1. Introduction and Terminology

All graphs in this paper will be finite and connected and will have no loops or parallel lines.

Let $n$ and $p$ be positive integers with $n \leq (p - 2)/2$ and let $G$ be a graph with $p$ points having a perfect matching. Graph $G$ is said to be $n$-extendable if every matching of size $n$ in $G$ extends to a perfect matching.

The concept of $n$-extendability for bipartite graphs was studied by Hetyei (1964). But the study of the more general family of $n$-extendable graphs which are not necessarily bipartite seems to have even earlier roots. In the late 1950's, Kotzig (1959a, 1959b, 1960) began to develop a decomposition theory for graphs with perfect matchings, but unfortunately these papers did not receive the attention that they deserve, due to the fact that they were written in Slovak. In the early 1960's, the study of decompositions of graphs in terms of their maximum matchings was begun by Gallai (1963, 1964) and independently by Edmonds (1965). One of the degenerate cases of their theory for maximum matchings, however, arises when the graphs in question have perfect matchings.

Motivated by these results of Kotzig, Hetyei, Gallai and Edmonds, Lovász (1972) extended and refined the canonical decompositions already extant.

In this same paper, Lovász also introduced the concept of a bicritical graph. A graph $G$ is said to be bicritical if $G - u - v$ has a perfect matching for every pair of distinct points $u$ and $v$ in $V(G)$. In the last ten years or so, the earlier work on decompositions of graphs in terms of their matchings has evolved further (see Lovász and Plummer (1986)) and today much attention continues to be focused upon the structure of bicritical graphs which are, in addition, 3-connected. Such graphs
have been christened bricks. (See, for example, the paper by Edmonds, Lovász and Pulleyblank (1982) and that of Lovász (1986).)

But what is the connection between n-extendability and bicriticality? One of the results presented in Plummer (1980) states that every 2-extendable graph is either bipartite or is a brick. (The reader should convince himself immediately that these two classes of graphs are disjoint.) Motivated by this result, the author has continued to study properties of n-extendable graphs (see (1985, 1986a, 1986b and 1986c)).

Let $S$ be a point cutset in graph $G$ and let $c(G-S)$ denote the number of components in $G-S$. Then, if $G$ is not complete, the toughness of $G$ is defined to be $\min \frac{|S|}{c(G-S)}$ where the minimum is taken over all point cutsets $S$ of $G$, whereas we define the toughness of $K_n$ to be $+\infty$ for all $n$. We denote the toughness of $G$ by tough$(G)$. We will also say that graph $G$ is $k$-tough if tough$(G) \geq k$. This parameter was introduced by Chvátal (1973a, 1973b) who was initially motivated by studies about Hamiltonian cycles in graphs. He noted, however, in (1973a) that every 1-tough graph with an even number of points has a perfect matching.

A generalization of both the concepts of Hamiltonian cycle and perfect matching is the idea of a $k$-factor of a graph. A $k$-factor of a graph $G$ is a spanning subgraph of $G$ which is regular of degree $k$. Thus a perfect matching is just a 1-factor and a Hamiltonian cycle is just a connected 2-factor. Chvátal conjectured in (1973a) that if $G$ is any graph on $p$ points and if $k$ is a positive integer such that $G$ is $k$-tough and $kp$ is even, then $G$ has a $k$-factor. This conjecture has only recently been settled in the affirmative by Enomoto, Jackson, Katerinis and A. Saito (1985).

In the present paper, we wish to treat some relationships between toughness of a graph and the n-extendability of the graph. In the next section we will prove two results. The first says essentially that if a graph has sufficiently high toughness (and has an even number of points) then it must be n-extendable. The second result applies to graphs with toughness less than one and presents an upper bound on the value of $n$ for which such a graph can be n-extendable.

In the final section, we compare and contrast these results with the $n$-factor results of Enomoto, Jackson, Katerinis and A. Saito.

Any graph terminology used, but not defined, in this paper may be found either in Bondy and Murty (1977) or Lovász and Plummer (1986).
2. Two results on toughness and \( n \)-extendability

In addition to the theorem of the author (1980) mentioned in the Introduction, there are two other results proved in that paper which we shall use here and hence we begin by stating them without proof.

