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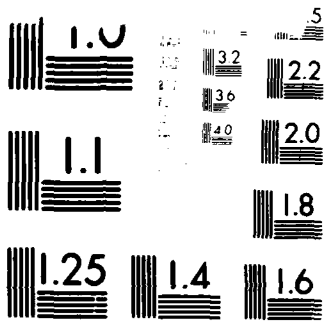
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EXACT TESTS FOR THE MAIN EFFECTS' VARIANCE COMPONENTS IN AN UNBALANCED RANDOM TWO-WAY MODEL

By

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Exact tests are presented for testing the significance of the random effects' variance components in an unbalanced random two-way classification model. A study is made concerning the power of the proposed tests, including a comparison with other approximate tests.

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SUMMARY

Exact tests are derived for testing hypotheses concerning the variance components of the main effects in an unbalanced random two-way crossed classification with interaction model.The tests are based on four sums of squares which are distributed independently as scalar multiples of chi-squared variates. These sums of squares can also be used to find an exact test concerning the interaction variance component, and to obtain simultaneous confidence intervals on all continuous functions of the model's variance components. A study is made concerning the power of the proposed tests, including a comparison with other approximate tests.

Key words: Variance components; Unbalanced random model; Two-way crossed classification with interaction model; Hypothesis testing; Power of exact tests.

1. Introduction

There are very few exact tests available concerning the variance components in an unbalanced random or mixed model. This is mainly attributed to the fact that in an unbalanced data situation, the traditional partitioning of the total sum of squares does not in general yield independent and chi-squared type sums of squares. Furthermore, such partitioning is not unique as is the case with balanced models.

Wald (1940, 1941) was the first author to introduce exact testing procedures for the unbalanced one-way and two-way crossed classification without interaction models. A generalization of Wald's tests was recently discussed by Seely and El-Bassiouni (1983). See also Harville and Fenech (1985). Using two different approaches, Spjøtvoll (1968) and Thomsen (1975) developed exact tests concerning the variance components in an unbalanced random two-way crossed classification with interaction model. However, unless the interaction variance component is zero, neither Spjøtvoll's tests nor those of Thomsen can be used to make inferences about the main effects' variance components. In this paper we managed to overcome this difficulty by producing exact tests concerning the latter variance components which apply even when the interaction variance component is different from zero. A comparison of the power of the exact tests with simulated powers of approximate ANOVA-based F tests shows that in most cases the exact tests are at least as powerful as the approximate tests. One main disadvantage of the latter tests is that their true critical values are unknown since they depend on variance components other than those under consideration. Thus, in practice the critical values of the approximate tests must be estimated using some procedure such as Satterthwaite's approximation. The simulation study shows that in some cases this approximation is highly unreliable for producing the critical values. Only in such cases were the approximate tests observed to be more powerful than the exact tests.

2. The Development of the Exact Tests

We shall adopt the same notation as in Thomsen (1975). Consider the unbalanced random two-way crossed classification model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk}, \qquad (2.1)$$

$i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$; $k = 1,2,\ldots,n_{ij}$, where μ is an unknown constant parameter; α_i, β_j, $(\alpha\beta)_{ij}$, and e_{ijk} are independent normally distributed random variables with zero means and variances σ^2_α, σ^2_β, $\sigma^2_{\alpha\beta}$, and σ^2_e, respectively. Alternatively, (2.1) can be written in matrix form as

$$\underset{\sim}{y} = \mu\, \underset{\sim}{1}_{n_{..}} + \underset{\sim}{X}_1\underset{\sim}{\alpha} + \underset{\sim}{X}_2\underset{\sim}{\beta} + \underset{\sim}{X}_3(\underset{\sim}{\alpha\beta}) + \underset{\sim}{e}, \qquad (2.2)$$

where $\underset{\sim}{y}$ is the vector of observations of dimension $n_{..} = \sum_{i,j} n_{ij}$, $\underset{\sim}{1}_{n_{..}}$ is a vector of ones of dimension $n_{..}$, $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$, and $\underset{\sim}{X}_3$ are matrices of zeros and ones of orders $n_{..}\times r$, $n_{..}\times s$, and $n_{..}\times rs$, respectively. The variance-covariance matrix of $\underset{\sim}{y}$, denoted by $\underset{\sim}{\Sigma}$, is

$$\underset{\sim}{\Sigma} = \underset{\sim}{X}_1\underset{\sim}{X}_1' \,\sigma_\alpha^2 + \underset{\sim}{X}_2\underset{\sim}{X}_2' \,\sigma_\beta^2 + \underset{\sim}{X}_3\underset{\sim}{X}_3' \,\sigma_{\alpha\beta}^2 + \underset{\sim}{I}_{n_{..}} \sigma_e^2, \tag{2.3}$$

where $\underset{\sim}{I}_{n_{..}}$ is the identity matrix of order $n_{..}\times n_{..}$.

Let $\bar{y}_{ij.}$ be the $(i,j)^{th}$ sample cell mean ($i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$). From (2.1) we have

$$\bar{y}_{ij.} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \bar{e}_{ij.}, \tag{2.4}$$

$i = 1,2,\ldots,r$; $j = 1,2,\ldots,s$, where $\bar{e}_{ij.} = \sum_{k=1}^{n_{ij}} e_{ijk}/n_{ij}$. In matrix form, (2.4) may be written as

$$\bar{\underset{\sim}{y}} = \mu \,\underset{\sim}{1}_{rs} + \underset{\sim}{B}_1\underset{\sim}{\alpha} + \underset{\sim}{B}_2\underset{\sim}{\beta} + \underset{\sim}{I}_{rs}(\underset{\sim}{\alpha\beta}) + \bar{\underset{\sim}{e}}, \tag{2.5}$$

where $\underset{\sim}{B}_1 = \underset{\sim}{I}_r \otimes \underset{\sim}{1}_s$, $\underset{\sim}{B}_2 = \underset{\sim}{1}_r \otimes \underset{\sim}{I}_s$, and $\otimes$ is the direct product symbol. The variance-covariance matrix of $\bar{\underset{\sim}{y}}$ is

$$\text{Var}\ \bar{\underset{\sim}{y}} = \underset{\sim}{A}_1\sigma_\alpha^2 + \underset{\sim}{A}_2\sigma_\beta^2 + \underset{\sim}{I}_{rs}\sigma_{\alpha\beta}^2 + \underset{\sim}{K}\,\sigma_e^2,$$

where $\underset{\sim}{A}_1 = \underset{\sim}{B}_1\underset{\sim}{B}_1'$, $\underset{\sim}{A}_2 = \underset{\sim}{B}_2\underset{\sim}{B}_2'$, and $\underset{\sim}{K} = \text{diag}(n_{11}^{-1}, n_{12}^{-1},\ldots,n_{rs}^{-1})$.

