A FINITE ALGORITHM FOR THE
LEAST TWO-NORM SOLUTION OF A
LINEAR PROGRAM

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ABSTRACT

By perturbing properly a linear program to a separable quadratic program it is possible to solve the latter in its dual variable space by iterative techniques such as sparsity-preserving SOR (successive overrelaxations). In this way large sparse linear programs can be handled.

In this paper we give a new computational criterion to check whether the solution of the perturbed quadratic programs provide the least 2-norm solution of the original linear program. This criterion improves on the criterion proposed in an earlier paper.

We also describe an algorithm for solving linear programs which is based on the SOR methods. The main property of this algorithm is that, under mild assumptions, it finds the least 2-norm solution of a linear program in a finite number of iterations.

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SIGNIFICANCE AND EXPLANATION

This paper provides the theory and practical algorithms for computing the unique smallest solution of a given linear program. The proposed method is capable of handling very large sparse linear programs that cannot be solved by the conventional simplex method.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.
1. INTRODUCTION

It was shown in [1] and [2] that the least 2-norm solution of a linear program can be computed by "properly" perturbing the linear program to a separable quadratic program and by solving the latter in its dual variable by iterative techniques such as SOR methods (see [3]). In this way large sparse linear programs, not solvable by standard pivotal methods can be handled (see [4], [1]). The main difficulty with this approach is that, until now, there is no easy way to know "a priori" if the perturbed quadratic problems is a "proper" perturbation of the original linear program.

In order to overcome this difficulty a computational criterion to check whether the solutions of the perturbed quadratic program was given in [5] and two algorithms where the perturbation parameter of the linear program is decreased during the computational procedure were proposed in [6] and [5].

We give now an outline of the paper. Section 2 contains the problem formulation. In Section 3 we characterize the least 2-norm solution of a linear program in terms of the constraints which are active at that point. This characterization will be utilized in subsequent sections of the paper. In Section 4 we turn our attention to the problem of the perturbation of a linear program and we propose a new criterion to check whether the solution of the perturbed problem is in fact the least 2-norm solution of the linear

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program. This criterion improves on that given in [5]. In the last section we describe an algorithm for solving linear programs which is based on the SOR method. The main property of this algorithm is that, under mild assumptions, it finds the least 2-norm solution of a linear program in a finite number of iterations.

We briefly describe the notation used. All the matrices and vectors are real. For the \( n \times n \) matrix \( A \) we denote row \( i \) by \( A_i \), column \( j \) by \( A_j \) and the element in row \( i \) and column \( j \) by \( A_{ij} \). For \( x \) in the real \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), \( x_i \) denotes element \( i \) for \( i = 1, \ldots, n \) and \( x_+ \) denotes the vector with components \( (x_+)_i = \max\{x_i, 0\} \) \( i = 1, \ldots, n \).

All vectors are column vectors unless transposed by \( T \). We will denote the 2-norm, \( \|x\|^2 = \left( \sum_{j=1}^{n} x_j^2 \right)^{1/2} \) by \( \|x\| \), while \( \mathbb{R}_+^n \) will denote the nonnegative orthant \( \{x : x \in \mathbb{R}^n, x \geq 0\} \). For a point \( c \) in \( \mathbb{R}^n \) and a closed set \( X \) in \( \mathbb{R}^n \) the 2-norm projection \( p_2(c,X) \) of the point \( c \) on \( X \) is defined

\[
\|c - p_2(c,X)\| = \min_{x \in X} \|c - x\|.
\]

For given index sets \( I, J \) we denote, as usual

\[
|I| = \text{number of indices in } I \\
I \cup J = \{i : i \in I, i \in J\} \\
I \setminus J = \{i : i \in I, i \notin J\}
\]

and \( 0|I| \) will be a null vector of dimension \( |I| \). Finally we denote the identity matrix by \( E \), \( A_I \) will be the submatrix of \( A \) with rows \( A_i \) \( i \in I \), \( A_{IJ} \) the submatrix of \( A \) with elements \( A_{ij} \), \( i \in I, j \in J \) and \( x_I \) will denote \( x_i \), \( i \in I \).
2. PROBLEM FORMULATION

We consider the linear program

\[
\begin{align*}
\text{Max} & \quad c^T x \\
\text{s.t.} & \quad Ax < b \\
& \quad x \geq 0
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m \) and \( X = \{ x : Ax < b, x \geq 0 \} \neq \emptyset \). Let \( \bar{X} \) denote the (possibly empty) optimal solution set of (1). The least 2-norm solution of (1), \( \rho_2(0, \bar{X}) \), can be characterized as the unique solution of the following problem

\[
\begin{align*}
\text{Min} & \quad \frac{1}{2} \| x \|^2 \\
\text{s.t.} & \quad Ax < b \\
& \quad x \geq 0
\end{align*}
\]

where \( \rho \) is the maximum value of (1).

Given any \( x \in X, u \in \mathbb{R}^m \) and \( v \in \mathbb{R}^n \) we define the following index sets:

\[
\begin{align*}
I_a(x) := \{ i : A_i x = b_i \}, & \quad J_a(x) := \{ i : x_i = 0 \} \\
I_n(x) := \{ i : A_i x < b_i \}, & \quad J_n(x) := \{ i : x_i > 0 \} \\
I_d(u) := \{ i : u_i > 0 \}, & \quad J_d(v) := \{ i : v_i > 0 \}.
\end{align*}
\]

Let \( I_b(x) \) and \( J_b(x) \) denote two index sets such that:

i) \( I_b(x) \subseteq I_a(x), J_b(x) \subseteq J_a(x) \).

ii) The vectors \( A_i^T, i \in I_b(x) \) and \( E_i^T, i \in J_b(x) \) (where \( E_i \) is the i-th row of the identity matrix) are linearly independent.

