_{\max})$, *then* γ *can be taken to be* 1.

Remark. Since N_i has independent increments, then F_s is independent of any sub σ-field generated by $N_i(t) - N_i(s), s \le t, i = 1, \ldots, m$, so that for $t \ge s$,

$$E[|y_x^\ell(t) - y_{\hat{x}}^\ell(t)| \ |F_s] \le |y_x^\ell(s) - y_{\hat{x}}^\ell(s)| e^{L(1+\lambda_{\max})(t-s)}.$$

Thus,

$$|u^d(y_x^\ell(s)) - u^d(y_{\hat{x}}^\ell(s))| \le C_\gamma |y_x^\ell(s) - y_{\hat{x}}^\ell(s)|^\gamma \text{ a.s.}$$

Remark. If $k_0 \ge ||f||/\beta$, then $u_0(x)$ is the optimal solution, i.e., no switching occurs.

We can obtain the following estimate by the method in [18][19].

Theorem 2.5. *If* $0 < k_0 < ||f||/\beta$, *then*

$$||u_\ell^d - u_\infty^d|| \leq ||u_0^d||\,(1 - \beta k_0 ||f||)^\ell. \tag{3.105}$$

Thus, $u_\ell \downarrow u_\infty$ *uniformly.*

Theorem 2.6.

$$u_\infty^d(x) = \inf_{\theta \geq 0} E\{\int_0^\theta f(y_x(s), d)e^{-\beta s}ds + M^d[u_\infty](y_x(\theta))e^{-\beta\theta}\}. \tag{3.106}$$

Theorem 2.7. *If* $\exists x_0$ *such that* $u^d(x_0) < M^d[u](x_0)$, *then* $\theta_1 > 0$ *a.s and*

$$u^d(x_0) = E\{\int_0^\theta f(y_{x_0}(s), d)e^{-\beta s}ds + u^d(y_{x_0}(\theta))e^{-\beta\theta}\} \tag{3.107}$$

for all $0 \leq \theta \leq \theta_1$.

Now, suppose we have a Holder continuous function u^d satisfying (3.21). We can define an optimal policy $\alpha^* = \theta_i, d_i \in A^d$ as follows.

$$\theta_0 = 0, d_0 = d,$$

If we are given θ_{i-1}, d_{i-1}, then set

$$\theta_i := \inf\{\text{ stopping time } \theta \geq \theta_{i-1} | u^{d_{i-1}}(y_x(\theta)) = M^{d_{i-1}}[u](y_x(\theta)) a.s.\} \tag{3.108}$$

If $\theta_i < \infty$, set

$$d_i = \text{ any } F_{\theta_i} - \text{ measurable random variable } \tilde{d} \in \{1, \ldots, m\}, \tilde{d} \neq d_{i-1}$$

such that

$$M^{d_{i-1}}[u](y_x(\theta_i)) = u^{\tilde{d}}(y_x(\theta_i)) + k(d, \tilde{d}) \text{ a.s.} \tag{3.109}$$

and

$$y_x(t) \text{ controlled by decision } d_{i-1} \text{ when } \theta_{i-1} \leq t < \theta_i.$$

Theorem 2.8. *The control policy* α^* *defined by* (3.108 *and* (3.109) *is optimal, i.e.,* $u^d(x) = J_x^d(\alpha^*) = \min_{\alpha \in A^d} J_x^d(\alpha)$. *In addition,* $\theta_i \to \infty$ a.s. *as* $i \to \infty$.

Corollary 2.9. *We have the additional estimate*

$$||u_\ell^d - u_\infty^d|| \leq \frac{||f||^2}{\beta^2 k_0(\ell + 1)}. \tag{3.110}$$

3.2.3 Viscosity Solutions of the Quasi-Variational Inequality (QVI).

We want to derive necessary and sufficient conditions for the optimal solution $u^d(x), x \in \mathcal{R}^n, d \in 1, \ldots, m$. Assume for the moment that the value functions $u^1, \ldots, u^m$ belong to $C^1(\mathcal{R}^n)$. Then by the necessary condition in Lemma 2.2, we have

$$E\left\{\frac{u^d(x) - u^d(y_x(t))}{t}\right\} \leq E\left\{\frac{1}{t}\int_0^t f(y_x(s), d)e^{-\beta s}ds + \left(\frac{e^{-\beta t} - 1}{t}\right) u^d(y_x(t))\right\} \tag{3.111}$$

and so, we obtain a differential form as $t \downarrow 0$,

$$-g(x,d) \cdot \nabla u^d(x) - \lambda_d[u^d(x + h(x,d)) - u^d(x)] \leq f(x,d) - \beta u^d(x) \tag{3.112}$$

$\forall x \in \mathcal{R}^n$ and $d \in 1, \ldots, m$. Combining (3.93), (3.107) and (3.112), we obtain a quasi-variational inequality (QVI)

$$\max\{\beta u^d - g^d \cdot \nabla u^d - \lambda_d[u^d(\cdot + h^d) - u^d] - f^d, u^d - M^d[u]\} = 0 \tag{3.113}$$

on $\mathcal{R}^n$, where

$$f^d(\cdot) := f(\cdot, d), g^d(\cdot) := g(\cdot, d), h^d(\cdot) := h(\cdot, d). \tag{3.114}$$

Note that (3.113) is a fully nonlinear first order partial differential equation which does not admit a differentiable solution in general. But, we can treat it using the method of viscosity solutions, which was introduced by M. G. Crandall and P. L. Lions [9], and which was used for deterministic switching problems by I. Capuzzo Dolcetta and L. C. Evans [12].

We denote by $BUC(\mathcal{R}^n)^m$, the *space of bounded, uniformly continuous $\mathcal{R}^m$-valued functions on $\mathcal{R}^n$*.

Definition. A function $u = (u^d, \ldots, u^m) \in C(\mathcal{R}^n)^m$ is said to be a *viscosity solution* of the (QVI) if for each $d \in \{1, \ldots, m\}$ and each $\phi \in C^1(\mathcal{R}^n)$ such that

(i) if $u^d - \phi$ attains a local maximum at $x_0 \in \mathcal{R}^n$, then

$$\max\{\beta u^d(x_0) - g^d(x_0) \cdot \nabla\phi(x_0) - \lambda_d[u^d(x_0 + h^d(x_0)) - u^d(x_0)] - f^d(x_0), u^d(x_0) - M^d[u](x_0)\} \leq 0 \tag{3.115}$$

and

(ii) if $u^d - \phi$ attains a local minimum at $z_0 \in \mathcal{R}^n$, then

$$\max\{\beta u^d(z_0) - g^d(z_0) \cdot \nabla\phi(z_0) - \lambda_d[u^d(z_0 + h^d(z_0)) - u^d(z_0)] - f^d(z_0),$$
$$u^d(z_0) - M^d[u](z_0)\} \geq 0. \tag{3.116}$$

Theorem 3.1. *Under the previous assumptions, the value function* $u = (u^1, \ldots, u^m)$ *with*

$$u^d(x) := \inf_{\alpha \in A^d} J_x^d(\alpha)$$

is a viscosity solution of the (QVI) (3.113).

Before discussion the existence of a solution to the (QVI), we consider the conditions under which (3.113) admits a unique solution, so that any functions constructed to satisfy (3.113) must be the optimal solution.

Lemma 3.2. *If* $u = (u^1, \ldots, u^m)$ *is any viscosity solution of* (3.113), *then*

$$u^d(x) \leq M^d[u](x), \forall x \in \mathcal{R}^n, d \in \{1, \ldots, m\}. \tag{3.117}$$

Theorem 3.3. *If* $u = (u^1, \ldots, u^m)$ *and* $v = (v^1, \ldots, v^m)$ *are viscosity solutions of* (3.113). *Then* $u \equiv v$.

3.2.4 Existence of Viscosity Solutions.

Now, we use a finite difference approximation to construct a sequence of solutions which converges to the solution of (3.113).

Let $\rho \in C^2(\mathcal{R}^n)$ such that

$$\begin{cases} \rho(x) = 0, & x \leq 0 \\ \rho(x) > 0, & x > 0 \\ 0 < \rho'(x) \leq 1, \rho''(x) > 0 & \text{for } x > 0 \end{cases} \tag{3.118}$$

and $\rho_\epsilon(x) = \rho(x/\epsilon)$, $x \in \mathcal{R}^n$, $\epsilon > 0$.