1980A. THEOREM. If \( n \geq 2 \) and \( G \) is \( n \)-extendable, then \( G \) is also \((n - 1)\)-extendable.

1980B. THEOREM. If \( G \) is \( n \)-extendable, then \( G \) is \((n + 1)\)-connected.

Our first result of the present paper follows in a straightforward way via Tutte's classical theorem characterizing graphs with perfect matchings.

2.1. THEOREM. Suppose that \( G \) is a graph with \( p = |V(G)| \) points with \( p \) even. Let \( n \) be a positive integer with \( p \geq 2n + 2 \). Then if \( \text{tough}(G) > n \), graph \( G \) is \( n \)-extendable. Moreover, this lower bound on \( \text{tough}(G) \) is sharp for all \( n \).

PROOF. First suppose that \( n = 1 \). Note that since \( \text{tough}(G) \geq 1 \), graph \( G \) has a perfect matching by Tutte's Theorem on perfect matchings.

Now suppose that for some line \( e = xy \in E(G) \), line \( e \) lies in no perfect matching for \( G \). Thus if \( G' = G - x - y \), by the above-mentioned theorem of Tutte there is a set \( S' \subseteq V(G') \) with \( |S'| < c_0(G' - S') \). (Note that here \( c_0(G' - S') \) denotes the number of components of \( G' - S' \) which have an odd number of points.) But then by parity, \( |S'| \leq c_0(G' - S') - 2 \).

Now let \( S_0 = S' \cup \{x, y\} \). Since \( G \) has a perfect matching, it follows that \( c_0(G - S_0) \leq |S_0| = |S'| + 2 \leq c_0(G' - S') \). But \( G - S_0 = G' - S' \) and so equality holds throughout and in particular, \( c_0(G - S_0) = |S_0| \).

(See Figure 1.)

But now

\[
\text{tough}(G) = \min_{S \subseteq V(G)} \frac{|S|}{c(G - S)} \leq \frac{|S_0|}{c(G - S_0)} \leq \frac{|S_0|}{c_0(G - S_0)} = 1,
\]

ccontradicting the hypothesis of this theorem. So the desired conclusion holds when \( n = 1 \).

Now suppose \( n \geq 2 \), and assume that \( G \) is not \( n \)-extendable. Let \( M = \{e_1, \ldots, e_n\} \) be a matching of size \( n \) which does not extend to a perfect matching. Denote \( e_i = x_iy_i \) for \( i = 1, \ldots, n \). Let \( G_1 = G - x_1 - \cdots - x_n - y_1 - \cdots - y_n \). Hence \( G_1 \) has no perfect matching.
and thus by Tutte's Theorem, there is a set $S_1 \subseteq V(G_1)$ such that $|S_1| < c_o(G_1 - S_1)$. Hence by parity, $|S_1| \leq c_o(G_1 - S_1) - 2$. (Note that $S_1$ might be empty.)

Now $G_2 = G - x_n - y_n$ has a perfect matching since we have already proved that $G$ is 1-extendable. Let $S_2 = S_1 \cup \{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$ and note that once again by Tutte's Theorem, $c_o(G_2 - S_2) \leq |S_2| = |S_1| + 2n - 2 \leq c_o(G_1 - S_1) + 2n - 4$. But $G_2 - S_2 = G_1 - S_1$ and so it follows that $c_o(G_2 - S_2) \geq |S_1| + 2$.

Now let $S_3 = S_1 \cup \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Then, since $G - S_3 = G_2 - S_2$, we have

$$\text{tough}(G) \leq \min_{S \subseteq V(G)} \frac{|S|}{c(G - S)} \leq \frac{|S_3|}{c(G - S_3)} = \frac{|S_3|}{c(G_2 - S_2)}$$

$$\leq \frac{|S_3|}{c_o(G_2 - S_2)} = \frac{|S_1| + 2n}{c_o(G_2 - S_2)}$$

$$\leq \frac{|S_1| + 2n}{|S_1| + 2} \leq \frac{n|S_1| + 2n}{|S_1| + 2} = n,$$

again a contradiction of the hypothesis.