iii) All the vectors \( A_i^T, i \in I_a(x) \setminus I_b(x) \) and \( E_i^T, i \in J_a(x) \setminus J_b(x) \) are linear combinations of vectors \( A_i^T, i \in I_b(x) \) and \( E_i^T, i \in J_b(x) \).
Furthermore we introduce the matrix $B(x) \in \mathbb{R}^{d \times d}$ and the vectors $q(x)$, $s(x) \in \mathbb{R}^d$, where $d = |I_b(x)| + |J_b(x)|$, as follows:

$$
B(x) := \begin{bmatrix} E_{I_b}(x) \\ A_{I_b}(x) \end{bmatrix}, \quad q(x) := \begin{bmatrix} 0 \\ \bar{b}_{I_b} \end{bmatrix}, \quad s(x) := (B(x)B(x)^T)^{-1}q(x) \tag{3}
$$

**Remark 1.** From iii) it follows that:

- For $\forall i \in I_a(x) \setminus I_b(x)$ there exists a $\tilde{z}_i \in \mathbb{R}^d$ such that
  $$A_i^T = B(x)_{I_b} \tilde{z}_i, \quad q(x)^T \tilde{z}_i = b_i$$

- For $\forall i \in J_a(x) \setminus J_b(x)$ there exists a $\bar{z}_i \in \mathbb{R}^d$ such that
  $$E_i^T = B(x)_{I_b} \bar{z}_i, \quad q(x)^T \bar{z}_i = 0.$$
3. PROPERTIES OF THE LEAST 2-NORM SOLUTION OF A LINEAR PROGRAM

In this section we show some elementary but important properties for the least 2-norm solution of a linear program.

PROPOSITION 1. Let \( \bar{x} \) be a solution of the linear program (1) and \((u, v)\) be a solution of its dual. Then the following statements hold:

a) If \( \bar{x} = p_2(0, \bar{x}) \) then \( \bar{x} \) is also the least 2-norm solution of the following system of equations

\[
B(x)x = q(x)
\]

that is \( \bar{x} = B(x)^T(B(x)B(x)^T)^{-1}q(x) = B(x)^Ts(x) \).

b) If \( \bar{x} = B(x)^T(B(x)B(x)^T)^{-1}q(x) + Kc \), for some \( K \in \mathbb{R} \), and

\[
\begin{align*}
I_b(\bar{x}) &< 0, \quad s(\bar{x}) > 0 \quad \text{if follows that} \quad \bar{x} = p_2(0, \bar{x}).
\end{align*}
\]

\[
\begin{align*}
I_b(\bar{x}) &\leq 0, \quad J_b(\bar{x}) &\leq 0,
\end{align*}
\]

\[
\begin{align*}
J_b(\bar{x}) &\leq 0, \quad J_d(\bar{x}) &\leq 0,
\end{align*}
\]

c) Moreover, if the gradients of active constraints are linearly independent

at \( \bar{x} \) then

\[
\begin{align*}
\langle s(\bar{x}) \rangle &< 0, \quad s(\bar{x}) > 0 \quad \text{if follows that} \quad \bar{x} = p_2(0, \bar{x}).
\end{align*}
\]

PROOF. First of all we set

\[
\begin{align*}
I_a := I_a(\bar{x}), \quad J_a := J_a(\bar{x}), \quad I_n := I_n(\bar{x}), \quad J_n := J_n(\bar{x}), \quad I_d := I_d(\bar{u}), \quad J_d := J_d(\bar{v}),
\end{align*}
\]

\[
\begin{align*}
I_b := I_b(\bar{x}), \quad J_b := J_b(\bar{x}), \quad B := B(\bar{x}), \quad q := q(\bar{x}), \quad s := s(\bar{x}).
\end{align*}
\]

a) We prove this part by contradiction. Suppose that exists \( \bar{x} \), with \( \bar{x} \neq \bar{x}, \) which is the least 2-norm solution of (4). Then all the points

\[
x(\theta) = \theta \bar{x} + (1-\theta)\bar{x}, \quad \forall \theta \in \mathbb{R},
\]

solve (4) and from the maximum principle it follows

\[
x^T(x(\theta) - \bar{x}) = 0.
\]

By taking into account Remark 1 we have

-5-
\[
\psi_i \in J_a \setminus J_b \quad x_i(\theta) = z_i^T x(\theta) = z_i^T q = 0
\]

\[
\psi_i \in I_a \setminus I_b \quad A_i x(\theta) = z_i^T B x(\theta) = z_i^T q = b_i
\]

Hence \( x_j(\theta) = 0 \), \( A_i x(\theta) = b_i \), \( \forall \theta \in R \). Since \( x_j > 0 \) and
\( A_i x(\theta) < b_i \) there exists a value \( \bar{\theta} \in (0,1) \) such that \( x_j(\bar{\theta}) > 0 \),
\( A_i x(\bar{\theta}) < b_i \). Therefore \( x(\bar{\theta}) \in X \) and it is, also, an optimal point for

(1): in fact from the KKT conditions for (1) it follows:
\[
c^T x(\bar{\theta}) = c^T x(\bar{\theta})_A = u^T I A_I x(\bar{\theta})_A = u^T b = c^T x.
\]

Hence we can conclude that \( x(\bar{\theta}) \) is feasible for problem (2). Now, by
applying the maximum principle to problem (2) we obtain
\[
\frac{-T}{x(x(\bar{\theta}) - x)} > 0
\]

Furthermore by (5) we have
\[
0 < \frac{-T}{x^T(x(\bar{\theta}) - x)} = \frac{-T}{x^T(\theta x + (1-\theta)x - x)}
\]
\[
= \bar{\theta}(\theta x - (1-\theta)x) = \bar{\theta}((1-\theta)x - (\theta x) < 0
\]

Therefore \( x \) is the least 2-norm solution of (3) that is (see [7])
\[
\tilde{x} = B^T (BB^T)^{-1} q = B^T s.
\]

b) By assumption we have
\[
\tilde{x} = B^T s + Kc = A^T I_b \setminus I_d s_{I_b \setminus I_d} + A^T I_b \setminus I_d s_{I_b \setminus I_d} + A^T J_b \setminus J_d s_{J_b \setminus J_d}
\]
\[
+ E^T J_d \setminus J_d s_{J_d \setminus J_d} + Kc
\]
and \( s_{I_b \setminus I_d} < 0, s_{J_b \setminus J_d} > 0 \).

