### REPORT DOCUMENTATION PAGE

**1. REPORT NUMBER**
ETL-R-079

**2. GOVT ACCESSION NO**

**4. TITLE (and Subtitle)**
Astrogeodetic Deflections of the Vertical from Stars Observations with the Danjon Astrolabe, or Similar Instruments

**5. TYPE OF REPORT & PERIOD COVERED**

**6. PERFORMING ORG. REPORT NUMBER**

**7. AUTHOR(S)**
Dr. Angel A. Baldini

**8. CONTRACT OR GRANT NUMBER(S)**

**9. PERFORMING ORGANIZATION NAME AND ADDRESS**
U. S. Army Engineer Topographic Laboratories
Fort Belvoir, VA 22060-5546

**10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS**

**11. CONTROLLING OFFICE NAME AND ADDRESS**

**12. REPORT DATE**

**13. NUMBER OF PAGES**

**15. SECURITY CLASS. (of this report)**

**16. DISTRIBUTION STATEMENT (of this Report)**
Approved for public release.
Distribution Unlimited.

**17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)**

**18. SUPPLEMENTARY NOTES**

**19. KEY WORDS (Continue on reverse side if necessary and identify by block number)**
Deflection of the Vertical,
vertex of isosceles triangles,
astrolabe observations,
latitude and longitude,
Danjon Astrolabe,
astro-geodetic observations.

**20. ABSTRACT (Continue on reverse side if necessary and identify by block number)**
The paper addresses a new observational method that allows one to compute the astronomic zenith distance of the astrolabe instrument as a function of the time when the stars cross the almucantar of the astrolabe and their right ascension and declination, therefore independent of the astronomic station coordinates.

---

Received by DTIC: APR 24 1983

**DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE**
Astrogeodetic Deflections of the Vertical
From Stars Observations with the Danjon
Astrolabe, or Similar Instruments

Dr. Angel A. Baldini
Research Institute
U.S. Army Engineer Topographic Laboratories (ETL)
Fort Belvoir, VA 22060-5546

BIOGRAPHICAL SKETCH
Dr. Angel A. Baldini joined ETL's predecessor organization in 1960. From 1957-1960 he was associated with the Georgetown University Observatory, Washington, D.C. Prior to 1957, he was Professor and Head, Department of Geodesy, La Plata University, Argentina. At ETL, Dr. Baldini works primarily in the field of astro-geodesy. Since 1974 he has been senior scientist in the Center for Geodesy, Research Institute, ETL. He authored 45 reports and papers since 1963 and presented the results at 25 Army, national, and international meetings. He received an Army Research and Development achievement award in 1969. He has been a member of the American Geophysical Union since 1960.

ABSTRACT
This report is a sequel to the report "Basic Formulae for Determining the Computation of the Deflection of the Vertical from Astronomic Star Observations," presented at the 1985 ASP-ACSM Convention, Washington, DC.

The paper addresses a new observational method that allows one to compute the astronomic zenith distance of the astrolabe instrument as a function of the time when the stars cross the almucantar of the astrolabe and their right ascension and declination, therefore independent of the astronomic station coordinates. The and deflection of the vertical components are obtained from an equation that relates the astronomic and geodetic zenith distance. The last is derived for each star as a function of the U.T.1 time and geodetic latitude and longitude.

INTRODUCTION
The first part of this paper deals with the definition of a point on a sphere as a common vertex of isosceles triangles, when the position of great circle arcs among the points on the sphere are known. One application of this concept is in astronomy of position. Herein it is applied to astrolabe observations to obtain the instrument constant zenith distance. Many isosceles triangles, depending on the number of stars used, can be considered. A constraint is that their common vertex is the zenith. In this particular case the almucantar zenith distance is obtained. For the second part of this report, vertical deflection components are obtained from astronomic, and the geodetic zenith distance of each observed star with respect to UT1 Time. Astronomic latitude and longitude may be obtained thereafter. A computation example is treated in some detail, employing data generated by a Danjon Astrolabe at the US Naval Observatory, Washington, D.C.
FUNDAMENTAL CONCEPTS

Consider a sphere of unit radius, with center at C, and three points 1, 2, 3 on it. Let these points be joined with great circles of arc each of which pass through two points, as shown in Figure 1, and let a portion of the great circles of arc be:

\[ a = 1.2 \]
\[ b = 2.3 \]
\[ c = 1.3 \]  

(1)

Figure 1.

On this sphere let point A be chosen in order to form two isosceles triangles. The two triangles are: 1-2-A and 2-3-A.