Let $\underset{\sim}{z} = \underset{\sim}{P}\,\bar{\underset{\sim}{y}}$, where $\underset{\sim}{P}$ is an orthogonal matrix of order $rs\times rs$ whose first row is $(rs)^{-\frac{1}{2}}\underset{\sim}{1}_{rs}'$ and simultaneously diagonalizes $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$. The vector $\underset{\sim}{z}$ can be partitioned as $(z_1,\underset{\sim}{z}_\alpha',\underset{\sim}{z}_\beta',\underset{\sim}{z}_{\alpha\beta}')'$, where z_1 is the first element of $\underset{\sim}{z}$; $\underset{\sim}{z}_\alpha, \underset{\sim}{z}_\beta$, and $\underset{\sim}{z}_{\alpha\beta}$ are vectors of dimensions $r-1$, $s-1$, and $(r-1)(s-1)$, respectively. The latter three vectors are normally distributed with zero means and have the following variance-covariance matrices (see Thomsen 1975, p. 259):

$$\text{Var } \underset{\sim}{z}_\alpha = (s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2)\ \underset{\sim}{I}_{r-1} + \underset{\sim}{K}_1\ \sigma_e^2$$

$$\text{Var } \underset{\sim}{z}_\beta = (r\sigma_\beta^2 + \sigma_{\alpha\beta}^2)\ \underset{\sim}{I}_{s-1} + \underset{\sim}{K}_2\ \sigma_e^2$$

$$\text{Var } \underset{\sim}{z}_{\alpha\beta} = \sigma_{\alpha\beta}^2\ \underset{\sim}{I}_{(r-1)(s-1)} + \underset{\sim}{K}_3\ \sigma_e^2,$$

where $\underset{\sim}{K}_1$, $\underset{\sim}{K}_2$, and $\underset{\sim}{K}_3$ are the submatrices of $\underset{\sim}{PKP}'$ which correspond to $\underset{\sim}{z}_\alpha$, $\underset{\sim}{z}_\beta$, and $\underset{\sim}{z}_{\alpha\beta}$, respectively. Let $\underset{\sim}{u}$ be the vector

$$\underset{\sim}{u} = (\underset{\sim}{z}_\alpha', \underset{\sim}{z}_\beta', \underset{\sim}{z}_{\alpha\beta}')'. \tag{2.6}$$

Then,

$$\text{Var } \underset{\sim}{u} = \text{diag}\left(\delta_1 \underset{\sim}{I}_{r-1}, \delta_2 \underset{\sim}{I}_{s-1}, \delta_3 \underset{\sim}{I}_{(r-1)(s-1)}\right) + \underset{\sim}{L}\ \sigma_e^2, \tag{2.7}$$

where δ_1, δ_2, and δ_3 are given by

$$\begin{aligned} \delta_1 &= s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2 \\ \delta_2 &= r\sigma_\beta^2 + \sigma_{\alpha\beta}^2 \\ \delta_3 &= \sigma_{\alpha\beta}^2, \end{aligned} \tag{2.8}$$

and $\underset{\sim}{L}$ is the $(rs-1)\times(rs-1)$ submatrix of $\underset{\sim}{PKP}'$ which corresponds to $\underset{\sim}{u}$ and is expressible in the form

$$\underset{\sim}{L} = \begin{bmatrix} \underset{\sim}{K}_1 & \underset{\sim}{K}_{12} & \underset{\sim}{K}_{13} \\ \underset{\sim}{K}_{12}' & \underset{\sim}{K}_2 & \underset{\sim}{K}_{23} \\ \underset{\sim}{K}_{13}' & \underset{\sim}{K}_{23}' & \underset{\sim}{K}_3 \end{bmatrix}, \tag{2.9}$$

where $\underset{\sim}{K}_{12}\sigma_e^2 = E(\underset{\sim}{z}_\alpha\, \underset{\sim}{z}_\beta')$, $\underset{\sim}{K}_{13}\sigma_e^2 = E(\underset{\sim}{z}_\alpha\, \underset{\sim}{z}_{\alpha\beta}')$, and $\underset{\sim}{K}_{23}\sigma_e^2 = E(\underset{\sim}{z}_\beta\, \underset{\sim}{z}_{\alpha\beta}')$. The matrix $\underset{\sim}{L}$ is of rank rs-1 and is, therefore, nonsingular.

The random vectors $\underset{\sim}{z}_\alpha$, $\underset{\sim}{z}_\beta$, and $\underset{\sim}{z}_{\alpha\beta}$ are not independent. However, they are independent of the error sum of squares,

$$Q = \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij.})^2, \tag{2.10}$$

which can also be written as

$$Q = \underset{\sim}{y}'\underset{\sim}{R}\,\underset{\sim}{y}, \tag{2.11}$$

where $\underset{\sim}{y}$ is the vector of observations and $\underset{\sim}{R}$ is the $n_{..}\times n_{..}$ matrix

$$\underset{\sim}{R} = \underset{\sim}{I}_{n_{..}} - \bigoplus_{i,j} (\underset{\sim}{J}_{n_{ij}}/n_{ij}). \tag{2.12}$$

In (2.12) $\underset{\sim}{J}_{n_{ij}}$ is the matrix of ones of order $n_{ij}\times n_{ij}$ ($i=1,2,\ldots,r$; $j=1,2,\ldots,s$) and the second term is the direct sum of the $\underset{\sim}{J}_{n_{ij}}/n_{ij}$'s. It is easy to verify that $\underset{\sim}{R}$ is idempotent of rank $n_{..} - rs$ and that

$$\underset{\sim}{R}\,\underset{\sim}{X}_1 = \underset{\sim}{R}\,\underset{\sim}{X}_2 = \underset{\sim}{R}\,\underset{\sim}{X}_3 = \underset{\sim}{0}, \tag{2.13}$$

where $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$, and $\underset{\sim}{X}_3$ are the matrices of zeros and ones in (2.2). Hence, Q/σ_e^2 has a chi-squared distribution with $n_{..} - rs$ degrees of freedom (see also Thomsen 1975, p. 260).

Since $\underset{\sim}{R}$ is symmetric it can be written as

$$\underset{\sim}{R} = \underset{\sim}{C}\,\underset{\sim}{\Lambda}\,\underset{\sim}{C}', \tag{2.14}$$

where $\underset{\sim}{C}$ is an orthogonal matrix and $\underset{\sim}{\Lambda}$ is a diagonal matrix of eigenvalues of $\underset{\sim}{R}$, both of order $n_{..} \times n_{..}$. We shall assume that

$$n_{..} > 2rs - 1. \tag{2.15}$$

This is not an unreasonable assumption and can, for example, be satisfied if each cell contains at least two observations. In this case and because $\underset{\sim}{R}$ is idempotent of rank $n_{..} - rs > rs - 1$, $\underset{\sim}{\Lambda}$ and $\underset{\sim}{C}$ can be partitioned as

$$\begin{aligned} \underset{\sim}{\Lambda} &= \operatorname{diag}\left(\underset{\sim}{I}_{\nu_1}, \underset{\sim}{I}_{\nu_2}, \underset{\sim}{0}\right) \\ \underset{\sim}{C} &= \left[\underset{\sim}{C}_1 : \underset{\sim}{C}_2 : \underset{\sim}{C}_3\right], \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} \nu_1 &= rs - 1 \\ \nu_2 &= n_{..} - 2rs + 1, \end{aligned} \tag{2.17}$$

$\underset{\sim}{0}$ is a zero matrix of order $rs \times rs$, and $\underset{\sim}{C}_1$, $\underset{\sim}{C}_2$, $\underset{\sim}{C}_3$ are of orders $n_{..} \times \nu_1$, $n_{..} \times \nu_2$, and $n_{..} \times rs$, respectively. Note that

$$\begin{aligned} \underset{\sim}{C}_i' \underset{\sim}{C}_i &= \underset{\sim}{I}, \; i = 1,2,3, \\ \underset{\sim}{C}_i' \underset{\sim}{C}_j &= \underset{\sim}{0}, \; i \neq j. \end{aligned} \tag{2.18}$$

Formula (2.14) can then be rewritten as

$$\underset{\sim}{R} = \underset{\sim}{C}_1 \underset{\sim}{C}_1' + \underset{\sim}{C}_2 \underset{\sim}{C}_2'. \tag{2.19}$$

From (2.11) and (2.19), the error sum of squares Q can be partitioned as

$$Q = Q_1 + Q_2, \tag{2.20}$$

where

$$Q_1 = \underset{\sim}{Y}' \underset{\sim}{C}_1 \underset{\sim}{C}_1' \underset{\sim}{Y} \tag{2.21}$$

$$Q_2 = \underset{\sim}{y}' \underset{\sim}{C}_2 \underset{\sim}{C}_2' \underset{\sim}{y}. \tag{2.22}$$

The sums of squares Q_1 and Q_2, and the random vector $\underset{\sim}{u}$ in (2.6) are independent. Furthermore, Q_1/σ_e^2 and Q_2/σ_e^2 have the chi-squared distribution with ν_1 and ν_2 degrees of freedom, respectively.

