AN ALGORITHM FOR POSITIVE DEFINITE LEAST SQUARE
ESTIMATION OF PARAMETERS (U) STANFORD UNIV CA SYSTEMS
OPTIMIZATION LAB H HU MAY 86 SOL-86-12
UNCLASSIFIED
F/G 12/1
AN ALGORITHM FOR POSITIVE DEFINITE LEAST SQUARE ESTIMATION OF PARAMETERS

by

Hui Bu

TECHNICAL REPORT SOL-86-12

May 1986

DTIC FILE COPY

Department of Operations Research
Stanford University
Stanford, CA 94305

This document has been approved
for public release and sale; its
distribution is unlimited.
AN ALGORITHM FOR POSITIVE DEFINITE LEAST SQUARE ESTIMATION OF PARAMETERS

by

Hui Bu

TECHNICAL REPORT SOL-86-12

May 1986

Research and reproduction of this report were partially supported by the National Science Foundation Grants DMS-8420623 and ECS-8312142; U.S. Department of Energy Contract DE-AA03-76SF00326, PA# DE-AS03-76ER72018; and Office of Naval Research Contract N00014-85-K-0343.

Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the author(s) and do NOT necessarily reflect the views of the above sponsors.

Reproduction in whole or in part is permitted for any purposes of the United States Government. This document has been approved for public release and sale; its distribution is unlimited.
AN ALGORITHM FOR POSITIVE DEFINITE LEAST SQUARE ESTIMATION OF PARAMETERS

Hui Hu

Abstract

We present an algorithm for positive definite least square estimation of parameters. This estimation problem arises from the PILOT dynamic macro-economic model and is equivalent to an infinite convex quadratic program. It differs from ordinary least square estimations in that the fitting matrix is required to be positive definite. The algorithm solves the infinite convex quadratic program by generating and solving a sequence of ordinary convex quadratic programs. By specifying a constant, the algorithm will find an approximate optimal solution after finitely many iterations, or will tend to an optimal solution in the limit. The algorithm is generalized to solve a class of infinite convex programs.

Key Words. Least square estimation, quadratic programming, positive definite matrix.

1. Introduction and Preliminaries

We resolve the following problem. Given two sequences of vectors \( a^t \) and \( b^t \) in \( \mathbb{R}^n \) with \( t = 1, \ldots, L \), a small positive number \( \epsilon \) and a large number \( K \gg \epsilon \), we want to find a real symmetric matrix \( X = (x_{ij}) \) such that \( \sum_{t=1}^{L} (a^t - b^t)^2 \) is minimized among all the real square matrices \( X \) satisfying conditions (a) and (b):
(a) $X^T = X$ and $-K \leq x_{ij} \leq K$ for all $i,j = 1, \ldots, n$;

(b) the smallest eigenvalue of $X$ is no less than $\epsilon$.

This problem differs from ordinary least square estimations in that the fitting matrix $X$ is required to be positive definite (Lemma 1). It arises from the PILOT dynamic macro-economic model designed to assess the long term impact of foreign competition, innovation, modernization and energy need (Dantzig [1]). It is a nonlinear optimization problem with matrix variables and constraints.

Throughout this paper, $S^{n-1} = \{ x \in \mathbb{R}^n : x^T x = 1 \}$ denotes the unit sphere in $\mathbb{R}^n$ and $\| \cdot \|$ denotes the Euclidean norm. For a real symmetric matrix $B$, $\lambda[B]$ stands for the smallest eigenvalue of $B$ and $v[B]$ a corresponding eigenvector of unit length. Superscripts on vectors are used to denote different vectors, while subscripts are used to denote components of a vector.

Before solving this problem, we state some lemmas about positive definite matrices and real symmetric matrices. These lemmas are either obvious or well-known; however, they play an important role throughout our discussion.