It remains only to exhibit extremal graphs for each value of $n \geq 1$. For each positive integer $n \geq 1$, define graph $H_n$ as follows. Let $N = \{e_1, \ldots, e_n\}$ be a set of $n$ independent lines. Join each of the $2n$ points of $N$ to each point of two disjoint copies of the complete graph $K_{2n+1}$. (See Figure 2.) Then $|V(H_n)| = 6n + 2$ and it is easy to see that
tough($H_n$) = $n$. However, the matching $N$ does not extend to a perfect matching.

Now let us begin to think of some type of converse to the above result. Let us remark at the outset that if a graph $G$ is $n$-extendable, there is no lower bound on the toughness of $G$. To see this, we construct the following family of graphs. Let $n$ and $k$ be any two positive integers. Let $S$ be a set of $2n$ independent points and let graph $J(n, k)$ be constructed by joining each point of set $S$ to both endpoints of each of $2n+k$
FIGURE 3. The extremal family \( \{ J(n, k) \}_{n-1, k-1}^{\infty} \)

It is easy to verify that \( J(n, k) \) is \( n \)-extendable for every value of \( k \). Clearly, \( \text{tough}(J(n, k)) \leq 2n/(2n + k) \), and hence \( \text{tough}(J(n, k)) \to 0 \) as \( k \to \infty \). Of course the number of points in graph \( J(n, k) \) is quite large and it makes sense to amend our search for some type of converse to Theorem 2.1 as follows. Again letting \( p = |V(G)| \), we may ask if there is a function \( f(p) \) such that if graph \( G \) is \( f(p) \)-extendable, then \( G \) is, say, 1-tough. The next result shows that the answer to this question is “yes”.

2.2. THEOREM. Let \( G \) be a graph with \( p \) points and let \( n \) be a positive integer. Suppose that \( G \) is \( n \)-extendable, but that \( \text{tough}(G) < 1 \). Then \( n \leq \lfloor \frac{p-2}{6} \rfloor \) and this bound is sharp for each \( n \).

PROOF. Since \( \text{tough}(G) < 1 \), there is a cutset \( S \) in \( G \) such that \( G - S \) has more than \( |S| = s \) components. Note that by Theorem 1980B, \( s \geq n + 1 \geq 2 \). Let the components of \( G - S \) be \( C_1, \ldots, C_{s+r} \), where \( r \geq 1 \).

Note that \( G - S \) must have at least one even component, for if not, by Tutte’s Theorem, \( G \) could not have a perfect matching, contradicting the hypothesis of the present theorem. So suppose that component \( C_1 \) is even and hence \( |V(C_1)| \geq 2 \). Since \( n \geq 1 \), \( G \) is 2-connected by Theorem 1980B, and hence there exists a line \( e_1 \) joining a point of \( C_1 \) to a point of \( S \). By hypothesis, \( G \) is \( n \)-extendable and \( n \geq 1 \), so by Theorem 1980A, \( G \) is 1-extendable. So extend line \( e_1 \) to a perfect matching \( F_1 \) of \( G \) and note that by parity, \( F_1 \) matches at least two points of component \( C_1 \).
2. TWO RESULTS ON TOUGHNESS AND N-EXTENDABILITY

into set $S$. It then follows that, in fact, $G - S$ must have at least 3 even components.

**Claim 1.** $G - S$ has at least $n$ even components.

If $1 \leq n \leq 3$, we are done. So we may suppose that $n \geq 4$.

Suppose, to the contrary, that $G - S$ has $t$ even components, where $3 \leq t \leq n - 1$. Relabeling these components if necessary, suppose that $C_1, \ldots, C_t$ are the even components of $G - S$. (Recall that altogether, $G - S$ has $s + r \geq s + 1 \geq n + 2$ components.)

We now construct a matching which contains two lines joining each of the components $C_1, \ldots, C_t$ to different points of $S$. Let $e_1$ be any line joining $C_1$ to a point $u_1$ in $S$, relabeling the points of $S$ if necessary. Now if all lines between $S$ and $C_2$ are incident with point $u_1$, it then follows that $u_1$ is a cutpoint of $G$, a contradiction of the fact that $G$ is 2-connected. Hence we can match points of $C_1$ and $C_2$ via lines $e_1$ and $e_2$ to distinct points $u_1$ and $u_2$ of $S$ say, where once again we relabel the points of $S$ if necessary.

