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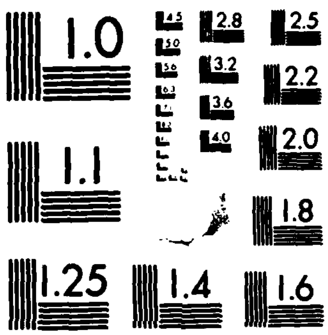
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The Pennsylvania State University

Department of Statistics

University Park, Pennsylvania

TECHNICAL REPORTS AND PREPRINTS

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A LARGE DEVIATIONS RESULT AND BAHADUR EFFICIENCY OF
TWO-SAMPLE TESTS BASED ON ONE-SAMPLE SIGN STATISTICS

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and

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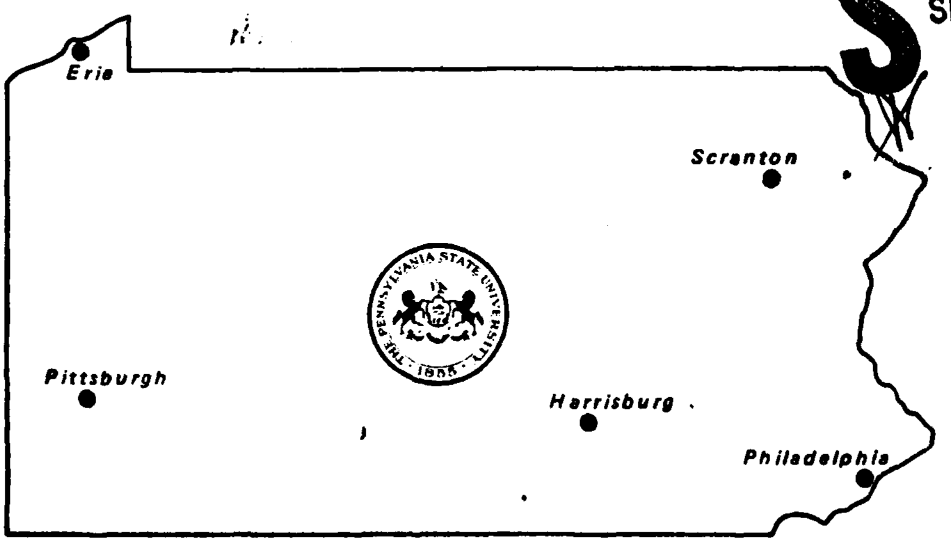

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SUMMARY

This paper discusses the two-sample test of location based on the comparison of two distribution free one-sample confidence intervals derived from sign statistics. This test procedure, first introduced by Hettmansperger (1986), rejects the null hypothesis of equal population medians when the two intervals are disjoint. He presents three different ways to select the two one-sample intervals and one choice leads to Mood's test. All solutions have the same Pitman efficiency. This paper shows that the choices can be distinguished on the basis of Bahadur's efficiency. We formulate the problem in terms of (asymptotically) fixed-width confidence intervals. In this context various median tests (including Mood's test) arise as special cases and they yield different performance. The solution that specifies equal asymptotic lengths for the one-sample intervals (which is different from Mood's test) is recommended.

Some key words: Bahadur efficiency; Fixed-width confidence interval; Pitman efficiency; Probability of large deviations; Sign statistic; Two-sample location problem.

1. INTRODUCTION

The two-sample test of location discussed in this paper is based on the comparison of two distribution free one-sample confidence intervals. The test rejects the null hypothesis of equal population medians if the intervals fail to overlap.

More precisely, let $\{X_i\}_{i=1}^{m}$ and $\{Y_i\}_{i=1}^{n}$ represent independent random samples from the respective populations $F_{\theta_x}(\cdot) = F(\cdot-\theta_x)$ and $F_{\theta_y}(\cdot) = F(\cdot-\theta_y)$ with unique medians θ_x and θ_y. Let ξ_p denote the p^{th} quantile of F, $0 < p < 1$. We assume that for all p

$$
\begin{aligned}
&F(\cdot) \text{ is twice differentiable at } \xi_p , \\
&\text{with } F'(\xi_p) = f(\xi_p) > 0 .
\end{aligned} \tag{1.1}
$$

Let the sign-interval on the X-sample be given by

$$[L_x, U_x] = [X_{(d_x)}, X_{(u_x)}] \tag{1.2}$$

where the endpoints are the d_x^{th} and u_x^{th} observations of the ordered sample

$$X_{(1)} \leqslant X_{(2)} \leqslant \ldots \leqslant X_{(m)}$$

with $u_x = m - d_x + 1$. The depth d_x, which specifies how deep into the ordered sample the endpoints lie, is defined by

$$d_x = m/2 + .5 - z_x m^{1/2}/2 \qquad (1.3)$$

where z_x is such that $\Phi(-z_x) = \alpha_x$ and Φ is the standard normal c.d.f.. We refer to this interval as a sign-interval since it can be derived by inverting the acceptance region of a size $2\alpha_x$ two-sided sign test. Similarly define the sign-interval $[L_y, U_y]$ on the Y-sample. Let $\Delta = \theta_y - \theta_x$. To test $H_o : \Delta = 0$ versus $H_A : \Delta \neq 0$, reject H_o if the sign-intervals are disjoint. That is,

$$\text{if} \quad U_x < L_y \quad \text{or} \quad U_y < L_x \quad . \qquad (1.4)$$

A two-sample confidence interval for Δ is given by

$$[L_y - U_x \, , \, U_y - L_x] \quad . \qquad (1.5)$$

This procedure has been introduced by Hettmansperger (1984) who derives the following two main limiting results: Suppose $m,n \to \infty$ such that $m/(m+n) \to \lambda$, $0 < \lambda < 1$.

