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<th>A NEGATIVE RESULT ABOUT SOME CONCEPTS OF NEGATIVE DEPENDENCE(U) FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS K JOAG-DEV ET AL. JUL 85 AFOSR-TR-86-0454 UNCLASSIFIED F49620-85-C-0007 F/G 12/1 NL</th>
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Theorem. Let \( X_1, X_2 \) be independent random variables and suppose there exist real numbers \( c, t_1, t_2 \) such that \( t_2 > t_1 \) and

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P[X_1 \geq c | X_1 + X_2 = t_1] > P[X_1 \geq c | X_1 + X_2 = t_2],
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where conditioning events have positive probability. Then there exists a random variable \( X_3 \) independent of \( X_1, X_2 \) such that the conditional distribution of \( (X_1, X_2, X_3) \) given the sum \( \sum_{i=1}^{3} X_i \) is not pairwise NQD.

Other negative results concerning negative dependence are presented.
A NEGATIVE RESULT ABOUT SOME CONCEPTS OF NEGATIVE DEPENDENCE

by

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FSU Statistics Report M703
AFOSR Technical Report No. 85-179

July, 1985

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Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF under Grant AFOSR 84-02081 and AFOSR 85-C-00072.

Key words: Qualitative dependence, negative dependence, \(\text{PF}_2\) functions, pairwise dependence, conditional negative dependence.

AMS Classification Nos.: Primary 62H05; Secondary 62H10.
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Abstract

The key result is:

Theorem. Let $X_1, X_2$ be independent random variables and suppose there exist real numbers $c, t_1, t_2$ such that $t_2 > t_1$ and

$$P[X_1 \geq c | X_1 + X_2 = t_1] > P[X_1 \geq c | X_1 + X_2 = t_2],$$

where conditioning events have positive probability. Then there exists a random variable $X_3$ independent of $X_1, X_2$ such that the conditional distribution of $(X_1, X_2, X_3)$ given the sum $\sum_{i=1}^3 X_i$ is not pairwise NQD.

Other negative results concerning negative dependence are presented.
1. **Introduction.**

Suppose Peter and Paul inherit a fortune. If $X_1$, $X_2$ are their shares, then it is clear that $(X_1, X_2)$ should be 'negatively dependent' according to every reasonable notion of negative dependence. We want to consider the question of a multivariate analog of this simple bivariate fact. For example, suppose $X_1$, $X_2$, $X_3$ are independent random variables representing the shares of Peter, Paul and Mary. Suppose that the sum $X_1 + X_2 + X_3$ is $t$. To what extent does the conditional distribution of $(X_1, X_2, X_3)$ exhibit negative dependence? If $X_i$ possess log concave density (also known as PF$_2$ condition) then utilizing a monotonicity result of Efron (1965), Joag-Dev and Proschan (1983) show that the above distribution does satisfy a strong negative dependence relation called 'negative association' (NA). Other concepts of negative dependence such as 'negative orthant dependence', 'reverse rule', etc. are established for a long list of multivariate distributions, such as multinomial, Dirichlet, hypergeometric, etc., by Block, Savits and Shaked (1982), Ebrahimi and Ghosh (1981), and others. Most of these examples can be perceived as the conditional distributions, obtained by fixing the sum of the independent PF$_2$ random variables. Note that assuming only finite variance, if the random variables are independent and identically distributed, then clearly the conditional covariance of every pair is negative when the total sum is fixed. A natural question would be: does the conditional joint distribution exhibit a condition such as pairwise negative quadrant dependence? Note that although this condition is stronger than negative covariance, it is weaker than 'negative upper orthant dependence' (NUOD) or 'negative lower orthant dependence' (NLOD) which in turn are weaker than NA (see the next section for the definitions). We show that
without the monotonicity resulting from $PF_2$, such a negative dependence condition does not hold.

2. Results.

Next we define some of the standard notions of negative dependence.

Let $Y = (Y_1, ..., Y_k)$ be a $k$ vector with real valued component random variables. The vector $Y$ is said to possess 'negative association' (NA) if for every partition of \(\{1, 2, ..., n\}\) into $A, \bar{A}$ and every pair of coordinatewise nondecreasing functions $f, g$,

\[
\text{COV}[f(Y_i, i \in A), g(Y_j, j \in \bar{A})] \leq 0.
\]

Condition (2.1) is stronger than 'negative upper orthant dependence' (NUOD) which requires

\[
P[Y_i \geq c_i, i = 1, ..., k] \leq \prod_{i=1}^{k} P[Y_i \geq c_i],
\]

for every set of constants $c_1, ..., c_k$. By reversing inequalities in the square brackets on both sides of (2.2) one obtains NLOD. It is easy to check that NA implies NLOD; however, between NUOD and NLOD neither implies the other for $k \geq 3$. For $k = 2$, the bivariate case, these two are equivalent and the condition is referred to as 'negative quadrant dependence' (NQD). It also follows that pairwise NQD condition is weaker than all the above.

