ERGODICITY AND MIXING IN A FIVE-MODE SPECTRAL MODEL OF TWO-DIMENSIONAL TURBULENCE(U) DAVID W TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER BET. M C MOSHER

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ERGODICITY AND MIXING IN A FIVE-MODE SPECTRAL MODEL
OF TWO-DIMENSIONAL TURBULENCE

by

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ERGODICITY AND MIXING IN A FIVE-MODE SPECTRAL MODEL OF TWO-DIMENSIONAL TURBULENCE

A dynamical system derived by truncating to five frequencies a Fourier representation of the Euler equations for incompressible, two-dimensional fluid flow is investigated. In using Fourier representations to model turbulence, investigators have assumed that the truncated systems are ergodic and mixing, although the validity of this assumption is an open question. The truncated Fourier system has seven independent constants of motion which define a three-dimensional torus. Since a component of the solution to the system is quasi-periodic, the flow of the system can never be mixing and, in particular, cannot be mixing on the three-dimensional torus defined by the constants of motion.
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ABSTRACT

A dynamical system derived by truncating to five frequencies a Fourier representation of the Euler equations for incompressible, two-dimensional fluid flow is investigated. In using Fourier representations to model turbulence, investigators have assumed that the truncated systems are ergodic and mixing, although the validity of this assumption is an open question. The truncated Fourier system has seven independent constants of motion which define a three-dimensional torus. Since a component of the solution to the system is quasi-periodic, the flow of the system can never be mixing and, in particular, cannot be mixing on the three-dimensional torus defined by the constants of motion.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

Truncated spectral representations of the Euler equations have been used in many theoretical and numerical investigations of inviscid, incompressible, two-dimensional fluid flow, especially, turbulent flow. The representations are measure-preserving dynamical systems which conserve energy and total vorticity or enstrophy. In analogy with statistical mechanics, the Fourier models are assumed to be ergodic and mixing on the set defined by the conserved energy and enstrophy, although the validity of this assumption is an open question. We will investigate a truncated system of five frequencies investigated by Kraichnan, Hald, and Glaz and, in particular, the question of whether the system is mixing.

The spectral models are derived by representing the vorticity of an inviscid, incompressible, two-dimensional flow field as a Fourier series with complex coefficients $w_k$, plugging this representation into the vorticity equation, and truncating

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*A complete listing of references is given on page 21.

**Definitions of ergodicity, mixing, invariant set, and quasi-periodic can be found in References 2 and 3.*
the resulting infinite system of ordinary differential equations after a finite number of frequencies. For each frequency \( k \) the resulting equation is

\[
dw_k/dt = 0.5 \sum_{p+q=k} (q^{-2} - p^{-2})|p,q|w_p w_q
\]

where \( p = (p_1, p_2) \), \( q = (q_1, q_2) \), \( p^2 = p_1^2 + p_2^2 \), \( |p,q| = p_1 q_2 - p_2 q_1 \), and \( d/dt \) is the derivative with respect to time \( t \). It is assumed that \( w_{-k} \) is the complex conjugate of \( w_k \), i.e., \( w_{-k} = w_k^* \). The summation \( p+q = k \) is extended over only a finite number of frequencies. These systems have at least two constants of motion, namely, the energy \( E \) and the squared vorticity or enstrophy \( \Omega \)

\[
E = 0.5 \sum_p p^{-2} w_p w_p^*, \quad \Omega = \sum_p w_p w_p^*
\]

Here the summation is taken over all the frequencies in the truncation. Using a truncation which retains the five frequencies \( k_1 = (1,1) \), \( k_2 = (2,1) \), \( k_3 = (3,0) \), \( k_4 = (2,-1) \), and \( k_5 = (1,-1) \), we obtain the system

\[
dw_1/dt = 8w_3 w_4^* \quad (1a)
\]
\[
dw_2/dt = 35w_3 w_5^* \quad (1b)
\]
\[
dw_3/dt = 27(w_1 w_4 - w_2 w_5) \quad (1c)
\]
\[
dw_4/dt = -35w_3^* w_1 \quad (1d)
\]
\[
dw_5/dt = -8w_3^* w_2 \quad (1e)
\]

The system has these constants of motion:

\[
\gamma_1 = 35r_1^2 + 8r_4^2 \quad (2a)
\]
\[
\gamma_2 = 35r_5^2 + 8r_2^2 \quad (2b)
\]
\[
\begin{align*}
\gamma_3 &= 35r_3^2 + 27r_2^2 + 27r_4^2 \\
I_1 &= w_1w_2 + w_4w_5 \\
I_2 &= 35w_1w_5 - 8w_2w_4 \\
I_3 &= \text{Im}(w_1w_3w_4 + w_2w_3w_5)
\end{align*}
\]

Here \( r_i = |w_i| \) for \( i=1,\ldots,5 \). The energy \( E = (\gamma_1 + \gamma_2)/140 + \gamma_3/630 \) and the enstrophy \( \Omega = (\gamma_1y_2 + \gamma_3)/35 \). However, \( \gamma_1, \gamma_2, \gamma_3, I_1, I_2, \) and \( I_3 \) have no clear physical interpretation.