Recall that $n \geq 4$. Suppose further that $C_1, \ldots, C_q, q < n$, have been matched into $S$ to points $u_1, \ldots, u_q$ respectively. If we cannot match a point of $V(C_{q+1})$ into $S$ at a point different from $u_1, \ldots, u_q$, then $\{u_1, \ldots, u_q\}$ is a cutset of $G$; that is, it separates $C_{q+1}$ from all the other components of $G - S$. Hence $\kappa(G) \leq q < n$, contradicting the fact that (by Theorem 1980B) graph $G$ is $(n + 1)$-connected. Thus we have the matching of size $n$ that we seek. Call it $M_1$.

Extend matching $M_1$ to a perfect matching $F_2$ of $G$. By parity, for each even component $C_1, \ldots, C_t$, matching $F_2$ must match at least one point to $S$ other than that matched by $M_1$. Without loss of generality, let us suppose that a point of $C_i$ is matched to point $u_{n+i}$, and that a point of $C_t$ is matched to point $u_{n+t}$. (See Figure 4.)

But now each of the odd components $C_{n+1}, \ldots, C_{s+t}$ must contain at least one point which is matched by perfect matching $F_2$ into the set $\{u_{n+t+1}, \ldots, u_s\}$ of $S$. Thus it follows that $s + r - n \leq s - (n + t)$ and so $r \leq -t < 0$, a contradiction and Claim 1 is proved.

Finally, we prove

**Claim 2.** Graph $G - S$ has at least $2n + r$ even components.

By Claim 1, graph $G - S$ has at least $n$ even components. Relabeling if necessary, suppose that they are $C_1, \ldots, C_n, \ldots, C_{n+q}$. Just as in the proof of Claim 1, since $G$ is $n$-connected, we can find a matching $M_2$ which joins exactly one point of component $C_i$ to a point $u_i$ in $S$ for $i = 1, \ldots, n$, where yet again, we relabel points in $S$ if necessary.

Since $G$ is $n$-extendable, let us extend matching $M_2$ to a perfect
matching \( F_3 \) of \( G \). By parity, each of \( C_1, \ldots, C_n \) has at least 2 points matched into \( S \) by \( F_3 \). So relabeling again if need be, assume that \( F_3 \) also matches a point \( u_{n+i} \) to a point in component \( C_i \) for \( i = 1, \ldots, n \).

(In particular at this point, we now know that \(|S| = s \geq 2n.|)

Thus \( F_3 \) must match the remaining \( s-2n \) points of \( S \) (if any) to some \( s-2n \) points in \( \bigcup_{i=1}^{s+r} V(C_i) \). So among the components \( C_{n+1}, \ldots, C_{s+r} \), at least \((s + r - n) - (s - 2n) = n + r\) must be even. These components, together with \( C_1, \ldots, C_n \) give the \( 2n + r \) even components of \( G - S \) as claimed.

Now we have

\[
|V(G)| = p = |S| + |V(C_1)| + \cdots + |V(C_{s+r})|
\geq s + 2(2n + r) + (s + r - (2n + r))
= s + 4n + 2r + s - 2n = 2s + 2n + 2r \geq 6n + 2r \geq 6n + 2.
\]

So \( n \leq (p - 2)/6 \) and since \( n \) is an integer, \( n \leq \lfloor \frac{p-2}{6} \rfloor \).

To show that the bound is sharp for all \( n \geq 1 \) consider the infinite family of graphs \( L_n \) where \( L_n = J(n, 1) \) and \( J(n, 1) \) is shown in Figure 3.

Note that \( p = |V(H_n)| = 2n + 2(2n + 1) = 6n + 2 \) and hence \( n = (p - 2)/6 \) and it is easy to check that graph \( H_n \) is \( n \)-extendable, but not \((n + 1)\)-extendable.

Of course, Theorem 2.2 can be restated as follows: if graph \( G \) is \( \lfloor \frac{p-2}{6} \rfloor + 1 \)-extendable, then \( G \) is 1-tough.
3. Comparisons with an \( n \)-factor theorem

Enomoto, Jackson, Katerinis and A. Saito (1985) have proved the following result.