(i) Then under $H_o : \Delta = 0$,

$$\alpha = P\{U_x < L_y\} + P\{U_y < L_x\} \to 2\Phi(-z) \tag{1.6}$$

where $z = (1-\lambda)^{1/2} z_x + \lambda^{1/2} z_y$. (1.7)

(ii) Let Λ denote the length of the two-sample confidence interval (1.5). If z_x and z_y satisfy the condition (1.7), then with probability 1

$$(m+n)^{1/2}\Lambda \to z/((\lambda(1-\lambda))^{1/2} f(0)) \ . \tag{1.8}$$

We note that the two one-sample intervals (for θ_x and θ_y) have respective approximate coverage probabilities $\gamma_x = 1 - 2\alpha_x$ and $\gamma_y = 1 - 2\alpha_y$. This follows from (1.3) and the normal approximation to the binomial distribution.

Now let α and λ , $0 < \lambda < 1$, be given and define z by $\alpha = 2\Phi(-z)$. Select z_x and z_y so that they satisfy (1.7). By (1.3) this determines the one-sample sign-intervals (that is, the depths). The resulting two-sample test is of approximate size α . Clearly there are infinitely many choices for z_x and z_y. Hettmansperger (1984) discusses three different choices. He recommends to select equal confidence coefficients $\gamma_x = \gamma_y$, or equivalently $z_x = z_y$, because these z values are essentially constant with respect to reasonable ratios of sample sizes. More precisely, by (1.7),

$$z_x = z_y = z(\lambda^{1/2}+(1-\lambda)^{1/2})^{-1} .$$

Another choice leads to Mood's (1950) median test. (For a discussion see Pratt (1964) and Gastwirth (1968)). Let, for simplicity, $m + n = 2r$, $m \geq n$. The Mood-interval for Δ is defined as follows:

$$[Y_{(d)} - X_{((m+n)/2-d+1)}, Y_{(n-d+1)} - X_{((m-n)/2+d)}] .$$

This interval is obtained by inverting the acceptance region of a two-sided test based on the Mood statistic which follows a hypergeometric distribution under $H_o : \Delta = 0$. From the normal approximation d is chosen so that an approximate size α test is achieved. That is,

$$d = n/2 + .5 - z(mn/(4(m+n-1)))^{1/2} \qquad (1.9)$$

where z is such that $\Phi(-z) = \alpha/2$. We can consider this interval as being constructed from two sign-intervals with depths $d_y = d$ and $d_x = (m-n)/2 + d_y$. Statement (1.9) is (asymptotically) equivalent to (1.3) if

$$z_y = z\lambda^{1/2} , \quad z_x = z(1-\lambda)^{1/2} ,$$

and the condition (1.7) is clearly satisfied.

The starting point for this paper is the observation that, according to (1.8), all choices of the z_x and z_y lead to the same Pitman efficiency, as long as (1.7) is satisfied. The choices can be distinguished, however, by an alternative notion which is Bahadur's efficiency. The analysis of this efficiency leads to a formulation of the problem in terms of (asymptotically) fixed-width confidence intervals. We compare the rates at which the Type I error probabilities tend to zero while the lengths remain fixed at (or tend to) a positive constant. In this context the various special choices (including Mood's test) yield different performance. On the basis of this efficiency criterion, we then recommend the solution that specifies equal asymptotic lengths for the one-sample intervals which is (except in the case of equal sample sizes) different from both the Mood solution and the equal confidence coefficients recommendation.

In Section 2 the exact size of the two-sample test is derived. In Section 3 the two-sample test procedure (1.4) is represented in terms of a sum statistic, and the probability distribution function (under H_0) of this statistic is derived using an urn model argument. A large deviations result is obtained and Bahadur efficiency is discussed in Section 4. Numerical evaluations and recommendations for the practitioner are given in the final section.

2. TYPE I ERROR PROBABILITY

Under $H_o : \Delta = 0$, the $\{X_i\}_{i=1}^m$ and $\{Y_i\}_{i=1}^n$ are independent random samples from the same population $F_\theta(x) = F(x-\theta)$, where $F(x)$ is a continuous cumulative distribution function with unique median 0 . Without loss of generality, we take $\theta = 0$. The exact size of the two-sample two-sided test (1.4) is obtained at once from the following theorem.