Hald\(^6\) discovered the constants of motion of Equations (2). Since the system has extra constants of motion in addition to \( E \) and \( \Omega \), the flow of the system cannot be ergodic or mixing on the set defined by \( E \) and \( \Omega \). However, the question remains of whether the system is ergodic or mixing on the set \( S \) defined by the new constants of motion, i.e., the set of points \( w_1, w_2, w_3, w_4, w_5 \) that satisfy Equations (2). Also, the topology of \( S \) is of interest.

We have obtained the following results. The eight constants of motion discovered by Hald are not independent. A parameterization of the set \( S \) defined by the remaining seven constants of motion shows that \( S \) is a three-dimensional torus. That \( S \) is a three-dimensional manifold implies the seven constants of motion are independent. The \( w_3 \) component of the solution to the system can be solved explicitly using a Jacobian elliptic function. Since the \( w_3 \) component is quasi-periodic, the flow of the system can never be mixing and, in particular, cannot be mixing on \( S \). The results of a numerical test are consistent with the parameterization of \( S \).

In the following sections of this report we will first parameterize the set \( S \) and then derive an explicit solution of the \( w_3(t) \) component of Equations (1). Mixing will be discussed next. Finally, the results of the numerical test will be described.
PARAMETERIZATION OF THE INVARIANT SET S

The parameterization of the set S defined by the constants of motion is similar to that of the four-mode case.\(^9\) Letting \(w_j = r_j e^{i\theta_j}\) for \(j = 1, \ldots, 5\) and \(i = \sqrt{-1}\), we will show that the projection of S onto \(r_2, r_4\) space is a set \(R\) homeomorphic to a rectangle. The variables \(r_1, r_3, r_5, \theta_1, \theta_3, \theta_4,\) and \(\theta_5\) can be parameterized by \(\theta_2\) and the \(r_2, r_4\) which lie in \(R\). We will first determine the set \(R'\) which will be the projection of the set defined by Equations (2a) through (2e) onto \(r_2, r_4\) space. We will find that these equations are not independent, but the following relation holds

\[\gamma_1 \gamma_2 - 280|I_1|^2 - |I_2|^2 = 0\]

Therefore, only seven of the eight Equations (2a) through (2f) can be independent. Next, we will show that Equation (2f), which is used to parameterize \(\theta_3,\) defines an annulus. The intersection of \(R'\) and the annulus will be \(R\).

Letting \(s = r_2\) and \(t = r_4\), we rewrite Equations (2d) and (2e) as follows:

\[\begin{align*}
-1 + r_1 s |I_1|^{-1} e^{i(n_1 - n_2 - \psi_1)} &= tr_5 |I_1|^{-1} e^{i(n_5 - n_4 - \psi_1 + \pi)} \quad (3a) \\
-1 + 35 r_1 r_5 |I_2|^{-1} e^{i(n_1 - n_5 - \psi_2)} &= 8 s t |I_2|^{-1} e^{i(n_2 - n_4 - \psi_2)} \quad (3b)
\end{align*}\]

We can interpret these equations by using triangles in the complex plane with \(i = \sqrt{-1}\). Let \(z_1 = x_1 + iy_1\) be the value of the right-hand side of Equation (3a). Then, \(s, t, r_1,\) and \(r_5\) satisfy Equation (3a) if and only if the triangle ABC in Figure 1 exists. We will call this the first triangle. Here AB is the complex number \(r_1 s |I_1|^{-1} e^{i(n_1 - n_2 - \psi_1)}\), CB is \(tr_5 |I_1|^{-1} e^{i(n_5 - n_4 - \psi_1 + \pi)}\), the angle \(\angle CAB = \arg(z_1 + 1) = \theta_1 - \theta_2 - \psi_1,\) and \(\arg(z_1) = \theta_5 - \theta_4 - \psi_1 + \pi.\) Similarly, let \(z_2 = x_2 + iy_2\) be the value of the right-hand side of Equation (3b). Then, \(s, t, r_1,\) and \(r_5\) satisfy Equation (3b) if and only if the triangle DEF in Figure 2 exists. We will call this the second triangle. Here DE is the complex number \(35 r_1 r_5 |I_2|^{-1} e^{i(n_1 - n_5 - \psi_2)}\), FE is \(8 s t |I_2|^{-1} e^{i(n_2 - n_4 - \psi_2)}\), the angle \(\angle FDE = \arg(z_2^* + 1) = \theta_1 - \theta_5 - \psi_2,\) and \(\arg(z_2^*) = \theta_2 - \theta_4 - \psi_2.\)
The equations for the angles obtained from the triangle representations of Equations (3), i.e.,

