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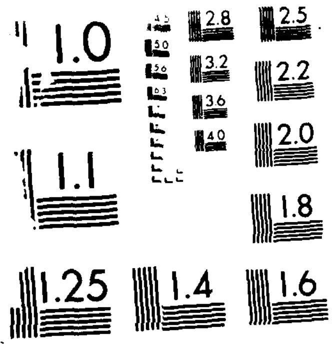

MICROCOPY RESOLUTION TEST CHART

A THEOREM ON MATCHINGS IN THE PLANE

Dedicated to the memory of Gabriel Dirac

by

Michael D. Plummer*
Vanderbilt University
Nashville, Tennessee, USA

1. Introduction and terminology

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The concept of n-extendability gradually evolved from the study of elementary bipartite graphs (which are 1-extendable) (see Hetyei (1964), Lovász and Plummer (1977)), and then of arbitrary 1-extendable (or "matching-covered") graphs by Lovász (1983). The study of n-extendability for arbitrary n was begun by the author (1980).

In this paper ~~we are~~ concerned with matchings in *planar* graphs. When we speak of an *imbedding* of planar graph G in the plane, we mean a topological imbedding in the usual sense (see White (1973)) and would remind the reader that such an imbedding is necessarily 2-*cell*. (See Youngs (1963).) If we wish to refer to a planar graph G together with an imbedding of G in the plane, we shall speak of the *plane* graph G.

The main result of this paper is to show that no planar graph is 3-extendable.

Throughout this paper, we will assume that all graphs are connected, that mindeg$(G) \geq 3$ and that mindeg$^*(G) \geq 3$, where mindeg$^*(G)$ denotes the size of a smallest face in an imbedding of G. For any additional terminology, we refer the reader to Harary (1969), to Bondy

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where the sum runs over the face angles at point v and x_i denotes the size of the ith face at v.

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PROOF. Let $p = |V(G)|$, $q = |E(G)|$ and r be the number of faces in any planar imbedding of G. Then

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by Euler's classical formula. ∎

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Since we are assuming that each $x_i \geq 3$, inequality (1) yields the following three diophantine inequalities:

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We shall need two basic results about n-extendable graphs. The proofs may be found in Plummer (1980).

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3.2. THEOREM. *If $n \geq 1$ and G is n-extendable, then G is $(n+1)$-connected.* ∎

Of course, since no planar graph can be 6-connected, this immediately tells us that no planar graph is 5-extendable. However, we now show that this result can be sharpened.

3.3. THEOREM. *No planar graph is 3-extendable.*

PROOF. Suppose G is a 3-extendable plane graph. Then by Theorem 3.2, graph G is 4-connected and hence $\operatorname{mindeg} v \geq 4$. But then by the results of Section 2, graph G must contain a control point v of degree four or five. The possible facial configurations about point v are listed in

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Section 2 and we proceed to treat each. (Note that since our graphs are, in particular, 3-connected here that the subgraph induced by the set of all points adjacent to our control point v is always a cycle.)

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Suppose there is a point $w \not\in \{u_2, u_3, u_4, u_5, v\}$, but w is adjacent to u_1. Then $\{u_1w, u_2u_3, u_4u_5\}$ is a matching of size three which cannot extend to a perfect matching, a contradiction. So the neighborhood of u_1, $N(u_1) \subseteq \{u_2, u_3, u_4, u_5, v\}$. We know that $\{u_2, v, u_5\} \subseteq N(u_1)$, but since G is 4-connected, we have that $\deg u_1 \geq 4$, and so u_1 is adjacent to at least one of u_3 and u_4. Suppose u_1 is adjacent to u_3. Then $\deg u_2 = 3$, a contradiction.

By symmetry, a similar contradiction is reached if u_1 is adjacent to u_4. ∎

Concluding remarks

In the decomposition theory of graphs with perfect matchings (see Lovász and Plummer (1985)), two important classes of "building blocks" are (1) 1-extendable *bipartite* graphs and (2) *bicritical* graphs. A graph G is **bicritical** if $G-u-v$ has a perfect matching for all choices of distinct points u and v. There is a nice relationship among 2-extendable graphs, 1-extendable bipartite graphs and bicritical graphs. In particular, we have the following result. For the proof, see Plummer (1980).

FIGURE 4.1. A 3-extendable toroidal graph

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(Note that no bicritical graph can be bipartite, so the two classes in the conclusion of the preceding theorem are disjoint.)