We now define the random vector $\underset{\sim}{\omega}$ as

$$\underset{\sim}{\omega} = \underset{\sim}{u} + (\lambda_{max} \underset{\sim}{I}_{\nu_1} - \underset{\sim}{L})^{\frac{1}{2}} \underset{\sim}{C}_1' \underset{\sim}{y}, \tag{2.23}$$

where $\underset{\sim}{L}$ is the matrix in (2.9) and λ_{max} is its largest eigenvalue. The matrix $\lambda_{max} \underset{\sim}{I}_{\nu_1} - \underset{\sim}{L}$ is positive semidefinite, hence the matrix $(\lambda_{max} \underset{\sim}{I}_{\nu_1} - \underset{\sim}{L})^{\frac{1}{2}}$ is well defined with eigenvalues equal to the square roots of the eigenvalues of $\lambda_{max} \underset{\sim}{I}_{\nu_1} - \underset{\sim}{L}$. Let $\underset{\sim}{\omega}$ be partitioned just like $\underset{\sim}{u}$ in (2.6) as

$$\underset{\sim}{\omega} = (\underset{\sim}{\omega}_\alpha', \underset{\sim}{\omega}_\beta', \underset{\sim}{\omega}_{\alpha\beta}')', \tag{2.24}$$

where the vectors $\underset{\sim}{\omega}_\alpha$, $\underset{\sim}{\omega}_\beta$, and $\underset{\sim}{\omega}_{\alpha\beta}$ are of dimensions r-1, s-1, and (r-1)(s-1), respectively.

<u>Lemma 1</u>

(i) $E\underset{\sim}{\omega}_\alpha = E\underset{\sim}{\omega}_\beta = E\underset{\sim}{\omega}_{\alpha\beta} = \underset{\sim}{0}$.

(ii) $\underset{\sim}{\omega}_\alpha$, $\underset{\sim}{\omega}_\beta$, and $\underset{\sim}{\omega}_{\alpha\beta}$ are independent normally distributed random vectors and have the following variance-covariance matrices:

$$\begin{aligned} \text{Var } \underset{\sim}{\omega}_\alpha &= (s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2 + \lambda_{max}\sigma_e^2)\, \underset{\sim}{I}_{r-1}, \\ \text{Var } \underset{\sim}{\omega}_\beta &= (r\sigma_\beta^2 + \sigma_{\alpha\beta}^2 + \lambda_{max}\sigma_e^2)\, \underset{\sim}{I}_{s-1}, \\ \text{Var } \underset{\sim}{\omega}_{\alpha\beta} &= (\sigma_{\alpha\beta}^2 + \lambda_{max}\sigma_e^2)\, \underset{\sim}{I}_{(r-1)(s-1)}. \end{aligned} \tag{2.25}$$

(iii) $\underset{\sim}{\omega}_\alpha$, $\underset{\sim}{\omega}_\beta$, and $\underset{\sim}{\omega}_{\alpha\beta}$ are independent of Q_2, where Q_2 is the sum of squares in (2.22).

Proof. (i) From (2.12) it can be seen that $\underset{\sim}{R}\,\underset{\sim}{1}_{n_{..}} = \underset{\sim}{0}$. Thus, by (2.19) we can write

$$(\underset{\sim}{C}_1\underset{\sim}{C}_1' + \underset{\sim}{C}_2\underset{\sim}{C}_2')\,\underset{\sim}{1}_{n_{..}} = \underset{\sim}{0}. \tag{2.26}$$

Using (2.18) in (2.26) we get $\underset{\sim}{C}_1'\,\underset{\sim}{1}_{n_{..}} = \underset{\sim}{0}$. It follows that $E(\underset{\sim}{C}_1'\underset{\sim}{y}) = \underset{\sim}{C}_1'\,\underset{\sim}{1}_{n_{..}}\,\mu = \underset{\sim}{0}$. Since $\underset{\sim}{u}$ in (2.6) has also a zero mean, we conclude that the mean of $\underset{\sim}{\omega}$ in (2.23) is zero.

(ii) It is obvious that $\underset{\sim}{\omega}$ in (2.23) is normally distributed. We now claim that $\underset{\sim}{u}$ is independent of $\underset{\sim}{C}_1'\underset{\sim}{y}$. To show this we note that the vector $\bar{\underset{\sim}{y}}$ in (2.5); which can be written as $\bar{\underset{\sim}{y}} = \underset{\sim}{D}\underset{\sim}{y}$, where $\underset{\sim}{D} = \underset{i,j}{\oplus}\, \underset{\sim}{1}'_{n_{ij}}/n_{ij}$; *is independent of* Q *and hence of* Q_1 (see 2.20 and 2.21). Consequently,

$$\underset{\sim}{D}\,\underset{\sim}{\Sigma}\,\underset{\sim}{C}_1\underset{\sim}{C}_1' = \underset{\sim}{0}, \tag{2.27}$$

where $\underset{\sim}{\Sigma}$ is the variance-covariance matrix of $\underset{\sim}{y}$ given in (2.3) (see Searle 1971, p. 59). From (2.18) and (2.27) we conclude that

$$\underset{\sim}{D}\,\underset{\sim}{\Sigma}\,\underset{\sim}{C}_1 = \underset{\sim}{0}. \tag{2.28}$$

Hence, $\mathrm{Cov}(\bar{\underset{\sim}{y}}, \underset{\sim}{y}'\underset{\sim}{C}_1) = \underset{\sim}{D}\,\underset{\sim}{\Sigma}\,\underset{\sim}{C}_1 = \underset{\sim}{0}$. Since $\underset{\sim}{u}$ is a subvector of $\underset{\sim}{z} = \underset{\sim}{P}\,\bar{\underset{\sim}{y}}$, then $\underset{\sim}{u}$ is independent of $\underset{\sim}{C}_1'\underset{\sim}{y}$ as was claimed.