\[ \theta_1 - \theta_2 = \arg(z_1 + 1) + \psi_1 \]  
\[ \theta_5 - \theta_4 = \arg(z_1) + \psi_1 - \pi \]  
\[ \theta_1 - \theta_5 = \arg(z_2 + 1) + \psi_2 \]  
\[ \theta_2 - \theta_4 = \arg(z_2) + \psi_2 \]

can be reduced to the following four equations:

\[ \theta_5 = \theta_2 + \arg(z_1 + 1) - \arg(z_2 + 1) + \psi_1 - \psi_2 \]  
\[ \theta_4 = \theta_2 - \arg(z_2) - \psi_2 \]  
\[ \theta_1 = \theta_2 + \arg(z_1 + 1) + \psi_1 \]  
\[ \arg(z_1) - \arg(z_1 + 1) + \arg(z_2 + 1) - \arg(z_2) = \pi \]  

Hence, angles \( \theta_1, \theta_4, \) and \( \theta_5 \) can be parameterized by \( \theta_2, z_1, \) and \( z_2. \) We will show that \( z_1 \) and \( z_2 \) can in turn be parameterized by \( s \) and \( t. \) First, however, we need to examine Equation (5d).

As seen in Figures 1 and 2,

\[ \arg(z_1) = \arg(z_1 + 1) + \text{angle ABC} \]  
\[ \arg(z_2) = \arg(z_2 + 1) + \text{angle DEF} \]

Using Equations (6), we convert Equation (5d) to the form
\[
\cos(\text{angle } ABC) + \cos(\text{angle } DEF) = 0
\]  

(7)

Using the law of cosines on the triangles in Figures 1 and 2, we obtain

\[
\cos(\text{angle } ABC) = \frac{(r_1^2 s^2 + r_5^2 t^2 - |I_{11}|^2)}{(2r_1 r_5 s t)}
\]

(8a)

\[
\cos(\text{angle } DEF) = \frac{(35^2 r_1^2 r_5^2 + 64s^2 t^2 - |I_{22}|^2)}{(560r_1 r_5 s t)}
\]

(8b)

Using Equations (8) with \( r_1 \) and \( r_5 \) expressed in terms of \( s \) and \( t \) by Equations (2a) and (2b), we transform Equation (7) to the form

\[
\gamma_1 \gamma_2 - 280|I_{11}|^2 - |I_{22}|^2 = 0
\]

(9)

Hence, only seven of the eight Equations (2a) through (2f) can be independent. Equation (9) can in turn be transformed into the following three equivalent equations which will be useful in parameterizing \( z_1 \) and \( z_2 \) in terms of \( s \) and \( t \):

\[
-8\gamma_2 + 4|I_{22}|^{-2}(\gamma_1 \gamma_2 + |I_{22}|^2)\gamma_2 = 1120\gamma_2 |I_{11}|^2 |I_{22}|^{-2}
\]

(10a)

\[
-8\gamma_1 + 4|I_{22}|^{-2}(\gamma_1 \gamma_2 + |I_{22}|^2)\gamma_1 = 1120\gamma_1 |I_{11}|^2 |I_{22}|^{-2}
\]

(10b)

\[
\frac{\gamma_1 \gamma_2 - |I_{22}|^{-2}(\gamma_1 \gamma_2 + |I_{22}|^2)^2/4}{-140^2} = -140^2 |I_{11}|^{-2} |I_{22}|^{-2}
\]

(10c)

To parameterize \( z_1 \) and \( z_2 \) in terms of \( s \) and \( t \), we use the following equations obtained from the triangle representations of Equations (2d) and (2e) shown in Figures 1 and 2:

\[
x_1^2 + y_1^2 = |I_{11}|^{-2} r_5^2
\]

(11a)

\[
(x_1 + 1)^2 + y_1^2 = |I_{11}|^{-2} r_1^2 s^2
\]

(11b)

\[
x_2^2 + y_2^2 = 64r_5^{-2} s^2 t^2
\]

