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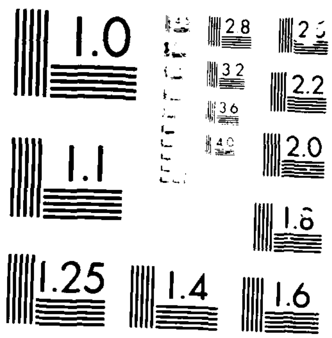
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DFR Property of First Passage Times
and Its Preservation
under Geometric Compounding

J. George Shanthikumar

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Abstract

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DFR Property of First Passage Times and Its Preservation under Geometric Compounding

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Abstract

It is shown that if the transition kernel of a discrete time Markov chain with state space $\{0, 1, \ldots\}$ is totally positive of order two (TP_2), the first passage time from state 1 to state 0 has decreasing failure rate (DFR). This result is used to show that (i) the sum of a geometric number (i.e., geometric compound) of i.i.d. DFR random variables is DFR and (ii) the number of customers served during a busy period in an M/G/1 queue with increasing failure rate service times is DFR. Recent results of Szekli (1986) and the closure property of i.i.d. DFR random variables under geometric compounding are combined to show that the stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (increasing mean residual) service times is DFR. We also provide sufficient conditions on the inter-renewal times under which the renewal function is concave. These results shed some light on a conjecture of Brown (1981).

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1. INTRODUCTION

Brown (1980) proved that the renewal function for a renewal process with decreasing failure rate (DFR) inter-renewal times is concave. In a subsequent paper (Brown 1981) he conjectured that DFR is also a necessary condition for concavity. A consequence of this conjecture, as pointed out by Brown, is that DFR distributions are closed under geometric compounding (i.e., the sum of a geometric number of i.i.d. DFR random variables is DFR). Thus a counterexample to this (possible) closure property of DFR random variables under geometric compounding would provide a counterexample to the concavity conjecture.

In an attempt to verify Brown's conjecture we obtained sufficient conditions for (i) the first passage time from state 1 to state 0 of a Markov chain with state space $\{0, 1, ...\}$ to be DFR and (ii) the renewal function to be concave. These results led to progress in two areas which shed light on Brown's conjecture:

(i) It is shown that DFR distributions are closed under geometric compounding,

(ii) An example is given which shows that the conjecture does not hold in the discrete time case.

The discrete time example does not generalize to the continuous case, and the truth of the conjecture is still unresolved. In the continuous case, however, it is shown that a consequence of Brown's conjecture, if true, is that the first passage time from state 1 to state 0 in a stochastically monotone Markov process with state space $\{0, 1, ...\}$ is DFR. This property is known to hold for birth and death processes (Keilson 1979) and in general for any Markov process that can be uniformized such that the embedded Markov chain has a totally positive of order two (TP_2) transition matrix (Assaf, Shaked and Shanthikumar 1985). But it is not known whether it holds for this larger class. For this class of Markov processes (i.e., stochastically monotone) it is easily verified that the first passage time from state 1 to state 0 has the new worse than used (NWU) property. The NWU property is weaker than the DFR property.

The results presented in this paper, apart from providing some useful insights into Brown's conjecture, have other applications. The DFR property of the first passage times of Markov chains is used to show that the number of customers served during a busy period in an M/G/1 queue with increasing failure rate service times is DFR. Geometric compounding of i.i.d. random variable arises naturally in many applied probability models. A recent paper of Gertsbakh (1984) discusses a wide range of applications in reliability theory. In the queueing theory context it is well known that the stationary waiting time in a GI/G/1 queue can be represented as a geometric compound of i.i.d. random variables (e.g., Feller 1971). Related results are that the class of completely monotone (CM) distributions is closed under geometric compounding and the stationary waiting time in an M/G/I queue with CM service times is CM (Keilson 1978) and that the distribution function of a geometric convolution of DFR distributions is concave and the stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (increasing mean residual) service times has a concave distribution function (Szekli 1986). We will strengthen the results of Szekli.

One aspect of our methodology which appears to be new is the consideration of a monotonicity property for a Markov chain which is stronger than stochastic monotonicity and weaker than TP_2. Stochastic monotonicity is based on the partial ordering of stochastic ordering, TP_2 on ordering by monotone likelihood ratio. An intermediate ordering we use is the hazard rate ordering (e.g., Ross 1983). This ordering is utilized by Brown (1980, 1983, 1984) to study properties of IMRL and DFR distributions.