From the KKT condition for (1) it follows
\[
\gamma(-c + A^T I_d \setminus I_d u - E^T J_d \setminus J_d v_j) = 0 \quad \forall \gamma \in R
\]

-6-
Now, by adding (6 and (7), we obtain

\[
\begin{align*}
\bar{x} &= (\gamma+K)c + A^T_{I_d}\gamma_{I_d} - s_{I_d} + A^T_{I_d}\gamma_{I_d} - s_{I_d} + A^T_{I_d}(-s_{I_d}) \\
&\quad - E^T_{J_d}\gamma_{J_d} + s_{J_d} - E^T_{J_d}\gamma_{J_d} - s_{J_d} s_{J_d} = 0.
\end{align*}
\]

If we choose a $\gamma > 0$ such that

\[
\gamma+K > 0, \quad \gamma_{J_d} + s_{J_d} > 0, \quad \gamma_{I_d} - s_{I_d} > 0
\]

and we set

\[
\begin{align*}
u^*_{I_d} &= \gamma_{I_d} - s_{I_d}, \quad u^*_{I_d} = \gamma_{I_d} - s_{I_d}, \quad u^*_{I_d} = -s_{I_d} \\
\nu^*_{J_d} &= \gamma_{J_d} + s_{J_d}, \quad v^*_{J_d} = \gamma_{J_d} + s_{J_d}, \quad v^*_{J_d} = s_{J_d}
\end{align*}
\]

\[
\gamma^* = \gamma + K, \quad u^*_{I_n} = 0, \quad v^*_{I_n} = 0
\]

we have that $(x, \gamma^*, u^*, v^*)$ satisfy the KKT conditions for (2).

c) $\implies$ It follows from part b)

$\Leftarrow$ By the assumption that the gradients of the active constraints are linearly independent at $\bar{x}$ we have

\[
I_a = I_b, \quad J_a = J_b.
\]

From part a) it follows

\[
\bar{x} = B^T s = A^T_{I_d} s_{I_d} + A^T_{I_d} s_{I_d} + E^T_{J_d} s_{J_d} + E^T_{J_d} s_{J_d}.
\]

Now $\bar{x}$ solve also problem (2) and, hence, there exists $u^* > 0, v^* > 0$ such that $(x, \gamma^*, u^*, v^*)$ satisfy the KKT conditions for (2), namely:

\[
\begin{align*}
\bar{x} &= A^T_{I_d}(-u^*) + E^T_{J_d} v^* + \gamma^* c \\
&= A^T_{I_d} u^* - E^T_{J_d} v^*
\end{align*}
\]

but by using the KKT condition for (1), that is

\[
c = A^T_{I_d} u^* - E^T_{J_d} v^*
\]

-7-
we obtain

\[
\bar{x} = A_d^T (\gamma u_d - u^*_d) + A_d I_d (\gamma u_d - u^*_d) + E_d^T (v^*_d - \gamma v_d)
\]

Then, by assumption, we have that the columns of \( B(x)^T \) are linearly independent and, hence, there must exist only one vector \( y \) such that:

\[
\bar{x} = B(x)^T y
\]

Therefore, from (8) and (9) we obtain

\[
s_I = \gamma u_d - u^*_d \quad \text{and} \quad s_J = v^*_d - \gamma v_d
\]

\[
s_b \geq 0 \quad \text{and} \quad s_b \leq 0
\]

and these last two inequalities complete the proof of part c).

We are now ready to establish a criterion for determining when the solution of the perturbed linear program is indeed the least 2-norm solution of the unperturbed linear program.
4. PERTURBATION OF LINEAR PROGRAMS

In [1], [2] it was shown that the least 2-norm solution of the linear program (1) can be obtained by solving a "proper" quadratic perturbation of the linear program. In particular we have the following important result.

**THEOREM 1.** Let the linear program (1) feasible. Then

\[ \langle \overline{X} \neq 0 \rangle \Rightarrow \langle 3 \varepsilon^* > 0 : p_2(\overline{C},X) = p_2(0,\overline{X}) \text{ for all } \varepsilon \in (0,\varepsilon^*) \rangle \]

\[ \langle x^* = p_2(0,X) \rangle \]

b) \[ \sup_{x \in X} c^T x = \langle \Rightarrow \langle 1 p_2(\overline{C},X) \rangle \rangle + \infty \text{ as } \varepsilon \to 0^+ . \]

**PROOF.** See [2].

We can note that \( p_2(\overline{C},X) \) is also a solution of the problem

\[
\begin{align*}
\text{Min} & \quad \frac{\varepsilon}{2} x^T x - c^T x \\
\text{s.t.} & \quad Ax < b \\
& \quad x > 0
\end{align*}
\]

and that the quadratic programming dual (see [8]) to (11) is

\[
\begin{align*}
\text{Min} & \quad \frac{1}{2} (A^T u - v) - c^T + \varepsilon b^T u \\
\text{s.t.} & \quad (u,v) > 0
\end{align*}
\]

where the primal and dual variable \( x \) and \( (u,v) \) are related by

\[
\begin{align*}
x = \frac{1}{\varepsilon} (-A^T u + v + c) .
\end{align*}
\]

Until now there was no simple way of determining "a priori" the parameter \( \varepsilon^* \) of Theorem 1. The next result which gives on "a posteriori" computational scheme for \( \varepsilon^* \) shows why it is difficult to compute \( \varepsilon^* \) "a priori". In fact
we can note that $\epsilon^*$ depends strictly on the least 2-norm solution of (1) (by means of the vector $s(x)$, defined by (3)) and on the optimal solution of the dual of the linear program (1).