Let $X_i, i = 1, ..., n$, be independent random variables from log concave densities. Efron (1965) shows that the conditional expectation,

\[
E[g(X_1, ..., X_n) | \sum X_i = s]
\]

is nondecreasing in $s$, where $g$ is an arbitrary coordinatewise nondecreasing function. This is the key tool in the proof given in Joag-Dev and Proschan (1983), to show NA for the conditional distribution of $\{X_i\}$ given $\sum X_i$. 
Theorem 2.1. Let $X_1, X_2$ be independent random variables and suppose there exist real numbers $c, t_1, t_2$ such that $t_2 > t_1$ and

$$P[X_1 \geq c | X_1 + X_2 = t_1] > P[X_1 \geq c | X_1 + X_2 = t_2],$$

(2.3)

where conditioning events have positive probability. Then there exists a random variable $X_3$, independent of $(X_1, X_2)$, such that the conditional distribution of $(X_1, X_2, X_3)$ given the sum $\sum_{i=1}^{3} X_i$, is not pairwise NQD.

Proof. Define $X_3$ to be a binary random variable such that

$$P[X_3 = 0] = P[X_3 = t_2 - t_1] = \frac{1}{2}.$$

Let $X_3$ be independent of $(X_1, X_2)$. Let the event $\sum_{i=1}^{3} X_i = t_2$ be denoted by $A$. The event $A$ may be written as a disjoint union of $A_1$ and $A_2$ denoting the events $[X_1 + X_2 = t_1, X_3 = t_2 - t_1]$ and $[X_1 + X_2 = t_2, X_3 = 0]$ respectively. Using this notation, we have

$$P[X_1 \geq c | A] = \frac{P[A_1]}{P[A]} P[X_1 \geq c | A_1] + \frac{P[A_2]}{P[A]} P[X_1 \geq c | A_2].$$

(2.4)

Using the independence of $X_3$ and $(X_1, X_2)$ in (2.4), we get

$$P[X_1 \geq c | A] = \alpha P[X_1 \geq c | X_1 + X_2 = t_1] + (1 - \alpha)P[X_1 \geq c | X_1 + X_2 = t_2],$$

where $\alpha = P[A_1]/P[A], 0 < \alpha < 1$. Due to assumption (2.3) and the independence of $X_3$, it follows that

$$P[X_1 \geq c | X_3 = t_2 - t_1, X_1 + X_2 = t_1] > P[X_1 \geq c | A].$$

(2.5)
However after some manipulation, we see that (2.5) is equivalent to

\[ P[X_1 < c, X_3 > t_2 - t_1 | A] > P[X_1 < c | A] P[X_3 > t_2 - t_1 | A]. \]

Inequality (2.6) shows that conditionally, \((X_1, X_3)\) is not NQD.

One may ask whether the assumption of a common distribution function will create NLOD. The following example shows that it does not. Let \(X_1, X_2, X_3\) be independent random variables having a common discrete distribution on 0, 2, 3, with corresponding probabilities \(P_0, P_2, P_3\) respectively. Let \(T\) denote the sum, \(\sum_{i=1}^{3} X_i\). Now

\[ P[X_1 \leq 2, i = 1, 2, 3 | T = 6] = \frac{p_2^3}{3p_0p_3^2 + p_2^2}, \]

while

\[ P[X_1 \leq 2 | T = 6] = \frac{p_2^3 + p_3p_0}{3p_0p_3^2 + p_2^2}. \]

Thus the NLOD condition would be violated if

\[ \left( \frac{p_2}{3p_0p_3^2 + p_2} \right)^3 > \left( \frac{p_2^3 + p_0p_3}{3p_0p_3^2 + p_2} \right)^3. \]

Put \(a = p_2^3, b = p_0p_3^2\). Then condition (2.7) is equivalent to

\[ \left( \frac{a}{a + 3b} \right)^3 > \left( \frac{a + b}{a + 3b} \right)^3 \iff a(a + 3b) > (a + b)^3 \iff 3a(a + 2b) > b^2. \]

The last inequality in (2.8) can easily be met when \(b\) is small or equivalently, \(p_2\) is large. For example if \(p_2 \geq \frac{1}{2}\) then it certainly holds.
3. Final Remark.

It seems that either a very strong negative dependence holds with the monotonicity condition while without it, even a somewhat weak condition does not hold. This brings out the crucial role played by the $\text{PF}_2$ property in conditional negative dependence.
References