(11c)
Expressing \( r_1 \) and \( r_5 \) in terms of \( s \) and \( t \) by Equations (2a) and (2b) and using Equations (9) and (10), we can solve Equations (11) for \( x_1 \), \( x_2 \), \( y_1 \), and \( y_2 \) in terms of \( s \) and \( t \):

\[
\begin{align*}
(x_2 + 1)^2 + y_2^2 &= 35^2 |I_2|^2 - 2 r_1^2 r_5^2 \\
(11d) &
\end{align*}
\]

\[
\begin{align*}
x_1 &= |I_1|^{-2}(y_1 s^2 - y_2 t^2 - 35|I_1|^2)/70 \\
(12a) &
\end{align*}
\]

\[
\begin{align*}
x_2 &= -4|I_2|^{-2}(y_1 s^2 + y_2 t^2 - 35|I_1|^2) \\
(12b) &
\end{align*}
\]

\[
\begin{align*}
y_1^2 &= -70^{-2}|I_1|^{-4}(y_1 s^2 + y_2 t^2 + 2|I_2|st - 35|I_1|^2) \\
(12c) &
\end{align*}
\]

\[
\begin{align*}
y_2^2 &= 16|I_2|^{-4}70^2|I_1|^{-4}y_1^2 \\
(12d) &
\end{align*}
\]

We see that the equations \( y_2^2 = 0 \) and \( y_1^2 = 0 \) are identical. The expression within each pair of parentheses of Equation (12c), when equated to zero, represents an ellipse. The length of the minor and major axes of the ellipses are, respectively,

\[
\begin{align*}
\beta_- &= 2(70)^{1/2} |I_1|/|y_1 + y_2 + (y_1 + y_2)^2/120|I_1|^2 |1/2 \\
(13a) &
\end{align*}
\]

\[
\begin{align*}
\beta_+ &= 2(70)^{1/2} |I_1|/|y_1 + y_2 - (y_1 + y_2)^2/120|I_1|^2 |1/2 \\
(13b) &
\end{align*}
\]

The graph of the equation \( y_2^2 = 0 \) is shown in Figure 3. \( R' \) is the set of \((s,t)\) for which the first and second triangles exist, or, equivalently, the set for which the right-hand side of Equation (12c) is non-negative. Since the right-hand side of Equation (12c) is negative at the origin and at infinity and \( s, t \) are non-negative, \( R' \) must be the set of points lying within one ellipse but in the complement of the other in the first quadrant of the \( s, t \) plane as seen in Figure 3.

At this point the angles \( \theta_1, \theta_4, \) and \( \theta_5 \) have been parameterized by \( z_1, z_2, \) and \( \theta_2 \) using Equations (5) which were derived from Equations (2d) and (2e). Since \( r_1, r_5, r_3, z_1, \) and \( z_2 \) were parameterized by the \( s, t \) in \( R' \) using Equations (2a), (2b),
(2c), (2d), and (2e), respectively, the set defined by Equations (2a) through (2e) is parameterized by s, t, and $\theta_2$ with the (s,t) lying in $R'$ and no constraint on the values of $\theta_2$. The only remaining variable is $\theta_3$ which will be parameterized by Equation (2f).

We will now determine the s, t which satisfy Equation (2f). Letting

$$A = r_1 t \sin(\theta_1 + \theta_4) - s r_5 \sin(\theta_2 + \theta_5)$$  \hspace{1cm} (14a)$$

$$B = -r_1 t \cos(\theta_1 + \theta_4) + s r_5 \cos(\theta_2 + \theta_5)$$  \hspace{1cm} (14b)$$

$$C = I_3 / r_3$$  \hspace{1cm} (14c)$$

$$x = \cos(\theta_3)$$  \hspace{1cm} (14d)$$

$$y = \sin(\theta_3)$$ \hspace{1cm} (14e)$$

We can write Equation (2f) as

$$C = Ax + By$$  \hspace{1cm} (15)$$

Solving for $\theta_3$, we have

$$\theta_3^+ = \arccos \left( \frac{AC + [B^2(A^2 + B^2 - C^2)]^{1/2}}{(A^2 + B^2)} \right)$$  \hspace{1cm} (16a)$$

$$\theta_3^- = \arccos \left( \frac{AC - [B^2(A^2 + B^2 - C^2)]^{1/2}}{(A^2 + B^2)} \right)$$  \hspace{1cm} (16b)$$