2. PRELIMINARIES

A random variable X or its distribution F on $[0,\infty)$ is said to be DFR (IFR) if its failure rate $r_X(t) \equiv f(t) / \bar{F}(t)$ is decreasing (increasing) in $t \in [0,\infty)$: f and $\bar{F}$ are the density and survival functions of X ("increasing" and "decreasing" are not used in the strict sense). X is said to be IMRL if $E(X - t \mid X > t)$ is increasing in $t \in [0,\infty)$.

Two non-negative random variables X_1 and X_2 or their distributions F_1 and F_2 are ordered in the sense of usual stochastic (hazard rate) ordering if $\bar{F}_1(t) \geq \bar{F}_2(t), t \geq 0$ ($\bar{F}_1(t) / \bar{F}_2(t)$ is increasing in $t \in [0,\infty)$). Denote $X_1 \geq_{st} X_2 (X_1 \geq_h X_2)$. Note that (e.g. Ross 1983) $X_1 \geq_h X_2$ implies $\{X_1 \mid X_1 > t\} \geq_{st} \{X_2 \mid X_2 > t\}, t \geq 0$.

A function $\underline{\underline{a}} = [a(i,j)]$ of two real variables ranging over linearly ordered sets X and Y, respectively is TP_2 if for any $n_1 < n_2$ and $m_1 < m_2$ ($n_i \in$ X, $m_i \in$ Y), $a(n_1, m_1) a(n_2, m_2) \geq a(n_1, m_2) a(n_2, m_1)$. Equivalently using the convention $0/0 = 0$ one has $\dfrac{a(n_2, m_2)}{a(n_2, m_1)} - \dfrac{a(n_1, m_2)}{a(n_1, m_1)} \geq 0$ (when defined; otherwise set the difference equal to zero).

Let $\underline{\underline{t}} = [t(i,j)]_{i,j \in N+}$; $t(i,j) = 1, i \geq j$ and $t(i,j) = 0$ otherwise. $\underline{\underline{a}} = [a(i,j)]_{i,j \in N+}$ and $\underline{\underline{b}} = [b(i,j)]_{i,j \in N+}$ are non-negative matrices (here $N+ = \{1, 2, \ldots\}$).

Lemma 2.1: $\underline{\underline{a}}\,\underline{\underline{t}} \in TP_2$, $\underline{\underline{r}}\,\underline{\underline{t}} \in TP_2$ and $\underline{\underline{t}}^{-1}\underline{\underline{r}}\,\underline{\underline{t}} \geq \underline{\underline{0}}$. Then $\underline{\underline{a}}\,\underline{\underline{r}}\,\underline{\underline{t}} \in TP_2$.

Proof: Let $\underline{\underline{A}} = \underline{\underline{a}}\,\underline{\underline{t}}$, $\underline{\underline{R}} = \underline{\underline{r}}\,\underline{\underline{t}}$, $\underline{\underline{B}} = \underline{\underline{a}}\,\underline{\underline{r}}\,\underline{\underline{t}}$ and for $1 \leq n_1 < n_2$, $1 \leq m_1 < m_2$ and $B(n_1, m_1) > 0$ consider

$$(2.1)\qquad \frac{B(n_1,m_2)}{B(n_1,m_1)}=\frac{\sum_{k=1}^{\infty}a(n_1,k)R(k,m_2)}{\sum_{\ell=1}^{\infty}a(n_1,\ell)R(\ell,m_1)}$$

$$\sum_{k=1}^{\infty}\frac{R(k,m_2)}{R(k,m_1)}\left[\frac{a(n_1,k)R(k,m_1)}{\sum_{\ell=1}^{\infty}a(n_1,\ell)R(\ell,m_1)}\right]$$

$$=\sum_{k=1}^{\infty}\left[\frac{R(k,m_2)}{R(k,m_1)}-\frac{R(k-1,m_2)}{R(k-1,m_1)}\right]\frac{\sum_{j=k}^{\infty}a(n_1,j)R(j,m_1)}{\sum_{\ell=1}^{\infty}a(n_1,\ell)R(\ell,m_1)}$$

where $\frac{R(0,m_2)}{R(0,m_1)}\equiv 0$. Since $\underline{\underline{R}}\in TP_2$, the expression in the square bracket in the right hand side of (2.1) is non-negative. Consider

$$(2.2)\qquad \frac{\sum_{j=k}^{\infty}a(n_1,j)R(j,m_1)}{\sum_{\ell=1}^{\infty}a(n_1,\ell)R(\ell,m_1)}=\frac{1}{1+\frac{\sum_{\ell=1}^{k-1}a(n_1,\ell)R(\ell,m_1)}{\sum_{j=k}^{\infty}a(n_1,j)R(j,m_1)}}$$