**Lemma 1.** For any real symmetric matrix $B$, the following are equivalent:

1. $B$ is positive definite;
2. $\lambda[B]$, the smallest eigenvalue of $B$, is positive;
3. there exists $\delta > 0$ such that $u^T Bu \geq \delta$ for all $u \in S^{n-1}$, where $S^{n-1} = \{ x \in \mathbb{R}^n : x^T x = 1 \}$ is the unit sphere in $\mathbb{R}^n$. $\square$
Lemma 2. For any real symmetric matrix $B$, $\lambda[B] = \min \{u^T Bu : u \in S^{n-1}\}$ (see, e.g., Wilkinson [4], p.98-99).

Lemma 3. For any real symmetric matrix $B$, $\lambda[B]$ is a continuous function of the elements of $B$ (see, e.g., Isaacson and Keller [3], p.136).

To solve this estimation problem, we first transform it into an equivalent (vector form) infinite convex quadratic program.

Given $a^t, b^t \in \mathbb{R}^n$ for $t = 1, \ldots , L$, let $A = (a_1^T \ldots a_L^T)^T$ and $B^i = (b_1^i \ldots b_L^i)^T$ for all $i = 1, \ldots , n$. Let $M$ be an $n^2 \times n^2$ block-diagonal matrix with diagonal blocks $A^T A$ and $E$ be an $nL \times n^2$ block-diagonal matrix with diagonal blocks $A$. Let $X_i$ be the $i$-th row of matrix $X$ for all $i = 1, \ldots , n$ and $F(\cdot)$ be a bijection from $\mathbb{R}^{nxn}$ to $\mathbb{R}^{n^2}$, $Y = F(X) = (X_1^T \ldots X_n^T)$. Then, in terms of vectors, condition (a) becomes:

(a) $Y_{(i-1)n+j} = x_{ij} = x_{ji} = Y_{(j-1)n+i}$ for all $i,j = 1, \ldots , n$ and

$-K \leq Y_{k} \leq K$ for all $k = 1, \ldots , n^2$.

By Lemma 2, condition (b) is equivalent to: $u^T X u \geq \varepsilon$ for all $u \in S^{n-1}$.

Thus, condition (b) becomes:

(b) $u^T X u = \sum_{i=1}^{n} u_i^T u (X_i^*)^T = (u_1^T \ldots u_n^T) Y \geq \varepsilon$ for all $u \in S^{n-1}$.
The objective function becomes:

\[
\begin{align*}
L & = \sum_{t=1}^{L} \|Xa_t - b_t\|^2 = \sum_{t=1}^{L} \sum_{i=1}^{n} (X_{i,t}a_t - b_t)^2 \\
& = \sum_{i=1}^{n} \sum_{t=1}^{L} (X_{i,t}a_t - b_t)^2 = \sum_{i=1}^{n} \|A(X_{i,t}) - B^i\|^2 \\
& = \sum_{i=1}^{n} (X_{i,t}^TA(X_{i,t})^T - 2(B^i)^TA(X_{i,t})^T + (B^i)^TB^i) \\
& = Y^TMY - 2((B^1)^T \cdots (B^n)^T)EY + \sum_{i=1}^{n} (B^i)^TB^i.
\end{align*}
\]

Consequently, the equivalent vector form optimization problem is:

(IQP): minimize $Y^TMY - 2((B^1)^T \cdots (B^n)^T)EY + \sum_{i=1}^{n} (B^i)^TB^i$

subject to

\[
(u_1^T \cdots u_n^T)Y \geq \varepsilon \quad \text{for all } u \in S^{n-1} \\
Y_{(i-1)n+j} - Y_{(j-1)n+i} = 0 \quad \text{for all } i,j = 1, \ldots, n \\
-K \leq Y_{i} \leq K \quad \text{for all } i = 1, \ldots, n^2.
\]

Remark 1.