Consider the penalized system for approximation.

$$\beta u_\epsilon^d(x) - \frac{1}{\epsilon}[u_\epsilon^d(x + \epsilon g^d(x)) - u_\epsilon^d(x)] - \lambda_d[u_\epsilon^d(x + h^d(x)) - u_\epsilon^d(x)]$$
$$+ \sum_{\hat{d} \neq d} \rho_\epsilon(u_\epsilon^d(x) - u_\epsilon^{\hat{d}}(x) - k(d, \hat{d})) = f^d(x) \tag{3.119}$$

or

$$u_\epsilon^d(x) - \frac{1}{\beta\epsilon}[u_\epsilon^d(x + \epsilon g^d(x)) - u_\epsilon^d(x)] - \frac{\lambda_d}{\beta}[u_\epsilon^d(x + h^d(x)) - u_\epsilon^d(x)]$$

$$+\frac{1}{\beta}\sum_{\hat{d}\neq d}\rho_\epsilon(u_\epsilon^d(x)-u_\epsilon^{\hat{d}}(x)-k(d,\hat{d}))=\frac{1}{\beta}f^d(x). \tag{3.120}$$

We define operators λ, Π_1, Π_2: $C(\mathcal{R}^n)^m \to C(\mathcal{R}^n)^m$ such that $\Lambda u = (\lambda^1 u, \ldots, \lambda^m u)$, $\Pi_1 u = (\Pi_1^1 u, \ldots, \Pi_1^m u)$ and $\Pi_2 u = (\Pi_2^1 u, \ldots, \Pi_2^m u)$ where

$$\lambda^d u(x) := -\frac{1}{\beta\epsilon}[u^d(x+\epsilon g^d(x))-u^d(x)] \tag{3.121}$$

$$\Pi_1^d u(x) := -\frac{\lambda_d}{\beta}[u^d(x+h^d(x))-u^d(x)] \tag{3.122}$$

$$\Pi_2^d u(x) := \frac{1}{\beta}\sum_{\hat{d}\neq d}\rho_\epsilon(u^d(x)-u^{\hat{d}}(x)-k(d,\hat{d})). \tag{3.123}$$

Definition. (i) An operator $S : X \to X$ with domain $D(S)$ is said to be *accretive* on the real Banach space X if

$$||x-\hat{x}+\gamma[S(x)-S(\hat{x})]|| \geq ||x-\hat{x}|| \tag{3.124}$$

for all x, $\hat{x} \in D(S)$, $\forall\gamma > 0$.

(ii) An operator S is said to be $m-$*accretive* on X if S is accretive on X and the range $R(I+\gamma S) = X$ for all $\gamma > 0$ (or equivalently for some $\gamma > 0$).

The following lemma is from Evans [17].

Perturbation Lemma 4.1. *If S is m-accretive on $X = C(\mathcal{R}^n)^m$ and T is accretive, Lipschitz continuous everywhere defined on X, then $(S+T)$ is m-accretive on X, in particular, the range $R(I+S+T) = C(\mathcal{R}^n)^m$.*

Lemma 4.2. *Λ is m-accretive on $C(\mathcal{R}^n)^m$.*

Lemma 4.3. *Π_1 and Π_2 are accretive.*

By the Perturbation Lemma 4.1, $\Lambda + \Pi$ is m-accretive and so, for each $\epsilon > 0$, we have a solution $u_\epsilon \in C(\mathcal{R}^n)^m$ of (3.120).

Theorem 4.4.

(i) $0 \leq u_\epsilon^d(x) \leq ||f||/\beta$, $\epsilon > 0$, $d \in 1, \ldots, m$.

(ii) *For each* $0 < \gamma < \min\left(\frac{\beta}{L(1+\lambda_{max})}, 1\right)$,

$$|u_\epsilon^d(x)-u_\epsilon^d(\hat{x})| \leq C_\gamma|x-\hat{x}|^\gamma, x \in \mathcal{R}^n, \epsilon > 0, d \in \{1, \ldots, m\}$$

with the same constant C_γ in (2.20). If $\beta > L(1+\lambda_{max})$, we can take $\gamma = 1$.

From the above lemma, u_ϵ^d are uniformly bounded and uniformly Holder continuous. Then by the Arzela-Ascoli Theorem, there exists a subsequence ϵ_ℓ such that $u_{\epsilon_\ell}^d \to u^d \in C(\mathcal{R}^n)$ for all $d \in 1, \ldots, m$. The convergence is uniform on each compact subset of $\mathcal{R}^n$. In fact, u is bounded and Holder continuous with the same Holder exponent γ.

Theorem 4.5. *$u_{\epsilon_\ell} \to u$ locally uniformly in $\mathcal{R}^n$ and u solves* (3.113) *in the viscosity sense.*

Remark. In general, u is only Holder continuous. If we know u has some regularity properties, say u' exists in some neighborhood, then one can show u satisfies (3.113) in the ordinary sense. The point is that the derivative of u is not continuous across characteristic curves.

3.2.5 The Case of Vanishing Switching Costs.

In the case when the switching costs vanish, $k(d, \hat{d}) = 0$ for some $\hat{d} \neq d$ in (3.19), then the dynamics may be switched at any time without incurring a cost; hence, the minimum cost does not depend on the initial control. That is,

$$u^1 = u^2 = \ldots = u^m := u \tag{3.125}$$

If we follow the arguments used in the previous sections, we can show that u is bounded and Holder continuous with the same Holder constant C_γ used in Lemma 2.4. If u were continuously differentiable on $\mathcal{R}^n$, then by the principle of dynamic programming, u would be (formally) a solution of the Hamiltonian - Jacobi - Bellman equation

$$\max_{d=1,\ldots,m} \{\beta u - g^d \cdot \nabla u - \lambda_d[u(\cdot + h^d) - u] - f^d\} = 0 \tag{3.126}$$

on $\mathcal{R}^n$. However, u is not always C^1. By invoking the same arguments used in subsection 4, we can show that u is the unique viscosity solution of (3.126) in the following sense:

Definition 5.1. A bounded and continuous function u on $\mathcal{R}^n$ is a *viscosity solution* of (3.126) if for each $\phi \in C^1(\mathcal{R}^n)$ such that

(i) if $u - \phi$ attains a local maximum at $x_0 \in \mathcal{R}^n$, then

$$\max_{d=1,\ldots,m} \{\beta u(x_0) - g(x_0)^d \cdot \nabla u(x_0) \tag{3.127}$$

$$-\lambda_d[u(x_0 + h^d(x_0)) - u(x_0)] - f^d(x_0)\} \leq 0$$

and

(ii) if $u - \phi$ attains a local minimum at $x_0 \in \mathcal{R}^n$, then

$$\max_{d=1,\dots,m} \{\beta u(z_0) - g(z_0)^d \cdot \nabla u(z_0) \tag{3.128}$$

$$-\lambda_d[u(z_0 + h^d(z_0)) - u(z_0)] - f^d(z_0)\} \geq 0$$

We now establish that the optimality system is *closed*; that is, each value function corresponding to non-zero switching costs will converge to u as the switching costs tend to zero. The result corresponds to a similar result in Capuzzo Dolcetta - Evans [12].

Theorem 5.1. *Suppose we have a set of switching costs* $\{k_\epsilon(d, \hat{d})\}$ *such that*

$$k_\epsilon(d, \hat{d}) > 0 \quad \forall d \neq \hat{d} \in \{1, \dots, m\} \tag{3.129}$$

$$k_\epsilon(d, \hat{d}) < k_\epsilon(d, \tilde{d}) + k_\epsilon(\tilde{d}, \hat{d}), d \neq \tilde{d} \neq \hat{d}$$

For each $\epsilon > 0$ *let* $u_\epsilon = (u_\epsilon^1, \dots, u_\epsilon^m)$ *be the unique viscosity solution of the corresponding QVI with switching costs* $\{k_\epsilon(d, \hat{d})\}$ *and let* u *be the unique viscosity solution of* (3.126). *If* $k_\epsilon(d, \hat{d}) \to 0$ *as* $\epsilon \to 0$ *for all* $d, \hat{d} \in \{1, \dots, m\}$, *then* $u_\epsilon^d \to u$ *as* $\epsilon \to 0$ *for all* $d \in \{1, \dots, m\}$.

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3.3 Almost Sure Stability of Linear Stochastic Systems with Poisson Process Coefficients

In this section we consider the problem of determining the sample path stability of a class of linear stochastic differential equations with point process coefficients. Necessary and sufficient conditions are obtained which are similar in spirit to those derived by Khas'minskii and Pinsky for diffusion processes. The conditions are based on the deep theorems of Furstenburg on the asymptotic behavior of products of random matrices. Estimates on the probabilities of large deviations for stable processes are also given; together with a result on the stabilization of unstable systems by feedback controls.

3.3.1 The Problem and Main Results.

Consider the linear stochastic system

$$dx(t) = Ax(t)dt + \sum_{i=0}^{m} B_i x(t) dN_i(t), \tag{3.130}$$

$$x(0) = x_0 \in \mathcal{R}^n \setminus \{0\}, t \geq 0,$$

on the underlying probability space (Ω, F, P) with A and B_i constant $n \times n$ real matrices, and $\{N_i(t), t \geq 0\}$, $i = 1, \ldots, m$, independent Poisson processes - specifically, one dimensional counting process with intensity $\lambda_i > 0$ and right-continuous paths. $N_i(t) \in \{0, 1, 2, \ldots\}$ counts the number of occurrences in $[0, t]$. We are interested in the almost sure stability properties of the solutions of (3.130). That is, if $|\cdot|$ is any norm on $\mathcal{R}^n$ ($||\cdot||$ is the induced matrix norm), we would like to characterize the asymptotic exponential growth rate

$$\lim_{t \uparrow \infty} \frac{1}{t} \log \left(\frac{|x(t)|}{|x_0|} \right) \tag{3.131}$$

if it exists.