Now for $A(n_1,k)>0$, set $R(0,m_1)=0$ and consider

$$\frac{\sum_{\ell=1}^{k-1} a(n_1,\ell)R(\ell,m_1)}{\sum_{j=k}^{\infty} a(n_1,j)R(j,m_1)}$$

$$=\frac{\sum_{\ell=1}^{k} A(n_1,\ell)\left[R(\ell,m_1)-R(\ell-1,m_1)\right]-A(n_1,k)R(k,m_1)}{\sum_{j=k}^{\infty} A(n_1,j)\left[R(j,m_1)-R(j-1,m_1)\right]+A(n_1,k)R(k-1,m_1)}$$

$$=\frac{\sum_{\ell=1}^{k} \frac{A(n_1,\ell)}{A(n_1,k)}\left[R(\ell,m_1)-R(\ell-1,m_1)\right]-R(k,m_1)}{\sum_{j=k}^{\infty} \frac{A(n_1,j)}{A(n_1,k)}\left[R(j,m_1)-R(j-1,m_1)\right]+R(k-1,m_1)}$$

$$\geq\frac{\sum_{\ell=1}^{k} \frac{A(n_2,\ell)}{A(n_2,k)}\left[R(\ell,m_1)-R(\ell-1,m_1)\right]-R(k,m_1)}{\sum_{j=k}^{\infty} \frac{A(n_2,j)}{A(n_2,k)}\left[R(j,m_1)-R(j-1,m_1)\right]+R(k-1,m_1)}$$

$$=\frac{\sum_{\ell=1}^{k-1} a(n_2,\ell)R(\ell,m_1)}{\sum_{j=k}^{\infty} a(n_2,j)R(j,m_1)}\ ,$$

since $\underline{\underline{t}}^{-1}\underline{\underline{r}}\,\underline{\underline{t}} \geq \underline{\underline{0}}$ implies $R(i,m_1)-R(i-1,m_1)\geq 0, i=1,2,\ldots$ and $\underline{\underline{A}}=\underline{\underline{a}}\,\underline{\underline{t}} \in TP_2$ implies $A(n_1,\ell)/A(n_1,k) \geq (\leq)\ A(n_2,\ell)/A(n_2,k)$ for $\ell \leq (\geq)\ k$ and $n_1<n_2$. Then from (2.2) one gets

$$(2.3)\qquad \frac{\sum_{j=k}^{\infty} a(n_1,j)R(j,m_1)}{\sum_{\ell=1}^{\infty} a(n_1,\ell)R(\ell,m_1)} \le \frac{\sum_{j=k}^{\infty} a(n_2,j)R(j,m_1)}{\sum_{\ell=1}^{\infty} a(n_2,\ell)R(\ell,m_1)} .$$

When $A(n_1,k)=0$, the left hand side of (2.3) is zero and the above inequality is trivially satisfied. Combining (2.1) and (2.3) one has

$$(2.4)\qquad \frac{B(n_1,m_2)}{B(n_1,m_1)} \le \frac{B(n_2,m_2)}{B(n_2,m_1)},\quad 1 \le n_1 < n_2;\ \ 1 \le m_1 < m_2 \ .$$

Since $\underline{\underline{a}}\,\underline{\underline{r}} \ge 0, B(i,j)$ is increasing in j and therefore $B(n_1,m_1)=0$ implies $B(n_1,m_2)=0$. Hence (2.4) in this case is trivially satisfied. $\square$

