REMARKS ON CERTAIN CRITERIA FOR DETECTION OF NUMBER OF SIGNALS

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November 1985

Technical Report No. 85-43

*This work is supported by Contract N00014-85-K-0292 of the Office of Naval Research and Contract F49620-85-C-0008 of the Air Force Office of Scientific Research. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.
ABSTRACT

In this note, we derive the asymptotic distribution of logarithm of the likelihood ratio statistic for testing the hypothesis that the number of signals is equal to q against the alternative that it is equal to k (specified) for a special case. This distribution is not chi-square. The above statistic also arises (see Wax and Kailath (1985)) in studying consistency property of MDL and AIC criteria for detection of the number of signals.
1. INTRODUCTION

In the area of signal processing, a model that is often used involves modeling the observation (complex) vector as a linear combination of the elements of the signal vector and vector of white noise. Under this model, the (unknown) number of signals is related to the multiplicity of the smallest eigenvalue of the covariance matrix of the observation vector. Recently, Wax and Kailath (1985) used Akaike's AIC criterion and Schwartz-Rissanen's MDL criterion for determination of the number of signals and stated that the AIC criterion is not consistent whereas the MDL criterion is consistent.

Parts of the proofs of the above statements are based upon the assumption that the asymptotic distribution of $-2 \log L_{tk}$ is chi-square where $L_{tk}$ denotes the likelihood ratio test statistic for testing $H_t$ against $H_k$ where $t$ and $k (k > t)$ are specified. In this note, we derive the asymptotic distribution of the above test statistic for trivariate case and point out that it is not chi-square. We will also derive analogous result when the underlying distribution is real multivariate normal.
2. SOME CRITERIA FOR DETECTION OF SIGNALS

Consider the model

\[ x(t) = A s(t) + n(t) \]  \hspace{1cm} (2.1)

where \( A = [A(s_1), \ldots, A(s_q)] \), \( s(t) = (s_1(t), \ldots, s_q(t))' \), \( s_i(t) \) is complex waveform associated with i-th signal, \( A(s_i) \) is a complex vector which depends upon the unknown vector associated with i-th signal, \( n(t) \) is noise vector. Also, \( s(t) \) and \( n(t) \) are distributed independently as complex multivariate normal with \( E(s(t)) = \mathbf{0}, E(n(t)) = \mathbf{0}, E(s(t)s(t)') = \mathbf{v} \) and \( E(n(t)n(t)') = \sigma^2 I_p \), where \( \sigma^2 \) is unknown. Here \( s' \) and \( \overline{s} \) respectively denote the transpose and conjugate of \( s \). The number of signals \( q \) is unknown. Also, \( x(t_1), \ldots, x(t_n) \) is a sample from a complex multivariate normal population with mean vector \( \mathbf{0} \) and covariance matrix \( \Sigma \). Now, let \( M_t \) denote the t-th model which states that the number of signals is t.

Wax and Kailath (1985) proposed using the AIC criterion and MDL criterion for the selection of the number of signals. According to the AIC criterion, we select the model for which

\[ \text{AIC}(t) = -2 \log L_t + v(t,p) \]  \hspace{1cm} (2.3)

is minimum where \( v(t,p) = t(2p - t) + 1 \). According to the MDL criterion, we select the model for which

\[ \text{MDL}(t) = - \log L_t + v(t,p) \log \frac{n}{2} \]  \hspace{1cm} (2.4)

is minimum. Now, let \( H_t \) denote the hypothesis that the number of signals is equal to t. The logarithm of the likelihood ratio
test statistic for testing $H_r$ against $H_k$ for specified values of $r$ and $k$ ($r < k$) is known to be $\log L_r - \log L_k$ where

$$\log L_t = -n\left(\sum_{i=t+1}^{p} \log l_i - \log\left(\sum_{i=t+1}^{p} l_i/(p - t)\right)\right). \quad (2.5)$$

Wax and Kailath (1985) pointed out that the MDL criterion is consistent whereas the AIC criterion is not consistent. In the proofs of the above statements, they have incorrectly assumed that $-2 \log L_{rk}$ is distributed asymptotically as chi-square.