This problem is the analog of the one considered by Khas'minshii [27] and Pinsky [28] for diffusion processes, and by Loparo and Blankenship [29] for systems with jump process coefficients. Like previous results, the expression given here for the growth rate is not an explicit, readily computable one, except in simple cases. The stability properties of the moments of the solution of (3.130) were considered by Marcus [30] [31] (see also [32]). Explicit stability criteria are possible for the moments. Related results on the optimal control and scheduling of systems with Poisson noises are given in [23] [24]. See also [26].

The system (3.130) is interpreted in terms of the integral equation

$$x(t) = x_0 + \int_0^t Ax(s)ds + \sum_{i=1}^{m} \int_0^t B_i x(s) dN_i(s) \tag{3.132}$$

with the stochastic integral defined by the calculus explained in [31][33].[5] Let $\{\tau_j^i, j \geq 1\}$ be the interarrival times and $t_j^i = \tau_1^i + \cdots + \tau_j^i$ be the occurrence time for the Poisson process $N_i(t)$. Then

$$\int_0^t B_i x(s) dN_i(s) \equiv \begin{cases} 0, & N_i(t) = 0 \\ \sum_{j=1}^{N_i(t)} B_i x(t_j^{i-}), & N_i(t) \geq 1. \end{cases} \tag{3.133}$$

Now, let $\{\tau_j, j \geq 1\}$ be the interarrival times of the sum process $N(t) = N_1(t) + \cdots + N_m(t)$ with intensity $\lambda = \lambda_1 + \cdots + \lambda_m$, and μ_j be the process indicating which N_i under went an increment at the occurrence time $t_j = \tau_1 + \cdots + \tau_j$. We assume the probability of multiple, simultaneous jumps is

[5] We could also treat some of the more complicated point process models in [31] [33], but the main ideas are best conveyed by the simple case considered here.

zero. The process $x(t), t \geq 0$ exists, has right continuous paths, and jumps at $t_j, j = 1, 2, \ldots$ If we set $D_i = I + B_i$, then

$$x(t) = e^{(A(t-t_{N(t)}))} D_{\tau_{N(t)}} \cdots D_{\tau_1} e^{(A\tau_1)} x_0. \tag{3.134}$$

This expression is the basis of our treatment of the almost sure stability problem. Its composition as a product of random matrices directed our attention to the work of Furstenberg and Kesten [35], Grenander [36] and Furstenberg [38]-[41] on the limits of products of random matrices.

Our main result is based on the following observations. First, for each $i = 1, \ldots, m$, the $\{\tau_j^i, j \geq 1\}$ are independent and exponentially distributed with parameter λ_i. The random processes $\{\tau_j, \mu_j, j \geq 1\}$ depend in a complex way on $\{\tau_j^i, i = 1, \ldots, m, j \geq 1\}$. However, $\{\tau_i, i \geq 1\}$ and $\{\mu_j, j \geq 1\}$ are independent and form independent, identically distributed sequences. This follows from the presumed independence of the $\{N_i(t), i = 1, \ldots, m\}$; see [25]. As a consequence, we have the following:

Theorem (Stability). *Consider the system (3.130) with the stated assumptions on the processes $N_i(t), i = 1, \ldots, m$. Then*

$$r = \lim_{k \to \infty} \frac{1}{k} E \log ||D_m u_k e^{A\tau_k} \ldots D_m u_1 e^{A\tau_1}|| < \infty \tag{3.135}$$

exists and

$$r = \lim_{k \to \infty} \frac{1}{k} \log ||D_m u_k e^{A\tau_k} \ldots D_m u_1 e^{A\tau_1}|| \quad \text{a.s.} \tag{3.136}$$

The quantity r is the *asymptotic exponential growth rate* of the process $x(t)$; that is,

$$\frac{|x(t)|}{|x(0)|} \approx e^{rt} \text{ for } t \text{ large}$$

Hence, $r > 0$ implies almost sure instability and $r < 0$ corresponds to almost sure asymptotic stability. This result is proved in section 3 of [25].

It is possible to obtain a more detailed description of the long term behavior of $\{x(t), t \geq 0\}$ by examining the behavior of products of random matrices acting on specific initial states $x(0) = 0$. The key questions are: Does the limit of

$$\frac{1}{k} \log ||D_{\mu_k} e^{A\tau_k} \cdots D_{\mu_1} e^{A\tau_1} x_0||$$

exist? If it does, how is it related to the rate r in (3.136)? To treat these questions, we generalize some results of Furstenburg, Kesten, Grenander and others on random walks on semi-simple Lie groups to general semi-groups (not necessarily groups since the terms D_k may be singular). This analysis is given in section 4 of [25]. The main result is as follows:

Suppose μ is the measure on the Borel sets $B(\mathcal{R}^{n\times n})$ defined by

$$\mu(\Gamma) \equiv P\{D_{\mu_1}e^{A\tau_1} \in \Gamma\}, \Gamma \in B(\mathcal{R}^{n\times n}).$$

Let SG be the closed semi-group generated by the support of μ, i.e.

$$SG \equiv \text{ smallest closed semi-group } \supset \{D_i e^{At}, 0 \le t < \infty, i = 1, \ldots, m\}.$$

Let ν be an invariant measure for μ; i.e., a solution of the integral equation

$$\mu \star \nu = \nu \tag{3.137}$$

Let Q_0 be the collection of extremal invariant probability measures of μ on $M \equiv S^{n-1} \cup \{0\}$.

Theorem *For all $\nu \in Q_0$,*

$$r_\nu \equiv \sum_{i=1}^{m} \lambda_i \int_M \int_0^\infty \log|D_i exp(At)u| e^{-\lambda t} dt d\nu(u) < infty \tag{3.138}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log\left(\frac{|x(t)|}{|x_0|}\right) = \lambda r_\nu \text{ a.s.} \tag{3.139}$$

for all $x_0 \in E_\nu^0$, an ergodic component corresponding to $\nu \in Q_0$. Indeed, there are only finite different values, say, $r_1 < r_2 < \ldots < r_\ell = r, \ell \le n$. Furthermore, if $\cup_{\nu\in Q_0} E_\nu^0$ contains a basis of $\mathcal{R}^n$, then the system (3.130) *is asymptotically stable almost surely if $r_\ell < 0$, while the system* (3.130) *is asymptotically unstable if $r_1 > 0$. In case $r_1 < 0$ and $r_\ell > 0$, then the stability of the system depends on the initial state x_0.*

To apply these theorems to a specific problem, one must determine r or at least its sign; or, more generally, the collection Q_0 must be constructed and r_ν computed. If the semi-group SG is *transient* or *irreducible*, then r_ν will be independent of ν (even though there may be many ergodic components). (See Theorem 4.10 and Corollary 4.11 of [25].) In this case a theorem of Furstenburg ([38], Theorem 8.6) may be used to determine the sign of $r_\nu = r$. Application of this result to specific systems requires a close analysis of the geometric structure of the semi-group associated with those systems. Several examples are given in the next subsection to illustrate the techniques.

Two final results of interest in engineering practice concern the occurence of *large deviations* in the paths of $\{x(t), t \ge 0\}$ of a stable system (3.130) and the ability to stabilize a system like (3.130) with feedback controls.

The following result is proved in section 5 of [25].

Theorem (Large deviations). *If the system* (3.130) *is asymptotically stable with* $r_\nu < 0$, *then there exist constants* $M(x_0, R)$ and $r_\nu\lambda < \gamma < 0$ *such that*

$$\mathbf{P}\{\sup s \geq t|x(s)| \geq R\} \leq M(x_0, R)e^{\gamma t}, t \geq 0. \tag{3.140}$$

The constants may be determined rather precisely, see [25] for details.

Theorem (Stabilisation). *The control system with state and control dependent Poisson noises*

$$dx(t) = Ax(t)dt + Bu(t)dt + Cx(t)dN_1(t) + Du(t)dN_2(t) \tag{3.141}$$

is stabilized by the linear feedback control $u(t) = -Kx(t)$ *almost surely where* K *is any matrix such that*

$$\lambda_1 \int_0^\infty \log||(I+C)e^{(A-BK)t}||e^{-\lambda t}dt \tag{3.142}$$

$$+\lambda_2 \int_0^\infty \log||(I-DK)e^{(A-BK)t}||e^{-\lambda t}dt < 0$$

where λ_i *is the intensity of* $N_i(t)$ *and* λ *is the intensity of* $N(t) = N_1(t) + N_2(t)$. *If* $D = 0$ *(no control dependent noise) and* (A, B) *is controllable, i.e.,*

$$\text{rank } ||[B, AB, \ldots, A^{n-1}B] = n$$

then (3.141) *is stabilized by any matrix* K *for which the eigenvalues of* $A-BK$ *lie to the left of* $\Re(s) = -\lambda|\log||I+C|||$ *in the complex plane.*

3.3.2 Examples and Applications.

We would like to use some examples to show how to apply our theorems to determine stability properties of specific systems. As we shall see, in many cases, it is difficult to find the necessary invariant measure because it is associated with an integral equation with shift arguments. It is difficult to evaluate a solution from this equation, although it exists.