Remark 2.2: Keilson and Kester (1978) show that if $\underline{\underline{a}}\,\underline{\underline{\iota}} \in TP_2$ and $\underline{\underline{\iota}}^{-1}\underline{\underline{r}}\,\underline{\underline{\iota}} \in TP_2$ then $\underline{\underline{a}}\,\underline{\underline{r}}\,\underline{\underline{\iota}} \in TP_2$. This follows from the observations that $\underline{\underline{a}}\,\underline{\underline{r}}\,\underline{\underline{\iota}} = \underline{\underline{a}}\,\underline{\underline{\iota}}\,\underline{\underline{\iota}}^{-1}\underline{\underline{r}}\,\underline{\underline{\iota}}$ and that the class of TP_2 matrices is closed under multiplication. Since $\underline{\underline{\iota}}^{-1}\underline{\underline{r}}\,\underline{\underline{\iota}} \in TP_2$ implies $\underline{\underline{r}}\,\underline{\underline{\iota}} \in TP_2$, our condition $\underline{\underline{r}}\,\underline{\underline{\iota}} \in TP_2$ and $\underline{\underline{\iota}}^{-1}\underline{\underline{r}}\,\underline{\underline{\iota}} \ge \underline{\underline{0}}$ is weaker than the TP_2 condition of $\underline{\underline{\iota}}^{-1}\underline{\underline{r}}\,\underline{\underline{\iota}}$.

3. DFR FIRST PASSAGE TIMES

Let $\underline{X} = \{X_n, n=0,1\ldots\}$ be a temporally homogeneous discrete time Markov chain with state space $\mathrm{N}=\{0,1,\ldots\}$ and transition probability matrix $\underline{\underline{P}} = [P(i,j)]_{i,j\in \mathrm{N}}$ $(P(i,j) = P\{X_n = j \mid X_{n-1} = i\}, i,j \in \mathrm{N})$. Define the first passage time

$$(3.1)\qquad T = \{\min[n: X_n = 0,\ \ n = 1,2,\ldots] \mid X_0 = 1\} .$$

Let $\underline{\underline{Q}} = \underline{\underline{P}}\,\underline{\underline{t}}$ be the transition kernel of $\underline{X}$ (i.e., $Q(i,j) = \sum_{k=j}^{\infty} P(i,k), i,j \in \mathrm{N}$). One has

Theorem 3.1: $\underline{\underline{Q}} \in TP_2$ implies $T \in$ DFR

Proof: The failure rate r_T of T is given by $r_T(n) \equiv P\{T = n \mid T \geq n\} = P\{X_n = 0 \mid X_{n-1} \geq 1\} = \sum_{i=1}^{\infty} P\{X_n = 0 \mid X_{n-1} = i, X_{n-1} \geq 1\} \; P\{X_{n-1} = i \mid X_{n-1} \geq 1\}$. That is

$$r_T(n) = 1 - E\left[Q(\widetilde{X}_{n-1}, 1)\right], \tag{3.2}$$

where $\widetilde{X}_{n-1} \overset{d}{=} \{X_{n-1} \mid X_{n-1} \geq 1\}$ ($\overset{d}{=}$ stands for equality in law.) Let $\underline{\underline{P}}_{\ell} = \left[P(i,j)\right]_{i,j \in N+}$ be the transition probability matrix of the lossy process $\underline{X}^{\ell} = \left\{X_n^{\ell}, n = 0, 1, \ldots\right\}$ of $\underline{X}$ on the state space N+ (e.g. Keilson 1979). Let $\underline{v}_n$ and $\underline{v}_n^{\ell}$ be the probability vector of $\widetilde{X}_n$ and X_n^{ℓ} respectively (i.e., $v_n(k) = P\left\{\widetilde{X}_n = k\right\}$ and $v_n^{\ell}(k) = P\left\{X_n^{\ell} = k\right\}$). Since state 0 is absorbing it is easily verified that

$$\underline{v}_n = \underline{v}_n^{\ell} \,/\, \underline{v}_n^{\ell}\underline{e} = \underline{v}_0^{\ell}\underline{\underline{P}}_{\ell}^{n} \,/\, \underline{v}_0^{\ell}\underline{\underline{P}}_{\ell}^{n}\underline{e}, \tag{3.3}$$

where $\underline{e} = (1, 1, \ldots)'$ and $\underline{v}_0 = \underline{v}_0^{\ell} = (1, 0, 0, \ldots)$. Now consider $(\underline{v}_n^{\ell}, n = 0, 1, \ldots)$. Clearly $\begin{pmatrix} \underline{v}_0^{\ell} \\ \underline{v}_1^{\ell} \end{pmatrix} \in TP_2$ and therefore from the closure property of TP_2 matrices under multiplication

$$\begin{pmatrix} \underline{v}_1^{\ell} \\ \underline{v}_2^{\ell} \end{pmatrix} \underline{\underline{t}} = \begin{pmatrix} \underline{v}_0^{\ell} \\ \underline{v}_1^{\ell} \end{pmatrix} \underline{\underline{P}}_{\ell}\,\underline{\underline{t}} \equiv \begin{pmatrix} \underline{v}_0^{\ell} \\ \underline{v}_1^{\ell} \end{pmatrix} \underline{\underline{Q}}_{\ell}$$