(1) $M$ is positive semidefinite because each of its diagonal blocks $A^TA$ is positive semidefinite. This implies that the objective function of (IQP) is convex. The feasible region of (IQP) is compact and convex, and is defined by an infinite number of linear constraints. Moreover, it is nonempty since $\bar{Y} = F(tI)$ is a feasible solution, where $I$ is the identity matrix. Therefore, (IQP) is a feasible infinite convex quadratic program and optimal solutions for (IQP) exist.
(2) Since $Y = \mathbf{F}(X) = (X_1 \cdots X_n)^T$ is a bijection, $\mathbf{F}^{-1}(\cdot)$ exists. Thus $X = \mathbf{F}^{-1}(Y)$ and $u^T X u = u^T \mathbf{F}^{-1}(Y) u = (u_1^T \cdots u_n^T) Y$. For convenience, we use $u^T \mathbf{F}^{-1}(Y) u$ and $(u_1^T \cdots u_n^T) Y$ interchangeably.

We have shown that this positive definite least square estimation problem is equivalent to the infinite convex quadratic program (IQP). In Section 2, we propose an algorithm for solving (IQP) and prove its convergence. In Section 3, we present computational results for randomly generated data. Finally, we show that the algorithm presented in Section 2 can be generalized to solve a class of infinite convex programs in Section 4.

2. An Algorithm and its Convergence

We propose an algorithm for solving the infinite quadratic program (IQP). This algorithm solves (IQP) by generating and solving a sequence of feasible convex quadratic programs (QP(k)) for $k = 1, 2, \ldots$. Each (QP(k)) has the same objective function as that of (IQP).

Algorithm 1

Step 1.

Let $k = 0$;

let $\alpha$ be a constant such that $\epsilon \leq \alpha < K$;

let (QP(k)) be the following quadratic program:
minimize $Y^TMY - 2((B^1)^T \cdots (B^n)^T)EY + \sum_i (B^i)^TB^i$

subject to

\[
Y(i-1)n+j - Y(j-1)n+i = 0, \; i,j = 1, \ldots , n \\
-K \leq Y \leq K, \; i = 1, \ldots , n^2.
\]

Step 2.

Find an optimal solution $Y^k$ of $(QP(k))$;

let $X^k = F^{-1}(Y^k)$, i.e., $x^k_{ij} = Y^k(i-1)n+j, \; i,j = 1, \ldots , n$;

calculate $\lambda[X^k]$ and $v[X^k]$;

if $\lambda[X^k] \geq \varepsilon$, go to Step 4.

Step 3.

Let $u^k = v[X^k]$;

form $(QP(k+1))$ by adding a cut, $(u^k)^TF^{-1}(Y)u^k \geq \alpha$, to $(QP(k))$;

$k := k + 1$;

go to Step 2.

Step 4.

If $\alpha > \varepsilon$, $Y^k$ is an approximate optimal solution of $(IQP)$; stop.

If $\alpha = \varepsilon$, $Y^k$ is an optimal solution of $(IQP)$; stop.

Comments.

(1) For any $k$, the feasible region of $(QP(k))$ is a nonempty polytope since $Y = P(AI)$ is a feasible solution. Therefore, optimal solutions exist for all $(QP(k))$. Furthermore, since the objective function of $(QP(k))$ is quadratic and convex, there are finite algorithms for finding its optimal solutions.
(2) Efficient finite algorithms for calculating eigenvalues and
eigenvectors of a matrix can be found in Wilkinson [4].

(3) $k$ counts the number of major iterations (step 2–step 3), or
equivalently, the number of cuts added before termination. Each major
iteration can be processed finitely.

(4) $\alpha$ is a constant and $\varepsilon < \alpha < K$. If $\alpha > \varepsilon$, then an
approximate optimal solution will be found after a finite number of
major iterations (see Theorem 1). If $\alpha = \varepsilon$, then any cluster point of
of the sequence $Y^0, Y^1, Y^2, \ldots$ is an optimal solution for (IQP) (see
Theorems 2 and 3).

Theorem 1. If $\alpha > \varepsilon$, then Algorithm 1 can find an approximate optimal
solution for (IQP) after a finite number of major iterations.