The variance-covariance matrix of $\underset{\sim}{\omega}$ in (2.23) can then be written as

$$\mathrm{Var}\,\underset{\sim}{\omega} = \mathrm{Var}\,\underset{\sim}{u} + (\lambda_{max}\underset{\sim}{I}_{\nu_1} - \underset{\sim}{L})^{\frac{1}{2}}\,\underset{\sim}{C}_1'\underset{\sim}{\Sigma}\underset{\sim}{C}_1(\lambda_{max}\underset{\sim}{I}_{\nu_1} - \underset{\sim}{L})^{\frac{1}{2}}. \tag{2.29}$$

But from (2.13) and (2.19) we have

$$(\underset{\sim}{C}_1\underset{\sim}{C}_1' + \underset{\sim}{C}_2\underset{\sim}{C}_2')\ \underset{\sim}{X}_i = \underset{\sim}{0}, \quad i = 1,2,3. \tag{2.30}$$

Using (2.18), equalities (2.30) yield

$$\underset{\sim}{C}_1'\ \underset{\sim}{X}_i = \underset{\sim}{0}, \quad i = 1,2,3. \tag{2.31}$$

It follows that

$$\underset{\sim}{C}_1'\ \underset{\sim}{\Sigma}\ \underset{\sim}{C}_1 = \underset{\sim}{C}_1'\ \underset{\sim}{C}_1\ \sigma_e^2 = \underset{\sim}{I}_{\nu_1}\sigma_e^2. \tag{2.32}$$

From (2.7), (2.29), and (2.32) we then obtain

$$\text{Var}\ \underset{\sim}{\omega} = \text{diag}\left(\delta_1\underset{\sim}{I}_{r-1}, \delta_2\underset{\sim}{I}_{s-1}, \delta_3\underset{\sim}{I}_{(r-1)(s-1)}\right) + \underset{\sim}{L}\sigma_e^2 + (\lambda_{max}\underset{\sim}{I}_{\nu_1} - \underset{\sim}{L})\sigma_e^2,$$

that is,

$$\text{Var}\ \underset{\sim}{\omega} = \text{diag}\left((\delta_1 + \lambda_{max}\sigma_e^2)\underset{\sim}{I}_{r-1}, (\delta_2 + \lambda_{max}\sigma_e^2)\underset{\sim}{I}_{s-1}, (\delta_3 + \lambda_{max}\sigma_e^2)\underset{\sim}{I}_{(r-1)(s-1)}\right). \tag{2.33}$$

From (2.33) we conclude that $\underset{\sim}{\omega}_\alpha$, $\underset{\sim}{\omega}_\beta$, and $\underset{\sim}{\omega}_{\alpha\beta}$ are independent and have the variance-covariance matrices described in (2.25).

(iii) Q_2 is independent of $\underset{\sim}{u}$ (since Q is) and is also independent of $\underset{\sim}{C}_1'\underset{\sim}{y}$ since $\underset{\sim}{C}_1'\ \underset{\sim}{\Sigma}\ \underset{\sim}{C}_2 = \underset{\sim}{0}$, which follows from (2.18) and (2.31) after noting the formula for $\underset{\sim}{\Sigma}$ in (2.3). Thus, Q_2 is independent of $\underset{\sim}{\omega}$.

From lemma1 we conclude that the sums of squares, $S_\alpha = \underset{\sim}{\omega}_\alpha'\underset{\sim}{\omega}_\alpha$, $S_\beta = \underset{\sim}{\omega}_\beta'\underset{\sim}{\omega}_\beta$, $S_{\alpha\beta} = \underset{\sim}{\omega}_{\alpha\beta}'\underset{\sim}{\omega}_{\alpha\beta}$, and Q_2 are distributed independently, and

$$S_\alpha/(s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2 + \lambda_{max}\sigma_e^2) \sim \chi_{r-1}^2$$

$$S_\beta/(r\sigma_\beta^2 + \sigma_{\alpha\beta}^2 + \lambda_{max}\sigma_e^2) \sim \chi_{s-1}^2$$

$$S_{\alpha\beta}/(\sigma^2_{\alpha\beta} + \lambda_{max}\sigma^2_e) \sim \chi^2_{(r-1)(s-1)}$$

$$Q_2/\sigma^2_e \sim \chi^2_{\nu_2},$$

where ν_2 is given in (2.17). A test statistic for testing the hypothesis $H_0:\sigma^2_\alpha = 0$ vs. $H_a:\sigma^2_\alpha \neq 0$ is, therefore, $F = MS_\alpha/MS_{\alpha\beta}$, where $MS_\alpha = S_\alpha/(r-1)$ and $MS_{\alpha\beta} = S_{\alpha\beta}/ (r-1)(s-1)$. Under H_0 this statistic has the F distribution with r-1 and (r-1)(s-1) degrees of freedom. The hypothesis H_0 can be rejected at the α-level of significance if $F \geq F_{\alpha,r-1,(r-1)(s-1)}$, the upper α100% point of the corresponding F distribution. Similarly, to test the hypothesis $H_0:\sigma^2_\beta = 0$ vs. $H_a:\sigma^2_\beta \neq 0$, we use the statistic $F=MS_\beta/MS_{\alpha\beta}$, where $MS_\beta = S_\beta/(s-1)$. Furthermore, the statistic $F=(\nu_2/\lambda_{max})(MS_{\alpha\beta}/Q_2)$ can be used to test the hypothesis $H_0:\sigma^2_{\alpha\beta} = 0$ vs. $H_a:\sigma^2_{\alpha\beta} \neq 0$. We do not, however, recommend using this test since it has fewer denominator degrees of freedom than the exact test for $\sigma^2_{\alpha\beta}$ given by Thomsen (1975).

We note that if the data set is balanced, then $\underset{\sim}{K} = \mathrm{diag}(n^{-1}_{11},\ldots,n^{-1}_{rs}) = \underset{\sim}{I}_{rs}/n$, where n is the number of observations per cell. Hence, $\underset{\sim}{PKP}' = \underset{\sim}{I}_{rs}/n$ and $\underset{\sim}{L} = \underset{\sim}{I}_{\nu_1}/n$, where ν_1 is given in (2.17). Consequently, $\lambda_{max} = 1/n$ and the vectors $\underset{\sim}{\omega}$ and $\underset{\sim}{u}$ in (2.23) become identical. Furthermore, the sums of squares nS_α, nS_β, and $nS_{\alpha\beta}$ reduce to the balanced ANOVA sums of squares associated with the main and interaction effects, respectively.

The following lemma is useful for the power study in Section 5 and is proved in the Appendix:

Lemma2

The largest eigenvalue, λ_{max}, of the matrix $\underset{\sim}{L}$ in (2.9) satisfies the double inequality

$$\frac{1}{rs} \sum_{i,j} \frac{1}{n_{ij}} < \lambda_{max} < \frac{1}{n^{(1)}} , \qquad (2.34)$$

where $n^{(1)}$ is the smallest cell frequency.

We note that the lower bound in (2.34) is the reciprocal of the harmonic mean of the cell frequencies.

3. A Numerical Example

Layton (1985) studied variation in fusiform rust in Southern pine tree plantations. Trees with female parents from different families were evaluated in several test locations. We extract data from five families and four test locations, and disregard the male parents (which were from a different set of families) for purpose of illustration. The number of plots in each family × test combination ranged from one to four. Proportions of symptomatic trees in each plot are recorded in the following table:

		Family number				
		288	352	19	141	60
		.804	.734	.967	.917	.850
	34	.967	.817	.930		
		.970	.833	.889		
			.304			
		.867	.407	.896	.952	.486
	35	.667	.511	.717		.467
		.793	.274			
		.458	.428			
Test number						
		.409	.411	.919	.408	.275
	40	.569	.646	.669	.435	.256
		.715	.310	.669	.500	
		.487		.450		
		.587	.304	.928	.367	.525
	41	.538	.428	.855		
		.961		.655		
		.300		.800		

We analyze variation due to family and test according to the model in (2.1), where y_{ijk} = arcsin-square root of the k^{th} observed proportion in family i and test j. The exact test will be used to test, for example, the null hypothesis H_o: $\sigma_\alpha^2 = 0$ regarding the family variance component.