is TP_2 since $\underline{\underline{Q}}_\ell = [Q(i,j)]_{i,j \in N+}$ is TP_2 ($\underline{\underline{Q}} \in TP_2$ implies $\underline{\underline{Q}}_\ell \in TP_2$). Now as an induction hypothesis assume that $\begin{pmatrix} \underline{v}_{n-1}^\ell \\ \underline{v}_n^\ell \end{pmatrix} \underline{\underline{\iota}} \in TP_2$. We have shown that this is true for $n = 1,2$.

Now observing that $\underline{\underline{\iota}}^{-1} \underline{\underline{P}}_\ell \underline{\underline{\iota}} \geq 0$, since $\underline{\underline{P}}\,\underline{\underline{\iota}}$ is TP_2 and $\underline{\underline{P}}$ is a transition probability matrix one has from Lemma 2.1

$$\begin{pmatrix} \underline{v}_n^\ell \\ \underline{v}_{n+1}^\ell \end{pmatrix} \underline{\underline{\iota}} = \begin{pmatrix} \underline{v}_{n-1}^\ell \\ \underline{v}_n^\ell \end{pmatrix} \underline{\underline{P}}_\ell \underline{\underline{\iota}}$$

is TP_2. Observe that this TP_2 property implies

$$\text{(3.4)} \qquad \frac{P\left\{X_{n+1}^\ell \geq k+1\right\}}{P\left\{X_n^\ell \geq k+1\right\}} \geq \frac{P\left\{X_{n+1}^\ell \geq k\right\}}{P\left\{X_n^\ell \geq k\right\}}, \quad k = 1,2,\ldots$$

Combining (3.3) and (3.4) one sees that

$$\text{(3.5)} \qquad \widetilde{X}_{n+1} \geq_h \widetilde{X}_n, \; n = 0,1,..$$

Since $\geq_h$ implies stochastic ordering and $Q(i,1)$ is increasing in i one has from (3.2) and (3.5), $r_T(n)$ is decreasing in $n \in N+$. $\square$

Remark 3.2: It is known that if $\underline{\underline{P}}$ is TP_2 then T has log-convex probability mass function which implies $T \in$DFR (Assaf, Shaked and Shanthikumar 1985). However, our condition $\underline{\underline{Q}}$ is TP_2 is weaker since $\underline{\underline{P}} \in TP_2$ implies $\underline{\underline{Q}}$ is TP_2 and not necessarily in the reverse direction.

Remark 3.3: Let $T_k^- = \min[n: X_n \le k-1,\ n = 1,2,...] \mid X_0 = k\}, (k \ge 1)$. Considering a modification of $\underline{X}$ such that its states $\{0, 1, ..., k-1\}$ are lumped into one absorbing state, from Theorem 3.1 one sees that $\underline{\underline{Q}} \in TP_2$ implies $T_k^- \in$ DFR.

Next consider an absorbing, right-continuous continuous time Markov chain $\underline{Y} = \{Y(t),\ t \ge 0\}$ with state space N, where 0 is the absorbing state. Let $\underline{\underline{\mu}} = [\mu(i,j)]_{i,j \in N}$ be the transition rate matrix of $\underline{Y}$ ($\mu(i,j)$ is the transition rate from state i to state j). $\mu_i = \sum_{j=0}^{\infty} \mu_{ij}$; $\underline{\underline{D}}_\mu = \text{diag}\,\{\mu_0, \mu_1, ...\}$.

$$(3.6) \qquad T^* = \left\{\inf\,[t: Y(t) = 0,\ t \ge 0] \mid Y(0) = 1\right\}$$

Define $\mu^* = \sup\{\mu_i, i = 0, 1, ...\}$. Then using uniformization (e.g., Keilson 1979, Chapter 2 or Assaf, Shaked and Shanthikumar 1985, Section 3) one has from Theorem 3.1

Corollary 3.4: If there exists a λ ($\mu^* \le \lambda < \infty$) such that $\left[\underline{\underline{I}} + \frac{1}{\lambda}(\underline{\underline{\mu}} - \underline{\underline{D}}_\mu)\right]\underline{\underline{t}}$ is TP_2 then $T^* \in$ DFR.