Example 2.1. Consider the simple system

$$dx(t) = \begin{pmatrix} k & \omega \\ -\omega & k \end{pmatrix} x(t)dt + \begin{pmatrix} -1 & \alpha \\ \alpha & -1 \end{pmatrix} x(t)dN(t) \tag{3.143}$$

where $N(t)$ is a Poisson process with intensity $\lambda > 0$. Then

$$e^{At} = e^{kt}\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}, \quad \omega > 0$$

$$D = I + B = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \alpha \neq 0.$$

In this case, $De^{At} \neq e^{At}D$ and

$$SG = \text{ smallest semi-group containing } De^{At}, 0 \leq t \leq \infty$$

where μ is the probability measure on SG with density function $\lambda e^{-\lambda t}, t \geq 0$ at each element De^{At}. Since D is non-singular, we can take $M = S^0$, the unit circle. In order to solve $\nu = \mu * \nu$, we let $\Gamma \in$ Borel set $B(S^0)$,

$$\nu(\Gamma) = \int_{SG\times S^0} \chi_\Gamma(g \circ x) d\mu(g) d\nu(x)$$

$$= \int_0^\infty \nu(e^{(-At)} D^{-1} \circ \Gamma) \lambda e^{-\lambda t} dt. \qquad (3.144)$$

For $x \in \Gamma, x = (\cos\theta, \sin\theta)^T$ for some $\theta \geq 0$ and let

$$y = exp(-At) D^{-1} x = e^{-kt} \begin{pmatrix} \cos\Omega t & -\sin\Omega t \\ \sin\Omega t & \cos\Omega t \end{pmatrix} \frac{1}{\alpha} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \frac{1}{\alpha} e^{-kt} \begin{pmatrix} -\sin(\Omega t - \theta) \\ \cos(\Omega t - \theta) \end{pmatrix}.$$

Let ϕ be an angle between the y and x_1-axis. Then

$$\tan\phi = \frac{\cos(\omega t - \theta)}{-\sin(\omega t - \theta)} = -\cot(\omega t - \theta). \qquad (3.145)$$

Differentiating (3.145), we get

$$\sec^2\phi d\phi = -\csc^2(\omega t - \theta) d\theta,$$

so that from (3.145)

$$\frac{d\phi}{d\theta} = \frac{-\csc^2(\omega t - \theta)}{\sec^2\phi} = \frac{-\csc^2(\omega t - \theta)}{1 + \cot^2(\omega t - \theta)} = -1.$$

Suppose ν has density function $f(\theta), 0 \leq \theta \leq 2\pi$. Thus, from (3.144),

$$f(\theta) = \int_0^\infty f(\phi) |\frac{d\phi}{d\theta}| \lambda e^{-\lambda t} dt = \int_0^\infty f(\phi) \lambda e^{-\lambda t} dt \qquad (3.146)$$

and so

$$f(\theta) = \frac{1}{2\pi}, 0 \le \theta \le 2\pi$$

satisfies (3.146). Since SG is transitive on S, then the Haar measure $\nu(\theta)$ with density $f(\theta)$ is a unique invariant measure of μ. Thus,

$$\begin{aligned} r_\nu &= \int_{SG\times S} \log|g \circ x| d\mu(g) d\nu(x) \\ &= \int_0^\infty \int_0^{2\pi} \log\left|De^{At} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right| \lambda e^{-\lambda t} \frac{1}{2\pi} d\theta dt \\ &= \int_0^\infty \int_0^{2\pi} \log\left|\alpha e^{kt} \begin{pmatrix} \sin(\theta - \omega t) \\ \cos(\theta - \omega t) \end{pmatrix}\right| \frac{\lambda}{2\pi} e^{-\lambda t} d\theta dt \\ &= \int_0^\infty \log|\alpha e^{kt}| \lambda e^{-\lambda t} dt \\ &= \log|\alpha| + \frac{k}{\lambda}. \end{aligned}$$

Consequently, if $k < \lambda \log|\alpha|$, the system (3.143) is asymptotically stable, while for $k > -\lambda \log|\alpha|$, the system (3.143) is asymptotically unstable.

Example 2.2 (Harmonic oscillator with damping).

Let $y(t)$ be a point process, regarded as the formal derivative of a Poisson process $N(t)$ with intensity λ. Consider the second order system

$$\ddot{z}(t) + y(t)\dot{z}(t) + [\omega^2 + ky(t)]z(t) = 0 \qquad (3.147)$$

$$z(0), \dot{z}(0) \text{ given}, t \ge 0, \omega > 0, k > 0.$$

Let $x_1(t) = \omega z(t)$, $x_2(t) = \dot{z}(t)$ and $x(t) = [x_1(t), x_2(t)]^T$. Then

$$dx(t) = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} x(t)dt + \begin{pmatrix} 0 & 0 \\ -(k/\omega) & -1 \end{pmatrix} x(t)dN(t) \qquad (3.148)$$

$$x(0) = \begin{pmatrix} \omega z(0) \\ \dot{z}(0) \end{pmatrix} \text{ given.}$$

Set

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ -(k/\omega) & -1 \end{pmatrix}$$

and

$$D = I + B = \begin{pmatrix} 1 & 0 \\ -(k/\omega) & 0 \end{pmatrix} \qquad expAt = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

Let SG be the smallest closed semi-group containing $De^{At}, t \geq 0$. The probability measure μ on SG has density $\lambda e^{-\lambda t}, t \geq 0$ at each element De^{At}. Since D is singular, we take $M = S^0 \cup 0$. It is easy to see that the only invariant set is

$$E = \left\{ P_1 = \left(\frac{\omega}{\sqrt{\omega^2 + k^2}}, \frac{-k}{\sqrt{\omega^2 + k^2}} \right), P_2 = \left(\frac{-\omega}{\sqrt{\omega^2 + k^2}}, \frac{k}{\sqrt{\omega^2 + k^2}} \right), (0,0) \right\}$$

with invariant measure ν of μ being defined by

$$\nu(P_i) = \frac{1}{2}, i = 1, 2 \text{ and } \nu(0) = 0.$$

Note that $SG \circ S^0 = E$ is invariant, so that the stability of the transient set $F = S^0 \setminus E$ *also depends on* r_ν *though* E *does not span* $\mathcal{R}^2$. (See [25].) Now, we calculate $r_\nu = r$ as follows.

$$\begin{aligned}
r_\nu &= \int_{SG \times M} \log |gx| d\mu(g) d\nu(x) \\
&= \frac{1}{2} \sum_{i=1}^{2} \int_0^\infty \log |De^{At} P_i| \lambda e^{-\lambda t} dt \\
&= \int_0^\infty \log |\cos\omega t - \frac{k}{\omega} \sin\omega t| \lambda e^{-\lambda t} dt \\
&= \frac{1}{2} \int_0^\infty \log \left[\cos^2\omega t - \frac{2k}{\omega} \cos\omega t \sin\omega t + \frac{k^2}{\omega^2} \sin^2\omega t \right] \lambda e^{-\lambda t} dt \\
&= \frac{1}{2} \int_0^\infty \log \left[\frac{1}{2}(1 + \frac{k^2}{\omega^2}) + \frac{1}{2}(1 - \frac{k^2}{\omega^2}) \cos 2\omega t - \frac{k}{\omega} \sin 2\omega t \right] \lambda e^{-\lambda t} dt \\
&= \frac{1}{2} \int_0^\infty \log \left[\frac{1}{2}(1 + \frac{k^2}{\omega^2}) + \frac{1}{2}(1 + \frac{k^2}{\omega^2}) \cos(2\omega t + \alpha) \right] \lambda e^{-\lambda t} dt \\
&= \frac{1}{2} \log \frac{1}{2} \left(1 + \frac{k^2}{\omega^2} \right) + \frac{1}{2} \int_0^\infty \log[1 + \cos(2\omega t + \alpha)] \lambda e^{-\lambda t} dt \qquad (3.149)
\end{aligned}$$

where

$$\tan\alpha = \frac{\omega k}{(\omega^2 - k^2)/2}, -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}.$$

Let

$$I_1 \equiv \int_0^\infty \log[1 + \cos(2\omega t + \alpha)]\lambda e^{-\lambda t}dt$$
$$= \int_\alpha^\infty \log[1 + \cos t]\frac{\lambda}{2\omega}e^{-\lambda(t-\alpha)/2\omega}dt. \tag{3.150}$$