PROPOSITION 2. Assume $\bar{x} \neq \emptyset$. Let the gradients of the active constraints $u$ be linearly independent at $\bar{x} = p_2(0,\bar{x})$ and let $(u^*, v^*)$ be the solution point of the dual of the linear program (1). Then it follows that

$$p_2^*(C, X) = \bar{x} = p_2(0, \bar{x}) \quad \forall \epsilon \in (0, \epsilon^*)$$

where $\epsilon^*$ is given by

$$\epsilon^* = \min \left\{ \left[ \min_{i \in I_d(u^*)} u^*_i / s(x)_i \right], \left[ \min_{i \in J_d(v^*)} - v^*_i / s(x)_i \right] \right\}$$

(14)

where $s(x)$ is defined by (3) and $I_d(u^*) = \{ i : u^*_i > 0 \}$ and $J_d(v^*) = \{ i : v^*_i > 0 \}$.

PROOF. We recall the definitions given in Section 2 and we set $I_a := I_a(\bar{x})$, $J_a := J_a(\bar{x})$, $I_n := I_n(\bar{x})$, $J_n := J_n(\bar{x})$, $I_d := I_d(u^*)$, $J_d := J_d(v^*)$, $s := s(\bar{x})$.

Now, from the KKT conditions for (1) we have

$$-c + A^T u^*_d - E^T v^*_d = 0 \quad (15)$$

By using part c) of Proposition 1 we obtain

$$\epsilon(\bar{x} - \bar{A}^T s_{I_d} - \bar{A}^T s_{I_d} I_d I_d - \bar{E}^T s_{J_d} J_d J_d - \bar{E}^T s_{J_d} J_d J_d - \bar{E}^T s_{J_d} J_d J_d) = 0 \quad (16)$$

where $s_{I_d} I_d < 0$ and $s_{J_d} J_d > 0$.

Hence, by adding (15) and (16), we obtain

$$\epsilon \bar{x} - c + \bar{A}^T (u^*_d - \epsilon s_{I_d}) + \bar{A}^T s_{I_d} (\epsilon s_{I_d} I_d I_d) - \bar{E}^T (v^*_d + \epsilon s_{J_d}) - \bar{E}^T s_{J_d} (\epsilon s_{J_d} J_d J_d) = 0$$
If we choose $\varepsilon^*$ as the largest value of $\varepsilon$ such that
\[
\begin{align*}
u^*_d - \varepsilon s^*_d > 0, & \quad v^*_d + \varepsilon s^*_d > 0 \\
(u(\varepsilon) = u^*_d - \varepsilon s^*_d, & \quad u(\varepsilon) = -\varepsilon s^*_d \quad \text{with } \Left(14) \Right) \\
\begin{array}{ll}
u_1(\varepsilon) = v^*_1 + \varepsilon s^*_1, & \quad v_1(\varepsilon) = \varepsilon s^*_1 \\

\end{array}
\end{align*}
\]

(that is if $\varepsilon^*$ is given by (14)) we can define $\forall \varepsilon \in (0, \varepsilon^*)$
\[
\begin{align*}
u(\varepsilon) = u^*_d - \varepsilon s^*_d, & \quad u(\varepsilon) = -\varepsilon s^*_d \\
\begin{array}{ll}
u(\varepsilon) = v^*_d + \varepsilon s^*_d, & \quad v(\varepsilon) = \varepsilon s^*_d \\

\end{array}
\end{align*}
\]

Then it turns out that $(x_u(\varepsilon), v(\varepsilon))$ is the unique solution of the KKT conditions for problem (11), $\forall \varepsilon \in (0, \varepsilon^*)$.

Therefore we have that
\[
\bar{x} = p_2(0, \bar{x}) = p_2(\bar{\varepsilon}, x) \quad \forall \varepsilon \in (0, \varepsilon^*)
\]
and that $(u(\varepsilon), v(\varepsilon))$ is the unique optimal solution of the dual problem (12).

In order to overcome the inherent difficulty of determining $\varepsilon^*$, "a priori" a practical computational criterion was proposed in [5] to check whether the solutions of the perturbed quadratic programs provide the least 2-norm solution of the original linear program. This criterion required two points $(x_i, u_i, v_i), i = 1, 2,$ that satisfy the KKT conditions for problem (11) with two different values for $\varepsilon$.

In the sequel we give an improved new criterion that requires only one point $(x, u, v)$ which satisfies the KKT conditions for problem (11). However this new criterion requires a matrix inversion. By contrast, no matrix inversion was required in [5].

In the next theorem we make use of the index sets $I_a(x), J_a(x), I_n(x), J_n(x), I_b(x), J_b(x), I_d(u), J_d(v)$, the matrix $B(x)$, and the vectors $q(x)$ and $s(x)$ which were defined in section 2. Furthermore we assume that
I_a(\bar{x}) U J_a(\bar{x}) \neq \emptyset \text{ where } \bar{x} \text{ is the solution of (11), because otherwise we have trivially that } \bar{x} \neq p_2(0,\bar{x}).

THEOREM 2. Let \( \bar{x} \) be a solution of problem (11), that is \( \bar{x} = p_2(\bar{c},\bar{X}) \), let \((\bar{u},\bar{v})\) be a solution of the dual problem (12) and let \( \varepsilon^* \) be the optimal value for \( \varepsilon \) given by (14). Then

\[
\begin{align*}
&\left( \bar{u} \succ -\varepsilon s(\bar{x}), \quad \bar{v} \succ \varepsilon s(\bar{x}) \right) \\
&\left( \begin{bmatrix} I_b(\bar{x}) & -I_b(\bar{x}) \\ J_b(\bar{x)} & -J_b(\bar{x)} \end{bmatrix} \right) \Rightarrow \left( \bar{x} = p_2(\bar{c},\bar{X}) = p_2(0,\bar{x}) \right)
\end{align*}
\]

for some \( \gamma < 1 \)

and \((u^*,v^*)\) are optimal for the dual of the linear program (1) where

\[
\begin{align*}
u^* &:= \frac{(\bar{u} + \varepsilon s(\bar{x}))}{1 - \gamma}, \quad u^* := \frac{\bar{u}}{1 - \gamma}, \quad u^* := 0 \\
I_b(\bar{x}) &:= \frac{I_a(\bar{x})}{1 - \gamma}, \quad J_b(\bar{x}) := \frac{J_a(\bar{x})}{1 - \gamma}, \quad I_b(\bar{x}) := 0
\end{align*}
\]