Theorem 2.1. Let $X_{(a)}$ denote the a^{th} ordered observation from $\{X_i\}_{i=1}^m$ and let $Y_{(b)}$ denote the b^{th} ordered observation from $\{Y_i\}_{i=1}^n$. Then

$$P(X_{(a)} < Y_{(b)}) = \sum_{t=a}^{m} \binom{m}{t} \cdot \frac{\Gamma(n+1)}{\Gamma(b)\Gamma(n-b+1)} \cdot \frac{\Gamma(b+t)\Gamma(m+n+1-b-t)}{\Gamma(n+m+1)} \qquad (2.1)$$

Proof. We note that

$$P(X_{(a)} < Y_{(b)} = P(F(X_{(a)}) < F(Y_{(b)})) = P(U_{1(a)} < U_{2(b)})$$

where $U_{1(a)} \sim \text{Beta}\ (a,m-a+1)$, $U_{2(b)} \sim \text{Beta}\ (b,n-b+1)$, and they are independent. Thus,

$P(U_{1(a)} < U_{2(b)})$

$$= \int_0^1 \int_0^y \frac{\Gamma(m+1)}{\Gamma(a)\Gamma(m-a+1)} x^{a-1}(1-x)^{m-a} \frac{\Gamma(n+1)}{\Gamma(b)\Gamma(n-b+1)} y^{b-1}(1-y)^{n-b} dxdy$$

$$= \int_0^1 \left(\sum_{t=a}^{m} \binom{m}{t} y^t (1-y)^{m-t} \right) \cdot \frac{\Gamma(n+1)}{\Gamma(b)\Gamma(n-b+1)} y^{b-1}(1-y)^{n-b} dy$$

$$= \sum_{t=a}^{m} \binom{m}{t} \frac{\Gamma(n+1)}{\Gamma(b)\Gamma(n-b+1)} \cdot \frac{\Gamma(b+t)\Gamma(n+m+1-b-t)}{\Gamma(m+n+1)}$$

$$\cdot \int_0^1 \frac{\Gamma(m+n+1)}{\Gamma(b+t)\Gamma(n+m+1-b-t)} y^{b+t-1}(1-y)^{n+m-b-t} dy \ .$$

The integrand is a beta probability density function with parameters $\alpha = b + t$ and $\beta = n + m - b - t + 1$. Hence, the integral is 1 . ■

<u>Corollary 2.1.</u> The exact size of the two-sample two-sided test (1.4), α , is given by

$$\alpha = P(U_x < L_y) + P(U_y < L_x)$$

$$= \sum_{t=m-d_x+1}^{m} \frac{\binom{m}{t}\binom{n}{d_y}}{\binom{m+n}{d_y+t}} \cdot \frac{d_y}{(d_y+t)} + \sum_{t=n-d_y+1}^{n} \frac{\binom{n}{t}\binom{m}{d_x}}{\binom{m+n}{d_x+t}} \cdot \frac{d_x}{(d_x+t)} \qquad (2.2)$$

Proof. For $P(U_x < L_y)$, let $a = m - d_x + 1$, $b = d_y$, apply (2.1), and some algebraic manipulation yields the first term in (2.2).

For $P(U_y < L_x)$, first interchange m with n in (2.1), then let $a = n - d_y + 1$, $b = d_x$ and (2.1) will, after some algebra, yield the second term of (2.2). ■

We emphasize that the size of the test depends on the depths d_x and d_y . A change in either one of the values alters the size. Once d_x and d_y have been selected, the corollary enables us to compute the exact probability of committing a Type I error. In the next section we show that $P(U_x < L_y) = P(U_y < L_x)$. Hence, each equals $\alpha/2$. We need only compute the first or second term of (2.2) and multiply by 2 to obtain α . In the one-sided situation, we reject $H_0 : \Delta = 0$ in favor of $H_A : \Delta > 0$ $(\Delta < 0)$ if $U_x < L_y$ $(U_y < L_x)$. Thus, the exact size of the one-sided test is given by either term. For a table which provides values for (d_x, d_y) for various low sample sizes (m,n) that yield useful one-sample confience coefficients (γ_x, γ_y) corresponding to a desirable confidence coefficient $\gamma = 1 - \alpha$ for the two-sample interval, see Tableman (1984, Table 1).

For sample sizes (m,n) not found in the table, one can use the normal approximation (1.6). To approximate the size, compute

$$\nu_x = (d_x - m/2 - .5)/(m^{1/2}/2) \ , \ \nu_y = (d_y - n/2 - .5)/n^{1/2}/2)$$

and evaluate $\Phi(\cdot)$ at

$$\nu = (n/(n+m))^{1/2}\nu_x + (m/(n+m))^{1/2}\nu_y \ .$$

Multiply by 2 for the two-sided test. For a second-order approximation of the size, which improves the normal approximation, see Tableman (1984, p. 28).

3. A SUM STATISTIC

In this section we present an equivalent formulation of the test procedure (1.4) in terms of a sum statistic, and obtain this statistic's null distribution. As will be seen in the next section, this form enables us to consider the problem of large deviations for use in stochastic comparisons (in the Bahadur sense), and facilitates the task of obtaining Bahadur slopes.

We first consider the one-sided situation. To test $H_o : \Delta = 0$ versus $H_A : \Delta > 0$, we reject H_o if $U_x < L_y$. Now,

$$X_{(m-d_x+1)} < Y_{(d_y)} \quad \text{if and only if}$$

$$\sum_{i=1}^{m} I\{X_i < Y_{(d_y)}\} \geq m - d_x + 1$$

where $I\{A\}$ is the indicator function of the event A. Let

$$S_x(d_y) = \sum_{i=1}^{m} I\{X_i < Y_{(d_y)}\} . \tag{3.1}$$

Then, we reject H_o if $S_x(d_y) \geq m - d_x + 1$. The next theorem gives the null distribution of $S_x(d_y)$.