Using the fact

$$\int_0^\pi \log(1 + \cos t)dt = -\pi \log 2,$$

we have

$$\int_\beta^{\beta+2\pi} \log(1 + \cos t)dt = -2\pi \log 2, \forall\beta.$$

Thus, let $p = \lambda/(2\omega)$,

$$I_1 > -2\pi p \log 2 \sum_{j=0}^\infty e^{-pj2\pi} = -\frac{2\pi p \log 2}{1 - e^{-2\pi p}} \tag{3.151}$$

and

$$I_1 < -2\pi p \log 2 \sum_{j=1}^\infty e^{-pj2\pi} = -2\pi p \log 2 \frac{e^{-2\pi p}}{1 - e^{-2\pi p}} \tag{3.152}$$
$$= -\frac{2\pi p \log 2}{e^{2\pi p} - 1}.$$

Thus, from (3.149), (3.150), (3.151) and (3.152), we have

$$-\frac{\pi p \log 2}{1 - e^{-2\pi p}} < r_\nu - \frac{1}{2}\log\frac{1}{2}(1 + \frac{k^2}{\omega^2}) < -\frac{\pi p \log 2}{e^{2\pi p} - 1}. \tag{3.153}$$

Hence, if $k \le \omega$, $r_\nu < 0$. What happens for $k > \omega$? We have to calculate k from (3.153) to determine the sign of r_ν. From (3.153), if

$$\frac{1}{2}\log\frac{1}{2}\left(1 + \frac{k^2}{\omega^2}\right) \ge \frac{\pi p \log 2}{1 - e^{-2\pi p}}$$

or

$$k \ge \omega\left[2exp\left(\frac{2\pi p \log 2}{1 - e^{-2\pi p}}\right) - 1\right]^{1/2} \tag{3.154}$$

then $r_\nu > 0$ and the system (3.148) is asymptotically unstable; while for

$$\frac{1}{2}\log\frac{1}{2}\left(1+\frac{k^2}{\omega^2}\right) \leq \frac{\pi p \log 2}{e^{2\pi p}-1}$$

or

$$k \leq \omega\left[2exp\left(\frac{2\pi p\log 2}{e^{2\pi p}-1}\right)-1\right]^{1/2}, \tag{3.155}$$

we have $r_\nu < 0$ and the system (3.148) becomes asymptotically stable.

Example 2.3 (Randomly coupled harmonic oscillators) (cf. [47] for $m = 1$). Let $y_{ij}(t), i,j = 1,\ldots,m$, be independent processes which are regarded as formal derivatives of independent Poisson processes $N_{ij}(t)$ with intensities λ_{ij}, respectively. Consider the following stochastic system of m coupled harmonic oscillators.

$$\ddot{z}_i(t) + \omega_i^2 z_i(t) = \sum_{j=1}^{m} b_{ij} y_{ij}(t) z_j(t) \tag{3.156}$$

$$z_i(0), \dot{z}_i(0) \text{ given}, t \geq 0, \omega_i > 0, i = 1,\ldots,m.$$

Let $x_{2i-1}(t) = \omega z_i(t)$, $x_{2i}(t) = \dot{z}_i(t)$ and $x = [x_1,\ldots,x_{2m}]^T$. Then in standard notation

$$dx(t) = Ax(t)dt + \sum_{i,j=1}^{m} B_{ij}x(t)dN_{ij}(t) \tag{3.157}$$

where

$$A = \text{diag}\,\{A_1,\ldots,A_m\}, A_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix},$$

and all the entries of B_{ij} are zero except the entry $e_{2i,2j-1} = b_{ij}/\omega_i$. Set

$$D_{ij} = I + B_{ij}.$$

Note that $\text{tr}(A) = 0$ and $\det(D_{ij}) = 1$, so we have $D_{ij}exp(At) \in SL(2m)$. We can define a measure μ on $SL(2m)$ with density $\lambda_{ij}e^{-\lambda t}$, $t \geq 0$, $\lambda = \sum_{i,j=1}^{m}\lambda_{ij}$ at each element $D_{ij}e^{At}$. In this case, it is difficult to determine an invariant measure because the corresponding integral equation is hard to solve. However, we can use a theorem of Furstenberg (Theorem 4.12 in [25]) to show the rate $r > 0$. Let

G = smallest subgroup containing $D_{ij}e^{At}$, $0 \leq t < \infty, i,j = 1,\ldots,m$
= smallest subgroup containing $D_{ij}, i,j = 1,\ldots,m; e^{At}, 0 \leq t < \infty$.

Then G may not be transitive on S^{2m-1}. If we assume no two ω_i are equal, then the commutant Σ of the smallest subgroup G_1 containing $e^{At}, t \geq 0$ is isomorphic to $\mathcal{C}$, i.e., $T \in \Sigma$ if

$$T = \text{diag}\,\{T_1, \dots, T_m\}$$

with

$$T_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}, \alpha_i, \beta_i \in \mathcal{R}.$$

Since $Te^{At} = e^{At}T$, and T and e^{At} are normal, they preserve their eigenspace. Thus, the invariant subspaces V of G_1 are of the form $\mathcal{R}^2_{j_1} \times \cdots \times \mathcal{R}^2_{j_\ell}$, $\ell < m$.

Before verifying the hypotheses of Furstenberg's theorem, we need a non-degeneracy assumption:

(A) For any index set $J = \{j_1, \dots, j_\ell\}$, $\ell < m$, there exists an $i \notin J$ such that $b_{ik} \neq 0$ for some $k \in J$.

By assumption (A), $\exists b_{ik} \neq 0$ so that the entry $e_{2i,2k-1}(D^j_{ik}) = jb_{ik}/\omega_i$ tends to infinity as $j \to \infty$. Thus, G is not compact.

Let an index set $J = \{j_1, \dots, j_\ell\}$. By assumption (A), $\exists i \notin J$ such that $b_{ik} \neq 0$ for some $k \in J$. Then $D_{ik}V \cap V$. Hence, G is irreducible.

Note that G_1 is connected. There is no finite index subgroup of G_1. Thus, any finite index subgroup H of G must contain G_1 and some mixed powers of $\{D_{ij}\}$. Moreover, the irreducibility of G is due to sufficiently more non-zero entries of D_{ij}, not the exact value b_{ij}, so H is also irreducible.

In the cases where some ω_i are equal. The commutant Σ properly contains $\mathcal{C}$ and the invariant subspaces of G_1 are much more complicated.

Consequently, by Furstenberg's Theorem ([38], Theorem 8.6), $r_\nu = r > 0$ and $x(t)$ grows exponentially a.s. This implies that all the states of all subsystems grow exponentially.

Remark. If assumption (A) does not hold, the system can be subdivided into proper subsystems Σ_i, which have property (A), and $\overline{\Sigma}$. States of Σ_i grow exponentially a.s. by the above arguments. The remaining subsystem $\overline{\Sigma}$ depends on Σ_i and its state thus grows exponentially a.s. Hence, the system of n coupled harmonic oscillators is asymptotically unstable.

Example 2.4 (Random telegraph wave).

Let $z(t)$ be random telegraph wave which takes on the value set $Z = \{-1, 1\}$ with transition probability satisfying

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ p_{-1} \end{pmatrix} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}\begin{pmatrix} p_1 \\ p_{-1} \end{pmatrix}.$$

Then the differential equation for $z(t)$ becomes

$$dz(t) = -2z(t)dN(t) \tag{3.158}$$
$$z(0) = \pm 1$$

where $N(t)$ is a Poisson process with intensity λ. If we consider the state process

$$dx(t) = [k + \omega z(t)]x(t)dt \tag{3.159}$$
$$x(0) = x_0, \omega > 0, t \geq 0,$$

then using (3.158), (3.159) and the fact $z^2(t) = 1$, we get

$$d(zx) = dzx + zdx \tag{3.160}$$
$$= -2zxdN + z(k + \omega z)xdt$$
$$= \omega xdt + kzxdt - 2zxdN.$$

Combining (3.159) and (3.160), we have

$$d\begin{pmatrix} x \\ zx \end{pmatrix} = \begin{pmatrix} k\omega \\ \omega k \end{pmatrix}\begin{pmatrix} x \\ zx \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ zx \end{pmatrix} dN(t). \tag{3.161}$$

Then,

$$expAt = e^{kt}\begin{pmatrix} \cosh\omega t & \sinh\omega t \\ \sinh\omega t & \cosh\omega t \end{pmatrix},$$

$$D = I + B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let SG be the smallest closed semi-group containing De^{At}, $0 \leq t < \infty$ and the measure μ is defined on SG with density $\lambda e^{-\lambda t}, t \geq 0$ at each element De^{At}. The corresponding invariant measure ν is difficult to calculate exactly and may not be unique since SG is not transitive on the circle S^0. However, SG is irreducible. By Furstenburg's Theorem, the rate r is independent of ν.