Since the arccos function is multivalued, it appears that $\theta_3$ could have four solutions. However, for fixed values of A, B, and C, Equation (15) defines a line in the variables x and y. The simultaneous solutions in x and y to Equations (14d), (14e), and (15) will be the intersection points of the line and the circle $x^2 + y^2 = 1$. Hence, for fixed values of A, B, and C, Equations (14d), (14e), and (15) can be solved by at most two values of $\theta_3$ which must be determined by Equations (16a) and (16b), respectively. Consequently, $\theta_3^+$ and $\theta_3^-$ are single-valued. When the line is tangent to the circle, $\theta_3$ has one solution and $\theta_3^+$ equals $\theta_3^-$. 
i.e., $\theta_3$ has two solutions if $A^2 + B^2 - C^2$ is positive and one solution if $A^2 + B^2 - C^2$ is zero.

Finding the $s,t$ which satisfy Equation (2f) is now reduced to finding the $s,t$ for which $A^2 + B^2 - C^2$ is non-negative. Defining $u = s^2 + t^2$ and $P(u) = A^2 + B^2 - C^2$, we have

$$P(u) = -8u^2/35 + (y_1 + y_2)u/35 - |I_1|^2 - 35|I_3|^2/(y_3 - 27u)$$

We will show that the values of $u$ satisfying Equations (2) must lie between two roots of $P(u)$ in the interval $[\beta_2^2, \beta_2^2] \cap (0, y_3/27)$. Let $Q(u) = (y_3 - 27u)P(u)$. Since $Q(u)$ is a cubic polynomial in $u$, it has three roots. Consequently, $P(u)$ also has three roots because $Q(y_3/27) = -35|I_3|^2$ with $I_3$ being, in general, nonzero. Since $P(u)$ is positive as $u$ approaches $y_3/27$ from the right but negative as $u$ approaches $\infty$, $P(u)$ has one real root greater than $y_3/27$.

However, the two remaining roots of $P(u)$ must lie in the interval $[\beta_2^2, \beta_2^2] \cap (0, y_3/27)$. In order that Equation (15) be solvable for nontrivial values of $\theta_3$, $P(u)$ must be positive for some value of $u'$ corresponding to a point $(s', t')$ in $R'$. In particular, $u'$ must be less than $y_3/27$ to satisfy Equation (2c). Hence, $P(u)$ must be positive for some values of $u$ in the interval $[\beta_2^2, \beta_2^2] \cap (0, y_3/27)$.

Since $P(u)$ is negative at both $\beta_2^2$ and $\beta_2^2$ and is also negative as $u$ approaches $y_3/27$ from the left, $P(u)$ must have its two remaining roots $u_1, u_2$ in $[\beta_2^2, \beta_2^2] \cap (0, y_3/27)$. Hence, $P(u)$ is non-negative for $(s, t)$ in $R'$ which lie in the annulus with inner radius $\sqrt{u_1}$ and outer radius $\sqrt{u_2}$ assuming $u_1$ is less than $u_2$.

We conclude that the $s,t$ for which $\theta_3$ is defined, i.e., the $s,t$ for which $A^2 + B^2 - C^2$ is non-negative, lie in the annulus with radii $\sqrt{u_1}$ and $\sqrt{u_2}$. The set $R$ shown in Figure 4 is the intersection of this annulus and $R'$. $R$ is the set of points in $s,t$ space which we use to parameterize the set $S$.

The coordinates $\theta_1$, $\theta_2$, $\theta_4$, and $\theta_5$ of the points in $S$ have now been parameterized by the coordinates $s,t$, and $\theta_2$ of points in the set $R \times [0, 2\pi]$. Equations (5) are used to parameterize $\theta_1$, $\theta_4$, and $\theta_5$ and Equations (16) are used for $\theta_3$. Equations (2a), (2b), and (2c) are used to parameterize $r_1$, $r_5$, and $r_3$, respectively. Collecting these equations, we have:
\[ r_1 = \left[ (y_1 - 8t^2)/35 \right]^{1/2} \] (17a)

\[ r_2 = s \] (17b)

\[ r_3 = \left[ (y_3 - 27(s^2 + t^2))/35 \right]^{1/2} \] (17c)

\[ r_4 = t \] (17d)

\[ r_5 = \left[ (y_2 - 8t^2)/35 \right]^{1/2} \] (17e)

\[ \theta_1 = \theta_2 + \arg(z_1 + 1) + \psi_1 \] (17f)

\[ \theta_2 = \theta_2 \] (17g)

\[ \theta_4 = \theta_2 - \arg(z_2) - \psi_2 \] (17h)

\[ \theta_5 = \theta_2 + \arg(z_1 + 1) - \arg(z_2 + 1) + \psi_1 - \psi_2 \] (17i)