(19)

\[
\begin{align*}
\begin{bmatrix} \bar{v} \succ -\varepsilon s(\bar{x}) \\ \bar{v} \succ \varepsilon s(\bar{x}) \end{bmatrix} \\
\begin{bmatrix} I_a(\bar{x}) & J_a(\bar{x}) \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
&\left( \bar{v} \succ -\varepsilon s(\bar{x}), \quad \bar{v} \succ \varepsilon s(\bar{x}) \right) \\
&\left( \begin{bmatrix} I_a(\bar{x}) & J_a(\bar{x}) \end{bmatrix} \right) \Rightarrow \left( \bar{x} = p_2(\bar{c},\bar{X}) = p_2(0,\bar{x}) \right)
\end{align*}
\]

(20)

b) If the gradients of the active constraints are linearly independent at \( \bar{x} \), then it follows that:

\[
\begin{align*}
\begin{bmatrix} \bar{u} \succ -\varepsilon s(\bar{x}) \\ \bar{v} \succ \varepsilon s(\bar{x}) \end{bmatrix} \\
\begin{bmatrix} I_a(\bar{x}) & J_a(\bar{x}) \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
&\left( \bar{u} \succ -\varepsilon s(\bar{x}), \quad \bar{v} \succ \varepsilon s(\bar{x}) \right) \\
&\left( \begin{bmatrix} I_a(\bar{x}) & J_a(\bar{x}) \end{bmatrix} \right) \Rightarrow \left( \bar{x} = p_2(\bar{c},\bar{X}) = p_2(0,\bar{x}) \right)
\end{align*}
\]

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and, if \( \bar{x} = p_2(0, \bar{x}) \), the optimal solution \((u^*, v^*)\) of the dual of the linear program (1) is given by (19) and (20) with \( y = 0 \), \( I_a(\bar{x}) = I_b(\bar{x}) \) and \( J_a(\bar{x}) = J_b(\bar{x}) \).

c) If the gradients of the active constraints are linearly independent at \( \bar{x} \) and the strict complementarity holds for the linear program (1) at the least 2-norm solution \( p_2(0, \bar{x}) \), then it follows that:

\[
\begin{pmatrix}
 u \\
 I_a(x) \\
 J_a(x)
\end{pmatrix} \succ -\varepsilon s(\bar{x}) \\
\begin{pmatrix}
 v \\
 I_b(x) \\
 J_b(x)
\end{pmatrix} \succ \varepsilon s(\bar{x})
\]

\[
\begin{pmatrix}
 [E-B(\bar{x})(B(\bar{x})B(\bar{x})^T)^{-1}B(\bar{x})]c = 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
 \bar{x} \neq \emptyset \\
 0 < \varepsilon < \varepsilon^*
\end{pmatrix}
\]

\[
\begin{pmatrix}
 \bar{x} = p_2(\varepsilon, x) = p_2(0, \bar{x})
\end{pmatrix}
\]

PROOF. Again we set \( I_a := I_a(\bar{x}), J_a := J_a(\bar{x}), I_n := I_n(\bar{x}), J_n := J_n(\bar{x}), I_b := I_b(\bar{x}), J_b := J_b(\bar{x}), I_d := I_d(\bar{u}), J_d := J_d(\bar{v}), B := B(\bar{x}), q := q(\bar{x}), s := s(\bar{x}) \).

a) Define \( \bar{y} := \varepsilon \bar{x} - c \).

Since the point \((\bar{x}, \bar{u}, \bar{v})\) satisfies the KKT condition for problem (11), the point \((\bar{y}, \bar{u}, \bar{v})\) satisfies the KKT conditions of the following problem

\[
\begin{aligned}
\text{Min } & \frac{1}{2} \| y \|^2 \\
\text{subject to } & Ay \leq \varepsilon b - Ac \\
& y \geq -c
\end{aligned}
\]

(21)

Therefore \( \bar{y} \) is the optimal solution of (21).

By repeating the same steps of the proof of part a) of Proposition 1 we have that \( \bar{y} \) is the least 2-norm solution of the following system of equations.
In fact, suppose not. Then there exists a \( \tilde{y}, \tilde{y} \neq \overline{y} \), which is the least norm solution of (22). Then \( \forall \theta \) the points \( y(\theta) = \theta \tilde{y} + (1-\theta)\overline{y} \) solve (22) and (recalling Remark 1)

\[
\begin{align*}
\forall i \in J_a \setminus J_b & \quad y_i(\theta) = \frac{z_i^T}{z_i^T} B y(\theta) = \frac{z_i^T}{z_i^T} (\epsilon q - B c) = -c_i \\
\forall i \in I_a \setminus I_b & \quad A_i y(\theta) = \frac{z_i^T}{z_i^T} B y(\theta) = \frac{z_i^T}{z_i^T} (\epsilon q - B c) = \epsilon b_i - A_i c.
\end{align*}
\]

Moreover, since \( \overline{y}_j > -c_j \) and \( A_i \overline{y} < \epsilon b_i - A_i c \), there exists a value \( \theta \in (0,1) \) such that \( y(\theta)_j > -c_j \) and \( A_i y(\theta) < \epsilon b_i - A_i c \). Therefore \( y(\theta) = \{y', \epsilon y' < \epsilon b - Ac, y' > -c\} \).