The first step is to obtain the matrix $\underset{\sim}{P}$. This can be done using the algorithm given by Graybill (1983, p. 406). Alternatively, $\underset{\sim}{P}$ may be constructed with rows 2 through 20 given by sets of normalized orthogonal contrasts for α effect, β effect and $\alpha\beta$ effect for balanced data. Computation of other components of $\underset{\sim}{\omega}$ in (2.23) are straightforward. For the present example, we obtain $MS_\alpha/MS_{\alpha\beta} = 0.1543/0.0415 = 3.718$, which has an observed significance level of 0.034.

4. Simultaneous Confidence Intervals on the Variance Components

One of the interesting consequences of Lemma1 is that simultaneous confidence intervals on all continuous functions of σ^2_α, σ^2_β, $\sigma^2_{\alpha\beta}$, and σ^2_e can be as easily obtained as in a balanced data situation. To see this, let us denote the expected values of MS_α, MS_β, $MS_{\alpha\beta}$, and Q_2/ν_2 by τ_α, τ_β, $\tau_{\alpha\beta}$, and τ_e, respectively. Then

$$\tau_\alpha = s\sigma^2_\alpha + \sigma^2_{\alpha\beta} + \lambda_{max}\,\sigma^2_e, \tag{4.1}$$

$$\tau_\beta = r\sigma^2_\beta + \sigma^2_{\alpha\beta} + \lambda_{max}\,\sigma^2_e, \tag{4.2}$$

$$\tau_{\alpha\beta} = \sigma^2_{\alpha\beta} + \lambda_{max}\,\sigma^2_e, \tag{4.3}$$

$$\tau_e = \sigma^2_e. \tag{4.4}$$

Individual $(1-\alpha)100\%$ confidence intervals on τ_α, τ_β, $\tau_{\alpha\beta}$, and τ_e are, respectively

$$C_\alpha = \{\tau_\alpha : S_\alpha/\chi^2_{\alpha/2,r-1} < \tau_\alpha < S_\alpha/\chi^2_{1-\alpha/2,r-1}\}$$

$$C_\beta = \{\tau_\beta : S_\beta/\chi^2_{\alpha/2,s-1} < \tau_\beta < S_\beta/\chi^2_{1-\alpha/2,s-1}\}$$

$$C_{\alpha\beta} = \{\tau_{\alpha\beta} : S_{\alpha\beta}/\chi^2_{\alpha/2,(r-1)(s-1)} < \tau_{\alpha\beta} < S_{\alpha\beta}/\chi^2_{1-\alpha/2,(r-1)(s-1)}\}$$

$C_e = \{\tau_e : Q_2/\chi^2_{\alpha/2,\nu_2} < \tau_e < Q_2/\chi^2_{1-\alpha/2,\nu_2}\}$, where $\chi^2_{\alpha,m}$ denotes the upper $\alpha 100\%$ point of the chi-squared distribution with m degrees of freedom. Since S_α, S_β, $S_{\alpha\beta}$, and Q_2 are independently distributed, the Cartesian product, $C = C_\alpha \times C_\beta \times C_{\alpha\beta} \times C_e$, represents an exact rectangular confidence region on $\underset{\sim}{\tau} = (\tau_\alpha, \tau_\beta, \tau_{\alpha\beta}, \tau_e)'$ with a confidence coefficient equal to $1-\alpha^* = (1-\alpha)^4$.

Let us now suppose that $\gamma = f(\sigma^2_\alpha, \sigma^2_\beta, \sigma^2_{\alpha\beta}, \sigma^2_e)$ is a continious function of the variance components. This function can be expressed as $\gamma = g(\tau_\alpha, \tau_\beta, \tau_{\alpha\beta}, \tau_e)$, where g is obtained from f by substituting the variance components by τ_α, τ_β, $\tau_{\alpha\beta}$, τ_e using equations (4.1) - (4.4). By the method described in Khuri (1981) for balanced data, the interval

$$B_g = \{\gamma : \min_{\underset{\sim}{\tau} \in C} g(\underset{\sim}{\tau}) \leq \gamma \leq \max_{\underset{\sim}{\tau} \in C} g(\underset{\sim}{\tau})\}$$

is a confidence interval on γ with a confidence coefficient greater than or equal to $1 - \alpha^*$. Furthermore, if g belongs to a family G of continuous functions of τ_α, τ_β, $\tau_{\alpha\beta}$, τ_e, then

$$P[\underset{\sim}{\gamma} \in B_g, \forall g \in G] \geq 1 - \alpha^*.$$

Thus, for $g \in G$, the intervals B_g are conservative simultaneous confidence intervals on the values of all continuous functions of the variance components for model (2.1).

5. The Power of the Exact Tests

Power values for each of the exact tests described in Section 2 can be easily computed using the F distribution. We shall only consider the power of the test concerning σ^2_α. A similar power study can be made regarding the test for σ^2_β.

Let Ψ denote the power of the test for σ^2_α. Then

$$\Psi = P\left[\frac{MS_\alpha}{MS_{\alpha\beta}} > F_{\alpha,\ r-1,\ (r-1)(s-1)} \mid H_a\right], \tag{5.1}$$

where $H_a : \sigma_\alpha^2 \neq 0$. Under H_a,

$$\frac{MS_\alpha}{s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2 + \lambda_{max}\ \sigma_e^2} \frac{\sigma_{\alpha\beta}^2 + \lambda_{max}\ \sigma_e^2}{MS_{\alpha\beta}} \sim F_{r-1,\ (r-1)(s-1)}.$$

Hence, (5.1) can be rewritten as

$$\Psi = P\left[F_{r-1,(r-1)(s-1)} > \frac{1}{1 + s\theta} F_{\alpha,r-1,(r-1)(s-1)}\right], \tag{5.2}$$

where

$$\theta = \sigma_\alpha^2/(\sigma_{\alpha\beta}^2 + \lambda_{max}\ \sigma_e^2).$$

From (5.2) it can be seen that Ψ is a function of the level of significance, α, λ_{max}, which depends on the design used, and the variance ratios $\sigma_\alpha^2/\sigma_e^2$, $\sigma_{\alpha\beta}^2/\sigma_e^2$ through θ. The latter variance ratio is considered a nuisance parameter. Since Ψ is a monotone increasing function of θ, it follows that Ψ is

i) a monotone increasing function of $\sigma_\alpha^2/\sigma_e^2$ for a fixed value of $\sigma_{\alpha\beta}^2/\sigma_e^2$ and a fixed design.

ii) a monotone decreasing function of the nuisance parameter $\sigma_{\alpha\beta}^2/\sigma_e^2$ for a fixed value of $\sigma_\alpha^2/\sigma_e^2$ and a fixed design.

iii)a monotone decreasing function of λ_{max} for fixed ratios of the variance components. Consequently, if $n_{..}$, the total of the cell frequencies, is fixed, then by Lemma2 higher power values are expected for smaller values of d, where d is

$$d = \frac{1}{n^{(1)}} - \frac{1}{rs} \sum_{i,j} \frac{1}{n_{ij}} .$$

For a balaned data set, $d = 0$ and maximum power is achieved. We can, therefore, regard the quantity $n^{(1)}d$, which belongs to the interval $[0,1)$, as a measure of imbalance. Small values of this measure are associated with designs that are nearly balanced. For a more general discussion concerning measures of imbalance for unbalanced models, the interested reader is referred to Khuri (1986).