Now let $\mathcal{Q}$ denote the parametric space. Under $H_t$, the supremum of the logarithm of the likelihood function, $L(\mathcal{Q})$, might be reached at a boundary point of $\mathcal{Q}$ and not in any inner point; such a point is $\lambda_1 = l_1, \ldots, \lambda_k = l_k, \lambda_{t+1} = \ldots = \lambda_p = \sum_{j=t+1}^{p} l_j/(p - t)$. So the conditions for $-2 \log L_{rk}$ to be distributed as chi-square asymptotically are not satisfied. In Section 3, we derive the asymptotic distribution of $-2 \log L_{rk}$ for the special case when $p = 3$. Analogous result is also derived when $x(t)$ is distributed as real multivariate normal with mean vector $\mathcal{Q}$ and covariance matrix $\Sigma_2$. 
3. DISTRIBUTIONS OF THE LRT STATISTICS
FOR DETECTION OF NUMBER OF SIGNALS

In this section, we will derive the asymptotic distribution of \(-2 \log L_{rk}\) when \(p = 3\), \(r = 0\) and \(k = 1\). The following definition of complex Gaussian matrix is needed in the sequel.

Let \(A = (a_{jk}) = R + iS\), where \(A\): \(p \times p\) is a Hermitian random matrix, \(R = (r_{jk})\) and \(S = (s_{jk})\). Assume \(s_{jj} = 0\), and the distinct elements of \(R\) and the upper-diagonal elements of \(S\) are independent real normal variables. Also, we assume that the variances of the off-diagonal elements of \(R\) and \(S\) are equal to 1 and the variances of the diagonal elements of \(R\) are equal to 2. Then, \(A = R + iS\) is known (see Krishnaiah (1976)) to be the central or noncentral complex Gaussian matrix accordingly as \(E(A) = 0\) or \(E(A) \neq 0\).

Now, let \(x_1, \ldots, x_n\) be distributed independently as complex trivariate normal with mean vector \(\mu\) and covariance matrix \(I_3\) where \(I_3\) is an identity matrix of order \(3 \times 3\). Also, let \(l_1 \geq l_2 \geq l_3\) be the roots of the equation.

\[
\left| \frac{1}{n} \sum_{j=1}^{n} x_j \overline{x}_j - \ell I_3 \right| = 0. \quad (3.1)
\]

Rewrite this equation as

\[
|\sqrt{n} \sum_{j=1}^{n} x_j \overline{x}_j - I_3 - \sqrt{n}(\ell - 1)I_3| = 0. \quad (3.2)
\]

By the central limit theorem, we have

\[
\mathcal{L}(\sqrt{n} \sum_{j=1}^{n} x_j \overline{x}_j - I_3) \rightarrow \mathcal{L}(A_3) \quad (3.3)
\]
as $n \to \infty$ where $A_3$ is $3 \times 3$ central complex Gaussian matrix. So the asymptotic distribution of $\sqrt{2n}(l_1-1, l_2-1, l_3-1)$ is the same as distribution of the eigenvalues $\tau_1 \geq \tau_2 \geq \tau_3$ of $A_3$. It is well known (see Wigner (1965)) that the joint density of $(\tau_1, \tau_2, \tau_3)$ is

$$h(t_1, t_2, t_3) = C \prod_{1 \leq j < m \leq 3} (t_j - t_m)^2 \exp(- \sum_{j=1}^{3} t_j^2/4),$$

where

$$C = 2^{-7\pi/2}.\) 

Since $\lim(l_1-1) = 0$ a.s., we can use Taylor's expansion and get from (2.5),

$$-2[\sup_{\theta \in \Theta_1} L(\theta) - \sup_{\theta \in \Theta_0} L(\theta)]$$

$$= [n(l_1-1)^2 + \frac{n}{2}(\sum_{j=2}^{3} (l_j-1))^2 - \frac{n}{3}(\sum_{j=1}^{3} (l_j-1))^2](1+o(1)) \text{ a.s.}$$

Here, $\Theta_k$ denotes the parameter space under $H_k$.

Write

$$W_n = -2[\sup_{\theta \in \Theta_1} L(\theta) - \sup_{\theta \in \Theta_0} L(\theta)].$$

Then

$$W_n - \frac{1}{2} \tau_1^2 + \frac{1}{4}(\tau_2+\tau_3)^2 - \frac{1}{6}(\tau_1+\tau_2+\tau_3)^2,$$

as $n \to \infty$. Now, let $\eta_1 = \frac{1}{\sqrt{2}} \tau_1, \eta_2 = \frac{1}{\sqrt{2}} \tau_2, \eta_3 = \frac{1}{\sqrt{2}} \tau_3$. Then

$$W_n - \eta_1^2 + \frac{1}{2}(\eta_2+\eta_3)^2 - \frac{1}{3}(\eta_1+\eta_2+\eta_3)^2 \overset{\Delta}{=} W,$$
where the joint density of $(\eta_1, \eta_2, \eta_3)$ is

\[ f_\eta(\eta_1, \eta_2, \eta_3) = 2^{-5/2} \pi^{-3/2} \exp\left(-\frac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2)\right) \prod_{1 \leq j < k \leq 3} (\eta_j - \eta_k)^2. \] (3.8)