We now present some applications of the above results. Let $\{Z_n,\ n = 1, 2, ...\}$ be a sequence of i.i.d. random variables with support N+ and K be a geometric random variable with $P\{K = k\} = (1-p)^{k-1}p$, $k \in$ N +. $\{Z_n\}$ and K are mutually independent.

Theorem 3.5: $Z_n \in$ DFR implies $Z^* = \sum_{n=1}^{K} Z_n \in$ DFR.

Proof: Let $\underline{X}$ be a temporally homogeneous Markov chain with state space N and transition probability matrix $\underline{\underline{P}}$ where $P(0,0) = 1$, $P(i, i+1) = 1 - r_Z(i)$, $P(i, 1) = (1-p)r_Z(i)$, $P(i, 0) = p r_Z(i)$, $i \in$ N + and all other entries are zero.

$r_Z(i) = P\{Z_n = i \mid Z_n \geq i\}$ is the failure rate of Z_n. Let T be as defined in (3.1) and define

(3.7) $$T_1 = \left\{\min\left[n{:}X_n \leq 1,\ n = 1,2,...\right] \mid X_0 = 1\right\}.$$

Simple calculation shows that

(3.8) $$P\{T_1 = k\} = \prod_{\ell=1}^{k-1} (1 - r_Z(\ell)) r_Z(k)$$

$$= P\{Z_n = k\},\ k = 1,2,...$$

It can also be verified that $P\left\{X_{T_1} = 1\right\} = (1-p)$ and $P\left\{X_{T_1} = 0\right\} = p$. Since state 0 is absorbing, employing the Markov property of $\underline{X}$, (3.8) and the fact that T_1 is a stopping time of $\underline{X}$, it is not hard to see that

(3.9) $$T \stackrel{d}{=} Z^* .$$

Computing $\underline{\underline{Q}}$ one sees that $Q(i,0) = 1, i \in \mathrm{N}, Q(i,1) = 1 - pr_Z(i)$, $Q(i,j) = 1 - r_Z(i)$, $2 \leq j \leq i$; $i \in \mathrm{N}+$ and all other entries are zero. In this case $\underline{\underline{Q}} \in TP_2$ as long as $r_Z(i)$ is decreasing in $i \in \mathrm{N}+$. The required result now follows from (3.9) and Theorem 3.1.

□

Consider a GI/G/1 queue at which customers arrive according to a renewal process with rate λ. The service times form a sequence of i.i.d. random variables with a common distribution function F. The arrival process and service times are mutually independent. We will first consider an M/G/1 queue which is a special case of GI/G/1 queue with Poisson arrival process.

Theorem 3.6: The number of customers served during a busy period of an M/G/1 queue with IFR service times is DFR.

Proof: Let $\tilde{X}_n$ be the number of customers in the M/G/1 queueing system just after the n-th customer departure. Then $\underline{\tilde{X}} = \{\tilde{X}_n,\ n = 0, 1, \ldots\}$ is a temporally homogeneous Markov chain with transition probabilities $\tilde{P}(i,j) = g(j + 1 - i),\ i \in N+;\ \tilde{P}(0,j) = g(j),\ j \in N$, where

$$g(k) = \begin{cases} 0 & k = -1, -2, \ldots \\ \displaystyle\int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!} dF(t), & k = 0, 1, \ldots \end{cases}$$

(e.g., Ross 1983). Consider a modification $\underline{X}$ of $\underline{\tilde{X}}$ such that state 0 is absorbing in $\underline{X}$ (i.e., $P(i,j) = \tilde{P}(i,j),\ i \in N+,\ j \in N$ and $P(0,0) = 1$). Then if T is as defined in (3.1), it is the number of customers served during the first busy period. Consider $Q(i,j) = \bar{G}(j + 1 - i),\ i, j \in N+$ $(Q(0,0) = 1,\ Q(0,j) = 0,\ j \in N+)$,

$$\bar{G}(k) = \begin{cases} 1 & k = -1, -2, \ldots \\ \displaystyle\sum_{\ell = k}^{\infty} g(\ell), & k = 0, 1, \ldots \end{cases}.$$

It is known that if $\bar{F}(t + x) / \bar{F}(t)$ is decreasing in t (i.e., F is IFR) then $\bar{G}(n + k) / \bar{G}(k)$ is decreasing in k (Block and Savits 1980). This observation leads to a straightforward verification that $\underline{\underline{Q}} \in TP_2$. The DFR property of T now follows from Theorem 3.1. □

4. CLOSURE OF DFR UNDER GEOMETRIC COMPOUNDING

$\{W_n,\ n=1,2,...\}$ is a sequence i.i.d. random variables with support $[0,\infty)$ and survival function $\bar{F}$ and K is a geometric random variable with $P\{K=k\}=(1-p)^{k-1}p,\ k=1,2,....$ $\{W_n\}$ and K are mutually independent. Then one has

Theorem 4.1: $W_n \in$ DFR implies $W^* = \sum_{n=1}^{K} W_n \in$ DFR.