Bicritical graphs — especially those which are 3-connected — are still not completely understood. Thus in light of Theorem 4.1 the study of graphs which are n-extendable, for $n \geq 2$, may help us to better understand the structure of 3-connected bicritical graphs, as well as being of interest in its own right.

The present paper is concerned with the planar case. Although we now know that no planar graph is 3-extendable, there are many such graphs which are 2-extendable. The dodecahedron, the icosahedron and the cube are but three familiar examples. We shall present a more detailed study of 2-extendable planar graphs in a subsequent paper.

Let us conclude by noting that there do exist 3-extendable graphs which can be imbedded on the surface of the torus. The Cartesian products of two even cycles $C_{2m} \times C_{2n}, (m, n \geq 2)$ are such graphs. See Figure 4.1 for an imbedding of $C_4 \times C_4$.

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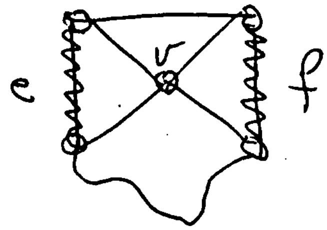
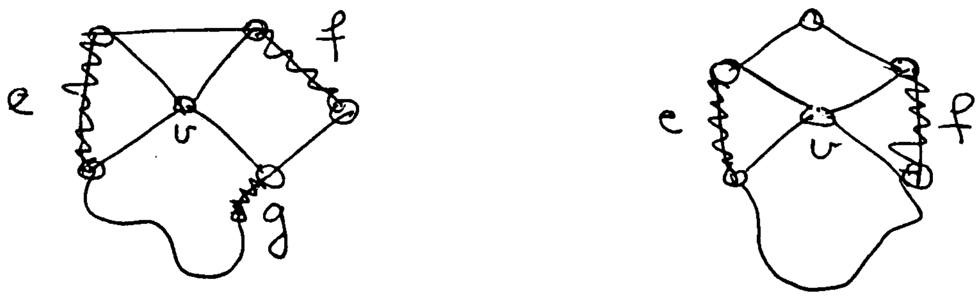


Figure 3.1.



(a) (b)

Figure 3.2.

(a) (b)

Figure 3.3.

Figure 3.4.

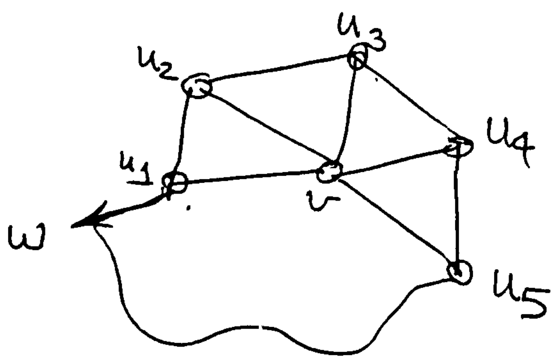


Figure 3.5

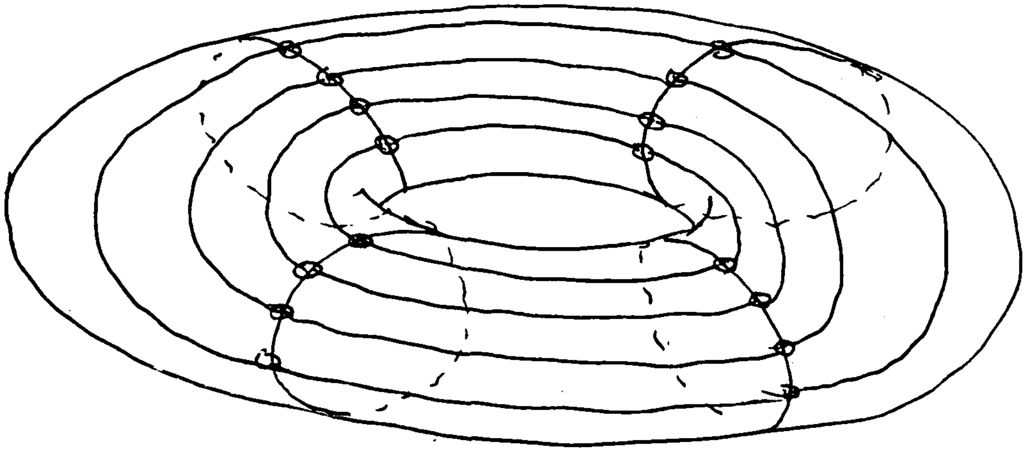


Figure 4.1.

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