Proof. Let $H(Y) = Y^T MY - 2((B^1)^T \cdots (B^n)^T)EY + \sum_{i=1}^{n}(B_i^i)^T B_i^i$ be the
objective functions of (IQP) and (QP(k)) for all $k$. Let
$C = \{Y: -K < Y_i < K, i = 1, \ldots, n^2\} \times S^{n-1}$ and $G(Y,u) = u^T F^{-1}(Y)u$.
Then $G(Y,u)$ is a continuous function and $C$ is a compact set. Hence,
$G(Y,u)$ is uniformly continuous on $C$, i.e., for any $\delta > 0$, there exists
$\eta > 0$ such that

- $(Y,u) - (\tilde{Y},\tilde{u}) < \eta$ implies $|G(Y,u) - G(\tilde{Y},\tilde{u})| < \delta$ for all
  $(Y,u)$ and $(\tilde{Y},\tilde{u})$ in $C$.

In particular, for $\delta = \alpha - \varepsilon > 0$, there exists $\bar{\eta} > 0$ such that (c)
holds. If Algorithm 1 goes on infinitely, then it generates $u^k \in S^{n-1}$
for $k = 0, 1, 2, \ldots$. Since $S^{n-1}$ is compact, for the $\bar{\eta} > 0$, there
exist $u^{k_1}$ and $u^{k_j}$ in the sequence such that $\|u^{k_1} - u^{k_j}\| < \bar{n}$.

Without loss of generality, we assume that $k_1 < k_j$. Since Algorithm 1 does not stop at iteration $k_j$, we have:

(d) $G(Y^{k_j}, u^{k_1}) = (u^{k_1})^T F^{-1}(y^{k_j}) u^{k_1} > \alpha$;

(e) $G(Y^{k_j}, u^{k_j}) = (u^{k_j})^T F^{-1}(y^{k_j}) u^{k_j} < \epsilon$.

However, (d) and (e) imply that $|G(Y^{k_j}, u^{k_1}) - G(Y^{k_j}, u^{k_j})| > \alpha - \epsilon = \delta$ while $\|y^{k_1} - y^{k_j}\| < \bar{n}$, which contradicts the uniform continuity of $G(Y, u)$ on $C$. Therefore, if $\alpha > \epsilon$, then Algorithm 1 terminates finitely. Suppose that it stops at iteration $k$. Then $\lambda[F^{-1}(Y^k)] \geq \epsilon$ and by Lemma 2, $Y^k$ is a feasible solution of (IQP).

However, since $\alpha > \epsilon$, the constraints $(u^i)^T F^{-1}(y^i) u^i \geq \alpha$ for $i = 0, 1, \ldots, k-1$ may be violated by certain feasible solutions of (IQP).

We can not guarantee that $H(Y^k) \leq H(Y)$ holds for all feasible $Y$.

But, if $Y \in \{Y \in \mathbb{R}^n: u^T F^{-1}(y)u \geq \alpha \text{ for all } u \in S^{n-1}, F^{-1}(Y)T = T^{-1}(Y), -K \leq Y^i \leq K \text{ for } i = 1, \ldots, n^2\}$, then $H(Y^k) \leq H(Y)$ is guaranteed to hold. Therefore, $Y^k$ is only an approximate optimal (or $\alpha$-suboptimal) solution in the case $\alpha > \epsilon$. 

**Remark 2.** When $\alpha$ increases, the number of cuts added before terminating decreases, but the final objective function value increases. For given data and choice of $\alpha$, if the algorithm does not terminate by a specified number of iterations, we can increase $\alpha$ and try again.

**Theorem 2.** If $\alpha = \epsilon$ and Algorithm 1 stops at a certain iteration $i$, then $Y^i$ is an optimal solution of (IQP).
Proof. Let \( H(Y) = Y^T M Y - 2((B^1)^T \cdots (B^n)^T) E Y + \sum_i (B^i)^T B^i \) be the objective functions of (IQP) and (QP(k)) for all \( k \). If Algorithm 1 stops at a certain iteration \( i \), then \( \lambda[X^i] \geq \epsilon \). By Lemma 2, \( Y^i \) is a feasible solution of (IQP). Next, suppose that \( Y \) is an arbitrary feasible solution of (IQP). Because \( \alpha = \epsilon \), \( Y \) is feasible for all (QP(k)). In particular, \( Y \) is feasible for (QP(i)). Since \( Y^i \) is an optimal solution of (QP(i)), we have \( H(Y^i) \leq H(Y) \). Therefore, \( Y^i \) is an optimal solution of (IQP). \( \square \)

**Theorem 3.** Suppose that \( \alpha = \epsilon \) and that Algorithm 1 does not stop finitely. Let the sequence \( Y^0, Y^1, Y^2, \ldots \) be generated by Algorithm 1. Then any cluster point is a solution for (IQP).