Proof: For some $\lambda > 0$ let $\bar{G}(k)=\bar{F}(\frac{k}{\lambda}),\ k=0,1,....$ Then if $\left\{Z_n^\lambda,\ n=1,2,...\right\}$ is a sequence of i.i.d. random variables with a common survival function $\bar{G}$, $r_{Z^\lambda}(k)=\left[\bar{G}(k-1)-\bar{G}(k)\right]/\bar{G}(k-1)$ is decreasing in $k=1,2,...$ (i.e., $Z_n^\lambda \in$ DFR). Let $\left\{E_n^\lambda,\ n=1,2,...\right\}$ be a sequence of i.i.d. exponential random variables with mean $\frac{1}{\lambda}$. Now define a sequence of i.i.d. random variables $\left\{W_n^\lambda,\ n=1,2,...\right\}$ such that $W_n^\lambda \stackrel{d}{=} \sum_{k=1}^{Z_n^\lambda} E_k^\lambda$. That is , if $\bar{F}^\lambda$ is the survival function of W_n^λ,

$$(4.1) \qquad \bar{F}^\lambda(t)=\sum_{k=0}^{\infty}\bar{F}(\frac{k}{\lambda})\frac{e^{-\lambda t}(\lambda t)^k}{k!}\ .$$

Then it is easily verified that

$$(4.2) \qquad \text{as } \lambda \to \infty,\ \bar{F}^\lambda(t) \to \bar{F}(t) \text{ at every continuity point of } \bar{F}$$

(Feller 1971). Then consider

$$\sum_{n=1}^{K} W_n^\lambda \stackrel{d}{=} \sum_{n=1}^{\sum_{k=1}^{K} Z_k^\lambda} E_n^\lambda\ .$$

Since $Z_k^\lambda \in$ DFR implies $\sum_{k=1}^{K} Z_k^\lambda \in$ DFR (Theorem 3.4) and a sum of DFR number of i.i.d. exponential random variables is DFR (Esary, Marshall and Proschan (1973), $\sum_{n=1}^{K} W_n^\lambda \in$ DFR. Since DFR property is preserved under limits as $\lambda \to \infty$ one has $\sum_{n=1}^{K} W_n \in$ DFR. $\square$

Remark 4.2: Szekli (1986) shows that W^* has convex survival function (say $\bar{F}^*$). However, $W^* \in$ DFR is equivalent to that $\bar{F}^*$ is log-convex: a stronger property than convexity.

The above result combined with the results fo Szekli (1986) lead to

Corollary 4.3: The stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (IMRL) service times is DFR.

5. CONCAVE RENEWAL FUNCTIONS

Consider a discrete time renewal process with the inter-renewal time having the first passage time distribution of $\underline{X}$ from state 1 to state 0. Let $\gamma(n)$ be the probability that a renewal occurs at time $n = 1, 2, \ldots$ (γ is the renewal density). Then the expected number of renewals $M(n)$ during $\{1, 2, \ldots, n\}$ is equal to $\sum_{k=1}^{n} \gamma(k)$ (M is the renewal function.)

Theorem 5.1: $\underline{\underline{t}}^{-1}\underline{\underline{Q}} \geq \underline{\underline{0}}$ implies $\gamma(n)$ is decreasing in $n \in \mathrm{N}+$ (i.e., M is concave on N+).