1985. **Theorem.** Let \( G \) be a graph with at least \( n + 1 \) points and suppose \( \text{tough}(G) \geq n \). Then, if \( |V(G)| \) is even, \( G \) has an \( n \)-factor.

This theorem answers in the affirmative a conjecture of Chvátal. In order to properly compare the conclusion of this result with that of our Theorem 2.1, let us try to state each result in as parallel a fashion as possible. Of course, if we were to define a graph to be "0-extendable" if it had a perfect matching, the two conclusions would say exactly the same thing when \( n = 1 \).

Now suppose \( n \geq 2 \) and consider the following two statements; the first being the result of Theorem 1985 and the second, our Theorem 2.1.

(A) \( \text{tough}(G) \geq n \Rightarrow G \) has an \( n \)-factor.

(B) \( \text{tough}(G) \geq n \Rightarrow G \) is \( (n - 1) \)-extendable.

We claim that the two conclusions are independent, in that neither implies the other.

First consider the family of graphs \( J(n, 1) \) already discussed above. Suppose \( n \geq 2 \). We already know that graph \( J(n, 1) \) is \( n \)-extendable. Hence by Theorem 1980A it is also \( (n - 1) \)-extendable. We claim it has no \( n \)-factor.

Suppose, to the contrary, that \( J(n, 1) \) does have an \( n \)-factor, \( F \). Then factor \( F \) must send \( 2n^2 \) lines from set \( S \) to \( G - S \). But each point of \( G - S \) must send at least \( n - 1 \) lines to set \( S \) and hence we have at least \( (n - 1)(4n + 2) = 4n^2 - 2n - 2 \) lines of factor \( F \) from \( G - S \) to \( S \). But then \( 2n^2 \geq 4n^2 - 2n - 2 \) and it follows that \( n = 1 \), a contradiction.

Finally, consider the infinite family of graphs \( \{M_n\}_{n=2}^{\infty} \) constructed as follows. Graph \( M_n \) is formed by taking two copies of the complete graph \( K_{n+1} \) and joining corresponding points of the two copies with a perfect matching. (This is, of course, just the prism over \( K_{n+1} \).) Graph \( M_n \) clearly has an \( n \)-factor consisting of precisely two components; namely, the two copies of \( K_{n+1} \). On the other hand, we claim that graph \( M_n \) is not \( (n - 1) \)-extendable. Let the points of the "top" \( K_{n+1} \) be \( u_1, \ldots, u_{n+1} \) and the corresponding points of the "bottom" \( K_{n+1} \) be \( v_1, \ldots, v_{n+1} \). (That is, \( u_i v_i, i = 1, \ldots, n + 1 \), is the perfect matching joining the top and bottom.)

In order to prove our assertion, let us consider the cases for \( n \) odd and even separately.
First suppose $n$ is odd. Select the matching $M_o$ consisting of $u_1v_1, u_2u_n, u_3u_{n-1}, \ldots$ together with line $v_nv_{n+1}$. Clearly $M_o$ is a matching of size $n - 1$, but it cannot be extended to a perfect matching for point $u_{n+1}$ could never be matched in such an extension.

Now suppose $n$ is even. First let us suppose also that $n \geq 4$. In this case, select the matching $M_e$ to consist of $u_1u_n, u_2u_{n-1}, \ldots$ together with line $v_1v_{n+1}$. Then matching $M_e$ has size $n - 1$, but it cannot be extended to a perfect matching since point $u_{n+1}$ could never be matched.

Finally, suppose $n = 2$. Let $M_2$ be the graph $K_4 - e$ for any line $e$ in $K_4$. Clearly $M_2$ is not 1-extendable, but it has a 2-factor.

We show the graphs $M_4$ and $M_5$ in Figure 5.
References

J. A. BONDY AND U. S. R. MURTY

V. CHVÁTAL

J. EDMONDS

J. EDMONDS, L. LOVÁSZ AND W. R. PULLEYBLANK

H. ENOMOTO, B. JACKSON, P. KATERINIS AND A. SAITO

T. GALLAI

G. HETYEI

A. KOTZIG
REFERENCES

L. LOVÁSZ

L. LOVÁSZ AND M. D. PLUMMER

M. D. PLUMMER
the odd components of $G' - S'$

Figure 1
Figure 3
Figure 4
Figure 5