Proof. Let \( Y^k \) and \( u^k \) for \( k = 0, 1, 2, \ldots \) be generated by Algorithm 1 with \( \alpha = \epsilon \). Since \( Y^0, Y^1, \ldots \) are in a compact set, there exist cluster points. Let \( \bar{Y} \) be a cluster point and without loss of generality we assume that \( \lim_k Y^k = \bar{Y} \). We claim that for any \( 0 < \beta < \epsilon \), there exists an integer \( N(\beta) \) such that \( \lambda[F^{-1}(Y^k)] \geq \beta \) for all \( k \geq N(\beta) \). Let \( H(Y), G(Y,u) \) and \( C \) be defined as before. Since \( G(Y,u) \) is uniformly continuous on \( C \), for \( \delta = \epsilon - \beta > 0 \), there exists \( \eta > 0 \) such that

\[(f) \quad \| (Y,u) - (\bar{Y},\bar{u}) \| < \eta \quad \text{implies} \quad |G(Y,u) - G(\bar{Y},\bar{u})| < \delta \quad \text{for all} \quad (Y,u) \text{ and } (\bar{Y},\bar{u}) \text{ in } C.\]

Suppose that the above claim is not true. Then there exist \( k_i \) and \( k_j \) (\( k_i < k_j \)) such that \( \lambda[F^{-1}(Y^{k_j})] < \beta \) and \( \| u^{k_i} - u^{k_j} \| < \eta \). Thus,
(g) \(G(Y^k_j, u^k_j) = (u^k_i)F^{-1}(Y^k_j)u^k_j \geq \epsilon;\)

(h) \(G(Y^k_j, u^k_j) = (u^k_j)F^{-1}(Y^k_j)u^k_j = \lambda[F^{-1}(Y^k_j)] < \beta.\)

However, (g) and (h) imply that \(|G(Y^k_j, u^k_i) - G(Y^k_j, u^k_j)| > \epsilon - \beta = \delta\)
while \(\|Y^k_j, u^k_i\| - \|Y^k_j, u^k_j\| < \eta,\) which contradicts (f). Therefore, the above claim is proved. By the claim and the continuity of \(F^{-1}(\cdot)\)
and \(\lambda[\cdot]\) (Lemma 3), we have \(\lambda[F^{-1}(\tilde{Y})] \geq \beta\) for any \(\beta\) satisfying \(0 < \beta < \epsilon.\) Consequently, \(\lambda[F^{-1}(\tilde{Y})] \geq \epsilon\) and \(\tilde{Y}\) is feasible for (IQP). Next, let \(Y\) be an arbitrary feasible solution of (IQP). Then \(Y\) is feasible for all (QP(k)). Because \(Y^k\) is an optimal solution of (QP(k)) and the objective functions of (QP(k)) and (IQP) are the same, we have \(H(Y^k) \leq H(Y)\) for \(k = 0, 1, 2, \ldots.\) It follows that \(H(\tilde{Y}) \leq H(Y)\) holds for all feasible \(Y\) and \(\tilde{Y}\) solves (IQP). \(\Box\)

**Remark 3.** Since \(\alpha = \epsilon,\) the feasible region of (QP(k)) contains that of (IQP). As the algorithm goes on, \(Y^k\) becomes more and more close to the feasible region of (IQP) and \(H(Y^k)\) tends increasingly to \(H(\tilde{Y})\).

3. **Computational Results**

We have coded Algorithm 1 in FORTRAN. We use the subroutine QPSOL (from the Systems Optimization Laboratory, Department of Operations Research, Stanford University) to solve (QP(k)) and the subroutine FO2ABF (from NAG Library, Stanford University) to calculate eigenvalues and eigenvectors.
The program was executed on a DEC 20 computer. All components of
the input data $\mathbf{a}$ and $\mathbf{b}$ are iid $U(-0.5,0.5)$.