<u>Theorem 3.1.</u> Under $H_o : \Delta = 0$, the probability distribution function of $S_x(d_y)$ is given by

$$P(S_x(d_y) = t) = \frac{\binom{m}{t}\binom{n}{d_y}}{\binom{m+n}{d_y+t}} \cdot \frac{d_y}{(d_y+t)}, \quad t = 0,1,\ldots,m . \tag{3.2}$$

<u>Proof.</u> Under H_o we may represent the probability space by a simple urn model with m x's and n y's. We draw the x's and y's out of the urn one at a time without replacement. Then the $P(S_x(d_y) = t)$ is the probability that after $d_y - 1 + t$ draws we have t x's and $(d_y - 1)$ y's and on the next draw we obtain a y. Hence

$$P(S_x(d_y) = t) = \frac{\binom{m}{t}\binom{n}{d_y-1}}{\binom{m+n}{d_y+t-1}} \cdot \frac{n-d_y+1}{m+n-d_y-t+1} .$$

After some algebraic manipulation, expression (3.2) is obtained. ∎

This probability distribution function previously appeared in (2.2).

We note that this distribution is not symmetric. If

$Y_{(d_y)}$ were replaced by the median of the Y sample, the statistic defined in (3.1) would be Mathisen's (1943) test statistic $\sum_{i=1}^{m} I\{X_i < \operatorname{med}_{j=1,\ldots,n} y_j\}$. When $n = 2k - 1$, the distribution of $S_x(d_y)$ is symmetric if and only if $d_y = k$. When $n = 2k$, there is no integer d_y for which $S_x(d_y)$ has a symmetric distribution.

Our final observation is stated as a corollary to Theorem 3.1.

<u>Corollary 3.1.</u>

$$P(U_x < L_y) = P(U_y < L_x) . \tag{3.3}$$

<u>Proof.</u> Now, $U_x < L_y$ iff $S_x(d_y) \geq m - d_x + 1$. Further, $U_y < L_x$ iff $\sum_{i=1}^{m} I\{X_i > Y_{(n-d_y+1)}\} \geq m - d_x + 1$. An argument similar to that given in the proof of (3.2) together with $P\{X_i = Y_{(n-d_y+1)}\} = 0$ gives

$$P\{\sum_{i=1}^{m} I\{X_i > Y_{(n-d_y+1)}\} = t\} = \frac{\binom{m}{t}\binom{n}{d_y}}{\binom{m+n}{d_y+t}} \cdot \frac{d_y}{(d_y+t)} .$$

The result follows. ■

4. A LARGE DEVIATIONS RESULT AND BAHADUR EFFICIENCY

Briefly, Bahadur (1967) efficiency is a comparison of the rates (called Bahadur slopes) at which the Type I error probabilities of two test procedures tend to zero while the Type II error probabilities remain fixed at (or tend to) a $\beta(\Delta)$, $0 < \beta(\Delta) < 1$, for fixed Δ. An alternative formulation is in terms of (asymptotically) fixed-width confidence intervals. That is, we compare the rates at which the Type I error probabilities tend to zero while the lengths remain fixed at (or tend to) a positive constant $L = 2a$ not depending on Δ. Such a formulation was first considered by Serfling and Wackerly (1976) for use in the construction and analysis of sequential confidence interval procedures.

Remark 1. The equivalence between the two formulations is seen in the following example: In the one-sample setting, consider the interval centered at the sample mean for the location parameter θ, i.e. $I_m = [\bar{X}_m \pm a]$, $a > 0$. For the sequence of intervals $\{I_m\}$, define the associated sequence of tests of $H_o : \theta = 0$ versus $H_A : \theta = a$ (or $-a$) by the rejection rule, reject H_o if $0 \notin I_m$. It is easily seen that the Type I error probability, $2\alpha_m = P\{0 \notin I_m\}$, tends to zero. In addition, note that the probability of a Type II error (covering 0 when a

or $-a$ attains) tends to $1/2$, which suffices to make the stochastic comparison. In general, let β_m represent the sequence of Type II error probabilities. As long as β_m tends to some quantity β, $0 < \beta < 1$, then if $-\log \alpha_m/m$ converges, it converges to $1/2$ of the Bahadur slope. (See Serfling, 1980, § 10.4.2.)

Since the length of the two-sample interval (1.5) is simply the sum of the lengths of the two one-sample intervals, the strategy we take is to first build a fixed-width two-sample interval from two fixed-width one-sample intervals, then use the sum statistic formulation of the test (3.1) to obtain the rate at which the Type I error (or equivalently the noncoverage) probability tends to zero. For ease of discussion we assume F is symmetric about zero. We also assume that F satisfies assumption (1.1) with $\xi_p = b$ or a, $b > 0$ and $a > 0$.

Consider the confidence interval (1.2) for θ_x. Define the depths as follows:

$$d(m) = m(1/2 - \varphi_x) \, , \; u(m) = m - d(m) + 1 \qquad (4.1)$$

where $\varphi_x = F_{\theta_x}(\theta_x+b) - 1/2$, $b > 0$ (see Figure 1). By symmetry then,

$$1/2 + \varphi_x = F_{\theta_x}(\theta_x + b) = F(b) \quad \text{and}$$
$$\tag{4.2}$$
$$1/2 - \varphi_x = F_{\theta_x}(\theta_x - b) = F(-b) \ .$$

Therefore, by construction, $\theta_x - b$ and $\theta_x + b$ correspond to lower and upper $(1/2 - \varphi_x)^{th}$ quantiles, respectively, of the distribution $F_{\theta_x}(x)$. Similarly define the depths for the endpoints of the confidence interval for θ_y , with

$$d(n) = n(1/2 - \varphi_y) \ , \ u(n) = n - d(n) + 1 \tag{4.3}$$

where $\varphi_y = F_{\theta_y}(\theta_y + a) - 1/2 = F(a) - 1/2$, $a > 0$.