6. A Power Comparison With Other Approximate Tests

In this section, we compare the power of the exact test statistic, $MS_{\alpha}/MS_{\alpha\beta}$, given by formula (5.1) with powers of tests that are most commonly used in practice, namely, the ANOVA-based approximate F tests.

There are several analyses of variance, each using a different method of computing sums of squares. Two of these methods, expressed in "reduction in SS" notation, are:

Source of Variation	Degrees of Freedom	Type I SS	Type II SS
A	r-1	$R(\alpha \mid \mu)$	$R(\alpha \mid \mu,\beta)$
B	s-1	$R(\beta \mid \mu,\alpha)$	$R(\beta \mid \mu,\alpha)$
A*B	(r-1)(s-1)	$R(\alpha\beta \mid \mu,\alpha,\beta)$	$R(\alpha\beta \mid \mu,\alpha,\beta)$
Residual	n.. - rs	Q	Q

See Searle (1971, Section 6.3) for a description of the "reduction" notation. The terminology, "Type I" and "Type II", is consistent with that of the SAS (1982) System of statistical software. An approximate F test statistic for H_0: $\sigma_\alpha^2 = 0$ based on the Type i SS (i = I, i = II) has the form $F(i) = MS_\alpha(i)/MS_{\alpha\beta}^*(i)$. The numerator, $MS_\alpha(i)$, is the Type i mean square for α. The denominator, $MS_{\alpha\beta}^*(i)$, is

$$MS_{\alpha\beta}^*(i) = \hat{\sigma}_e^2 + k_1(i)\ \hat{\sigma}_{\alpha\beta}^2 + k_2(i)\ \hat{\sigma}_\beta^2,$$

where $k_1(i)$ and $k_2(i)$ are the coefficients in the expected mean square

$$E[MS_\alpha(i)] = \sigma_e^2 + k_1(i)\ \sigma_{\alpha\beta}^2 + k_2(i)\ \sigma_\beta^2 + k_3(i)\ \sigma_\alpha^2$$

and $\hat{\sigma}_e^2$, $\hat{\sigma}_{\alpha\beta}^2$, and $\hat{\sigma}_\beta^2$ are, respectively, the analysis of variance estimators of σ_e^2, $\sigma_{\alpha\beta}^2$, and σ_β^2, based upon Q, $R(\alpha\beta|\mu,\alpha,\beta)$, and $R(\beta|\mu,\alpha)$. (Note: $k_2(II) = 0$).

Powers of the approximate test statistics F(I) and F(II) were estimated via computer simulation. The simulation study required two steps; the first to estimate critical values of F(I) and F(II) under H_0: $\sigma_\alpha^2 = 0$; the second to estimate the power for $\sigma_\alpha^2 > 0$. All simulations were conducted using PROC MATRIX of the SAS (1982) System. The SAS functions RANNOR and RANGAM were used to generate pseudo-random normal and chi-squared variates, respectively. Powers were estimated for 25 combinations of values of the variance components and six n_{ij} patterns, making 150 cases in all. Without loss of generality, $\sigma_e = 1.0$ was used in all combinations. Values of $\sigma_{\alpha\beta}$ and σ_β constituted a "response surface design" containing a 2×2 factorial and an interior point, namely, (0.2,0.2), (0.2,5.0), (5.0,0.2), (5.0,5.0) and (1.0,1.0). For each of these five combinations, five values of σ_α = 0.2, 0.5, 1.0, 2.0 and 5.0 were considered to produce the 25 combinations of $\sigma_{\alpha\beta}$, σ_β, and σ_α.

The six n_{ij} patterns contained three "near balance" patterns (NB) and three "highly unbalanced" patterns (HU), each containing 5×5, 5×10 and 10×5 arrays. The six patterns, with rows representing levels of factor A and columns representing levels of factor B, are:

	NB (near balance)					HU (highly unbalanced)				
	5	5	5	5	6	9	2	9	1	2
r=5	4	4	6	4	5	10	1	2	9	10
	6	4	4	4	4	1	8	1	2	2
s=5	4	6	5	5	6	9	10	1	9	3
	6	5	4	5	6	8	3	2	10	1

	5	5	5	5	6	9	2	9	1	2
	4	4	6	4	5	10	1	2	9	10
	6	4	4	4	4	1	8	1	2	2
r=10	4	6	5	5	6	9	10	1	9	3
	6	5	4	5	6	8	3	2	10	1
s=5	5	5	5	5	6	9	2	9	1	2
	4	4	6	4	5	10	1	2	9	10
	6	4	4	4	4	1	8	1	2	2
	4	6	5	5	6	9	10	1	9	3
	6	5	4	5	6	8	3	2	10	1

	5	5	5	5	6	5	5	5	5	6	9	2	9	1	2	9	2	9	1	2
r=5	4	4	6	4	5	4	4	6	4	5	10	1	2	9	10	10	1	2	9	10
	6	4	4	6	4	6	4	4	6	4	1	8	1	2	2	1	8	1	2	2
s=10	4	6	5	5	6	4	6	5	5	6	9	10	1	9	3	9	10	1	9	3
	6	5	4	5	6	6	5	4	5	6	8	3	2	10	1	8	3	2	10	1

The distributions of F(I) and F(II) depend on the true values of $\sigma_{\alpha\beta}$ and σ_β, even under the null hypothesis H_0: $\sigma_\alpha^2 = 0$. Therefore, it was necessary to estimate the critical values of F(I) and F(II) for all values of $\sigma_{\alpha\beta}$ and σ_β, and all n_{ij} patterns involved in the power study. This was done as follows: For each of the five combinations of $\sigma_{\alpha\beta}$ and σ_β and each of the six n_{ij} patterns (thirty cases in all), 1000 sets of cell means $\bar{y}_{ij.}$ and Q values were generated according to the model $\bar{y}_{ij.} = \mu + \beta_j + (\alpha\beta)_{ij} + \bar{e}_{ij.}$, where β_j, $(\alpha\beta)_{ij}$, and $\bar{e}_{ij.}$ are independently distributed as normal variates with zero means and variances σ_β^2, $\sigma_{\alpha\beta}^2$, and σ_e^2/n_{ij}, respectively, Q/σ_e^2 has the chi-squared distribution with $n_{..} - rs$ degrees of freedom, and, without loss of generality, $\mu = 0$. (Note the absence of α_i in the model, corresponding to $\sigma_\alpha^2 = 0$). For each set of $\bar{y}_{ij.}$ and Q values, F(I) and F(II) were calculated, and the 95% sample quantiles of F(I) and F(II) were recorded from the 1000 sets. This process was repeated ten times, and the mean and standard deviation of the ten F(I) and F(II) quantiles were computed to estimate the upper 5% critical values for F(I) and F(II). These are reported in Tables 6.1 and 6.2.

The estimated critical values in Tables 6.1 and 6.2 demonstrate the degree of dependence of the null distributions of F(I) and F(II) upon the nuisance parameters, $\sigma_{\alpha\beta}$ and σ_β. The most serious disturbance of the distributions is for small values of $\sigma_{\alpha\beta}$ (=0.2), especially in "highly unbalanced" cases (Table 6.2).