First, we compute one problem six times with different $\alpha$ values
to demonstrate the influence of $\alpha$ on the number of major iterations
and on the final objective values, see Table 1.

Given data:

$L = 8, \quad \varepsilon = 1.0, \quad K = 10^9, \quad n^2 = 16$;

$$
\begin{align*}
a^1 &= (-0.3052, 0.1087, -0.3915, -0.4383) \\
a^2 &= (0.1379, 0.1707, -0.1208, 0.3839) \\
a^3 &= (0.2999, -0.4803, 0.1790, -0.2021) \\
a^4 &= (-0.1334, 0.1864, -0.0431, 0.4557) \\
a^5 &= (-0.0681, 0.4627, -0.1384, 0.0547) \\
a^6 &= (-0.4691, 0.0743, 0.3823, 0.1650) \\
a^7 &= (-0.2117, -0.3549, 0.4991, -0.1264) \\
a^8 &= (-0.0865, 0.0886, -0.4886, -0.3304) \\

b^1 &= (0.2325, -0.1774, -0.3115, 0.2133) \\
b^2 &= (-0.4512, -0.1078, 0.0383, -0.0906) \\
b^3 &= (-0.0641, -0.3664, -0.1086, -0.3182) \\
b^4 &= (-0.3645, -0.1941, -0.1331, -0.3830) \\
b^5 &= (-0.2327, -0.0301, 0.0613, 0.2470) \\
b^6 &= (-0.3909, 0.3732, -0.0953, -0.1953) \\
b^7 &= (-0.1478, -0.2652, -0.3996, 0.3307) \\
b^8 &= (-0.2671, 0.3283, 0.0569, -0.3668)
\end{align*}
$$
Next, we solve a number of problems in different dimensions, see Table 2. Again, all components of the input data $a^t$ and $b^t$ are iid $U(-0.5,0.5)$.

### Table 1

<table>
<thead>
<tr>
<th>value of $\alpha$</th>
<th>no. of major iterations</th>
<th>final objective function value</th>
<th>CPU time (second)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>4</td>
<td>5.2713</td>
<td>1.89</td>
</tr>
<tr>
<td>1.01</td>
<td>7</td>
<td>4.8017</td>
<td>2.77</td>
</tr>
<tr>
<td>1.001</td>
<td>13</td>
<td>4.7585</td>
<td>4.94</td>
</tr>
<tr>
<td>1.0001</td>
<td>14</td>
<td>4.7537</td>
<td>5.44</td>
</tr>
<tr>
<td>1.00001</td>
<td>15</td>
<td>4.7532</td>
<td>5.88</td>
</tr>
<tr>
<td>1.000001</td>
<td>16</td>
<td>4.7532</td>
<td>6.63</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>problem dimension $(n^2, L)$</th>
<th>value of $\alpha$ and $\varepsilon$ $(\alpha, \varepsilon)$</th>
<th>no. of major itera.</th>
<th>value of final obj. function</th>
<th>CPU time (second)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(16, 6)</td>
<td>(1.01, 1)</td>
<td>7</td>
<td>3.3775</td>
<td>2.90</td>
</tr>
<tr>
<td>(16, 10)</td>
<td>(1.01, 1)</td>
<td>8</td>
<td>5.9837</td>
<td>3.23</td>
</tr>
<tr>
<td>(36, 8)</td>
<td>(1.01, 1)</td>
<td>15</td>
<td>9.0326</td>
<td>25.18</td>
</tr>
<tr>
<td>(36, 12)</td>
<td>(1.01, 1)</td>
<td>17</td>
<td>14.1864</td>
<td>28.19</td>
</tr>
<tr>
<td>(64, 12)</td>
<td>(1.01, 1)</td>
<td>33</td>
<td>19.1114</td>
<td>219.29</td>
</tr>
<tr>
<td>(64, 18)</td>
<td>(1.01, 1)</td>
<td>26</td>
<td>29.6196</td>
<td>175.32</td>
</tr>
</tbody>
</table>
4. A Generalization

Herein we show that Algorithm 1 presented in Section 2 can be
generalized to solve the following infinite convex program (ICP).