With the depths so defined we can appeal to Bahadur's almost sure representation of the central order statistic. (See Serfling, 1980, p. 93.) We state this representation for the endpoints $X_{(d(m))}$, $X_{(u(m))}$.

With probability 1 ,

$$X_{(d(m))} = \theta_x - b + [(1/2 - \varphi_x) - F_m(\theta_x - b)]/f(b) + o(m^{-1/2})$$
$$\tag{4.4}$$
$$X_{(u(m))} = \theta_x + b + [(1/2 + \varphi_x) - F_m(\theta_x + b)]/f(b) + o(m^{-1/2})$$

where F_m is the empirical distribution function. Let

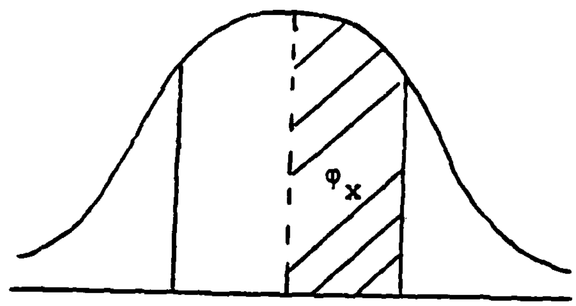

Figure 1. Description of the $(1/2-\varphi_x)^{th}$ quantiles: $\varphi_x = F_{\theta_x}(\theta_x+b) - 1/2$, $b > 0$.

Λ_m, Λ_n, and $\Lambda_{m,n}$ denote the lengths of the intervals (1.2), $[L_y, U_y]$, (1.5) respectively with depths defined as in (4.1, 4.3). Then it immediately follows that as $m,n \to \infty$, with probability 1

$$\Lambda_m \to 2b, \quad \Lambda_n \to 2a, \quad \text{and} \quad \Lambda_{m,n} \to 2a + 2b. \tag{4.5}$$

The Type I error probability of the two-sample test (1.4) is given by

$$\begin{aligned} 2\alpha_{m,n} &= P_o\{X_{(u(m))} < Y_{(d(n))}\} + P_o\{Y_{(u(n))} < X_{(d(m))}\} \\ &= 2P_o\{X_{(u(m))} < Y_{(d(n))}\} \end{aligned}$$

where the last equality follows from the symmetry established in Corollary 3.1. It follows from the sum statistic formulation of the test (3.1) that

$$\alpha_{m,n} = P_o\{S_x(d(n)) \geq m - d(m) + 1\} \tag{4.6}$$

where the null distribution of $S_x(d(n))$ is given in Theorem 3.1. Suppose that $m,n \to \infty$ so that $m/(m+n) \to \lambda$, $0 < \lambda < 1$. Then (by a straightforward argument) under $\Delta = 0$,

$$S_x(d(n))/(m+n) \to \lambda F(-a) \quad \text{in probability}$$

and from (4.1)

$$(m-d(m)+1)/(m+n) \to \lambda F(b) > \lambda F(-a)$$

since both a and b are positive. Therefore

$$\alpha_{m,n} \to 0 \quad \text{as} \quad m,n \to \infty .$$

The following lemma establishes the probability of large deviations for the sum statistic $S_x(d(n))$. The proof is given in the appendix.

<u>Lemma 4.1.</u> Assume $m/N \to \lambda$, $0 < \lambda < 1$, $N = m+n$, as $n,m \to \infty$. Without loss of generality, take $m \geq n$. Then for τ such that $\lambda/2 < \tau < \lambda$, with $\rho = 1 - \lambda$,

$$\lim_{n,m\to\infty} N^{-1}\log P_o\{S_x(d(n)) \geq N\tau\}$$

$$= \tau\log((1-\rho)/\tau)+(1-\rho-\tau)\log((1-\rho)/(1-\rho-\tau))$$

$$+ \rho\log 2 - (\rho(1-2\varphi_y)/2)\log(1-2\varphi_y) - (\rho(1+2\varphi_y)/2)\log(1+2\varphi_y)$$

$$- \log 2 + ((2\tau+\rho(1-2\varphi_y))/2)\log(\rho(1-2\varphi_y)+2\tau)$$

$$+ ((2-2\tau-\rho(1-2\varphi_y))/2)\log(2-2\tau-\rho(1-2\varphi_y)) ,$$

where φ_y is given in (4.3).

The theorem that follows establishes that the Type I error probability of the two-sample test based on the comparison of two fixed-width one-sample sign-intervals converges to zero at an exponential rate. We refer to this rate as the index of exponential convergence and denote it by $e(a,b)$ as it depends on the choices of a and b as well as the distribution F.