In practice, the true critical values of F(I) and F(II) would not be available because $\sigma_{\alpha\beta}$ and σ_β are not known. Instead, the calulated values of F(I) and F(II) would typically be referred to an F distribution with denominator degrees of freedom given by a Satterthwaite-type approximation such as illustrated by Milliken and Johnson (1984, Section 20.1.2). Actual Rejection probabilities (type 1 error rates) corresponding to a nominal α = .05 for F(I) and F(II) using these approximate degrees of freedom were estimated in the simulation study. These are also reported in Tables 6.1 and 6.2. The results show that the Satterthwaite-type approximate procedures produce true type 1 error rates that are far less than the nominal .05 for some cases, particularly those with small values of $\sigma_{\alpha\beta}$ in the highly unbalanced situation.

Estimation of power for the statistics F(I) and F(II) followed a process similar to that used to estimate the critical values. For each of the 25 selected combinations of $\sigma_{\alpha\beta}$, σ_β and σ_α and each of the six n_{ij} patterns, 2000 sets of cell means $\bar{y}_{ij.}$ and Q values were generated according to model (2.4) with μ taken equal to zero. The statistics F(I) and F(II) were calculated for each set of $\bar{y}_{ij.}$ and Q values. The proportion of times, out of the 2000, that F(I) and F(II) exceeded the estimated critical values in Tables 6.1 and 6.2 was computed. These proportions are estimates of the powers of F(I) and F(II) for testing H_0: $\sigma_\alpha^2 = 0$, and are recorded in Tables 6.3 and 6.4. Powers of the exact test statistic $MS_\alpha/MS_{\alpha\beta}$ are also reported in the latter tables. These results show that, with a few exceptions, the power of the exact test is better or essentially as good as the power of either of the approximate procedures. The exceptions are for small values of σ_α (0.2 and 0.5) and small values of $\sigma_{\alpha\beta}$ (0.2).

It must be remembered that the approximate tests whose powers are shown in Tables 6.3 and 6.4 could not be computed in practice because their critical values depend on the unknown $\sigma_{\alpha\beta}$ and σ_β. The dependence is most severe for small values of $\sigma_{\alpha\beta}$. These are the same values of $\sigma_{\alpha\beta}$ for which the power of the exact test was inferior to the approximate tests. Therefore, the power of the exact test appears to be generally as good or better than powers of the approximate tests except in cases for which valid critical values of the approximate tests are most unreliable.

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APPENDIX

Proof of Lemma2

Consider the orthogonal matrix $\underset{\sim}{P}$ whose first row is $(rs)^{-\frac{1}{2}}\underset{\sim}{1}'_{rs}$ and simultaneously diagonalizes $\underset{\sim}{A}_1 = \underset{\sim}{B}_1\underset{\sim}{B}_1'$, $\underset{\sim}{A}_2 = \underset{\sim}{B}_2\underset{\sim}{B}_2'$, where $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ are the matrices in (2.5). Let $\underset{\sim}{P}_1$ be the submatrix of $\underset{\sim}{P}$ obtained by deleting the first row. It is easy to verify that

$$\underset{\sim}{P}_1\underset{\sim}{P}_1' = I_{rs-1}, \tag{A.1}$$

$$\underset{\sim}{P}_1'\underset{\sim}{P}_1 + \frac{1}{rs}\underset{\sim}{J}_{rs} = \underset{\sim}{I}_{rs}. \tag{A.2}$$

Now, λ_{max} is the largest eigenvalue of $\underset{\sim}{L} = \underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1'$, that is, $\lambda_{max} = e_{max}(\underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1')$. If $n^{(1)}$ is the smallest cell frequency, then the matrix $(1/n^{(1)})\underset{\sim}{P}_1\underset{\sim}{P}_1' - \underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1'$ is positive semidefinite. Using (A.1) we get

$$e_{max}(\underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1') < \frac{1}{n_{(1)}} e_{max}(\underset{\sim}{P}_1\underset{\sim}{P}_1') = \frac{1}{n_{(1)}}. \qquad (A.3)$$

It is also true that λ_{max} is greater than or equal to the sum of the eigenvalues of $\underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1'$ divided by rs-1 (this is the average eigenvalue of $\underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1'$). Thus,

$$e_{max}(\underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1') > \frac{1}{(rs-1)} tr(\underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1'). \qquad (A.4)$$

But,

$$\begin{aligned} tr(\underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1') &= tr(\underset{\sim}{P}_1'\underset{\sim}{P}_1\underset{\sim}{K}) \\ &= tr\left[(\underset{\sim}{I}_{rs} - \frac{1}{rs}\underset{\sim}{J}_{rs})\underset{\sim}{K}\right] \text{ (using A.2)} \\ &= tr(\underset{\sim}{K}) - \frac{1}{rs} tr(\underset{\sim}{1}_{rs}\underset{\sim}{1}_{rs}'\underset{\sim}{K}) \\ &= tr(\underset{\sim}{K}) - \frac{1}{rs}\underset{\sim}{1}_{rs}'\underset{\sim}{K}\underset{\sim}{1}_{rs} \\ &= \sum_{i,j}\frac{1}{n_{ij}} - \frac{1}{rs}\sum_{i,j}\frac{1}{n_{ij}}. \qquad (A.5) \end{aligned}$$

From (A.4) and (A.5) we conclude that

$$e_{max}(\underset{\sim}{P}_1\underset{\sim}{K}\underset{\sim}{P}_1') > \frac{1}{rs}\sum_{i,j}\frac{1}{n_{ij}}. \qquad (A.6)$$

The proof of Lemma2 follows from inequalities (A.3) and (A.6).

Table 6.1

Estimated 95% quantiles of F(I) and F(II), and estimated type 1 error rates for F(I) and F(II) using Satterthwaite's approximate degrees of freedom for a nominal $\alpha = .05$ for "near balance" cases

Design	$\sigma_{\alpha\beta}$	σ_{β}	95% Quantiles		Type 1 Error Rates	
			F(I)	F(II)	F(I)	F(II)
	0.2	0.2	3.01	3.01	0.048	0.047
	0.2	5.0	2.32	3.07	0.005	0.051
5×5	1.0	1.0	2.90	2.92	0.046	0.045
	5.0	0.2	2.98	2.97	0.048	0.048
	5.0	5.0	2.94	2.94	0.049	0.047
	0.2	0.2	2.18	2.18	0.053	0.052
	0.2	5.0	2.04	2.16	0.004	0.089
10×5	1.0	1.0	2.17	2.20	0.054	0.055
	5.0	0.2	2.16	2.15	0.051	0.052
	5.0	5.0	2.15	2.17	0.051	0.051
	0.2	0.2	2.67	2.68	0.052	0.053
	0.2	5.0	2.52	2.66	0.029	0.051
5×10	1.0	1.0	2.59	2.61	0.048	0.049
	5.0	0.2	2.63	2.63	0.050	0.050
	5.0	5.0	2.58	2.63	0.049	0.049

Table 6.2
Estimated 95% quantiles of F(I) and F(II), and estimated type 1 error rates for F(I) and F(II) using Satterthwaite's approximate degrees of freedom for a nominal $\alpha = .05$, for "highly unbalanced" cases

Design	$\sigma_{\alpha\beta}$	σ_{β}	95% Quantiles		Type 1 Error Rates	
			F(I)	F(II)	F(I)	F(II)
	0.2	0.2	4.45	4.76	0.003	0.002
	0.2	5.0	2.11	4.64	0.000	0.002
5×5	1.0	1.0	2.81	3.26	0.037	0.049
	5.0	0.2	3.00	3.00	0.054	0.049
	5.0	5.0	2.76	3.08	0.037	0.053
	0.2	0.2	2.61	2.67	0.019	0.018
	0.2	5.0	2.03	2.70	0.000	0.017
10×5	1.0	1.0	2.12	2.23	0.035	0.050
	5.0	0.2	2.20	2.21	0.057	0.056
	5.0	5.0	2.10	2.16	0.036	0.052
	0.2	0.2	2.97	3.04	0.035	0.036
	0.2	5.0	2.58	3.15	0.009	0.041
5×10	1.0	1.0	2.57	2.70	0.046	0.051
	5.0	0.2	2.67	2.70	0.055	0.055
	5.0	5.0	2.54	2.65	0.046	0.052

Table 6.3
Estimated powers of F(I) and F(II) and
exact powers (E) of $MS_{\alpha}/MS_{\alpha\beta}$ for "near balance" cases.