(IPC): \[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x,u) \geq 0 \text{ for all } u \in U \\
& \quad x \in S
\end{align*}
\]

where \( U \) and \( S \) are compact convex sets, \( f(x) \) is a convex function on
\( S \), \( g(x,u) \) is continuous on \( S \times U \) and is concave in \( x \) when \( u \) is
fixed and convex in \( u \) when \( x \) is fixed.

Definition (\( \varepsilon \)-optimal solution) A vector \( \bar{x} \in S \) is an \( \varepsilon \)-optimal
solution of (IPC) if \( g(\bar{x},u) \geq -\varepsilon \) for all \( u \in U \) and
\( f(\bar{x}) \leq v(\text{IPC}) \), where \( v(\text{IPC}) \) is the optimal objective function value
of (IPC).

Algorithm 2

Step 1.

Let \( k := 0; \)

let \( \text{(CP}(k) ) \) be the following convex program:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in S.
\end{align*}
\]
Step 2.
If \((CP(k))\) is infeasible, go to Step 4;
find an optimal solution \(x^k\) of \((CP(k))\);
if \(g(x^k, u) \geq 0\) for all \(u \in U\), go to Step 5.

Step 3.
Find a \(u^k \in U\) satisfying \(g(x^k, u^k) < 0\).
form \((CP(k+1))\) by adding a cut \(g(x, u^k) \geq 0\) to \((CP(k))\);
\(k = k + 1;\)
go to Step 2.

Step 4.
\((ICP)\) is infeasible, stop.

Step 5.
\(x^k\) is an optimal solution of \((ICP)\), stop.

Notice that \(g(x, u)\) is uniformly continuous on \(S \times U\), it is not hard to prove the following theorems.

**Theorem 4** (finite \(\varepsilon\)-convergence) For any \(\varepsilon > 0\), Algorithm 2 can find an \(\varepsilon\)-optimal solution of \((ICP)\) after finitely many iterations.

**Theorem 5** (convergence) If Algorithm 2 does not stop finitely, then any cluster point of the sequence \(x^k\) for \(k = 1, 2, \ldots\) is an optimal solution of \((ICP)\).
Acknowledgements. The author is very grateful to her dissertation advisor, Professor G.B. Dantzig, who suggested the problem and provided guidance throughout the research. She also would like to thank Professor A.J. Hoffman for his helpful suggestions.

References


**Title:** An Algorithm for Positive Definite Least Square Estimation of Parameters

**Type of Report & Period Covered:** Technical Report

**Performing Organizational Name and Address:**
Department of Operations Research - SOL
Stanford University
Stanford, CA 94305

**Controlling Office Name and Address:**
Office of Naval Research - Dept. of the Navy
800 N. Quincy Street
Arlington, VA 22217

**Report Date:** May 1986

**Number of Pages:** 15 pp.

**Security Class. (of this report):** UNCLASSIFIED

**DISTRIBUTION STATEMENT (of this Report):**

This document has been approved for public release and sale; its distribution is unlimited.

**Distribution Statement (of the abstract entered in Block 20, if different from Report):**

**Supplementary Notes:**

**Key Words:**
- Least square estimation
- Quadratic programming
- Positive definite matrix

**Abstract:** We present an algorithm for positive definite least square estimation of parameters. This estimation problem arises from the PILOT dynamic macro-economic model and is equivalent to an infinite convex quadratic program. It differs from ordinary least square estimations in that the fitting matrix is required to be positive definite. The algorithm solves the infinite convex quadratic program by generating and solving a sequence of ordinary convex quadratic programs. By specifying a constant, the algorithm will find an approximate optimal solution after finitely many iterations, or will tend to an optimal solution in the limit. The algorithm is generalized to solve a class of infinite convex programs.
END

D T I C

8 - 86