<u>Theorem 4.1.</u> Under the same assumptions as those given in Lemma 4.1, for the sequence of intervals (1.5) with depths defined by (4.1) and (4.3), the index of exponential convergence of $\alpha_{m,n}$ (4.6) is

$$
\begin{aligned}
-e(a,b) &= \lim_{n,m\to\infty} N^{-1}\log\alpha_{m,n} \\
&= -(1-\rho)F(b)\log F(b) - (1-\rho)(1-F(b))\log(1-F(b)) \\
&\quad + \rho\log 2 - \log 2 \\
&\quad -\rho(1-F(a))\log(2(1-F(a))) - \rho F(a)\log 2F(a) \qquad (4.7) \\
&\quad +((1-\rho)F(b) + \rho(1-F(a)))\log(2(1-\rho)F(b)+2\rho(1-F(a))) \\
&\quad +(1-(1-\rho)F(b) - \rho(1-F(a)))\log(2-2(1-\rho)F(b)-2\rho(1-F(a)))
\end{aligned}
$$

<u>Proof.</u> From (4.1) and (4.2), we have

$$m - d(m) + 1 = N(\lambda F(b) + o(1)) , \quad b > 0 .$$

Let τ_N denote $\lambda F(b) + o(1)$, and τ denote $\lambda F(b)$. Then

$$\tau_N \to \tau \quad \text{as} \quad n,m \to \infty , \quad \text{and}$$

$$\lambda/2 < \lambda F(b) < \lambda .$$

From (4.3),

$$(1-\lambda)(1-2\varphi_y)/2 = (1-\lambda)F(-a) = \rho F(-a) .$$

Hence, Lemma 4.1 applies with τ replaced by $\lambda F(b)$. After some algebraic manipulation, the expression (4.7) is obtained. ■

<u>Remark 2.</u> Four interesting cases are the following:

(a) If $a = b$, the index is symmetric in ρ and $1 - \rho$; (i.e. in $1 - \lambda$ and λ) .

(b) If $a = b$ and $m = n$, the index reduces to the index of Mood's test. (See Woodworth, 1970.)

(c) If a and b are related via the relationship

$$\lambda F(b) + (1-\lambda)F(-a) = 1/2 \quad , \qquad (4.8)$$

then the index is again the index of Mood's test.

(d) Suppose that the asymptotic length of one interval vanishes, e.g. $a = 0$. Then the index reduces to that of Mathisen's statistic (Killeen, et al., 1972).

(e) If $m = n$ then for $a + b = c$, the index is maximized by $a = b = c/2$ which yields Mood's statistic. On the other hand, the index is a minimum for $a + b = c$ just when a or b is 0 which yields Mathisen's statistic. Hence, for equal sample sizes Mood's test is best and Mathisen's test is worst. However, for more extreme sample size ratios, Mathisen's test has a larger index than Mood's test; (see Killeen, et al., 1972).

These remarks are crucial in that they show the intricate relationship of the special Mood and Mathisen-intervals to that of the general two-sample interval constructed from two arbitrarily chosen (asymptotically) fixed-width sign-intervals.

5. NUMERICAL COMPARISONS AND DISCUSSION

Thus, various median tests arise as special cases as a result of formulating the problem in terms of (asymptotically) fixed-width intervals. In this context we are able to distinguish between the two-sample test based on the Mood-interval and any other solution to the condition (1.7).

In order to make efficiency comparisons we specify a constant $c > 0$ and then consider values a and b such that $a + b = c$ with specified ratio a/b. For the Mood-interval, however, we are not free to do this. The relationship (4.8) in terms of c is $\lambda F(b) + (1-\lambda)F(b-c) = 1/2$. Once c is specified, b and hence a are determined by this additional constraint. The (Bahadur) asymptotic efficiency as $m,n \to \infty$ (with $m/(m+n) \to \lambda$) of Procedure $\mathcal{A}$ relative to Procedure $\mathcal{B}$ is then

$$\mathrm{eff}(\mathcal{A},\mathcal{B}) = \mathrm{index}(\mathcal{A})/\mathrm{index}(\mathcal{B}) .$$

Table 1 provides numerical evaluation of the indices of exponential convergence. We select values of $1/2$, $1/4$, $1/8$ for $\rho = 1 - \lambda$; and values of 1, $2/3$, and $3/2$ for the ratio a/b. Without loss of generality, we take

(a, ρ) to correspond to the interval formed on the Y-sample. Evaluation of the indices is done at the standard normal distribution. For tables with indices evaluated at the logistic and Laplace distributions see Tableman (1984). These tables reveal similar information and thus are omitted. Figure 2 supplies a graphical display of the efficiencies of the equal asymptotic lengths $(a = b)$ solution relative to the Mood-interval.

Based on the information displayed in the table and figure, and with economic considerations in mind, we recommend taking $a = b$ for a specified c. For if observations from each population are equal in cost, selecting equal sample sizes yields the more efficient procedure (as always). (From Remark 2 (b), this solution is asymptotically equal to the Mood procedure.) On the other hand, if one population is more expensive to sample from than the other, then taking two sign-intervals with equal asymptotic lengths will provide the more efficient procedure for more extreme values of ρ; and, as was noted in Remark 2 (a), the index is symmetric in ρ and $(1-\rho)$. Therefore, an experimenter can adjust the ratio of sample sizes to meet cost constraints (for example), pick $a = b$, and obtain a more (Bahadur) efficient procedure than if he had chosen the Mood-interval procedure.