If we write

\[
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{pmatrix} = \begin{pmatrix}
\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{pmatrix},
\]

then \( \eta_1 - \eta_2 = \sqrt{3} Y_1 - \sqrt{1/2} Y_2, \eta_2 - \eta_3 = \sqrt{\frac{1}{2}} Y_2, \eta_1 - \eta_3 = \sqrt{3} Y_1 + \frac{1}{\sqrt{2}} Y_2, \) and

\[
w = \left[ \frac{2}{\sqrt{6}} \eta_1 - \frac{1}{\sqrt{6}}(\eta_2 + \eta_3) \right]^2 = Y_1^2. \quad (3.9)
\]

Thus the joint density of \((Y_1, Y_2, Y_3)\) is

\[
f(y_1, y_2, y_3) = 2^{-7/2} \pi^{-3/2} (3y_1^2 - y_2^2)^2 y_2^2 \exp\left(-\frac{1}{2}(y_1^2+y_2^2+y_3^2)\right) 
\cdot \sqrt{3} y_1 > y_2 > 0, \quad -\infty < y_3 < \infty.
\] (3.10)

Hence, the density of \(Y_1\) is

\[
f_1(y_1) = 2^{-3} \pi^{-1} \left[ \sqrt{3} y_1 \right] \left(3y_1^2 - y_2^2\right)^2 y_2^2 \exp\left(-\frac{1}{2}(y_1^2+y_2^2)\right) dy_2, \quad y_1 > 0.
\] (3.11)
So the density of $\gamma_1^2$ is

$$a(u) = \frac{1}{2} \ u^{-1/2} f_1(\sqrt{u}) = 2^{-4/2} \pi^{-1/2} e^{-u/2} \int_0^{\sqrt{3u}} (3u-y^2) y^2 e^{-y^2/2} \, dy, \ u > 0. \quad (3.12)$$

Now, let

$$J_k(x) = \int_0^x y^{2k} e^{-y^2/2} \, dy, \ k = 0, 1, 2, 3.$$ 

Then we have

$$J_k(x) = -x^{2k-1} e^{-x^2/2} + 2(k-1)J_{k-1}(x) \quad (3.13)$$

for $k = 1, 2, 3$. Thus,

$$J_3(x) = (-x^5 - 5x^3 - 15x)e^{-x^2/2} + 15J_0(x),$$

$$J_2(x) = (-x^3 - 3x)e^{-x^2/2} + 3J_0(x), \quad (3.14)$$

$$J_1(x) = -xe^{-x^2/2} + J_0(x).$$

So, we have

$$a(u) = \frac{1}{16\pi} u^{-1/2} e^{-u/2} [J_3(\sqrt{3u}) - 6uJ_2(\sqrt{3u}) + 9u^2J_1(\sqrt{3u})]$$

$$= \frac{1}{16\pi} u^{-1/2} e^{-u/2} [((3u)^{3/2} - 15\sqrt{3u})e^{-3u/2}$$

$$+ (9u^2 - 18u + 15)J_0(\sqrt{3u})] \quad (3.15)$$

$$= \frac{1}{16\pi} (3\sqrt{3u} - 15\sqrt{3}) e^{-2u} + \frac{1}{16\pi} u^{-1/2} e^{-u/2} (9u^2 - 18u + 15)J_0(\sqrt{3u}),$$

$$(u > 0).$$
where
\[ J_0(\sqrt{3u}) = \int_0^\infty e^{-y^2/2} dy. \]  
(3.16)

From (3.7), (3.9) and (3.15), it follows that the asymptotic distribution of \( W_n = -2[\sup_{\theta \in \Theta_1} L(\theta) - \sup_{\theta \in \Theta_0} L(\theta)] \) is not chi-square.

In view of the above counter example, the strong consistency of the MDL criterion does not follow from the argument of Wax and Kailath (1985). But, it follows from the results given in a companion paper by Zhao, Krishnaiah and Bai (1985).