Design	$\sigma_{\alpha\beta}$	σ_{β}		σ_{α} = 0.2	0.5	1.0	2.0	5.0
			I	0.200	0.740	0.965	0.997	1.000
5×5	0.2	0.2	II	0.200	0.739	0.964	0.997	1.000
			E	0.196	0.720	0.960	0.996	0.999
			I	0.124	0.479	0.841	0.980	0.999
5×5	0.2	5.0	II	0.209	0.737	0.959	0.997	0.999
			E	0.196	0.720	0.960	0.996	0.999
			I	0.082	0.266	0.688	0.952	0.998
5×5	1.0	1.0	II	0.090	0.269	0.696	0.953	0.997
			E	0.076	0.252	0.673	0.948	0.998
			I	0.053	0.060	0.084	0.211	0.733
5×5	5.0	0.2	II	0.053	0.062	0.085	0.215	0.736
			E	0.051	0.057	0.082	0.203	0.732
			I	0.063	0.059	0.085	0.196	0.727
5×5	5.0	5.0	II	0.066	0.063	0.097	0.207	0.732
			E	0.051	0.057	0.082	0.203	0.732
			I	0.329	0.962	1.000	1.000	1.000
10×5	0.2	0.2	II	0.331	0.962	1.000	1.000	1.000
			E	0.317	0.943	0.999	1.000	1.000
			I	0.154	0.649	0.999	0.999	1.000
10×5	0.2	5.0	II	0.351	0.952	0.998	1.000	1.000
			E	0.317	0.943	0.999	1.000	1.000
			I	0.088	0.415	0.909	0.997	1.000
10×5	1.0	1.0	II	0.080	0.405	0.910	0.997	1.000
			E	0.091	0.410	0.914	0.998	1.000
			I	0.049	0.053	0.110	0.312	0.946
10×5	5.0	0.2	II	0.050	0.058	0.110	0.317	0.947
			E	0.051	0.060	0.101	0.322	0.945
			I	0.050	0.058	0.104	0.317	0.924
10×5	5.0	5.0	II	0.051	0.062	0.106	0.323	0.927
			E	0.051	0.060	0.101	0.322	0.945

Table 6.3 (continued)

Design	$\sigma_{\alpha\beta}$	σ_β		σ_α				
				0.2	0.5	1.0	2.0	5.0
			I	0.416	0.920	0.991	0.999	1.000
5×10	0.2	0.2	II	0.410	0.920	0.991	0.999	1.000
			E	0.398	0.907	0.991	0.999	1.000
			I	0.182	0.630	0.930	0.995	0.999
5×10	0.2	5.0	II	0.428	0.923	0.992	1.000	1.000
			E	0.398	0.907	0.991	0.999	1.000
			I	0.119	0.485	0.887	0.988	0.998
5×10	1.0	1.0	II	0.117	0.492	0.895	0.988	0.998
			E	0.117	0.493	0.884	0.988	0.999
			I	0.057	0.062	0.125	0.392	0.912
5×10	5.0	0.2	II	0.055	0.061	0.128	0.392	0.912
			E	0.052	0.067	0.133	0.409	0.912
			I	0.057	0.074	0.136	0.415	0.919
5×10	5.0	5.0	II	0.055	0.075	0.136	0.418	0.924
			E	0.052	0.067	0.133	0.409	0.912

Table 6.4

Estimated powers of F(I) and F(II) and exact powers (E) of $MS_\alpha/MS_{\alpha\beta}$ for "highly unbalanced" cases.

Design	$\sigma_{\alpha\beta}$	σ_β		σ_α				
				0.2	0.5	1.0	2.0	5.0
			I	0.142	0.529	0.895	0.980	0.990
5×5	0.2	0.2	II	0.126	0.498	0.873	0.976	0.987
			E	0.101	0.413	0.837	0.981	0.999
			I	0.063	0.127	0.296	0.602	0.927
5×5	0.2	5.0	II	0.132	0.518	0.881	0.975	0.987
			E	0.101	0.413	0.837	0.981	0.999
			I	0.067	0.179	0.511	0.685	0.995
5×5	1.0	1.0	II	0.059	0.178	0.520	0.901	0.998
			E	0.069	0.198	0.578	0.919	0.996
			I	0.040	0.054	0.070	0.154	0.606
5×5	5.0	0.2	II	0.036	0.062	0.077	0.154	0.609
			E	0.051	0.057	0.082	0.201	0.727
			I	0.059	0.055	0.065	0.136	0.551
5×5	5.0	5.0	II	0.051	0.044	0.065	0.150	0.609
			E	0.051	0.057	0.082	0.201	0.727
			I	0.218	0.853	0.997	1.000	1.000
10×5	0.2	0.2	II	0.231	0.854	0.996	1.000	1.000
			E	0.136	0.667	0.983	0.999	1.000
			I	0.063	0.144	0.363	0.713	0.989
10×5	0.2	5.0	II	0.227	0.861	0.998	1.000	1.000
			E	0.136	0.667	0.983	0.999	1.000
			I	0.078	0.261	0.736	0.991	1.000
10×5	1.0	1.0	II	0.081	0.294	0.794	0.996	1.000
			E	0.079	0.314	0.844	0.996	1.000
			I	0.052	0.056	0.088	0.216	0.854
10×5	5.0	0.2	II	0.051	0.054	0.085	0.214	0.852
			E	0.051	0.060	0.100	0.317	0.943
			I	0.046	0.081	0.080	0.182	0.792
10×5	5.0	5.0	II	0.052	0.070	0.091	0.220	0.852
			E	0.051	0.060	0.100	0.317	0.943

Table 6.4 (continued)

Design	$\sigma_{\alpha\beta}$	σ_β		σ_α 0.2	0.5	1.0	2.0	5.0
			I	0.330	0.865	0.983	0.999	1.000
5×10	0.2	0.2	II	0.311	0.860	0.985	0.999	1.000
			E	0.190	0.711	0.958	0.996	0.999
			I	0.061	0.107	0.292	0.668	0.966
5×10	0.2	5.0	II	0.289	0.856	0.984	0.999	1.000
			E	0.190	0.711	0.958	0.996	0.999
			I	0.087	0.311	0.754	0.967	0.999
5×10	1.0	1.0	II	0.095	0.373	0.801	0.975	0.999
			E	0.099	0.406	0.833	0.981	0.999
			I	0.050	0.053	0.095	0.303	0.814
5×10	5.0	0.2	II	0.057	0.059	0.099	0.309	0.815
			E	0.052	0.067	0.131	0.405	0.910
			I	0.046	0.056	0.089	0.227	0.773
5×10	5.0	5.0	II	0.051	0.060	0.107	0.275	0.834
			E	0.052	0.067	0.131	0.405	0.910

END

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