Now, by applying again the maximum principle, we obtain

\[
0 < \frac{-y^T}{y(\tilde{y}) - \overline{y}} = \theta (y^T\tilde{y} - 1\overline{y}^2) = \theta (1\overline{y}^2 - 1\overline{y}^2) < 0.
\]

Hence we can conclude that \( \overline{y} \) is the least norm solution of (22), namely

\[
\overline{y} = B^T (BB)^{-1} [\epsilon q - B c]
\]

from which

\[
\epsilon x = [E - B^T (BB)^{-1} B] c + \epsilon B^T (BB)^{-1} q
\]

\[
= [E - B^T (BB)^{-1} B] c + \epsilon B^T s.
\]

Now by recalling the KKT conditions for (11) we have

\[
\epsilon x - c + A_i^T \overrightarrow{u}_i - \overrightarrow{v}_j = 0
\]

from which we obtain

\[
[E - s^T (BB)^{-1} B] c + \epsilon B^T s - c + A_i^T \overrightarrow{u}_i - \overrightarrow{v}_j = 0
\]

\[
[E - B^T (BB)^{-1} B] c - c + A_i^T (\overrightarrow{u}_i + \epsilon s_i) + A_i (\overrightarrow{u}_i) - \overrightarrow{v}_j = 0
\]

\[
- \overrightarrow{v}_j \epsilon s_j - \overrightarrow{v}_j \epsilon s_j - \overrightarrow{v}_j \epsilon s_j = 0
\]

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By assumptions we have \([E - B^T(BB^T)^{-1}B] = \gamma c\) with some \(\gamma < 1\)

\[ \overline{u}_{I_b} + \epsilon s_{I_b} > 0, \quad \overline{v}_{J_b} - \epsilon s_{J_b} > 0. \]

Therefore if we set

\[
\begin{align*}
\overline{u}^*_{I_b} := \frac{\overline{u}_{I_b} + \epsilon s_{I_b}}{1 - \gamma}, & \quad \overline{u}^*_{I_a \setminus I_b} := \frac{\overline{u}_{I_a \setminus I_b}}{1 - \gamma}, & \quad \overline{u}^*_{I_n} := 0, \\
\overline{v}^*_{J_b} := \frac{\overline{v}_{J_b} - \epsilon s_{J_b}}{1 - \gamma}, & \quad \overline{v}^*_{J_a \setminus J_b} := \frac{\overline{v}_{J_a \setminus J_b}}{1 - \gamma}, & \quad \overline{v}^*_{J_n} := 0,
\end{align*}
\]

we have that \((x, u^*, v^*)\) satisfy the KKT conditions for the linear program \((1)\).

b) \((\Rightarrow)\) The proof of this implication follows directly from part a).

\((\Leftarrow)\) By using the KKT conditions for linear program \((1)\) we know that there exist \((\tilde{u}, \tilde{v})\) such that

\[-c + A^T\tilde{u} - E^T\tilde{v} = 0\]

from which

\[ c = B^T\begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix}. \]

Therefore we have

\[ [E - B^T(BB^T)^{-1}B]c = [E - B^T(BB^T)^{-1}B]B^T\begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix} = 0. \]

Now, by repeating the same steps of proof of Proposition 2 and by using (17) and (18) we obtain that

\[ \overline{u}_{I_a} > -\epsilon s_{I_a}, \quad \overline{v}_{J_a} > \epsilon s_{J_a}. \]
c) The proof of part c) follows from the proof of part b) and the proof of Proposition 2 (see (17) and (18)).

REMARK 2. Part b) of Theorem 2 characterizes the solvability of a linear program in terms of the solvability of a convex quadratic function minimization on the non negative orthant (problem (12)) without any "a priori" assumptions regarding the solvability of the linear program (1).

We now turn our attention to computational procedures for determining the least 2-norm solution of a linear program.
5. AN ALGORITHM WITH FINITE TERMINATION FOR LINEAR PROGRAMMING

The quadratic programming problem (12) can be solved by using a sparsity-preserving SOR algorithm introduced in [4]. More specifically we have the following algorithm where we have assumed that \( \lambda_j \neq 0, \forall j = 1, \ldots, m. \)

**LPSOR Algorithm**

Choose \((u^0, v^0) \in \mathbb{R}^{m+n}, \omega \in (0, 2)\) and \(\epsilon > 0.\)

Having \((u^k, v^k)\) compute \((u^{k+1}, v^{k+1})\) as follows:

\[
u_j^{k+1} = v_j^k - \frac{\omega}{\lambda_j^2} \left( \lambda_j \left( \sum_{j=1}^{m} (A_j^T)_{jk} u_j^k + \sum_{j=1}^{m} (A_j^T)_{jk} u_j^k - v_j^k - c \right) \right), \quad k = 1, \ldots, m
\]

\[
u_j^{k+1} = \left( v_j^k - \omega (-A_j u_j^{k+1} + v_j^k + c) \right)_+,
\]

The principal and computationally-distinguishing features of this SOR algorithm are that it preserves the sparsity structure of the problem and require only simple operations, and, hence, very large problems can be tackled.

We refer to [2] and [4] for a more complete description of sparsity-preserving SOR algorithm, here we only recall the following convergence result for the LPSOR Algorithm

**Proposition 3.** Assume that \(\mathbf{x} \neq \emptyset\) and that the gradients of the active constraints of the linear program (1) at the optimal point \(\mathbf{x} = p(0, X)\) are linearly independent. Let \(\{(u^k, v^k)\}\) be the sequence generated by the LPSOR Algorithm. Then

a) There exists a real positive number \(\epsilon^*\) such that for each \(\epsilon \in (0, \epsilon^*)\), the sequence \(\{(u^k, v^k)\}\) converges to a point \((u(\epsilon), v(\epsilon))\) which solves problem (12) and the corresponding \(x(\epsilon)\) determined by (13) is independent of
\( \varepsilon \) and \( x(\varepsilon) = \bar{x} = p_2(0, X) \).

b) Moreover, if strict complementarity holds for problem (1) at the least 2-norm solution \( \bar{x} = p_2(0, X) \), then there exists a real positive number \( \varepsilon^{**} \), \( \varepsilon^{**} > \varepsilon^* \), such that for each \( \varepsilon \in (0, \varepsilon^{**}) \) there exists a \( k^* \) such that for all \( k > k^* \) the following hold:

\[
\begin{align*}
\{ i : u^k_i > 0 \} &= \{ i : u^*_i > 0 \} = I_a(\bar{x}) \\
\{ i : u^k_i = 0 \} &= \{ i : u^*_i = 0 \} = I_n(\bar{x}) \\
\{ i : v^k_i > 0 \} &= \{ i : v^*_i > 0 \} = J_a(\bar{x}) \\
\{ i : v^k_i = 0 \} &= \{ i : v^*_i = 0 \} = J_n(\bar{x})
\end{align*}
\]

where \((u^*, v^*)\) is the optimal solution of the dual of the linear program (1). \((I_a(x), I_n(x), J_a(x), J_n(x))\) were defined in section 2.)