Let

$$X(t) = De^{At} = e^{kt}\begin{pmatrix} \cosh\omega t & \sinh\omega t \\ -\sinh\omega t & -\cosh\omega t \end{pmatrix},$$

then

$$||X(t)||_2 = e^{kt}(\cosh 2\omega t + \sinh 2\omega t)^{1/2} = e^{(k+\omega)t},$$

and

$$r_1 = \int_0^\infty \log ||X(t)||_2 \lambda e^{-\lambda t} dt$$
$$= \int_0^\infty (k+\omega) t \lambda e^{-\lambda t} dt$$
$$= \frac{k+\omega}{\lambda}.$$

Again, we calculate

$$X(t_2)X(t_1) = e^{k(t_1+t_2)} \begin{pmatrix} \cosh\omega(t_1-t_2) & \sinh\omega(t_1-t_2) \\ \sinh\omega(t_1-t_2) & \cosh\omega(t_1-t_2) \end{pmatrix}$$

with

$$||X(t_2)X(t_1)||_2 = e^{k(t_1+t_2)}[\cosh\omega(t_1-t_2) + \sinh\omega(t_1-t_2)]$$
$$= e^{k(t_1+t_2)} e^{\omega(t_1-t_2)},$$

so that

$$r_2 = \int_0^\infty \int_0^\infty \log ||X(t_2)X(t_1)||_2 \lambda e^{-\lambda t_1} dt_1 \lambda e^{-\lambda t_2} dt_2$$
$$= \int_0^\infty \int_0^\infty [k(t_1+t_2) + \omega(t_1-t_2)] \lambda e^{-\lambda t_1} dt_1 \lambda e^{-\lambda t_2} dt_2$$
$$= 2\frac{k}{\lambda}.$$

In general,

$$r_\ell = \int_0^\infty \cdots \int_0^\infty \log ||X(t_\ell)\cdots X(t_1)||_2 \lambda e^{-\lambda t_1} dt_1 \cdots \lambda e^{-\lambda t_\ell} dt_\ell$$
$$= \begin{cases} \ell\frac{k}{\lambda} + \frac{\omega}{\lambda}, & \ell \text{ is odd} \\ \ell\frac{k}{\lambda}, & \ell \text{ is even} \end{cases}$$

Thus,

$$r = \lim_{l\to\infty} \frac{r_\ell}{\ell} = \frac{k}{\lambda}.$$

From (3.161), we know that stability of (3.159) is equivalent to that of (3.161). Hence, the system (3.159) is asymptotically stable for $k < 0$ while it is asymptotically unstable for $k > 0$. This result shows that the random telegraph process $z(t)$ does not affect the stability of the corresponding deterministic system.

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Chapter 4

Simultaneous Detection and Estimation for Diffusion Process Signals

4.1 Abstract

We consider the problem of simultaneous detection and estimation when the signals corresponding to the M different hypotheses can be modelled as outputs of M distinct stochastic dynamical systems of the Ito type. Under very mild assumptions on the models and on the cost structure, we show that there exists a set of sufficient statistics for the simultaneous detection-estimation problem that can be computed recursively by linear equations. Furthermore, we show that the structure of the detector and estimator is completely determined by the cost structure. The methodology used employs recent advances in nonlinear filtering and stochastic control of partially observed stochastic systems of the Ito type. Specific examples and applications in radar tracking and discrimination problems are discussed.

4.2 Introduction

In a typical present day radar environment, the radar receiver is subjected to radiation from various sources. A very important function of the radar receiver is its ability to discriminate between the various waveforms received and select the desired one for further processing. Furthermore, an equally important function of the receiver is to estimate important parameters of the radiating source from the received waveforms. Thus the receiver is required often to perform a "combined detection and estimation" function.

An abstract formulation of the combined detection and estimation problem in the language of statistical decision theory has been developed by Middleton and Esposito in [1]. They correctly point out that optimal processing in such problems often requires the mutual coupling of the detection and estimation algorithms. Although from the mathematical point of view estimation may be considered as a generalized detection problem, from an operational point of view, the two procedures are different: e.g., one usually selects different cost functions for each and obtains different data processors as a result. It is then correctly argued in [1] that it is practically appropriate to retain the usual distinction between detection and estimation. There are various ways that the detector and estimator can be coupled leading to a hierarchy of complex processors. We describe here some important cases.

4.3 Detection-Oriented Estimation

Here, the detection operation is optimized with a priori knowledge of the existence of an estimator following it. The estimator is dependent on the detector's decision by being gated on only if the detector decides that the desired signal is present. Here, the coupling is via cost terms that assess the performance deterioration when the estimator is turned off while the signal is present $C_{e,1}$, or the estimator is turned on while the signal is not present $C_{e,0}$. Therefore, the average risks corresponding to the operations of detection and estimation can be minimized separately. This leads to a detection test that is a modified generalized likelihood test. If the cost terms $C_{e,1}$, $C_{e,0}$ are constant, the coupling just reduces to a modification of the threshold [1]. Since the detector's decision rule does not depend on the estimate, the structure of the optimal estimator is not a function of the data region specified by the decision rule of the detector's operation, when the detector's decision is to accept the signal. ***In practical terms, this means that we can choose to estimate only when the detector has decided that the desired signal is present.***

4.4 Coupled Detection-Estimation with Decision Rejection

Here, detection and estimation run in parallel and are followed by rejection of the estimate if the detector's decision is not to accept the signal. Here, the detector's cost depends on the value of the estimate. Typically, one solves the detection problem knowing the estimator. Then a second optimization is performed over all estimators. This case usually results in relatively simple estimators and complex highly nonlinear detectors [1].

Motivation for these problems stems from distributed target problems, see in particular [2]-[7].

We concentrate in this section on a two hypotheses detection formulation, but it is clear that the methods can be easily extended to M-ary detection problems. The two hypotheses are H_0 = the received signal is a process y_{0t} plus noise, H_1 = the received signal is a process y_{1t} (different from y_{0t}) plus noise. Both processes are modeled as outputs of stochastic dynamical systems of the diffusion type. The noise is the same in both cases. Due to this fact, we can assume that noise is eliminated from the mathematical formulation of the problem of detection, while as we shall see its presence may be crucial for the estimation problem.

We did not study detectors with "learning", and we suggest this is a promising extension of the results reported here. We note, however, that our formalism includes general "learning" algorithms. Most of the work on detectors with "learning" is problem specific and does not utilize dynamical system models for the signals as we do. The major criticism for the work of Middleton and Esposito [1] is that although they used a Bayesian approach to the estimation problem, they considered nonrecursive solutions and detection was coupled to estimation through cost structure which explicitly considers coupling of the detection and estimation costs. Clearly nonrecursive solutions are not appropriate for advanced sensors employed in guided platforms. Furthermore, it would be unrealistic to assume that the designer has such explicit knowledge of the functional couplings between detection and estimation costs.

Several other authors have analyzed the problem. Scharf and Lytle [13] studied detection problems involving Gaussian noise of unknown level, thus including noise parameters in the problem. As in [1], their solution is also nonrecursive and focuses on the existence of uniformly most powerful tests. Spooner [14], [15] considered in detail unknown parameters in the noise model. Jaffer and Gupta [16], [17] consider the recursive Bayesian problem using a quadratic cost, Gauss-Markov processes and estimating only signal parameters. Birdsall and Gobien [18] considered the problem of simultaneous detection and estimation from a Bayesian viewpoint. This work is close in spirit with our approach, although the class of problems we can analyze by our methods is significantly wider. We also follow a Bayesian methodology during the initial phase of analysis. It becomes clear that by using Bayesian methods one can analyze the problems under consideration in an inherently intuitive, simple conceptual manner which can be easily obscured in highly structured methodologies utilizing specific detector structures and cost relationships. As a result, one can analyze the special problems described earlier as specializations of a wider picture and framework. The results reported in [16] are limited by two important assumptions: (a) the observed data have densities that display finite dimensional sufficient statistics under both hypotheses for the unknown parameters, and (b) the unknown parameters form a finite-dimensional vector. Both nonsequential and sequential problems are analyzed in [18]. The most important result of [18] is the proof that through a Bayesian approach both estimation and detection occur simultaneously, with the detector using the a posteriori densities generated by two separate estimators, one for each hypothesis. A particularly attractive feature is that no assumptions are made on the estimation criterion and very flexible assumptions are made on the detection criterion. When finite-dimensional sufficient statistics exist, the optimum processor partitions naturally into three parts: a "primary" processor which is totally independent of a priori distributions on the parameters, a "secondary" processor which modifies the output according to the priors and solves the detection problem, and an estimator which uses the output of the other two in estimating the unknown parameters. Only the estimator structure depends on cost functionals.

Since dynamical system models are not utilized to represent signals in [18], there is great difficulty in analyzing the far more interesting sequential problem. It is for this reason that one is forced to make the limiting assumptions mentioned above. In our approach, we consider diffusion type models for the signals, and we utilize modern methods from nonlinear filtering and stochastic control to analyze the problem [19]-[23]. Corresponding results for Markov chain models can be easily obtained, but we only give brief comments for such problems here.

4.5 Nomenclature and Formulation of the Sequential Problem

In this section, we present a general formulation for the continuous time, sequential, simultaneous detection and estimation problem when the signals can be represented as outputs of diffusion type processes [20]. To simplify notation, terminology and subsequent computations, we consider only the scalar observation case here. All results extend to vector observations in a straight-forward manner. The observed data $y(t)$ constitute, therefore, a real-valued scalar stochastic process.