Proof: Let $\underline{X}^* = \{X_n^*, n = 0, 1, \ldots\}$ be a modification of $\underline{X}$ such that as soon as $\underline{X}$ reaches 0 it is placed back to state 1 (representing a renewal). Then $Q^*(i,j) = Q(i,j)$, $i \in \mathrm{N}+$, $j = 2, 3, \ldots$, and $Q^*(i, 1) = 1$, $i \in \mathrm{N}+$. So $\underline{\underline{t}}^{-1}\underline{\underline{Q}} \geq \underline{\underline{0}}$ implies $\underline{\underline{t}}^{-1}\underline{\underline{Q}}^* \geq \underline{\underline{0}}$ and therefore $\underline{X}^*$ is stochastically *monotone. Specifically* $\{X_n^* \mid X_0^* = 1\} \geq_{\mathrm{st}} \{X_{n-1}^* \mid X_0^* = 1\}$, $n = 1, 2, \ldots$ *(note that the state space* of $\underline{X}^*$ is $N+$). Then the observation that

$$\gamma(n) = 1 - E\left\{Q(X^{*}_{n-1}, 1)\right\}$$

and that $Q(i, 1)$ is increasing in i leads to the desired conclusion. $\square$

We will next see that the inter-renewal time need not be DFR for the renewal function to be concave. This will be achieved by showing that $\underline{\underline{\iota}}^{-1}\underline{\underline{Q}} \geq \underline{\underline{0}}$ is not sufficient for the DFR property of T.

Counterexample 5.2: Consider the first passage time T (defined in 3.1) of a Markov chain with transition probability matrix

$$\underline{\underline{P}} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

Then

$$\underline{\underline{Q}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & \frac{1}{4} \end{bmatrix}$$

and $\underline{\underline{\iota}}^{-1}\underline{\underline{Q}} \geq \underline{\underline{0}}$. So the renewal density is decreasing (Theorem 5.1). Computing one gets $r_T(1) = \frac{3}{4}$; $r_T(2) = 0$; $r_T(3) = \frac{9}{16}$. So T is not DFR. Then one sees that in the discrete time case the *DFR property of inter-renewal times is sufficient but not necessary* for a concave renewal function.

We next turn our attention to the continuous time case. The inter-renewal times have the first passage time distribution of $\underline{Y}$ from state 1 to state 0. γ and M are the renewal density and renewal function, respectively, of the renewal process. Similar to the discrete time case one has

Theorem 5.3: $\underline{Y}$ is stochastically monotone implies $\gamma(t)$ is decreasing in $t \in [0,\infty)$ (i.e., M is concave on $[0,\infty)$).

Remark 5.4: An interesting consequence of Theorem 5.3 and the conjecture of Brown (1981), if true, is that the first passage time T^* of a stochastically monotone Markov process $\underline{Y}$ with state space N (i.e., for any $k \in$ N, $\sum_{j\geq k} \mu(i,j)$ is increasing in $i<k$ and $\sum_{j\leq k} \mu(i,j)$ is decreasing in $i>k$) from state 1 to state 0 is DFR.

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Bibliography

1. Assaf, D., Shaked, M. and Shanthikumar, J.G. (1985), First-passage times with PF_r densities, *J. Appl. Prob. 22*, 185-196.

2. Block, H.W. and Savits, T.H. (1980), Laplace transforms for classes of life distributions, *Ann. Prob. 8*, 465-474.

3. Brown, M. (1980), Bounds, inequalities, and monotonicity properties for some specialized renewal processes, *Ann. Prob. 8*, 227-240.

4. Brown, M. (1981), Further monotonicity properties for specialized renewal processes, *Ann. Prob. 9*, 891-895.

5. Brown, M. (1983), Approximating IMRL distributions by exponential distributions, with applications to first passage times, *Ann. Prob. 11*, 419-427.

6. Brown, M. (1984), Inequalities for distributions with increasing failure rate, Report No. MB 84-01, City College, CUNY.

7. Esary, J.D., Marshall, A.W. and Proschan, F. (1973), Shock models and wear processes, *Ann. Prob. 1*, 627-649.

8. Feller, W. (1971), *An Introduction to Probability Theory and its Appilcations, Vol. II, 2nd Edition*, John-Wiley, New York.

9. Gertsbakh, I.B. (1984), Asymptotic methods in reliability theory: A review, *Adv. Appl. Prob. 16*, 147-175.

10. Keilson, J. (1978), Exponential spectra as a tool for the study of server-systems with several classes of customers, *J. Appl. Prob. 15*, 162-170.

11. Keilson, J. (1979), *Markov Chain Models: Rarity and Exponentiality*, Springer-Verlag, New York.

12. Keilson, J. and Kester, A. (1978), Unimodality preservation in Markov chains, *Stoch. Proc. Appl. 7*, 179-190.

13. Ross, S.M. (1983), *Stochastic Processes*, Wiley, New York.

14. Szekli, R. (1986), On the concavity of the waiting time distribution in some GI/G/1 queues, *J. Appl. Prob.* *22* (forthcoming).

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