PROOF. See [5].

In applying part i) of Proposition 2 we must be able to select a value of \( \varepsilon \) such that \( \varepsilon < \varepsilon^* \). In order to ensure that \( \varepsilon < \varepsilon^* \) we may have to choose very small values for \( \varepsilon \) yield a very slow convergence of the LPSOR Algorithm when applied to the problem (12).

In order to overcome both the lack of a practical "a priori" selection procedure for the parameter \( \varepsilon^* \) and the need to solve problem (12) exactly many times for a decreasing sequence of \( \varepsilon \) values (until one of the criteria of Theorem 2, Corollary 1 or Theorem 2 of [5] is satisfied) we propose an algorithm which is based on the results of the preceding sections.

The key idea of this algorithm is similar to that of the algorithms proposed in [5], [6], where problem (12) is solved only approximately after which \( \varepsilon \) is decreased in a prescribed manner.
The main distinguishing property of the present algorithm is that, under mild assumptions, it finds the least 2-norm solution of the linear program (1) in a finite number of iterations. The algorithm described below makes use of two preselected sequences \( \{\epsilon^k\} \) and \( \{N^k\} \) such that
\[
\lim_{k \to +\infty} \epsilon^k = 0 \quad \text{and} \quad \lim_{k \to +\infty} N^k = +\infty.
\]

**ALGORITHM I**

Choose \((u^0, v^0) \in R^{m+n}_+\). Having \((u^{k-1}, v^{k-1})\), \((u^k, v^k)\) is obtained by applying \(N^k\) iterations of the SOR algorithm to the problem (12) with \(\epsilon = \epsilon^k\) and by using \((u^{k-1}, v^{k-1})\) as a starting point.

**THEOREM 3.** Assume that:

a) The linear program (1) is solvable.

b) The gradients of the active constraints are linearly independent at the optimal point \(x = p_2(0, \bar{x})\).

c) The strict complementarity holds for the linear program (1) at the optimal point \(x = p_2(0, \bar{x})\).

Let \((u^k, v^k)\) be the sequence produced by the algorithm and let
\[
i^k := \{i : u^k_i > 0\}, \quad j^k := \{i : v^k_i > 0\}, \quad B^k := \begin{bmatrix} E^k \\ J^k \\ \vdots \\ A^k \\ I^k \end{bmatrix}.
\]

Then there exist a \(\bar{k}\) such that for all \(k \geq \bar{k}\) it follows that the matrix \(B^k\) has full rank and
\[
x^k := (B^k)^{-1} s^k = \bar{x} = p_2(0, \bar{x})
\]
where
\[ s^k := [B^k(B^k)^T]^{-1} \begin{pmatrix} 0 \\ |J^k| \\ b \\ I^k \end{pmatrix} \]  \hspace{1cm} (25)

PROOF. By using Proposition 3 we have that there exist a \( \varepsilon^{**} \) and a \( k^* \) such that \( \forall \varepsilon \in (0, \varepsilon^{**}) \) and \( \forall k > k^* \) we have (by recalling the definitions given in section 2)

\[ i^k = I_a(\overline{x}) = I_b(\overline{x}), \hspace{0.5cm} j^k = J_a(\overline{x}) = J_b(\overline{x}) \]

and, hence

\[ b^k = B^k(\overline{x}), \hspace{0.5cm} s^k = s(\overline{x}) . \]

Therefore, there exists a \( k \) large enough such that \( \varepsilon \overline{k} < \varepsilon^{**} \) and \( \overline{N}^k > k^* \)

and, hence, the proof of theorem follows from part a) of Proposition 1.

REMARK 3. Practically we do not need to compute the matrix \( (B^k(s^k)^T)^{-1} \), we only have to solve the system

\[ (B^k(B^k)^T)s^k = \begin{pmatrix} 0 \\ |J^k| \\ b \\ I^k \end{pmatrix} . \]

Therefore any efficient sparsity preserving method for solving linear systems of equations can be used.

THEOREM 4. Assume that:

a) The linear program (1) has a unique solution \( \overline{x} \).

b) The gradients of the active constraints are linearly independent at the optimal point \( \overline{x} \).

Let \( \{u^k, v^k\} \) be the sequence produced by the algorithm and let
\[ i_k^k = \{i : u_i^k > 0\}, \quad j_k^k = \{i : v_i^k > 0\} \]

\[ \tilde{i}_k^k = \{i : u_i^k = 0\}, \quad \tilde{j}_k^k = \{i : v_i^k = 0\} \]

Then there exist \( \nu, k \) such that for all \( k > \tilde{k} \) the matrix \( A_{i,j}^k \) is a square non singular matrix and

\[ x^k := (A_{i,j}^k)^{-1}_i b_i^k = \bar{x} \]

**Proof.** By the assumptions a) and b) we have that the linear program (4) and its dual have unique optimal solutions and that the strict complementarity holds (see page 26 of [11]). Now, by using again Proposition 3 we have that there exist \( \varepsilon^{**} \) and \( \kappa^* \) such that \( \nu \varepsilon \in (0, \varepsilon^{**}] \) and \( \nu k > \kappa^* \) we have

\[ i_k^k = I_a(\bar{x}) = I_b(\bar{x}) = I_d(\bar{u}), \quad j_k^k = J_a(\bar{x}) = J_b(\bar{x}) = J_d(\bar{u}) \]

\[ \tilde{i}_k^k = I_n(\bar{x}) = \{i : u_i = 0\}, \quad \tilde{j}_k^k = J_n(\bar{x}) = \{i : v_i = 0\} \]

where \( (\bar{u}, \bar{v}) \) is the optimal solution of the dual of the linear program (1).