Let the angles at point 2, be:

\[ B_1 = 1-2-A \]
\[ B_2 = 3-2-A \]  

(2)

If the arcs a, b, and c, are known, the angles B₁ and B₂ can be computed as a function of them, hence the point A can be defined. Consequently the portion of the great arcs between A and the points 1, 2, and 3, must be equal. The sum of \( B_1 + B_2 \) can be known through the equation

\[ \cos (B_1 + B_2) = \frac{\cos C - \cos \alpha \cos \beta}{\sin \alpha \sin \beta} \]  

(3)

We have found that these angles can be obtained from the equations
To obtain the arc distance from A to the points 1, 2, and 3 we have:

\[ y = A-1 = A-2 = A-3 \]

then \( y \) can be obtained from the following equations:

\[ \tan y = \frac{\tan \frac{\theta}{2} - \tan \frac{\alpha}{2} \cos \beta}{\tan \frac{\alpha}{2} \sin (\theta + \beta)} \]  

(4)

\[ \tan \alpha = \frac{\tan \frac{\theta}{2} a - \tan b \sin (\theta + \beta)}{\tan \frac{\beta}{2} \sin (\beta + \theta)} \]  

(5)

APPLICATION OF THE FUNDAMENTAL CONCEPT TO ASTROLABE OBSERVATIONS

The main application of the preceding concept is for observations of stars at an equal altitude, because many isosceles triangles can be considered. This is the case when the Danj n Astrolabe or similar instrument is employed. A theodolite may be used in the same way as an astrolabe, provided that during observations no movement should change the angle between the telescope and the bubble. To this case point 1, 2, 3, are replaced by images of the stars and the point A will be the zenith, the pole of the horizon. The arcs \( a, b, c \), are related to the meridian of the stars and the astronomical meridian. The arc \( y \) then will be the almucantar zenith distance. If we observe more than three stars at the same altitude, we have more than sufficient data for the determination of the almucantar zenith.

Figure 2. Unit Sphere Projected from Above.
The arc of the great circle that passes between two stars is computed from the general equation:

$$\cos \delta_i \cos \delta_j \sin^2 \delta_i - \sin \delta_i \sin \delta_j + \cos \delta_i \cos \delta_j \cos \delta \cos \delta_i = 0 \quad (7)$$

Let us designate by the angle of the isosceles triangles as seen in figure 2. The zenith distance then can be computed according to equation (7).

Let us now find out the relationship of a star position to the astronomical and geodetic points, to obtain the deflections of the vertical as functions of the astronomical and geodetic star zenith distance.

**DEFLECTIONS OF THE VERTICAL AS FUNCTION OF THE ASTRONOMIC AND GEODETIC ZENITH DISTANCES OF A STAR.**

For this purpose we use a general formula that links a star position to its zenith distance and hour angle, setting it as follows in the astronomical system:

$$2 \tan \delta = 2 \cot \delta \sin \phi + \cos \phi + \cos \phi \theta$$

A similar equation can be set for the station in the local geodetic system as follows:

$$2 \tan \delta = 2 \cot \delta \sin \beta + \cos \beta + \cos \beta \theta$$

$$Z_i - \text{station in the local geodetic system}$$

We have to remember that \( \phi \) and \( \lambda \) are unknowns, while \( \beta \) and \( \lambda \) are knowns, but they differ, with respect to \( \phi \) and \( \lambda \), by small amount, \( \varepsilon \) and \( \eta \), the deflections of the vertical. From equations (9) and (10) we shall derive an equation for obtaining \( \varepsilon \) and \( \eta \) as a function of the astronomical and geodetic zenith distance. Subtracting equation (10) from (9) we have

$$-4 \sin \frac{1}{2} (Z - \lambda) \sin \frac{1}{2} (Z - \lambda) = 4 \tan \delta \sin \frac{1}{2} (\phi - \beta) \cos \frac{1}{2} (\phi \theta)$$

Now let

$$X_1 = \frac{1}{2} (\phi + \varepsilon)$$

$$X_2 = \frac{1}{2} (\beta - \eta)$$

so it can be set

$$\frac{1}{2} (X_1 + X_2) = \frac{1}{2} (\phi + \varepsilon + \beta - \eta)$$

$$\frac{1}{2} (X_1 - X_2) = \frac{1}{2} (\phi - \beta + \varepsilon - \eta)$$
then it follows at once that the corresponding terms on the right hand side of equation (11) can be expressed as follows:
\[
\cos \chi, - \cos \chi = -2 \left[ \sin \left( \frac{\theta - \beta}{2} \right) \cos \left( \frac{\theta - \beta}{2} \right) + \cos \left( \frac{\theta - \beta}{2} \right) \sin \left( \frac{\theta - \beta}{2} \right) \right] \\
\left( \sin \left( \frac{\theta - \beta}{2} \right) \cos \left( \frac{\theta - \beta}{2} \right) + \cos \left( \frac{\theta - \beta}{2} \right) \sin \left( \frac{\theta - \beta}{2} \right) \right)
\]
(12)