\[ \theta_3^+ = \begin{cases} \arccos_1(\Phi^+) & \text{if } \lambda(\arccos_1(\Phi^+)) = 0 \\ \arccos_2(\Phi^+) & \text{if } \lambda(\arccos_2(\Phi^+)) = 0 \end{cases} \] (17j)

\[ \theta_3^- = \begin{cases} \arccos_1(\Phi^-) & \text{if } \lambda(\arccos_1(\Phi^-)) = 0 \\ \arccos_2(\Phi^-) & \text{if } \lambda(\arccos_2(\Phi^-)) = 0 \end{cases} \] (17k)

\[ \Phi^+ = (AC + [B^2(A^2 + B^2 - C^2)]^{1/2})/(A^2 + B^2) \]

\[ \Phi^- = (AC - [B^2(A^2 + B^2 - C^2)]^{1/2})/(A^2 + B^2) \]

\[ \lambda(\theta) = C - A \cos(\theta) - B \sin(\theta) \]

Here \( \lambda(\theta) = 0 \) is Equation (2f) and \( \arccos_i \) for \( i = 1,2 \) are the two branches of the \( \arccos \) function taking values between 0 and \( \pi \) and between \( \pi \) and 2\( \pi \), respectively.
The variables $A$, $B$, and $C$ are defined by Equations (14a), (14b), and (14c), respectively, and $z_1$ and $z_2$ are defined by Equations (12).

Equations (17), however, cannot be used in their present form as a one-to-one parameterization of $S$ because $\theta_3$ has multiple values. In addition, the signs of $y_1$ and $y_2$ are undetermined implying both branches of the arg function must be used in Equations (17f), (17h), and (17i). As indicated in the derivation of Equations (16), $\theta_3$ has two values for any $< s, t, \theta_2 >$ in $R \times [0, 2\pi]$, i.e., $\theta^+_3$ and $\theta^-_3$.

Also, for any $< s, t >$ in $R$ Equations (17) do not specify the signs of $y_1(s,t)$ and $y_2(s,t)$ implying $\arg(z_1)$ and $\arg(z_1+1)$ for $i = 1, 2$ will each have two values for any $< s, t >$ in $R$. However, only the sign of $y_1$ needs to be determined because we can show that $y_1$ and $y_2$ have opposite signs for any $s, t$ in $R$. The argument proceeds as follows. Since $\arg(z_1) = \arg(z_1+1) + \angle ABC$ as seen in Figure 1, we have

$$\arg(z_1) - \arg(z_1+1) \geq 0 \text{ for } y_1 \geq 0$$

and

$$\arg(z_1) - \arg(z_1+1) \leq 0 \text{ for } y_1 \leq 0$$

Exactly the same inequalities hold with $z_1$, $y_1$, and angle $ABC$ replaced with $z_2$, $y_2$, and angle $DEF$, respectively, in Figure 2. If $y_1 \geq 0$, the angle $ABC$ lies between $0$ and $\pi$ implying by Equation (5d) that

$$\arg(z_2) - \arg(z_2+1) = \angle DEF \leq 0$$

and, therefore, $y_2 \leq 0$. Similarly, if $y_1 \leq 0$, then $y_2 \geq 0$.

To obtain a one-to-one parameterization of $S$, we can account for the two values of $\theta_3$ and the sign of $y_1$ by reformulating Equations (17) as four maps $\psi_1$, ..., $\psi_4$ corresponding to four cases:

$$\theta_3 = \theta^+_3, \ y_1 \geq 0 \quad \text{for case 1} \quad (18a)$$

$$\theta_3 = \theta^+_3, \ y_1 \leq 0 \quad \text{for case 2} \quad (18b)$$

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\[ \theta_3 = \theta_3^+, y_1 \geq 0 \quad \text{for case 3} \]  
\[ \theta_3 = \theta_3^-, y_1 \leq 0 \quad \text{for case 4} \]

Each \( \psi_i \) is a one-to-one map from \( R \times [0, 2\pi] \) to \( S \)

\[ \psi_i : R \times [0, 2\pi] \rightarrow S \]

for \( i = 1, \ldots, 4 \). The map \( \psi_1 \) is defined by Equations (17) with \( \theta_3 = \theta_3^+ \) and with the arg function in Equations (17f), (17h), and (17i) taking values between 0 and \( \pi \), i.e., \( y_1 \geq 0 \). The map \( \psi_2 \) will also have values \( \theta_3 \) defined by \( \theta_3^+ \) but the arg function will take values between \( \pi \) and \( 2\pi \), i.e., \( y_1 \leq 0 \). However, \( \psi_3 \) will have values \( \theta_3 \) defined by \( \theta_3^- \) and the arg function defined for \( y_1 \geq 0 \). The map \( \psi_4 \) will have values \( \theta_3 \) defined by \( \theta_3^- \) and the arg function defined for \( y_1 \leq 0 \).