Therefore for values of \( k \) large enough (such that \( \varepsilon^k < \varepsilon^{**} \) and \( N^k > \kappa^* \)) the proof of theorem follows by using usual arguments of the duality theory for linear programs (see [11], page 45).

In Algorithm I different stopping criterions can be used. As examples we propose two algorithms. In Algorithm II the stopping criterion is based on Theorem 2 and Theorem 3 and it uses the properties of the multiplier function \( \lambda(x) \) for the linear program (1) (we refer to [9] and [11]) for the definition and the properties of the multiplier function \( \lambda(x) \), here we recall only that \( \lambda(*) \) is a function from \( \mathbb{R}^n \) to \( \mathbb{R}^{m+n} \) such that if \( \bar{x} \) is optimal for the linear program (1) \( \lambda(\bar{x}) \) gives the optimal solution of the dual of the linear program).
Algorithm III is based on Theorem 4 and the duality theory for the linear programs.

Again Algorithm II and Algorithm III make use of the preselected sequences \( \{e^k\} \) and \( \{N^k\} \) such that (23) holds and the sequence \( \{I^k, J^k, \tilde{I}^k, \tilde{J}^k, B^k\} \) and \( \{s^k\} \) given by (24), (25) and (26).

**ALGORITHM II**

Data: \((u^0, v^0) \in \mathbb{R}^{m+n}\)

Step 0: Set \( k = 1 \).

Step 1: Compute \((u^k, v^k)\) by applying \(N^k\) iterations of the SOR algorithm to the problem (12) with \( \epsilon = \epsilon^k \) and by using \((u^{k-1}, v^{k-1})\) as starting point.

Step 2: If \( s^k \) has full rank, \( u^k > -\epsilon s^k \) and \( v^k > \epsilon s^k \), go to step 3; else go to step 5.

Step 3: Set \( x^k = (B^k)^Ts^k \) and \( \lambda^k = \lambda(x^k) \).

Step 4: If \((x^k, \lambda^k)\) satisfy the KKT conditions for the linear program (1) stop; else go to step 5.

Step 5: Set \( k = k+1 \), go to step 1.

**REMARK 4.** Step 2 is not necessary, it serves only to reduce the numbers of computations of the multiplier function. If the computation of the matrix \((B^k(B^k)^T)^{-1}\) is too expensive we can replace step 2 with

Step 2': If \( k = iM \), with \( i = 1, 2, \ldots \), go to step 3; else go to step 5.

where \( M \) is a fixed positive number.

**COROLLARY 1.** Under the assumption a), b) and c) of Theorem 3, Algorithm II terminates in a finite number of iterations and the produced point \( x^v \) is the
least 2-norm solution of the linear program (1) and $\lambda^v$ is the optimal solution of its dual.

PROOF. The proof follows by repeating similar arguments of proof of Theorem 3 and by using part c) of Theorem 2 and the properties of the multiplier function.

ALGORITHM III

Data: $(u^0, v^0) \in \mathbb{R}^{m+n}$.

Step 0: Set $k = 1$.

Step 1: Compute $(u^k, v^k)$ by applying $N^k$ iteration of the SOR algorithm to the problem (12) with $\varepsilon = \varepsilon^k$ and by using $(u^{k-1}, v^{k-1})$ as starting point.

Step 2: If $A$ is a square nonsingular matrix go to step 3; else go the step 5;

Step 3: Set $z^k = (A^{-1})^k (A^{-1})^T b^k$, $w^k = (A^{-1})^T c^k$.

Step 4: If $z^k < 0, (A^{-1})^T b^k < 0, w^k > 0, (A^{-1})^T c^k > 0$ stop; else go to step 5.

Step 5: Set $k = k+1$, go to step 1.

COROLLARY 2. Under the assumption a) and b) of Theorem 4, Algorithm III terminates in a finite number of iterations and if $(z^v, w^v)$ is the produced point and if $(x^v, u^v)$ is defined as follows

$$
\begin{align*}
x_i^v & = \begin{cases} 
z_i^v & \text{if } i \in \bar{J}^k \\
0 & \text{if } i \in J^k 
\end{cases} \\
u_i^v & = \begin{cases} 
w_i^v & \text{if } i \in \bar{I}^k \\
0 & \text{if } i \in I^k 
\end{cases}
\end{align*}
$$
Then $x^*$ is the unique optimal solution of the linear program (1) and $u^*$ is the optimal solution of its dual.

PROOF. The proof follows from Theorem 4 and from the duality theory for linear programs (see [11], pages 45-50).

For brevity's sake, here we do not discuss an interesting algorithm which is the conjunction of Algorithm I of this paper with the algorithm proposed in [6]. In fact such an algorithm would be linearly or superlinearly convergent under assumptions weaker than those of Theorem 3 and would also have finite termination under the assumption a) - c) of Theorem 3.

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A finite algorithm for the least two-norm solution of a linear program

Summary Report - no specific reporting period

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Linear programming, Quadratic programming

By perturbing properly a linear program to a separable quadratic program it is possible to solve the latter in its dual variable space by iterative techniques such as sparsity-preserving SOR (successive overrelaxations). In this way large sparse linear programs can be handled.

In this paper we give a new computational criterion to check whether the solution of the perturbed quadratic programs provide the least 2-norm solution of the original linear program. This criterion improves on the criterion

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proposed in an earlier paper.

We also describe an algorithm for solving linear programs which is based on the SOR methods. The main property of this algorithm is that, under mild assumptions, it finds the least 2-norm solution of a linear program in a finite number of iterations.