Due to the small values of \( \frac{\theta - \beta}{2} \) and \( \frac{\theta - \beta}{2} \), we set for practical purposes:
\[
\cos \frac{\theta - \beta}{2} = \cos \frac{\theta - \beta}{2} = 1
\]

hence equation (12) becomes:
\[
\cos \chi = -(\theta - \beta) \sin \theta \cos \theta - 2 \sin \left( \frac{\theta - \beta}{2} \right) \sin \left( \frac{\theta - \beta}{2} \right)
\]
(13)

In a similar way, let
\[
m = \phi - \ell \\
n = \phi + \tau
\]
from which it is obtained for the third and fourth term on the right side of equation (11)
\[
\cos m - \cos n = -(\theta - \beta) \sin \beta \cos \beta + 2 \sin \left( \frac{\theta - \beta}{2} \right) \sin \left( \frac{\theta - \beta}{2} \right)
\]
(14)

adding to equation (11), equations (13) and (14), and using the constraints
\[
\phi - \ell = E \\
\theta - \tau = L - \lambda = n \sec \delta
\]
(15)

as well as the fact that \( Z - \zeta \) is small, we obtain the final expression for computing the \( E \) and \( N \) deflections of the vertical components:
\[
(Z - \zeta) \sin \frac{Z}{\cos \delta} = E \left( \tan \delta \cos \beta - \sin \beta \cos \tau \right) + n \sin \tau
\]
(16)

where
\[
\zeta = UT (t + c) + \theta_0 - (\alpha - \ell)
\]
(17)

and
\[
UT = \text{universal time} \\
\theta_0 = \text{apparent sidereal time at \( \Thetah \) U.T.} \\
L = \text{geodetic longitude} \\
B = \text{geodetic latitude} \\
\kappa = \text{geodetic zenith distance} \\
Z = \text{astronomic zenith distance}
\]

The geodetic zenith distance is known by computation from
\[
\cos Z = \sin B \sin \delta + \cos B \cos \delta \cos \tau
\]
(18)

Each star provides an observation equation of the form:
\[
a \cdot E + b \cdot N = (Z - \zeta) \sin \zeta \sec \delta
\]
The set of four stars used in this example, according to its azimuth is: 5-16-12-4, which we reorder with numbers: 1-2-3-4. Geodetic coordinates, arbitrarily chosen, are

\[
\begin{align*}
B &= 39^\circ \ 54' \ 00'' \\
L &= -5^h \ 08^m \ 13^s.576
\end{align*}
\]

Table 1, shows the evaluation of zenith distance of the Danjon Astrolabe. The first column indicates the two angles opposed to the zenith, as shown in Figure 2, column 2, the corresponding values of these angles are computed according to equation (3). Column 3, 4, 5, and 6, show the values of the angles \( \psi_{ij} \) computed according to equation (4).

Table 2, shows the values of the geodetic zenith distances computed by using equation (16). The geodetic hour angle \( \alpha \), is obtained from (17), the values of which are shown in column 2. In column 3, are the values of \( \phi \), and in the fourth column the difference between geodetic and astronomic zenith distance.
Table 3, shows the values of the coefficients a, b, c, computed from:

\[
\begin{align*}
    a &= \tan \delta \cos \theta - \sin \theta \cos \tau \\
    b &= \sin \theta \sin \tau \\
    c &= \sin Z / \cos \delta
\end{align*}
\]

In the last column are the values of the left member of equation (16).

Table 4, shows the evaluation of the deflection components according to equation (16). The evaluation of \( \xi \) and \( \eta \) are computed from the equation:

\[
\alpha \cdot \xi + b \cdot \eta = c (\xi - Z)
\]

taken clockwise we use two equations, in the following order: 1-2; 2-3; 3-4; 4-1.