To construct the parameterization \( \psi \) of \( S \) from \( \psi_1, \ldots, \psi_4 \) we need to paste together four copies of \( R \times [0, 2\pi] \). To facilitate the construction of \( \psi \), we can view \( R \times [0, 2\pi] \) as a box because \( R \) is homeomorphic to a square. That is, \( R \) can be mapped by a homeomorphism onto a square \( R_{sq} \) in a space with new coordinates \( s', t' \) as shown in Figure 5; the two sides of \( R \) determined by the ellipses in Figure 4 are mapped to opposite sides of the square and the two sides determined by the annulus are mapped to the other two sides of the square. The four copies of \( R_{sq} \times [0, 2\pi] \) viewed as boxes can be pasted together with corresponding sides matched as shown in Figure 6 to form a composite box with periodic sides, i.e., a three-dimensional torus. Each map \( \psi_i \) will be defined on the \( i \)th box in Figure 6 for \( i = 1, \ldots, 4 \). On the surfaces of the boxes the \( \psi_i \) for \( i = 1, \ldots, 4 \) have values in common because 
\[ \theta_3^+ = \theta_3^- \] for any \( < s, t, \theta_2 > \) with \( < s, t > \) on the circles defined by \( s^2 + t^2 = u_1 \) and \( s^2 + t^2 = u_2 \) in Figure 4. Also, the two branches of the arg function (for \( y_1 \geq 0 \) and \( y_1 \leq 0 \)) agree for any \( < s, t, \theta_2 > \) with \( < s, t > \) on the ellipses defined by \( y_1^2 = 0 \) in Figure 4. Since each \( \psi_i \) is one-to-one and continuous on the \( i \)th box and the \( \psi_i \) have values in common only on the surfaces of the boxes for \( i = 1, \ldots, 4 \), we conclude that the \( \psi_1, \ldots, \psi_4 \) define a homeomorphism \( \psi \) from the three-dimensional torus consisting of four copies of \( R_{sq} \times [0, 2\pi] \) pasted together to the set \( S \) defined by the constants of motion.
EXPLICIT SOLUTION OF $w_3(t)$

An explicit solution of the $w_3(t)$ component of Equations (1) will be derived. We start by differentiating Equation (1c) with respect to time $t$. Using Equations (2a), (2b), and (2c) we obtain

$$\frac{d^2 w_3}{dt^2} = (\alpha + \beta r_3^2) w_3$$  \hspace{1cm} (19)$$

where $\alpha = 16\gamma_3 - 27(\gamma_1 + \gamma_2)$ and $\beta = -560$. Converted to polar coordinates, Equation (19) can be represented as two equations

$$\frac{d^2 r_3}{dt^2} - \beta r_3^3 - ar_3 - r_3\left(\frac{d\theta_3}{dt}\right)^2 = 0$$  \hspace{1cm} (20a)$$

$$2\left(\frac{dr_3}{dt}\right)\left(\frac{d\theta_3}{dt}\right) + r_3 \left(\frac{d^2 \theta_3}{dt^2}\right) = 0$$  \hspace{1cm} (20b)$$

Integrating the second equation with respect to $t$, we obtain

$$\frac{d\theta_3}{dt} = r_3^{-2} c_1$$  \hspace{1cm} (21)$$

where $c_1$ is a constant. If we substitute $d\theta_3/dt$ into the first equation, multiply by $dr_3/dt$, integrate and multiply the result by $r_3^2$, the first equation becomes

$$(dy/dt)^2 = 2\beta y^3 + 4ay^2 - 8c_2y - 4e$$  \hspace{1cm} (22)$$

where $y = r_3^2$ and $c_2$ is a constant from the integration. Equation (22) can be written in the form

$$(dy/dt)^2 = h^2(y-\lambda_1)(y-\lambda_2)(y-\lambda_3)$$  \hspace{1cm} (23)$$

where $h^2 = 2\beta$ and $\lambda_1$, $\lambda_2$, and $\lambda_3$ are the roots of the right-hand side. This is an elliptic equation. It has an explicit solution\textsuperscript{10}

$$r_1 = y = \lambda_3 + (\lambda_2 - \lambda_3)sn^2(hM(t+t_0),k)$$  \hspace{1cm} (24)$$

where $h^2 = 0.25(\lambda_1 - \lambda_3)$, $k^2 = (\lambda_2 - \lambda_3)/(\lambda_1 - \lambda_3)$, and $sn$ is a Jacobian elliptic function. Hence, $r_1$ is periodic with period $\rho = 2K/hM$, where
\[ k = \frac{1}{\pi} \left( \int_0^1 \frac{dx}{(1-x^2)(1-k^2x^2)} \right)^{1/2} \]

Now \( \theta_3(t) \) can be obtained by integrating Equation (21) with respect to \( t \) using Equation (24).