Table 1. Index of exponential convergence $\times 10^3$:

Standard normal c.d.f.

		c				
ρ	a/b	.01	.1	1	2	4
	Mood 1/1	.008	.795	75.2	256	585
1/2	2/3	.008	.795	74.9	252	562
	3/2	.008	.795	74.9	252	562
	Mood	.006	.596	53.9	155	215
	(b=)*	.0025	.025	.234	.383	.431
1/4	1/1	.006	.597	56.8	196	466
	2/3	.006	.597	57.1	199	465
	3/2	.006	.596	56.0	188	430
	Mood	.0035	.348	29.9	76.1	95.5
	(b=)	.00125	.0125	.1122	.168	.18
1/8	1/1	.0035	.348	33.4	118	300
	2/3	.0035	.348	33.8	122	309
	3/2	.0035	.348	32.8	111	267

* b determined by $\lambda F(b) + (1-\lambda)F(b-c) = 1/2$.

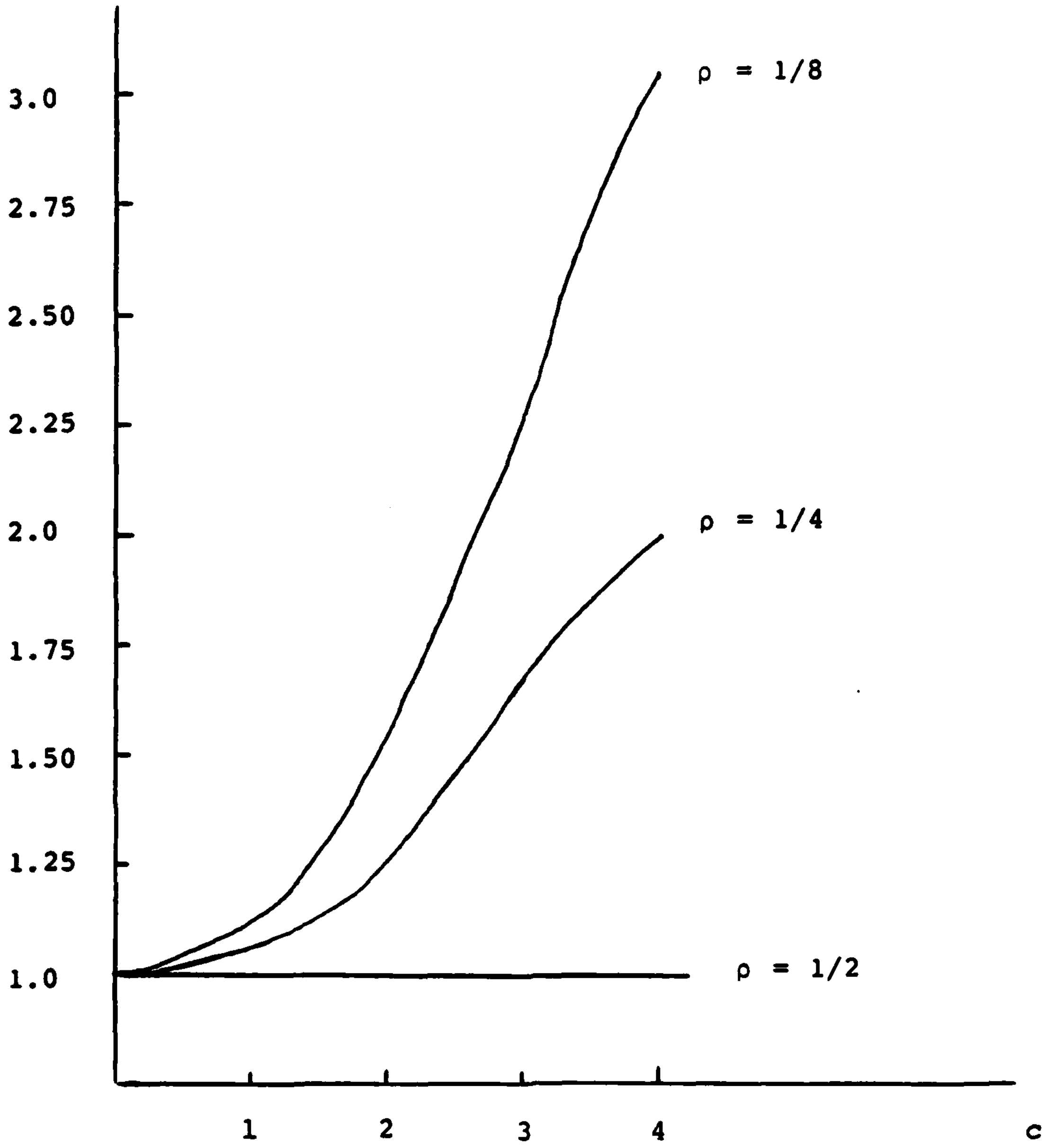

Figure 2. Bahadur efficiencies of equal asymptotic lengths (a=b) solution with respect to Mood-interval evaluated at the standard normal.