Table 1. Evaluation of Astronomic Zenith Distance

<table>
<thead>
<tr>
<th>Reference Angle</th>
<th>Evaluation Angle</th>
<th>( W_{12} )</th>
<th>( W_{13} )</th>
<th>( W_{23} )</th>
<th>( W_{34} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{12} = W_{23} )</td>
<td>105° 03' 31.88</td>
<td>65° 55' 10.75</td>
<td>39 08 20.54</td>
<td>39 06 19.71</td>
<td>52° 53' 53.21</td>
</tr>
<tr>
<td>( W_{12} = W_{13} )</td>
<td>105° 03' 31.84</td>
<td>65° 55' 11.10</td>
<td>39 08 20.54</td>
<td>39 06 19.71</td>
<td>52° 53' 53.21</td>
</tr>
<tr>
<td>( W_{12} = W_{24} )</td>
<td>39 02 12.92</td>
<td>12.92</td>
<td>39 08 20.54</td>
<td>39 06 19.71</td>
<td>52° 53' 53.21</td>
</tr>
<tr>
<td>( W_{23} = W_{34} )</td>
<td>50 02 15.52</td>
<td>50 02 15.52</td>
<td>39 08 20.54</td>
<td>39 06 19.71</td>
<td>52° 53' 53.21</td>
</tr>
</tbody>
</table>

Mean

\[
\begin{align*}
    \alpha &= 65° 55' 10.93 \\
    \beta &= 39° 08 20.54 \\
    \gamma &= 37° 08 21.13 \\
    \delta &= 52° 53' 53.21
\end{align*}
\]

Table 2. Evaluation of Geodetic Zenith Distance and

<table>
<thead>
<tr>
<th>STA</th>
<th>( \xi )</th>
<th>( \zeta )</th>
<th>( \xi - Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>29° 56' 56.09</td>
<td>30° 00' 25.28</td>
<td>-2°47</td>
</tr>
<tr>
<td>2</td>
<td>39 56 32.50</td>
<td>30 01 34.59</td>
<td>66.84</td>
</tr>
<tr>
<td>3</td>
<td>-39 40 10.64</td>
<td>30 00 47.21</td>
<td>19.75</td>
</tr>
<tr>
<td>4</td>
<td>-20 38 0210</td>
<td>29 59 10.04</td>
<td>-71.75</td>
</tr>
</tbody>
</table>

Table 3. Evaluation of Coefficients a, b, c

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>( c(\xi - Z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2121</td>
<td>0.4992</td>
<td>0.5432</td>
<td>-1°34</td>
</tr>
<tr>
<td>0.3913</td>
<td>0.6428</td>
<td>0.7515</td>
<td>50.23</td>
</tr>
<tr>
<td>0.4506</td>
<td>-0.0838</td>
<td>0.7812</td>
<td>15.43</td>
</tr>
<tr>
<td>-0.3709</td>
<td>-0.0352</td>
<td>0.5176</td>
<td>-40.24</td>
</tr>
</tbody>
</table>
Table 4. Evaluation of Deflection of a Vertical

<table>
<thead>
<tr>
<th>Station Pair Clockwise</th>
<th>$\xi$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>77°75</td>
<td>30.82</td>
</tr>
<tr>
<td>2-3</td>
<td>77.83</td>
<td>30.76</td>
</tr>
<tr>
<td>3-4</td>
<td>77.68</td>
<td>30.66</td>
</tr>
<tr>
<td>4-1</td>
<td>77.60</td>
<td>30.75</td>
</tr>
</tbody>
</table>

The $\xi$ and $\eta$ values are large because an arbitrary geodetic point was used.

CONCLUSIONS

The classical method of determining deflections of the vertical compares astronomic and geodetic latitudes and longitudes from the relations:

$$\xi = \lambda - B$$

$$\eta = (\lambda - L) \cos B$$

By astronomic observations, latitude $\lambda$ and longitude $\lambda$ are determined. The corresponding geodetic values $B$ and $L$ are known from geodetic measurements.

The observational method developed in this paper with a solution for vertical deflections from astronomic and geodetic zeroith distance can be employed advantageously. By contrast with the classical method, this method does not require the knowledge of the astronomic coordinates, hence reducing the time and cost of field work.

Results from several independent sets of four stars have shown that the determination of the almucantar zenith distance is less prone to errors than for the determination of latitude and longitude. If astronomic coordinates are needed, then the geodetic values $B$ and $L$ can be used as an approximate value of latitude $\lambda$ and longitude $\lambda$. The $\xi$ and $\eta$ values must be replaced by

$$\xi = d \lambda \cos \beta$$

$$\eta = d \lambda \sec \beta$$

hence, the astronomic station coordinates shall be

$$\varphi = B + \xi$$

$$\lambda = L + \eta \sec \beta$$

The $\xi$ and $\eta$ values must be corrected for the effect of polar migration, as follows

$$d \xi = -x \cos \lambda + y \sin \lambda \sin \varphi$$

$$d \eta = -(x \sin \lambda + y \cos \lambda) \sin \varphi$$

The rectangular coordinates $X$ and $Y$ are published by the International Latitude Service.