**MIXING**

We will show that the real and imaginary components of \( w_3(t) \) are quasi-periodic. Letting \( x_3 \) and \( y_3 \) be the real and imaginary parts of \( w_3 \), respectively, we can convert Equation (19) into two second order equations

\[
\begin{align*}
\frac{d^2x_3}{dt^2} &= (\alpha + \beta r_3^2)x_3 \\
\frac{d^2y_3}{dt^2} &= (\alpha + \beta r_3^2)y_3
\end{align*}
\]

Letting \( v_1 = x_3 \) and \( v_2 = dx_3/dt \), we now convert (25a) into a first-order system of two equations

\[ \frac{dv}{dt} = H(t)v \]  

with \( v = (v_1, v_2) \) and \( H(t) \) a \( 2 \times 2 \) matrix. Because \( r_3^2(t) \) is periodic with period \( \rho \), \( H(t) \) is periodic with the same period. By Floquet theory there exists a periodic nonsingular matrix \( P(t) \) of period \( \rho \) and a constant matrix \( R \) such that a fundamental matrix \( \Phi(t) \) of the system (26) can be represented as

\[ \Phi(t) = P(t)e^{tR} \]

However, the eigenvalues of the matrix \( e^R \) must be pure imaginary because \( r_3(t) = r_3(-t) \) making \( x_3(t) = x_3(-t) \) by Equation (25a) and both \( x_3 \) and \( dx_3/dt \) are bounded as seen by inspection of Equations (1c), (2a), (2b), and (2c). Transforming \( e^R \) into Jordan canonical form and representing the transformed matrix in exponential form \( e^{R'} \), we see that the new fundamental matrix \( \Phi'(t) = P'(t)e^{tR'} \) (\( P' \) being the transformed matrix \( P \)) must be quasi-periodic. Hence, \( x_3(t) \) is quasi-periodic as is \( y_3(t) \) by the same argument applied to Equation (25b).

Since \( w_3(t) \) is quasi-periodic, the solution to Equations (1) can never, in general, be mixing. In particular, since \( r_3(t) \) is periodic, the solution can never be mixing on the set \( S \) defined by the constants of motion.
NUMERICAL TEST

The Gear-Hindmarsh\textsuperscript{13} ordinary differential equation solver was used to solve the system of Equations (1) for the initial point $w_1 = 1$, $w_2 = 1$, $w_3 = 1$, $w_4 = 1$, and $w_5 = \sqrt{-1}$. The parameter EPS was set at $10^{-7}$. At every 10 time steps the solution was plugged into Equations (2) to ensure that the resulting constants of motion agreed to four digits with those of the initial point, namely, $Y_1 = 43$, $Y_2 = 43$, $Y_3 = 89$, $I_1 = 1 + \sqrt{-1}$, $I_2 = -8 - 35\sqrt{-1}$, and $I_3 = -1$. An Apollo DN420 computer was used with fifteen digits of accuracy (double precision).

When the system was integrated to $t = 1$, the trajectory of the system wound around the torus passing repeatedly through the four boxes in Figure 6. The trajectory projected onto $s,t$ space intersects repeatedly the boundary of $R$ but never crosses it as seen in Figure 7. That the trajectory appears in Figure 7 to be tangent to the boundary of $R$ while never crossing it is evidence that the derivation of the region $R$ is correct.
Figure 1 - First Triangle

Figure 2 - Second Triangle
Figure 3 - Graph of $y_1^2 = 0$; $R'$ is the Shaded Region

Figure 4 - The Set $R$ Defined by the Intersection of $R'$ and the Annulus with Inner Radius $\sqrt{u_1}$ and Outer Radius $\sqrt{u_2}$
Figure 5 - The Square $R_{sq}$ in $s', t'$ Space Homeomorphic to $R$ with Sides Labeled by Equations Defining Corresponding Sides of $R$

Figure 6 - Four Copies of the Box $R_{sq} \times [0, 2\pi]$ Homeomorphic to a Three-Dimensional Torus
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