The statistics of $y(\cdot)$ are not completely known. More specifically, they depend on some parameters and some hypotheses. For simplicity, we shall consider here only the binary hypotheses detection problem. Extensions to M-ary detection are trivial. We shall denote by H_0, H_1 the two mutually exclusive and exhaustive hypotheses.

Under hypothesis H_0, the received data $y(t)$ can be represented as:

$$\begin{aligned} dy(t) &= h^0(s^0(t), \theta^0)dt + dv(t) \\ dx^0(t) &= f^0(x^0(t), \theta^0)dt + g^0(x^0(t), \theta^0)dw^0(t) \end{aligned} \tag{4.1}$$

where θ^0 is a vector-valued unknown parameter that may be assumed fixed or random throughout the problem. Here $v(\cdot)$, $w(\cdot)$ are independent, 1-dimensional and n_0-dimensional, respectively, standard Wiener processes [20]. In other words, when hypothesis H_0 is true, the received data can be thought of as the output of a stochastic dynamical system, corrupted by white Gaussian noise. h^0, f^0, g^0, θ^0 parameterize the nonlinear stochastic system.

Similarly, when hypothesis H_1 is true, the received data $y(t)$ can be modelled as

$$\begin{aligned} dy(t) &= h^1(x^1(t), \theta^1)dt + dv(t) \\ dx^1(t) &= f^1(t), \theta^1)dt + g^1(x^1(t), \theta^1)dw^1(t) \end{aligned} \tag{4.2}$$

where now x^1 is n_1-dimensional. The vector parameters θ^0, θ^1 may have common components. For instance, in the classical "noise or signal-plus-noise" problem, any noise parameters clearly appear in both hypotheses and would thus be common to θ^0, θ^1.

We note that we have the same "observation noise" $v(\cdot)$ under both hypotheses. This is clearly the case in radar applications (see [6]). On the other hand, when one is faced with state and parameter dependent observation noises, a simple transformation translates the two models in the form (4.1) (4.2). We shall assume that $h^i, f^i, g^i, i = 0, 1$, have sufficient properties to guarantee existence and uniqueness of probability distribution functions for $y(\cdot)$ under either hypothesis. As a minimal hypothesis, we assume that the martingale problems for (4.1) and (4.2) are well posed [24] for all values of θ^0, θ^1 in appropriate compact sets Θ^0, Θ^1, respectively. Furthermore, neither (4.1) nor (4.2) exhibit explosions [24] for any value of the parameters. Often we shall make stronger assumptions such as existence of strong solutions to (4.1) (4.2), or smoothness of $f^i, g^i, h^i, i = 0, 1$, or existence of classical probability densities for y_i under either hypothesis.

We shall denote by $p_y^i(\cdot, t \mid \theta^i), i = 0, 1$, the probability density of $y(t)$ under hypothesis H^i and when the parameter obtains the value $\theta^i, i = 0, 1$. We shall denote the probability measures corresponding to y under H^0 or H^1 by μ_y^0, respectively. As is well known, these are measures on the space of continuous functions [24]. Finally, we note that although we have assumed time invariant stochastic models in (4.1), (4.2) the results extend easily to the time varying case.

Following a Bayesian approach, we assume a priori densities for the two parameters θ^0, θ^1 which will be denoted by $p_\theta^i(\cdot, 0), i = 0, 1$ respectively. Similarly initial densities for $x^0(0)$ and $x^1(0)$ are assumed known and independent of θ^0, θ^1, respectively. They will be denoted by $p_x^i(\cdot, 0)$. The choice of these a priori densities is frequently a very interesting problem in applications, as they represent the designer's a priori knowledge about the models used.

With these preliminaries, we can now formulate the problem. Let y^t denote as usual the portion of the observed sample path "up to time t", i.e., $y^t = \{y(s), s \leq t\}$. Given the observed data y^t, we wish to design a processor which at time t will optimally select simultaneously which of the two hypotheses H_0 or H_1 is true, and optimal estimates for the parameters θ^0 and θ^1. Moreover, the processor should operate recursively so as to permit real-time implementation.

To complete the problem formulation, we need to specify costs for detection and estimation. Let $c_i(\hat{\theta}^i(t), \theta^i), i = 0, 1$ be the penalty for "estimating" θ^i, by $\hat{\theta}^i(t)$ at time t. If c_i is quadratic, we have the well known minimum variance estimates. Similarly, let $\gamma(t)$ denote the decision, at time t, of wheter we declare hypothesis H_0 or H_1 to hold. Then $k(\gamma(t), i), i = 0, 1$ will denote the penalty when the true hypothesis is H_i and we decide $\gamma(t)$, at time t. Obviously, there are infinitely many variations on the possible choice for a cost function. We shall consider only two possibilities in this report.

Finite time average integral cost

$$J_f = E\{\int_0^T \lambda_e c_0(\hat{\theta}^0(t), \theta^0) X\{t, \gamma(t) = 0\} + c_1(\hat{\theta}^1(t), \theta^1) X\{t, \gamma(t) = 1\} dt + \lambda_d k(\gamma(t), i) dt\} \tag{4.3}$$

and infinite time average discounted cost.

$$J_d = E\{\int_0^\infty C(\gamma, \hat{\theta}^0, \hat{\theta}^1, x) e^{-\alpha t} dt\} \tag{4.4}$$

where $C(\gamma, \hat{\theta}^0, \hat{\theta}^1, x)$ is the integrand in (4.3) and α the discount rate. λ_e, λ_d are weights. The reasons for the characteristic functions appearing in (4.3), (4.4) are rather obvious. The estimator will contribute cost only when utilized, and it will be utilized for θ^0 only when $\gamma(t) = 0$. We would like to point out that this does not preclude both estimators from running continuously. This scheme is used only to assess costs properly.

The appropriate formulation of the problem is as a partially observable stochastic control problem. The admissible controls are

$$\begin{aligned} \gamma &: R \to \{0, 1\} \\ \hat{\theta}^0 &: R \to \Theta^0 \\ \hat{\theta}^1 &: R \to \Theta^1 \end{aligned} \tag{4.5}$$

where all functions are nonanticipative with respect to y; i.e., measurable w.r. to F_t^y:

$$\gamma(\cdot), \hat{\theta}^0(\cdot), \hat{\theta}^1(\cdot) \in F_t^y \tag{4.6}$$

The cost is either (4.3) or (4.4). For the system dynamics, we proceed as follows. The state equations are mixed consisting of the continuous components

$$\begin{aligned} dx^0(t) &= f^0(x^0(t), \theta^0(t))dt + g^0(x^0(t), \theta^0(t))dw^0(t) \\ dx^1(t) &= f^1(x^1(t), \theta^1(t))dt + g^1(x^1(t), \theta^1(t))dw^1(t) \\ d\theta^0(t) &= 0 \\ d\theta^1(t) &= 0 \end{aligned} \tag{4.7}$$

and the discrete component $z(t)$ which can take only the values 0 or 1 and is constant. The initial densities for $x^0, x^1, \theta^0, \theta^1$ have already been described. The initial probability vector for $z(t)$ (which tracks which hypothesis is true) is

$$Pr\{z(0) = 0\} = P_0, Pr\{z(0) = 1\} = P_1 \tag{4.8}$$

The observations are

$$dy(t) = (1 - z(t)h^0(x^0(t), \theta^0)dt + z(t)h^1(x^1(t), \theta^1)dt + dv(t) \tag{4.9}$$

Since (4.7) are degenerate, there are some technical minor difficulties, which can be circumvented, however, using recent techniques. This completes the formulation of the problem.

4.6 Structure of the Optimal Processor

Following recent results [25]-[29] in stochastic optimal control theory, we have obtained first the following results that reduce the partially observed stochastic control problem described in Section 4.5 to an equivalent, infinite dimensional fully observed problem.

Theorem 1: There exist optimal $\gamma, \hat{\theta}^0, \hat{\theta}^1$ for the stochastic optimal control problem (4.3) - (4.9).

Proof: This follows from the results of Fleming and Pardoux [27] and Bismut [29]. The only difference is that due to the structure of the dynamics here (i.e., they do not depend on the controls $\gamma, \hat{\theta}^0, \hat{\theta}^1$) we can show that optimal controls exist in the class of strict sense controls as specified in Section 4.5 (i.e., $\gamma(t), \hat{\theta}^0(t), \hat{\theta}^1(t)$ are measurable with respect to F_f^y).

We then introduce as in Fleming and Pardoux [27] the associated "separated" stochastic control problem. In the separated stochastic control problem, the state at time t is a measure Λ_t on R^N (where $N = n_0 + n_1 + 2$), which is un unnormalized conditional distribution of the state $x(t) \equiv [x_0(t), x_1(t), \theta_0(t), \theta_1(t), z(t)]^T$ of the problem formulated in Section 4.5. The dynamics of the measure-valued process Λ_t obey the Zakai equation of nonlinear filtering [26]-[31], and [20].