Now we suppose that \( x_1, \ldots, x_n \) are \( pxl \) i.i.d. real normal random vectors with \( E x_1 = 0 \) and \( E x_1 x_1' = \Sigma > 0 \). We point out that, the asymptotic distribution of \(-2[\sup_{\theta \in \Theta_1} L(\theta) - \sup_{\theta \in \Theta_0} L(\theta)]\) here also is not chi-square.

Now let \( R = (r_{jm}) \): \( pxp \) be a symmetric random matrix. We assume that \( \{r_{jm}; j \leq m = 1, \ldots, p\} \) are independent normal variables with means zero, \( \text{var}(r_{jj}) = 1 \) and \( \text{var}(r_{jl}) = 1/2 \) for all \( j < l \). Then \( R \) is known to be distributed as (real) central or noncentral Gaussian matrix accordingly as \( E(R) = 0 \) or not.

Let \( l_1 \geq l_2 \geq l_3 \) be the eigenvalues of \( \frac{1}{n} \sum_{j=1}^n x_j x_j' \) and \( \tau_1 \geq \tau_2 \geq \tau_3 \) be the eigenvalues of 3x3 central (real) Gaussian matrix. Then, using the same argument as before, the joint distribution of \( \sqrt{2}(l_1-1, l_2-1, l_3-1) \) tends to that of \( (\tau_1, \tau_2, \tau_3) \).

It is well known (e.g., see Anderson (1984)) that the joint density of \( (\tau_1, \tau_2, \tau_3) \) is given by
\[
\begin{align*}
h(t_1, t_2, t_3) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \sum_{j=1}^{3} t_j^2 \right) \prod_{1 \leq i < j \leq 3} (t_i-t_j) \\
& \quad (\ast > t_1 > t_2 > t_3 > -\ast).
\end{align*}
\] (3.17)

Let \( Y_1 = \frac{2}{\sqrt{6}} \tau_1 - \frac{1}{\sqrt{6}} (\tau_2 + \tau_3) \), \( Y_2 = \frac{1}{\sqrt{2}} (\tau_2 - \tau_3) \), \( Y_3 = \frac{1}{\sqrt{3}} (\tau_1 + \tau_2 + \tau_3) \).

Then the joint density of \( (Y_1, Y_2, Y_3) \) is given by

\[
f(y_1, y_2, y_3) = \frac{1}{2\pi} (3y_1^2 - y_2^2) y_2 \exp\left(-\frac{1}{2} (y_1^2 + y_2^2 + y_3^2) \right),
\] (3.18)

\[
\sqrt{3}y_1 > y_2 > 0, \quad -\ast < y_3 < \ast.
\]

Using the same argument, we have

\[
W_n \rightarrow W = y_1^2. \quad (3.19)
\]

By (3.18), the density of \( Y_1 \) is

\[
f_1(y_1) = \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{3}y_1} (3y_1^2 - y_2^2) y_2 \exp\left(-\frac{1}{2} (y_1^2 + y_2^2) \right) dy_2
\]

\[
= \frac{1}{\sqrt{2\pi}} (3y_1^2 - 2) e^{-y_1^2/2} + \frac{2}{\sqrt{2\pi}} e^{-2y_1^2}, \quad y_1 > 0.
\] (3.20)

Thus, the density of \( W \) is

\[
\frac{1}{2} u^{-1/2} f_1(\sqrt{u}) = \frac{1}{2\sqrt{2\pi}} (3u^{1/2} - 2u^{-1/2}) e^{-u/2} + \frac{1}{\sqrt{2\pi}} u^{-1/2} e^{-2u},
\] (3.21)

\[
0 < u < \ast.
\]
From (3.19) and (3.21), it follows that the asymptotic distribution of $W_n = -2[\sup_{\theta \in \Theta_1} L(\theta) - \sup_{\theta \in \Theta_0} L(\theta)]$ is not chi-square distribution.
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## Remarks on certain criteria for detection of number of signals

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**Report Date:** November 1985

**Number of Pages:** 12

**Distribution Statement:** Approved for public release; distribution unlimited

**Key Words:** Information theoretic criteria, multivariate distribution theory, signal detection.

**Abstract:**

In this note, we derive the asymptotic distribution of logarithm of the likelihood ratio statistic for testing the hypothesis that the number of signals is equal to q against the alternative that it is equal to k (specified) for a special case. The distribution is not chi-square. The above statistic also arises (see Wax and Kailath (1985)) in studying consistency property of MDL and AIC criteria for detection of the number of signals.