APPENDIX

Proof of Lemma 4.1. We show that conditions of Theorem 2.2 of Killeen, et al. (1972) are satisfied. Let $[x]$ denote the greatest integer $\leq x$. From Theorem 3.1,

$$\lim_{m,n\to\infty} N^{-1}\log P_0\{S_x(d(n)) = [N\tau]\}$$

$$= \lim N^{-1}\log\binom{m}{[N\tau]} + \lim N^{-1}\log\binom{n}{d(n)} - \lim N^{-1}\log\binom{N}{d(n)+[N\tau]}$$

$$+ \lim N^{-1}\log(d(n)/(d(n)+[N\tau])) .$$

(1) With $d(n)$ defined by (4.3),

$$d(n)/(d(n)+[N\tau]) \to ((1-\lambda)(1-2\varphi_y)/2)/((1-\lambda)(1-2\varphi_y)/2+\tau) .$$

Therefore, $\lim N^{-1}\log(d(n)/(d(n)+[N\tau])) = 0$.

(2) In the next three steps, we use the following:

If $\lim_{n\to\infty} a/n = \alpha$, $\lim_{n\to\infty} b/n = \beta$, $0 < \beta < \alpha < \infty$ where a,b are integers, then it follows from Sterling's formula that

$$\lim_{n\to\infty} n^{-1}\log\binom{a}{b} = \beta\log(\alpha/\beta) + (\alpha-\beta)\log(\alpha/(\alpha-\beta)) .$$

(3) $m/N \to \lambda$, $[N\tau]/N\to\tau$; and by assumption, $0 < \tau < \lambda$. Therefore, by (2)

$$\lim N^{-1}\log\binom{m}{[N\tau]} = \tau\log(\lambda/\tau) + (\lambda-\tau)\log(\lambda/(\lambda-\tau)) .$$

(4) $n/N \to (1-\lambda)$; by (4.3),

$$d(n)/N \to (1-\lambda)(1-2\varphi_y)/2 < (1-\lambda) .$$

Therefore, by (2)

$$\lim N^{-1}\log\binom{n}{d(n)} = \rho\log 2 - (\rho(1-2\varphi_y)/2)\log(1-2\varphi_y)$$

$$- (\rho(1+2\varphi_y)/2)\log(1+2\varphi_y)$$

where $\rho = 1 - \lambda$.

(5) $N/N = 1$; $(d(n) + [N\tau])/N \to (1-\lambda)(1-2\varphi_y)/2 + \tau < 1$.
Therefore, by (2) and after some algebra

$$-\lim N^{-1}\log\binom{N}{d(n)+[N\tau]}$$

$$= -\log 2 + ((2\tau+\rho(1-2\varphi_y))/2)\log(\rho(1-2\varphi_y)+2\tau)$$

$$+ ((2-2\tau-\rho(1-2\varphi_y))/2)\log(2-2\tau-\rho(1-2\varphi_y)) .$$

Summing up (1), (3), (4), and (5), we obtain

$$\lim_{n,m\to\infty} N^{-1}\log P_o\{S_x(d(n)) = [N\tau]\}$$

= the expression stated in Lemma.

This along with the fact that

$\lim N^{-1}\log P_o\{S_x(d(n)) \geq \exp N^{1/2}\} = -\infty$ implies Condition 2.2 (of Theorem 2.2) is satisfied. Now,

$$P_o\{S_x(d(n)) = [N\tau] + 1\}/P_o\{S_x(d(n)) = [N\tau]\}$$

$$= ((m-[N\tau])/([N\tau]+1))((d(n)+[N\tau])/(N-d(n)-[N\tau]))$$

$$\to ((\lambda-\tau)/\tau)(((1-\lambda)\ (1-2\varphi_y)/2+\tau)/(1-(1-2\varphi_y)(1-\lambda)/2-\tau)) \text{ as } m,n\to\infty$$

which is positive and finite.

Therefore,

$$N^{-1}\log(P_o\{S_x(d(n)) = [N\tau]+1\}/P_o\{S_x(d(n)) = [N\tau]\}) \to 0 \text{ as } m,n\to\infty .$$

Condition 2.1 is satisfied.

To check the non-increasing property: Let $x > \phi_N = N\tau$. Since $\lambda/2 < \tau < \lambda$, we only need to check for x such that

$$N\tau < x < N\lambda .$$

Now,

$$P\{S_x(d(n)) = [x] + 1\}/P\{S_x(d(n)) = [x]\}$$

$$= ((m-[x])/[x+1])((d(n)+[x])/(N-d(n)-[x])) .$$

Need to show that for sufficiently large N, this ratio is less than 1. This follows immediately from the fact that

$$\lambda(1-2\varphi_y)/2 < \lambda/2$$

and that $\lambda/2 < \tau < \lambda$. Therefore, by Theorem 2.2 of Killeen, et. al.,

$$\lim_{n,m\to\infty} N^{-1}\log P_o\{S_x(d(n)) \geq N\tau\} = \lim_{n,m\to\infty} N^{-1}\log P_o\{S_x(d(n)) = [N\tau]\} .$$

■

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper discusses the two-sample test of location based on the comparison of two distribution free one-sample confidence intervals derived from sign statistics. This test procedure, first introduced by Hettmansperger (1986), rejects the null hypothesis of equal population medians when the two intervals are disjoint. He presents three different ways to select the two one-sample intervals and one choice leads to Mood's test. All solutions have the same Pitman efficiency. This paper shows that the choices can be distinguished on the basis of Bahadur's efficiency. We formulate the

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20. ABSTRACT Continued:

problem in terms of (asymptotically) fixed-width confidence intervals. In this context various median tests (including Mood's test) arise as special cases and they yield different performance. The solution that specifies equal asymptotic lengths for the one-sample intervals (which is different from Mood's test) is recommended.

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