In the sequel, we assume that all functions appearing in (4.1) - (4.9) are bounded and continuous and that g^0, f^0, g^1, f^1 are Lipschitz in $x^0, \theta^0, x^1, \theta^1$, respectively. Due to the discrete component $z(t)$ of the state $x(t)$, we have to consider a two-dimensional measure valued process Λ^0, Λ^1, where Λ^i is the unnormalized conditional distribution of the state

$$x(t) \equiv [x_0(t), x_1(t), \theta_0(t), \theta_1(t)]$$

(slight abuse of notation here) when hypothesis H_i is true, $i = 0, 1$. We further assume that for $i = 0, 1$, the corresponding Zakai equation has a unique solution which is absolutely continuous with respect to Lebesque measure; i.e., we assume the existence of conditional unnormalized probability densities for $x(t) \in R^N$ given y^t. For results on this, see [30], [31].

Let $u^i(x,t)$ denote the conditional probability density of $x(t)$ given y^t when hypothesis H_i holds. Then $u^i(\cdot,\cdot)$ satisfies the Zakai equation

$$du^i = L_i^* u^i dt + dy(t) h^i u^i, i = 0, 1 \tag{4.10}$$

where L^* is the formal adjoint to the infinitesimal generator of the i^{th} component of (4.7); i.e., it has the form

$$L = \frac{1}{2} \sum_{1,j=1}^{N} a_{ij}^i(x) \frac{\partial^2}{x_i x_j} + \sum_{i=1}^{N} b_i^i(x) \frac{\partial}{\partial x_i} \tag{4.11}$$

Here

$$a^i = \sigma^i(\sigma^i)^T, \sigma^i = \begin{bmatrix} g^i & 0 \\ 0 & 0 \end{bmatrix} \quad b^i = \begin{bmatrix} f^i & 0 \\ 0 & 0 \end{bmatrix} \tag{4.12}$$

To complete the description of the "separated" stochastic control problem, let $C(\gamma, \hat{\theta}^0, \hat{\theta}^1, x)$ denote the integrand in the cost definition (4.3). Then if we let

$$u(x,t) = \begin{bmatrix} u^0 & (x_0, \theta_0, t) \\ u^1 & (x_1, \theta_1, t) \end{bmatrix} \tag{4.13}$$

we can rewrite the cost (4.3) as

$$J_f(\pi) = E_y\{\int_0^T \int C(\gamma, \hat{\theta}^0, \hat{\theta}^1, x)[u(x,t)^T \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}] dx dt\} \tag{4.14}$$

where π is the policy corresponding to a particular selection of $\gamma(\cdot)$, $\theta^0(\cdot)$, $\theta^1(\cdot)$, and E_y is expectation with respect to y. Note that u depends explicitly on y.

The separated problem is to choose a policy π which is a function of u^0, u^1 to minimize (4.14). This is a fully observed problem since u^0, u^1 satisfy (4.10) and enter directly into (4.14). We then have the following very important result:

Theorem 2: Under the above assumptions, the optimal $\gamma, \hat{\theta}^0, \hat{\theta}^1$ (which exist according to Theorem 1) are functions of u^0, u^1 only. That is, they depend on y^t only through the unnormalized conditional densities u^0, u^1.

Proof: The proof follows from appropriate modifications of the results in [25]-[29] and will appear elsewhere. The significance of the result is that it provides the basic structure of the optimal processor by identifying u^0, u^1 as the sufficient statistics for the original problem. Furthermore, the result is free from structural assumptions on the detection and estimation costs and can be established in far greater generality than the results presented here may indicate.

In Figure 1 below, we give a pictorial illustration of the result. We basically have to run two "filters" in parallel, one for each hypothesis. The output of each filter (which, by the way, is represented by the bilinear stochastic p.d.e. (4.10)) is the unnormalized conditional probability density of x^0, θ^0 or

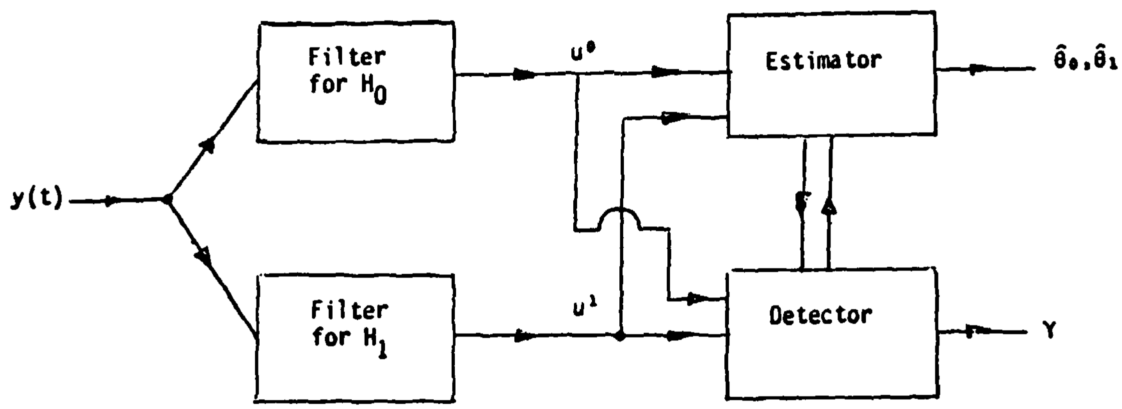

Figure 1

Figure 1 illustrates the generic structure of the optimal processor x^1, θ^1 given H^0 or H^1. Each filter is driven directly by the observations.

The estimator, detector and their coupling will depend on the explicit cost structure. They are problem dependent. Their explicit functioning can be computed as our final result indicates.

Theorem 3: The explicit dependence of γ (which is discrete valued), $\hat{\theta}_0, \hat{\theta}_1$ on u^0, u^1 can be determined by solving a variational inequality on the space of solutions of (4.10).

Proof: The result is rather technical. A complete proof will be given elsewhere. It follows by appropriate modifications to the results of [26], [32].

This result opens the way for promising electronic implementation of the optimal processor by the following steps: (1) solve numerically the resulting variational inequality using the methods of [33], (2) implement the resulting numerical algorithm by a special purpose, multiprocessor, VLSI device along the lines of [34]. In simple cost cases, explicit solutions of the variational inequality can be obtained, of course.

4.7 Motivation and Examples from Radar Tracking Loops

The primary motivation for the mathematical problem studied in Section 4.6 comes from design consideration of advanced (smart) sensors in guided platforms. To be more specific, let us consider radar sensors.

The radar return from a scatterer carries (depending on the radar sophistication) significant information about a scatterer. For example, range, Doppler extend, shape and extend, motion, of a scatterer can be extracted from a radar return by appropriate processing. In today's dense environment, a very important function of an advanced processor is classification of scatterers. This function is required, for example, by sensors participating in a surveillance network (since threats must be classified, so that appropriate response can be applied), in electronic warfare (since decoys and other counter-measures can be designed to emulate target characteristics) and in tracking radars (since the sensor often must develop a tracking path for a designated priority target).

A related equally important function of a radar receiver is the estimation of parameters embedded in the return signal. For example, pulse length, pulse repetition frequency, amplitude scintillation spectrum, conical scan frequency, antenna pointing, surface roughness. The two problems of detection and estimation are indeed closely related, as explained earlier.

In our earlier work [2]-[5], we have developed statistical models for distributed scatterers which can represent accurately phenomena characteristic of distributed scatterer radar returns such as amplitude scintillation and angle noise or glint. In addition, we have developed similar statistical models for the effects of multipath on radar returns, for sea clutter returns and for chaff cloud returns. The models developed in [2]-[5] are of the form

$$\begin{aligned} dx(t) &= A(t,\theta)x(t)dt + B(t,\theta)dw(t) \\ dy(t) &= h(t,x(t),\theta)dt + dv(t) \end{aligned} \tag{4.15}$$

Furthermore, A, B, h are piecewise constant with respect to time since the models developed in [2]-[5] are piecewise stationary. For example in [2], we used models like (4.5) to describe the RCS scintillation for ships. The same type models can be used for other distributed targets such as tanks or armored vehicles. For example, when the return appears spiky, indicating higher probability of strong return, an appropriate model is provided by a lognormal process, where $x(\cdot)$ in (4.15) is scalar and h is chosen to be an exponential function of x. For chaff clouds, a more appropriate model is provided by a Rayleigh process, where $x(\cdot)$ is two dimensional, with the two components being identically distributed, independent Gaussian random processes and

$$h(t,x(t),\theta) = \sqrt{x_1^2(t) + x_2^2(t)}$$

Clearly then, in target discrimination problems with distributed targets of this type, one encounters problems like those treated in Section 4.6. It is important to note that since the first of (4.15) is linear, the corresponding filtering and stochastic control problems described in Section 4.6 are definitely more tractable. For further examples of this type, we refer the reader to [2]-[5].

Further research is needed to apply the powerful results of Section 4.6 to specific problems in order to evaluate current design principles and more importantly, in order to suggest new electronic implementations capable of performing in a dense, hostile environment. In particular, the methodology developed in Section 4.6 can be used to identify the cost structures that lead to the specific hierarchies suggested in the introduction.

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